Turkish Journal of Mathematics

Volume 48 | Number 3 Article 6

5-1-2024

Approximation by operators Involving Δh -Gould-Hopper Appell polynomials

BİLGE ZEHRA SERGİ YILMAZ
bilgezsergi@gazi.edu.tr

GÜRHAN İÇÖZ
gurhanicoz@gazi.edu.tr

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2024) 48: 423 – 447
© TÜBİTAK
doi:10.55730/1300-0098.3517

Approximation by operators Involving Δ_h-Gould-Hopper Appell polynomials

Bilge Zehra SERGİ YILMAZ[1,*], **Gürhan İÇÖZ**[2]

[1]Graduate S33chool of Natural and Applied Science, Gazi University, Ankara, Turkiye

[2]Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkiye

Received: 05.05.2023 • **Accepted/Published Online:** 14.02.2024 • **Final Version:** 10.05.2024

Abstract: The present paper deals with the approximation properties of the linear positive operators, including Δ_h-Gould-Hopper Appell polynomials. We investigate some theorems for convergence of the operators and their approximation degrees with the help of the classical approach, Peetre's $\mathcal{K}$-functional, Lipschitz class and Voronovskaja-type theorem. In the last section of the paper, we introduce the Kantorovich form of the operators and examine the approximation degree.

Key words: Szász operators, Appell polynomials, Voronovskaja-type theorem, Kantorovich operators

1. Introduction

The approximation theory has an extensive using area in mathematical physics. For this subject, firstly, Weierstrass [16] considered the approximation on the continuous functions problems in 1885. He expressed that every function on the closed interval $[a,b]$ is approachable by means of the polynomials. This work has been an inspiration to most of mathematicians for studying on polynomials. Later, Szász [13] introduced the linear positive operators as

$$S_m\left(z;x\right)=e^{-mx}\sum_{u=0}^{\infty}\frac{\left(mx\right)^u}{u!}z\left(\frac{u}{m}\right) \tag{1.1}$$

for each $m\in\mathbb{N}$. Here $x\in[0,\infty)$, $z\in C\left[0,\infty\right)$ and $C\left[0,\infty\right)$ is the space of continuous functions. In the light of these studies, Jakimovski and Leviatan [9] found a generalization of the operators (1.1) with the help of the Appell polynomials. Let

$$D(v)=\sum_{u=0}^{\infty}a_uv^u, D(1)\neq 0, \tag{1.2}$$

$v\in\mathbb{R}$ (or $v\in\mathbb{C}$) and $|v|<R$ $(R>1)$. From (1.2), the generating function of the Appell polynomials as follows

$$D(v)e^{vx}=\sum_{u=0}^{\infty}\mathcal{P}_u(x)v^u. \tag{1.3}$$

*Correspondence: bilgezsergi@gazi.edu.tr

2010 *AMS Mathematics Subject Classification:* 41A25, 41A36, 33C45

In addition, by taking into consideration (1.2) and (1.3), the formal series of Appell polynomials are defined as

$$\mathcal{P}_m(x) = \sum_{u=0}^{\infty} a_m \frac{x^{m-u}}{(m-u)!} \tag{1.4}$$

(see [3]). With the help of the above expressions, Jakimovski and Leviatan obtained the operators

$$K_m\left(z;x\right) = \frac{e^{-mx}}{D(1)} \sum_{u=0}^{\infty} \mathcal{P}_u(mx) z\left(\frac{u}{m}\right). \tag{1.5}$$

Here, $z \in T$ and $T := \left\{z \in C[0,\infty) : |z(x)| \leq ae^{\lambda x} \text{ for constant } a > 0, \lambda > 0\right\}$. Then, Ismail [7] considered a new operator via Sheffer polynomials as follows

$$Q_m(z;x) = \frac{e^{-mxS(1)}}{D(1)} \sum_{u=0}^{\infty} \mathcal{E}_u(mx) z\left(\frac{u}{m}\right), m \in \mathbb{N} \tag{1.6}$$

and Sheffer polynomials have the generating functions as

$$D(t)e^{xS(t)} = \sum_{u=0}^{\infty} \mathcal{E}_u(x)t^u, |t| < R. \tag{1.7}$$

In here, $S(t) = \sum_{u=1}^{\infty} g_u t^u (g_1 \neq 0)$ is analytic function. These operators (1.6) are more general than the operators (1.1) and (1.5).

Remark 1.1 *If we take $S(t) = t$ in (1.6), then we achieve (1.5). Moreover, for $S(t) = t$ and $D(t) = 1$, we get the operator (1.1).*

Later, Varma et al. [14] introduced a different operator as follows

$$L_m(z;x) = \frac{1}{D(1)B(mx)} \sum_{u=0}^{\infty} \mathcal{J}_u(mx) z\left(\frac{u}{m}\right) \tag{1.8}$$

for each $m \in \mathbb{N}$. Here, $x \geq 0$, $B(t)$ and $D(t)$ are the analytic functions and $\mathcal{J}_u(x)$ is Brenke type polynomials which have the generating functions as

$$D(t)B(xt) = \sum_{u=0}^{\infty} \mathcal{J}_u(x)t^u. \tag{1.9}$$

Remark 1.2 *If one writes $B(t) = e^t$ in (1.8), then this operator is reduced to (1.5). Additively, if one take $B(t) = e^t$ and $D(t) = 1$, then the Szàsz operators (1.1) is obtained.*

Besides, İçöz et al. [8] presented the linear positive operators by means of the generalized Appell polynomials $\mathcal{P}_u(x)$. These polynomials have the generating function

$$D(r(t))E(xr(t)) = \sum_{u=0}^{\infty} \mathcal{P}_u(x)t^u \tag{1.10}$$

and this operator as follows

$$\mathcal{M}_m(z;x) = \frac{1}{D(r(1))E(mxr(1))} \sum_{u=0}^{\infty} \mathcal{P}_u(mx) z\left(\frac{u}{m}\right). \tag{1.11}$$

Remark 1.3 *For $r(t) = t$ and $E(t) = e^t$, the operators (1.11) reduce to (1.5). In addition, if one take $D(t) = 1$, then we get the Szàsz operators in (1.1).*

Then, Özarslan and Yaşar [11] introduced the Δ_h-Gould-Hopper Appell polynomials by the equality

$$D(t;h)(1+ht)^{\frac{x}{h}}(1+ht^2)^{\frac{y}{h}} = \sum_{u=0}^{\infty} \mathcal{A}_u(x,y;h)\frac{t^u}{u!}, \tag{1.12}$$

where $D(t;h)$ is an analytic function with the formal series as follows

$$D(t;h) = \sum_{u=0}^{\infty} \sigma_{u,h}\frac{t^u}{u!}, \ \ \sigma_{0,h} \neq 0. \tag{1.13}$$

For example from [11], when they take $D(t;h)$ as

$$D(t;h) = \frac{2t}{(1+ht)^{\frac{1}{h}}+1} = \sum_{u=0}^{\infty} \mathcal{G}_{u,h}\frac{t^u}{u!}$$

instead of $D(t;h)$ in (1.13), then they find the degenerate Genocchi numbers $\mathcal{G}_{u,h} := \mathcal{G}_u(0,0;h)$ which is obtained from the degenerate Gould-Hopper Genocchi polynomial sequence $\{\mathcal{G}_u(x,y;h)\}_{u\in\mathbb{N}}$. Here, the degenerate Gould-Hopper Genocchi polynomial has the generating function as follows

$$\frac{2t}{(1+ht)^{\frac{1}{h}}+1}(1+ht)^{\frac{x}{h}}(1+ht^2)^{\frac{y}{h}} = \sum_{u=0}^{\infty} \mathcal{G}_u(x,y;h)\frac{t^u}{u!}.$$

For the another example from [11], when they take $D(t;h)$ as

$$D(t;h) = \frac{ht}{\ln(1+ht)} = \sum_{u=0}^{\infty} \mathcal{B}^{II}_{u,h}\frac{t^u}{u!}$$

instead of $D(t;h)$ in (1.13), then they find the Δ_h-Bernoulli numbers of the second kind $\mathcal{B}^{II}_{u,h} := \mathcal{B}^{II}_u(0,0;h)$ which is obtained from the Δ_h-Gould-Hopper Bernoulli polynomial sequence of the second kind $\left\{\mathcal{B}^{II}_u(x,y;h)\right\}_{u\in\mathbb{N}}$. Here, the Δ_h-Gould-Hopper Bernoulli polynomial has the generating function as follows

$$\frac{ht}{\ln(1+ht)}(1+ht)^{\frac{x}{h}}(1+ht^2)^{\frac{y}{h}} = \sum_{u=0}^{\infty} \mathcal{B}^{II}_u(x,y;h)\frac{t^u}{u!}.$$

Besides, Biricik et al. [4] obtained the sequences of twice-iterated Δ_h-Gould-Hopper Appell polynomials as follows

$$D_2(t;h)D_1(t;h)(1+ht)^{\frac{x}{h}}(1+ht^2)^{\frac{y}{t}} = \sum_{u=0}^{\infty}\mathcal{K}_u^{[2]}\frac{t^u}{u!}$$

where $D_1(t;h)$ and $D_2(t;h)$ are defined by (1.13).

After these studies, Sergi et al. [12] presented the linear positive operators

$$T_m(z;x) = \frac{1}{D(1;h)(1+h)^{\frac{mx}{h}}}\sum_{u=0}^{\infty}\mathcal{A}_u(mx;h)\frac{1}{u!}z\left(\frac{u(1+h)}{m}\right) \tag{1.14}$$

where $h\in(-1,0)$, $|t|<R$ $(R>1)$ and

$$D(t;h)(1+ht)^{\frac{x}{h}} = \sum_{u=0}^{\infty}\mathcal{A}_u(x;h)\frac{t^u}{u!} \tag{1.15}$$

is generating function of the degenerate Appell polynomials (see [5]). In addition, $D(1;h)$ has limit by this means $\lim_{h\to 0^-} D(1;h) = D^*(1)$ and D^* is analytic function with $D^*(t)=\sum_{u=0}^{\infty}a_u t^u$, $D^*(1)\neq 0$.

Remark 1.4 *If we take limit of the operators (1.14) as $h\to 0^-$, then we get (1.5) and if we take $D(t;h)=1$, then we obtain the Szász operators in (1.1).*

Lemma 1.5 *[12] For $e_j=t^j$, $j=0,1,2$, the operators T_m have the following equalities*

$$\begin{aligned}
T_m(e_0;x) &= 1,\\
T_m(e_1;x) &= x+\frac{D'(1;h)}{mD(1;h)}(1+h),\\
T_m(e_2;x) &= x^2+\frac{x}{m}\left(\frac{2D'(1;h)}{D(1;h)}(1+h)+1\right)+\frac{1}{m^2}\left(\frac{D''(1;h)+D'(1;h)}{D(1;h)}(1+h)^2\right).
\end{aligned}$$

Lemma 1.6 *[12] For the operators T_m in (1.14) and $x\in[0,\infty)$, the following equalities are satisfied*

$$T_m(t-x;x) = \frac{(1+h)D'(1;h)}{mD(1;h)}, \tag{1.16}$$

$$T_m((t-x)^2;x) = \frac{x}{m}+\frac{(1+h)^2}{m^2}\left(\frac{D''(1;h)+D'(1;h)}{D(1;h)}\right). \tag{1.17}$$

By taking into consideration the above studies, we can define a generalization of the linear operators containing Δ_h-Gould-Hopper Appell polynomials as

$$T_{m,n}(z;x,y) := \frac{1}{[D(1;h)]^2(1+h)^{\frac{(m+1)x+(n+1)y}{h}}}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{\mathcal{A}_u(mx,y;h)}{u!}\frac{\mathcal{A}_v(x,ny;h)}{v!}z\left(\frac{u(1+h)}{m},\frac{v(1+h)}{2n}\right) \tag{1.18}$$

for $I=[0,\infty), I^2=I\times I,\ z\in C\left(I^2\right)$ and $m,n\in\mathbb{N}$. Here, the space $C\left(I^2\right)$ is the space of all real-valued continuous functions on I^2. Under the following conditions, we can say that the operator $T_{m,n}$ is a linear and positive operator:

$$\begin{array}{ll} (i) & D(1;h)\neq 0, h\in(-1,0), \\ (ii) & \text{For all } u\in\mathbb{N},\ \sigma_{u,h}/D(1;h)\geq 0, \\ (iii) & \text{The power series at (1.12) and (1.13) are convergent in } |t|<R(R>1). \end{array} \tag{1.19}$$

In this paper, we obtain the approximation properties of the operators $T_{m,n}$. In Section 2, we consider the order of approximation with the help of some lemmas and theorems. Later, in Section 3, we will express the Kantorovich variant of the operators $T_{m,n}$, its approximation properties and approximation degree.

2. Approximation of $T_{m,n}$ operators

In this section, we will introduce some theorems for the degree of approximation of the operators $T_{m,n}$ by means of complete modulus of continuity, partial modulus of continuity, Peetre's $\mathcal{K}$-functional and Lipschitz class functions. Then, we will give the Voronovskaja-type theorem for the operators $T_{m,n}$.

Before considering the proof of theorems for the operators (1.18), it is necessary to mention the following lemmas.

Lemma 2.1 *The following equalities are satisfied*

$$\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(mx,y;h)}{u!} = D(1;h)(1+h)^{\frac{mx+y}{h}}, \tag{2.1}$$

$$\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(x,ny;h)}{u!} = D(1;h)(1+h)^{\frac{x+ny}{h}}, \tag{2.2}$$

$$\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(mx,y;h)}{u!}u = (mx+2y)D(1;h)(1+h)^{\frac{mx+y}{h}-1}+D'(1;h)(1+h)^{\frac{mx+y}{h}}, \tag{2.3}$$

$$\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(x,ny;h)}{u!}u = (x+2ny)D(1;h)(1+h)^{\frac{x+ny}{h}-1}+D'(1;h)(1+h)^{\frac{x+ny}{h}}, \tag{2.4}$$

$$\begin{aligned}\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(mx,y;h)}{u!}u^2 &= (mx)^2D(1;h)(1+h)^{\frac{mx+y}{h}-2}+mx\Big[\left(2D'(1;h)+D(1;h)\right)(1+h)^{\frac{mx+y}{h}-1} \\ &\quad +\left(4y-h\right)D(1;h)(1+h)^{\frac{mx+y}{h}-2}\Big]+4y\left(y-h\right)D(1;h)(1+h)^{\frac{mx+y}{h}-2} \\ &\quad +4y\left(D'(1;h)+D(1;h)\right)(1+h)^{\frac{mx+y}{h}-1}+\left(D''(1;h)+D'(1;h)\right)(1+h)^{\frac{mx+y}{h}},\end{aligned} \tag{2.5}$$

$$\begin{aligned}\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(x,ny;h)}{u!}u^2 &= (2ny)^2D(1;h)(1+h)^{\frac{x+ny}{h}-2}+4ny\Big[\left(D'(1;h)+D(1;h)\right)(1+h)^{\frac{x+ny}{h}-1} \\ &\quad +\left(x-h\right)D(1;h)(1+h)^{\frac{x+ny}{h}-2}\Big]+x\left(x-h\right)D(1;h)(1+h)^{\frac{x+ny}{h}-2} \\ &\quad +x\left(2D'(1;h)+D(1;h)\right)(1+h)^{\frac{x+ny}{h}-1}+\left(D''(1;h)+D'(1;h)\right)(1+h)^{\frac{x+ny}{h}},\end{aligned} \tag{2.6}$$

$$\begin{aligned}
\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(mx,y;h)}{u!}u^3 &= (mx)^3D(1;h)(1+h)^{\frac{mx+y}{h}-3}+(mx)^2\Big[3(D'(1;h)+D(1;h))(1+h)^{\frac{mx+y}{h}-2}\\
&-3hD(1;h)(1+h)^{\frac{mx+y}{h}-3}\Big]\\
&+mx\Big[\left(3D''(1;h)+6D'(1;h)+D(1;h)\right)(1+h)^{\frac{mx+y}{h}-1}\\
&-3h\left(D'(1;h)+D(1;h)\right)(1+h)^{\frac{mx+y}{h}-2}+2h^2D(1;h)(1+h)^{\frac{mx+y}{h}-3}\Big]\\
&+(2y)^3D(1;h)(1+h)^{\frac{mx+y}{h}-3}\\
&+(2y)^2\Big[3\left(D'(1;h)+2D(1;h)\right)(1+h)^{\frac{mx+y}{h}-2}-6hD(1;h)(1+h)^{\frac{mx+y}{h}-3}\Big]\\
&+2y\Big[\left(3D''(1;h)+9D'(1;h)+4D(1;h)\right)(1+h)^{\frac{mx+y}{h}-1}\\
&-6h\left(D'(1;h)+2D(1;h)\right)(1+h)^{\frac{mx+y}{h}-2}+8h^2D(1;h)(1+h)^{\frac{mx+y}{h}-3}\Big]\\
&+2mxy\Big[\left(6D'(1;h)+9D(1;h)\right)(1+h)^{\frac{mx+y}{h}-2}\\
&+3\left(mx+2y-3h\right)D(1;h)(1+h)^{\frac{mx+y}{h}-3}\Big]\\
&+\left(D'''(1;h)+3D''(1;h)+D'(1;h)\right)(1+h)^{\frac{mx+y}{h}}, \qquad (2.7)
\end{aligned}$$

$$\begin{aligned}
\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(x,ny;h)}{u!}u^3 &= (2ny)^3D(1;h)(1+h)^{\frac{x+ny}{h}-3}\\
&+(2ny)^2\Big[3\left(D'(1;h)+2D(1;h)\right)(1+h)^{\frac{x+ny}{h}-2}\\
&-6hD(1;h)(1+h)^{\frac{x+ny}{h}-3}\Big]\\
&+2ny\Big[\left(3D''(1;h)+9D'(1;h)+4D(1;h)\right)(1+h)^{\frac{x+ny}{h}-1}\\
&-6h\left(D'(1;h)+2D(1;h)\right)(1+h)^{\frac{x+ny}{h}-2}+8h^2D(1;h)(1+h)^{\frac{x+ny}{h}-3}\Big]\\
&+x^3D(1;h)(1+h)^{\frac{x+ny}{h}-3}+x^2\Big[3(D'(1;h)+D(1;h))(1+h)^{\frac{x+ny}{h}-2}\\
&-3hD(1;h)(1+h)^{\frac{x+ny}{h}-3}\Big]\\
&+x\Big[\left(3D''(1;h)+6D'(1;h)+D(1;h)\right)(1+h)^{\frac{x+ny}{h}-1}\\
&-3h\left(D'(1;h)+D(1;h)\right)(1+h)^{\frac{x+ny}{h}-2}+2h^2D(1;h)(1+h)^{\frac{x+ny}{h}-3}\Big]\\
&+2xny\Big[\left(6D'(1;h)+9D(1;h)\right)(1+h)^{\frac{x+ny}{h}-2}\\
&+3\left(x+2ny-3h\right)D(1;h)(1+h)^{\frac{x+ny}{h}-3}\Big]\\
&+\left(D'''(1;h)+3D''(1;h)+D'(1;h)\right)(1+h)^{\frac{x+ny}{h}}, \qquad (2.8)
\end{aligned}$$

$$\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(mx,y;h)}{u!}u^4 = (mx)^4D(1;h)(1+h)^{\frac{mx+y}{h}-4}+(mx)^3\Big[\left(4D'(1;h)+6D(1;h)\right)(1+h)^{\frac{mx+y}{h}-3}$$
$$-6hD(1;h)(1+h)^{\frac{mx+y}{h}-4}\Big]$$
$$+(mx)^2\Big[\left(6D''(1;h)+18D'(1;h)+7D(1;h)\right)(1+h)^{\frac{mx+y}{h}-2}$$
$$-6h\left(2D'(1;h)+3D(1;h)\right)(1+h)^{\frac{mx+y}{h}-3}+11h^2D(1;h)(1+h)^{\frac{mx+y}{h}-4}\Big]$$
$$+mx\Big[\left(4D'''(1;h)+18D''(1;h)+14D'(1;h)+D(1;h)\right)(1+h)^{\frac{mx+y}{h}-1}$$
$$-h\left(6D''(1;h)+18D'(1;h)+7D(1;h)\right)(1+h)^{\frac{mx+y}{h}-2}$$
$$+4h^2\left(2D'(1;h)+3D(1;h)\right)(1+h)^{\frac{mx+y}{h}-3}-6h^3D(1;h)(1+h)^{\frac{mx+y}{h}-4}\Big]$$
$$+(2y)^4D(1;h)(1+h)^{\frac{mx+y}{h}-4}$$
$$+(2y)^3\left[4\left(D'(1;h)+3D(1;h)(1+h)^{\frac{mx+y}{h}-3}-3hD(1;h)(1+h)^{\frac{mx+y}{h}-4}\right)\right]$$
$$+(2y)^2\Big[\left(6D''(1;h)+30D'(1;h)+28D(1;h)\right)(1+h)^{\frac{mx+y}{h}-2}$$
$$-12h\left(2D'(1;h)+6D(1;h)\right)(1+h)^{\frac{mx+y}{h}-3}+44h^2D(1;h)(1+h)^{\frac{mx+y}{h}-4}\Big]$$
$$+2y\Big[\left(4D'''(1;h)+24D''(1;h)+43D'(1;h)+8D(1;h)\right)(1+h)^{\frac{mx+y}{h}-1}$$
$$-2h\left(6D''(1;h)+30D'(1;h)+28D(1;h)\right)(1+h)^{\frac{mx+y}{h}-2}$$
$$+16h^2\left(2D'(1;h)+6D(1;h)\right)(1+h)^{\frac{mx+y}{h}-3}-48h^3D(1;h)(1+h)^{\frac{mx+y}{h}-4}\Big]$$
$$+2mxy\Big[\left(12D''(1;h)+48D'(1;h)+32D(1;h)\right)(1+h)^{\frac{mx+y}{h}-2}$$
$$-12h\left(3D'(1;h)+7D(1;h)\right)(1+h)^{\frac{mx+y}{h}-3}+52h^2D(1;h)(1+h)^{\frac{mx+y}{h}-4}$$
$$+6mx(2D'(1;h)+4D(1;h))(1+h)^{\frac{mx+y}{h}-3}-4hD(1;h)(1+h)^{\frac{mx+y}{h}-4}$$
$$+4y\left(6D'(1;h)+15D(1;h)(1+h)^{\frac{mx+y}{h}-3}-15hD(1;h)(1+h)^{\frac{mx+y}{h}-4}\right)$$
$$+4\left(3mxy+(mx)^2+4y^2\right)D(1;h)(1+h)^{\frac{mx+y}{h}-4}\Big]$$
$$+\left(D^{(4)}(1;h)+6D'''(1;h)+7D''(1;h)+D'(1;h)\right)(1+h)^{\frac{mx+y}{h}} \qquad (2.9)$$

and

$$\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(x,ny;h)}{u!}u^4 = (2ny)^4D(1;h)(1+h)^{\frac{x+ny}{h}-4}$$
$$+(2ny)^3\left[4\left(D'(1;h)+3D(1;h)(1+h)^{\frac{x+ny}{h}-3}-3hD(1;h)(1+h)^{\frac{x+ny}{h}-4}\right)\right]$$

$$+(2ny)^2\Big[\left(6D''(1;h)+30D'(1;h)+28D(1;h)\right)(1+h)^{\frac{x+ny}{h}-2}$$

$$-12h\left(2D'(1;h)+6D(1;h)\right)(1+h)^{\frac{x+ny}{h}-3}+44h^2D(1;h)(1+h)^{\frac{x+ny}{h}-4}\Big]$$

$$+2ny\Big[\left(4D'''(1;h)+24D''(1;h)+43D'(1;h)+8D(1;h)\right)(1+h)^{\frac{x+ny}{h}-1}$$

$$-2h\left(6D''(1;h)+30D'(1;h)+28D(1;h)\right)(1+h)^{\frac{x+ny}{h}-2}$$

$$+16h^2\left(2D'(1;h)+6D(1;h)\right)(1+h)^{\frac{x+ny}{h}-3}-48h^3D(1;h)(1+h)^{\frac{x+ny}{h}-4}\Big]$$

$$+x^4D(1;h)(1+h)^{\frac{x+ny}{h}-4}+x^3\Big[\left(4D'(1;h)+6D(1;h)\right)(1+h)^{\frac{x+ny}{h}-3}$$

$$-6hD(1;h)(1+h)^{\frac{x+ny}{h}-4}\Big]$$

$$+x^2\Big[\left(6D''(1;h)+18D'(1;h)+7D(1;h)\right)(1+h)^{\frac{x+ny}{h}-2}$$

$$-6h\left(2D'(1;h)+3D(1;h)\right)(1+h)^{\frac{x+ny}{h}-3}+11h^2D(1;h)(1+h)^{\frac{x+ny}{h}-4}\Big]$$

$$+x\Big[\left(4D'''(1;h)+18D''(1;h)+14D'(1;h)+D(1;h)\right)(1+h)^{\frac{x+ny}{h}-1}$$

$$-h\left(6D''(1;h)+18D'(1;h)+7D(1;h)\right)(1+h)^{\frac{x+ny}{h}-2}$$

$$+4h^2\left(2D'(1;h)+3D(1;h)\right)(1+h)^{\frac{x+ny}{h}-3}-6h^3D(1;h)(1+h)^{\frac{x+ny}{h}-4}\Big]$$

$$+2xny\Big[\left(12D''(1;h)+48D'(1;h)+32D(1;h)\right)(1+h)^{\frac{x+ny}{h}-2}$$

$$-12h\left(3D'(1;h)+7D(1;h)\right)(1+h)^{\frac{x+ny}{h}-3}+52h^2D(1;h)(1+h)^{\frac{x+ny}{h}-4}$$

$$+6x(2D'(1;h)+4D(1;h))(1+h)^{\frac{x+ny}{h}-3}-4hD(1;h)(1+h)^{\frac{x+ny}{h}-4}$$

$$+4ny\Big(6D'(1;h)+15D(1;h)(1+h)^{\frac{x+ny}{h}-3}-15hD(1;h)(1+h)^{\frac{x+ny}{h}-4}\Big)$$

$$+4\left(3xny+x^2+4(ny)^2\right)D(1;h)(1+h)^{\frac{x+ny}{h}-4}\Big]$$

$$+\Big(D^{(4)}(1;h)+6D'''(1;h)+7D''(1;h)+D'(1;h)\Big)(1+h)^{\frac{x+ny}{h}}. \tag{2.10}$$

Proof If we take $t=1$ and write mx instead of x in (1.12), we can find

$$D(1;h)(1+h)^{\frac{mx+y}{h}}=\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(mx,y;h)}{u!}.$$

Similarly, if we take $t=1$ and write ny instead of y in (1.12), we can obtain

$$D(1;h)(1+h)^{\frac{x+ny}{h}}=\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(x,ny;h)}{u!}$$

which are corresponding to (2.1) and (2.2) respectively.

By taking the first-order derivative with respect to t for the generating function (1.12), we have

$$D'(t;h)(1+ht)^{\frac{x}{h}}(1+ht^2)^{\frac{y}{h}}+xD(t;h)(1+ht)^{\frac{x}{h}-1}(1+ht^2)^{\frac{y}{h}}+2tyD(t;h)(1+ht)^{\frac{x}{h}}(1+ht^2)^{\frac{y}{h}-1}$$
$$=\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(x,y;h)}{u!}ut^{u-1}.$$

Then, if we take $t=1$ and write mx instead of x in the above equality, we get (2.3) that

$$D'(1;h)(1+h)^{\frac{mx+y}{h}}+mxD(1;h)(1+h)^{\frac{mx+y}{h}-1}+2yD(1;h)(1+h)^{\frac{mx+y}{h}-1}$$
$$=\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(mx,y;h)}{u!}u. \tag{2.11}$$

Besides, if we take $t=1$ and write ny instead of y, then we obtain (2.4) that

$$D'(1;h)(1+h)^{\frac{x+ny}{h}}+xD(1;h)(1+h)^{\frac{x+ny}{h}-1}+2nyD(1;h)(1+h)^{\frac{x+ny}{h}-1}$$
$$=\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(x,ny;h)}{u!}u. \tag{2.12}$$

By taking the second-order derivative with respect to t for the generating function (1.12), we obtain

$$D''(t;h)(1+ht)^{\frac{x}{h}}(1+ht^2)^{\frac{y}{h}}+xD'(t;h)(1+ht)^{\frac{x}{h}-1}(1+ht^2)^{\frac{y}{h}}+2tyD'(t;h)(1+ht)^{\frac{x}{h}}(1+ht^2)^{\frac{y}{h}-1}$$
$$+xD'(t;h)(1+ht)^{\frac{x}{h}-1}(1+ht^2)^{\frac{y}{h}}+xh\left(\frac{x}{h}-1\right)D(t;h)(1+ht)^{\frac{x}{h}-2}(1+ht^2)^{\frac{y}{h}}$$
$$+2xtyD(t;h)(1+ht)^{\frac{x}{h}-1}(1+ht^2)^{\frac{y}{h}-1}+2yD(t;h)(1+ht)^{\frac{x}{h}}(1+ht^2)^{\frac{y}{h}-1}$$
$$+2tyD'(t;h)(1+ht)^{\frac{x}{h}}(1+ht^2)^{\frac{y}{h}-1}+2xtyD(t;h)(1+ht)^{\frac{x}{h}-1}(1+ht^2)^{\frac{y}{h}-1}$$
$$+4t^2hy\left(\frac{y}{h}-1\right)D(t;h)(1+ht)^{\frac{x}{h}}(1+ht^2)^{\frac{y}{h}-2}=\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(x,y;h)}{u!}u(u-1)t^{u-2}.$$

If we take $t=1$, write mx instead of x and considering (2.11) in the last equality, then we find (2.5) as follows

$$\left(D''(1;h)+D'(1;h)\right)(1+h)^{\frac{mx+y}{h}}+4y\left(D'(1;h)+D(1;h)\right)(1+h)^{\frac{mx+y}{h}-1}$$
$$+4yh\left(\frac{y}{h}-1\right)D(1;h)(1+h)^{\frac{mx+y}{h}-2}+(mx)^2D(1;h)(1+h)^{\frac{mx+y}{h}-2}$$
$$+mx\left(\left(2D'(1;h)+D(1;h)\right)(1+h)^{\frac{mx+y}{h}-1}+\left(4y-h\right)D(1;h)(1+h)^{\frac{mx+y}{h}-2}\right)=\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(mx,y;h)}{u!}u^2.$$

Later, if we take $t=1$, write ny instead of y and considering (2.12) in the last equality, then we find (2.6) as follows

$$\left(D''(1;h)+D'(1;h)\right)(1+h)^{\frac{x+ny}{h}}+x\left(2D'(1;h)+D(1;h)D(1;h)\right)(1+h)^{\frac{x+ny}{h}-1}$$
$$+x\left(\frac{x}{h}-1\right)D(1;h)(1+h)^{\frac{x+ny}{h}-2}+(2ny)^2D(1;h)(1+h)^{\frac{x+ny}{h}-2}$$
$$+4ny\left(\left(D'(1;h)+D(1;h)\right)(1+h)^{\frac{x+ny}{h}-1}+\left(x-h\right)D(1;h)(1+h)^{\frac{x+ny}{h}-2}\right)=\sum_{u=0}^{\infty}\frac{\mathcal{A}_u(x,ny;h)}{u!}u^2.$$

Similarly to this process, we can see that (2.7)–(2.10) is obtained.

Lemma 2.2 *Let* $T_{m,n}(e_{i,j};x,y), e_{i,j}=t^i s^j$ $(i,j=0,1,2,3,4)$ *and* $x,y\in[0,\infty)$. *Then, we obtain*

$$\begin{aligned}
T_{m,n}(e_{0,0};x,y) &= 1,\\
T_{m,n}(e_{1,0};x,y) &= x+\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right),\\
T_{m,n}(e_{0,1};x,y) &= y+\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right),\\
T_{m,n}(e_{2,0};x,y) &= x^2+\frac{x}{m}\left(\frac{2D'(1;h)}{D(1;h)}(1+h)+1+4y\right)\\
&\quad+\frac{1}{m^2}\left(\frac{D''(1;h)+D'(1;h)}{D(1;h)}(1+h)^2+\frac{4yD'(1;h)}{D(1;h)}(1+h)+4y(y+1)\right),\\
T_{m,n}(e_{0,2};x,y) &= y^2+\frac{y}{n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+1+x\right)\\
&\quad+\frac{1}{4n^2}\left(\frac{D''(1;h)+D'(1;h)}{D(1;h)}(1+h)^2+\frac{2xD'(1;h)}{D(1;h)}(1+h)+x(x+1)\right),\\
T_{m,n}(e_{3,0};x,y) &= x^3+\frac{3x^2}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+1\right)\\
&\quad+\frac{x}{m^2}\left(\frac{3\left(D''(1;h)+2D'(1;h)\right)}{D(1;h)}(1+h)^2-\frac{3hD'(1;h)}{D(1;h)}(1+h)+1-h\right)\\
&\quad+\frac{8y^3}{m^3}+\frac{12y^2}{m^3}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2\right)+\frac{2y}{m^3}\left(\frac{3\left(D''(1;h)+3D'(1;h)\right)}{D(1;h)}(1+h)^2\right.\\
&\quad\left.-\frac{6hD'(1;h)}{D(1;h)}(1+h)+4(1-h)\right)\\
&\quad+\frac{1}{m^3}\left(\frac{D'''(1;h)+3D''(1;h)+D'(1;h)}{D(1;h)}(1+h)^3\right)\\
&\quad+\frac{2xy}{m^2}\left(\frac{6D'(1;h)}{D(1;h)}(1+h)+3(3+mx+2y)\right),\\
T_{m,n}(e_{0,3};x,y) &= y^3+\frac{3y^2}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2\right)\\
&\quad+\frac{y}{4n^2}\left(\frac{3\left(D''(1;h)+3D'(1;h)\right)}{D(1;h)}(1+h)^2-\frac{6hD'(1;h)}{D(1;h)}(1+h)+4(1-h)\right)\\
&\quad+\frac{x^3}{8n^3}+\frac{3x^2}{8n^3}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+1\right)+\frac{x}{8n^3}\left(\frac{3\left(D''(1;h)+2D'(1;h)\right)}{D(1;h)}(1+h)^2\right.\\
&\quad\left.-\frac{3hD'(1;h)}{D(1;h)}(1+h)+1-h\right)\\
&\quad+\frac{1}{8n^3}\left(\frac{D'''(1;h)+3D''(1;h)+D'(1;h)}{D(1;h)}(1+h)^3\right)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{xy}{4n^2}\left(\frac{6D'(1;h)}{D(1;h)}(1+h)+3(3+x+2ny)\right), \\
T_{m,n}(e_{4,0};x,y) \quad = \quad & x^4+\frac{2x^3}{m}\left(\frac{2D'(1;h)}{D(1;h)}(1+h)+3\right)+\frac{x^2}{m^2}\left(\frac{6\left(D''(1;h)+3D'(1;h)\right)}{D(1;h)}(1+h)^2\right. \\
& \left.-\frac{12hD'(1;h)}{D(1;h)}(1+h)+7-4h\right)+\frac{x}{m^3}\left(\frac{2\left(2D'''(1;h)+9D''(1;h)+7D'(1;h)\right)}{D(1;h)}(1+h)^3\right. \\
& \left.-\frac{6h\left(D''(1;h)+3D'(1;h)\right)}{D(1;h)}(1+h)^2+\frac{8h^2D'(1;h)}{D(1;h)}(1+h)+h^2-4h+1\right) \\
& +\frac{16y^4}{m^4}+\frac{16y^3}{m^4}\left(\frac{2D'(1;h)}{D(1;h)}+6\right)+\frac{4y^2}{m^4}\left(\frac{6\left(D''(1;h)+5D'(1;h)\right)}{D(1;h)}(1+h)^2\right. \\
& \left.-\frac{24hD'(1;h)}{D(1;h)}(1+h)+4(7-4h)\right) \\
& +\frac{2y}{m^4}\left(\frac{4D'''(1;h)+24D''(1;h)+43D'(1;h)}{D(1;h)}(1+h)^3\right. \\
& \left.-\frac{6h\left(2D''(1;h)+5D'(1;h)\right)}{D(1;h)}(1+h)^2+\frac{32h^2D'(1;h)}{D(1;h)}(1+h)+8(h^2-4h+1)\right) \\
& +\frac{1}{m^4}\left(\frac{D^{(4)}(1;h)+6D'''(1;h)+7D''(1;h)+D'(1;h)}{D(1;h)}(1+h)^4\right) \\
& +\frac{2xy}{m^3}\left(\frac{12\left(D''(1;h)+4D'(1;h)\right)}{D(1;h)}(1+h)^2-\frac{36hD'(1;h)}{D(1;h)}(1+h)+4(8-5h)\right. \\
& +12mx\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2\right)+12y\left(\frac{2D'(1;h)}{D(1;h)}(1+h)+5\right) \\
& \left.+4\left((mx)^2+3mxy+4y^2\right)\right), \\
T_{m,n}(e_{0,4};x,y) \quad = \quad & y^4+\frac{y^3}{n}\left(\frac{2D'(1;h)}{D(1;h)}(1+h)+6\right)+\frac{y^2}{4n^2}\left(\frac{6\left(D''(1;h)+5D'(1;h)\right)}{D(1;h)}(1+h)^2\right. \\
& \left.-\frac{24hD'(1;h)}{D(1;h)}(1+h)+4(7-4h)\right) \\
& +\frac{y}{8n^3}\left(\frac{4D'''(1;h)+24D''(1;h)+43D'(1;h)}{D(1;h)}(1+h)^3\right. \\
& \left.-\frac{6h\left(2D''(1;h)+5D'(1;h)\right)}{D(1;h)}(1+h)^2+\frac{32h^2D'(1;h)}{D(1;h)}(1+h)+8(h^2-4h+1)\right) \\
& +\frac{x^4}{16n^4}+\frac{x^3}{8n^4}\left(\frac{2D'(1;h)}{D(1;h)}(1+h)+3\right)+\frac{x^2}{16n^4}\left(\frac{6\left(D''(1;h)+3D'(1;h)\right)}{D(1;h)}(1+h)^2\right. \\
& \left.-\frac{12hD'(1;h)}{D(1;h)}(1+h)+7-4h\right)+\frac{x}{16n^4}\left(\frac{2\left(2D'''(1;h)+9D''(1;h)+7D'(1;h)\right)}{D(1;h)}(1+h)^3\right. \\
& \left.-\frac{6h\left(D''(1;h)+3D'(1;h)\right)}{D(1;h)}(1+h)^2+\frac{8h^2D'(1;h)}{D(1;h)}(1+h)+h^2-4h+1\right)
\end{aligned}
$$

$$+\frac{1}{16n^4}\left(\frac{D^{(4)}(1;h)+6D'''(1;h)+7D''(1;h)+D'(1;h)}{D(1;h)}(1+h)^4\right)$$
$$+\frac{xy}{8n^3}\left(\frac{12\left(D''(1;h)+4D'(1;h)\right)}{D(1;h)}(1+h)^2-\frac{36hD'(1;h)}{D(1;h)}(1+h)+4(8-5h)\right.$$
$$+12x\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2\right)+12ny\left(\frac{2D'(1;h)}{D(1;h)}(1+h)+5\right)$$
$$\left.+4\left(x^2+3xny+4(ny)^2\right)\right).$$

Proof If we take into consideration the equalities (2.1)–(2.10), respectively, then we achieve the proof of lemma.

Lemma 2.3 *Let $x,y\in[0,\infty)$. Then, the following equalities are satisfied*

$$T_{m,n}(t-x;x,y) = \frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right), \tag{2.13}$$

$$T_{m,n}(s-y;x,y) = \frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right), \tag{2.14}$$

$$T_{m,n}((t-x)^2;x,y) = \frac{x}{m}+\frac{1}{m^2}\left[\frac{D''(1;h)+D'(1;h)}{D(1;h)}(1+h)^2\right.$$
$$\left.+4y\left(\frac{D'(1;h)}{D(1;h)}(1+h)+1+y\right)\right], \tag{2.15}$$

$$T_{m,n}((s-y)^2;x,y) = \frac{y}{n}+\frac{1}{4n^2}\left[\frac{D''(1;h)+D'(1;h)}{D(1;h)}(1+h)^2\right.$$
$$\left.+x\left(\frac{2D'(1;h)}{D(1;h)}(1+h)+1+x\right)\right], \tag{2.16}$$

$$T_{m,n}((t-x)^4;x,y) = 3\frac{x^2}{m^2}+\frac{x}{m^3}\left[\frac{2\left(3D''(1;h)+(5-4h+12y)D'(1;h)\right)}{D(1;h)}(1+h)^2\right.$$
$$\left.+\frac{8h\left(h-9y+6\right)D'(1;h)}{D(1;h)}(1+h)+h^2+28h+8y\left(8-5h+3y\right)-31\right]$$
$$+\frac{1}{m^4}\left[\frac{D^{(4)}(1;h)+6D'''(1;h)+7D''(1;h)+D'(1;h)}{D(1;h)}(1+h)^4\right.$$
$$+2y\left(\frac{4D'''(1;h)+24D''(1;h)+43D'(1;h)}{D(1;h)}(1+h)^3\right.$$
$$\left.-\frac{6h\left(2D''(1;h)+5D'(1;h)\right)}{D(1;h)}(1+h)^2+\frac{32h^2D'(1;h)}{D(1;h)}(1+h)^2+8(h^2-4h+1)\right)$$
$$+4y^2\left(\frac{3\left(2D''(1;h)+5D'(1;h)\right)}{D(1;h)}(1+h)^2-\frac{24hD'(1;h)}{D(1;h)}(1+h)+4(7-4h)\right)$$
$$\left.+16y^3\left(\frac{2D'(1;h)}{D(1;h)}+6\right)+16y^4\right] \tag{2.17}$$

and

$$
\begin{aligned}
T_{m,n}((s-y)^4;x,y) \quad = \quad & 3\frac{y^2}{n^2}+\frac{y}{8n^3}\left[\frac{12D''(1;h)+(39+9h+24x)D'(1;h)}{D(1;h)}(1+h)^2\right. \\
& +\frac{4h(8h-9x+3)D'(1;h)}{D(1;h)}(1+h) \\
& \left.+8h^2-28h+4x\left(8-5h+3x\right)+4\right] \\
& +\frac{1}{16n^4}\left[\frac{D^{(4)}(1;h)+6D'''(1;h)+7D''(1;h)+D'(1;h)}{D(1;h)}(1+h)^4\right. \\
& +x\left(\frac{2\left(2D'''(1;h)+9D''(1;h)+7D'(1;h)\right)}{D(1;h)}(1+h)^3\right. \\
& -\frac{6h\left(D''(1;h)+3D'(1;h)\right)}{D(1;h)}(1+h)^2 \\
& \left.+\frac{8h^2D'(1;h)}{D(1;h)}(1+h)+h^2-4h+1\right) \\
& +x^2\left(\frac{6\left(D''(1;h)+3D'(1;h)\right)}{D(1;h)}(1+h)^2-\frac{12hD'(1;h)}{D(1;h)}(1+h)+7-4h\right) \\
& \left.+2x^3\left(\frac{2D'(1;h)}{D(1;h)}(1+h)+3\right)+x^4\right]. \qquad (2.18)
\end{aligned}
$$

In this paper, we will define that $C\left(I_{ab}\right)$ is the space of all bounded and continuous functions on $I_{ab}=[0,a]\times[0,b]$ endowed with

$$
\|z\|_{C(I_{ab})}=\sup_{(x,y)\in I_{ab}}|z(x,y)|\,. \qquad (2.19)
$$

Besides, we will express the set of weighted function as $N:=\left\{z:|z(x,y)|\le C_z\alpha(x,y),\ \alpha(x,y)=1+x^2+y^2\right\}$ where C_z is fixed and depends just on z.

Theorem 2.4 *For $z\in C\left(I^2\right)\cap N$, we obtain that*

$$
\lim_{m,n\to\infty}T_{m,n}(z;x,y)=z(x,y)
$$

uniformly on each compact subset I_{ab} of I^2.

Proof Considering Lemma 2.2, we have

$$
\lim_{m,n\to\infty}T_{m,n}(e_{p,r})=e_{p,r},(p,r)\in\{(0,0),(1,0),(0,1)\},
$$

$$
\lim_{m,n\to\infty}T_{m,n}(e_{20}+e_{02})=e_{20}+e_{02}
$$

uniformly on I_{ab}. If we apply the Volkov theorem in [15], then we complete the proof.

Now, for estimating the order of approximation, we consider some definitions.

Definition 2.5 *[2] Let $z \in C(I_{ab})$ and $\delta_1, \delta_2 > 0$. For $(x,y),(t,s) \in I_{ab}$, the complete modulus of continuity $\omega(z;\delta_1,\delta_2)$ of the function z is defined by*

$$\omega(z;\delta_1,\delta_2) := \sup\left\{|z(t,s)-z(x,y)| : (t,s),(x,y) \in (I_{ab}), |t-x| \le \delta_1, |s-y| \le \delta_2\right\}.$$

The modulus of continuity has these properties:

$$(i)\ \omega(z;\delta_1,\delta_2) \to 0,\ \text{if } \delta_1 \to 0,\ \delta_2 \to 0,$$
$$(ii)\ |z(t,s)-z(x,y)| \le \omega(z;\delta_1,\delta_2)\left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right).$$

In addition, for $\delta > 0$, partial modulus of continuity are given by

$$\begin{aligned}\omega^{(1)}(z;\delta) &= \sup\left\{|z(x_1,y)-z(x_2,y)| : y \in [0,b],\ |x_1-x_2| \le \delta\right\},\\ \omega^{(2)}(z;\delta) &= \sup\left\{|z(x,y_1)-z(x,y_2)| : x \in [0,a],\ |y_1-y_2| \le \delta\right\}.\end{aligned}$$

Definition 2.6 *[6] Let $C^2(I_{ab})$ be the space of all functions $z \in C(I_{ab})$ and belongs to the space $C(I_{ab})$ such that $\frac{\partial^i z}{\partial x^i}, \frac{\partial^i z}{\partial y^i}, (i=1,2)$. The norm on the space $C^2(I_{ab})$ is defined as follows*

$$\|z\|_{C^2(I_{ab})} = \|z\|_{C(I_{ab})} + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i z}{\partial x^i}\right\|_{C(I_{ab})} + \left\|\frac{\partial^i z}{\partial y^i}\right\|_{C(I_{ab})}\right). \tag{2.20}$$

For $z \in C(I_{ab})$ and $\delta > 0$, the Peetre's $\mathcal{K}$-functional is defined by

$$\mathcal{K}(z;\delta) = \inf_{g \in C^2(I_{ab})}\left\{\|z-g\|_{C(I_{ab})} + \delta\,\|g\|_{C(I_{ab})}\right\}.$$

Besides,

$$\mathcal{K}(z;\delta) \le C\left[\omega_2\left(z;\sqrt{\delta}\right) + min(1,\delta)\,\|z\|_{C(I_{ab})}\right] \tag{2.21}$$

where C is a constant independent of z and δ. In addition, $\omega_2\left(z;\sqrt{\delta}\right)$ is the second modulus of continuity which is the similar form as the one variable case in [10] as follows

$$\omega_2(z;\delta) := \sup_{\sqrt{t^2+s^2}\le\delta}\|z(\cdot+2t,\cdot+2s) - 2z(\cdot+t,\cdot+s) + z(\cdot,\cdot)\|_{C(I_{ab})}.$$

We will find the error of approximation as the estimation of the difference of $|T_{m,n}(z;x,y)-z(x,y)|$. To obtain the quantitative results, we can consider this difference in some function spaces.

Theorem 2.7 *For $z \in C(I_{ab})$ and $(x,y) \in I_{ab}$, the inequality*

$$|T_{m,n}(z;x,y)-z(x,y)| \le 4\omega\left(z;\sqrt{\delta_m(x)},\sqrt{\delta_n(y)}\right)$$

is satisfied. Here, $\delta_m(x)$ and $\delta_n(y)$ are given as (2.15) and (2.16), respectively.

Proof From the Definition 2.5, we find

$$\begin{aligned} |T_{m,n}(z;x,y)-z(x,y)| &\leq T_{m,n}\left(|z(t,s)-z(x,y)|\,;x,y\right) \\ &\leq \omega(z;\sqrt{\delta_m(x)},\sqrt{\delta_n(y)})\left(1+\frac{1}{\sqrt{\delta_m(x)}}T_{m,n}\left(|t-x|\,;x,y\right)\right) \\ &\quad\times\left(1+\frac{1}{\sqrt{\delta_n(y)}}T_{m,n}\left(|s-y|\,;x,y\right)\right). \end{aligned} \tag{2.22}$$

With the help of the Cauchy-Schwarz inequality and linearity of the operators $T_{m,n}$, we have

$$\begin{aligned} T_{m,n}(|t-x|\,;x,y) &\leq \left(T_{m,n}\left((t-x)^2;x,y\right)\right)^{\frac{1}{2}}\left(T_{m,n}\left(1;x,y\right)\right)^{\frac{1}{2}} \\ &= \sqrt{\delta_m(x)}, \\ T_{m,n}(|s-y|\,;x,y) &\leq \left(T_{m,n}\left((s-y)^2;x,y\right)\right)^{\frac{1}{2}}\left(T_{m,n}\left(1;x,y\right)\right)^{\frac{1}{2}} \\ &= \sqrt{\delta_n(y)}. \end{aligned}$$

By using these inequalities in (2.22),we complete the proof.

Theorem 2.8 *Let $z\in C\left(I_{ab}\right)$ and $\delta_m(x)$ and $\delta_n(y)$ are defined as (2.15) and (2.16), respectively. Then the following inequality is satisfied*

$$|T_{m,n}(z;x,y)-z(x,y)|\leq 2\left[\omega^{(1)}\left(z;\sqrt{\delta_m(x)}\right)+\omega^{(2)}\left(z;\sqrt{\delta_n(y)}\right)\right].$$

Proof According to definition of the partial modulus of continuity and Cauchy-Schwarz inequality, we can find

$$\begin{aligned} |T_{m,n}\left(z;x,y\right)-z(x,y)| &\leq T_{m,n}\left(|z(t,s)-z(x,y)|\,;x,y\right) \\ &\leq T_{m,n}\left(|z(t,y)-z(x,y)|\,;x,y\right)+T_{m,n}\left(|z(t,s)-z(t,y)|\,;x,y\right) \\ &\leq T_{m,n}\left(\omega^{(1)}\left(z;|t-x|\right);x,y\right)+T_{m,n}\left(\omega^{(2)}\left(z;|s-y|\right);x,y\right) \\ &\leq \omega^{(1)}\left(z;\sqrt{\delta_m(x)}\right)\left(1+\frac{1}{\sqrt{\delta_m(x)}}T_{m,n}\left(|t-x|\right);x,y\right) \\ &\quad+\omega^{(2)}\left(z;\sqrt{\delta_n(y)}\right)\left(1+\frac{1}{\sqrt{\delta_n(y)}}T_{m,n}\left(|s-y|\right);x,y\right) \\ &\leq \omega^{(1)}\left(z;\sqrt{\delta_m(x)}\right)\left(1+\frac{1}{\sqrt{\delta_m(x)}}\sqrt{\delta_m(x)}\right) \\ &\quad+\omega^{(2)}\left(z;\sqrt{\delta_n(y)}\right)\left(1+\frac{1}{\sqrt{\delta_n(y)}}\sqrt{\delta_n(y)}\right) \end{aligned}$$

and therefore, the proof is completed.

Theorem 2.9 *For $z \in C(I_{ab})$, the operators $T_{m,n}$ satisfy the inequality as follows*

$$
\begin{aligned}
&|T_{m,n}(z;x,y) - z(x,y)| \\
\leq\ & M\left[\omega_2\left(z;\frac{1}{2}\sqrt{\delta_m(x)+\delta_n(y)+\left(\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right)\right)^2+\left(\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right)^2}\right)\right. \\
&+\min\left\{1,\delta_m(x)+\delta_n(y)+\left(\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right)\right)^2\right. \\
&\left.\left.+\left(\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right)^2\right\}\|z\|_{C(I_{ab})}\right] \\
&+\omega\left(z;\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right),\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right).
\end{aligned}
$$

Here, M is a positive constant, ω_2 is the second order modulus of continuity and $\delta_m(x)$ and $\delta_n(y)$ are defined by (2.15) and (2.16), respectively.

Proof Firstly, we will consider the auxiliary operators as follows

$$
\begin{aligned}
\tilde{T}_{m,n}(z;x,y) &= T_{m,n}(z;x,y) - z\left(x+\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right), y+\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right) \\
&\quad + z(x,y). \qquad (2.23)
\end{aligned}
$$

From Lemma 2.3, we have

$$
\tilde{T}_{m,n}(t-x;x,y) = 0,\ \tilde{T}_{m,n}(s-y;x,y) = 0. \qquad (2.24)
$$

On the other hand, for $g \in C^2(I_{ab})$, $(t,s) \in I_{ab}$, using the Taylor expansion formula, we find

$$
\begin{aligned}
g(t,s) - g(x,y) &= g(t,y) - g(x,y) + g(t,s) - g(t,y) \\
&= \frac{\partial g(x,y)}{\partial x}(t-x) + \int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du + \frac{\partial g(x,y)}{\partial y}(s-y) + \int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv.
\end{aligned}
$$

Applying $\tilde{T}_{m,n}$ for the above equality and considering the equalities (2.24), we obtain

$$
\begin{aligned}
\tilde{T}_{m,n}(g;x,y) - g(x,y) &= \tilde{T}_{m,n}\left(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;x,y\right) + \tilde{T}_{m,n}\left(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;x,y\right) \\
&= T_{m,n}\left(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;x,y\right) \\
&\quad -\int_x^{x+\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right)}\left(x+\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right)-u\right)\frac{\partial^2 g(u,y)}{\partial u^2}du \\
&\quad +T_{m,n}\left(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;x,y\right) \\
&\quad -\int_y^{y+\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)}\left(y+\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)-v\right)\frac{\partial^2 g(x,v)}{\partial v^2}dv.
\end{aligned}
$$

Hence,

$$\begin{aligned}\left|\tilde{T}_{m,n}(g;x,y)-g(x,y)\right| &\leq T_{m,n}\left(\left|\int_x^t |t-u|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right|;x,y\right)\\&+\left|\int_x^{x+\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right)}\left|x+\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right)-u\right|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right|\\&+T_{m,n}\left(\left|\int_y^s |s-v|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|;x,y\right)\\&+\left|\int_y^{y+\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)}\left|y+\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)-v\right|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|\\&\leq \left[T_{m,n}\left((t-x)^2;x,y\right)+\left(x+\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right)-x\right)^2\right]\|g\|_{C^2(I_{ab})}\\&+\left[T_{m,n}\left((s-y)^2;x,y\right)+\left(y+\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)-y\right)^2\right]\|g\|_{C^2(I_{ab})}.\end{aligned}$$

Moreover, from the equality (2.23), we have

$$\begin{aligned}\left|\tilde{T}_{m,n}(z;x,y)\right| &\leq |T_{m,n}(z;x,y)|\\&+\left|z\left(x+\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right),y+\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right)\right|+|z(x,y)|\\&\leq 3\|z\|_{C(I_{ab})}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}|T_{m,n}(z;x,y)-z(x,y)| &= \Big|\tilde{T}_{m,n}(z;x,y)-z(x,y)\\&+z\left(x+\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right),y+\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right)-z(x,y)\Big|\\&\leq \left|\tilde{T}_{m,n}(z-g;x,y)\right|+\left|\tilde{T}_{m,n}(g;x,y)-g(x,y)\right|+|g(x,y)-z(x,y)|\\&+\left|z\left(x+\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right),y+\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right)-z(x,y)\right|.\end{aligned}$$

Then, for $\delta_m(x)=T_{m,n}((t-x)^2;x,y)$ and $\delta_n(y)=T_{m,n}((s-y)^2;x,y)$, we get

$$
\begin{aligned}
|T_{m,n}(z;x,y)-z(x,y)| &\leq 4\left\|z-g\right\|_{C(I_{ab})}+\left|\tilde{T}_{m,n}(g;x,y)-g(x,y)\right| \\
&+\left|z\left(x+\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right),y+\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right)\right. \\
&\left.-z(x,y)\right| \\
&\leq \left\{4\left\|z-g\right\|_{C(I_{ab})}+\left[\delta_m(x)+\delta_n(y)+\left(\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right)\right)^2\right.\right. \\
&\left.\left.+\left(\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right)^2\right]\left\|g\right\|_{C^2(I_{ab})}\right\} \\
&+\omega\left(z;\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right),\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right) \\
&\leq 4\mathcal{K}\left(z;\frac{1}{4}\left[\delta_m(x)+\delta_n(y)+\left(\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right)\right)^2\right.\right. \\
&\left.\left.+\left(\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right)^2\right]\right) \\
&+\omega\left(z;\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right),\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right).
\end{aligned}
$$

From the definition of Peetre's $\mathcal{K}$-functional, we obtain

$$
\begin{aligned}
&|T_{m,n}(z;x,y)-z(x,y)| \\
\leq\ & M\left[\omega_2\left(z;\frac{1}{2}\sqrt{\delta_m(x)+\delta_n(y)+\left(\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right)\right)^2+\left(\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right)^2}\right)\right. \\
&+\min\left\{1,\delta_m(x)+\delta_n(y)+\left(\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right)\right)^2\right. \\
&\left.\left.+\left(\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right)^2\right\}\left\|z\right\|_{C(I_{ab})}\right] \\
&+\omega\left(z;\frac{1}{m}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right),\frac{1}{2n}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right)\right)
\end{aligned}
$$

where M is a positive constant. Therefore, desired result is obtained.

Now, we consider the approximation degree of the operators $T_{m,n}$ in terms of the Lipschitz class. For $0<b_1,b_2\leq 1$, $z\in C(I_{ab})$ and $L>0$, Lipschitz class is shown as $Lip_L(b_1,b_2)$ and this class is defined as follows

$$|z(t,s)-z(x,y)|\leq L\left|t-x\right|^{b_1}\left|s-y\right|^{b_2}.$$

Now, from this definition, we can obtain the following theorem.

Theorem 2.10 *Let the function z be in $Lip_L(b_1, b_2)$. The following inequality is satisfied*

$$|T_{m,n}(z;x,y) - z(x,y)| \leq L\,(\theta_m(x))^{\frac{b_1}{2}}\,(\theta_n(y))^{\frac{b_2}{2}}$$

where $L > 0$ and $\theta_m(x)$ and $\theta_n(y)$ are defined by (1.17).

Proof Since $z \in Lip_L(b_1, b_2)$ and the operators $T_{m,n}$ are linear positive operators, we find

$$\begin{aligned} |T_{m,n}(z;x,y) - z(x,y)| &\leq T_{m,n}\left(|z(t,s) - z(x,y)|\,;x,y\right) \\ &\leq LT_m(|t-x|^{b_1}\,;x)T_n(|s-y|^{b_2}\,;y) \end{aligned}$$

where the operators T_m and T_n are defined by (1.14). Taking into consideration the Hölder inequality and the equations in Lemma 1.6, we have

$$\begin{aligned} |T_{m,n}(z;x,y) - z(x,y)| &\leq L\left(T_m((t-x)^2;x)\right)^{\frac{b_1}{2}}(T_m(1;x))^{\frac{2-b_1}{2}}\left(T_n((s-y)^2;y)\right)^{\frac{b_2}{2}}(T_n(1;y))^{\frac{2-b_2}{2}} \\ &= L\,(\theta_m(x))^{\frac{b_1}{2}}\,(\theta_n(y))^{\frac{b_2}{2}}\,. \end{aligned}$$

Therefore, the proof is completed.

Theorem 2.11 *For $z \in C^1(I_{ab})$, we have*

$$\|T_{m,n}(z) - z\| \leq \|z_x\|\sqrt{\delta_m(x)} + \|z_y\|\sqrt{\delta_n(y)}$$

where $\delta_m(x)$ and $\delta_n(y)$ are defined by (2.15) and (2.16) respectively and $C^1(I_{ab})$ is the space of first order continuously differentiable functions.

Proof For arbitrary $(t,s), (x,y) \in I_{ab}$, we get

$$z(t,s) - z(x,y) = \int_x^t z_u(u,s)du + \int_y^s z_v(x,v)dv.$$

If we apply the operators $T_{m,n}$ on both sides of the above equation and make the arrangements, then we have

$$|T_{m,n}(z;x,y) - z(x,y)| \leq T_{m,n}\left(\left|\int_x^t z_u(u,s)du\right|;x,y\right) + T_{m,n}\left(\left|\int_y^s z_v(x,v)dv\right|;x,y\right).$$

Since

$$\left|\int_x^t z_u(u,s)du\right| \leq \|z_x\|\,|t-x| \text{ and } \left|\int_y^s z_v(x,v)dv\right| \leq \|z_y\|\,|s-v|\,,$$

we get

$$|T_{m,n}(z;x,y) - z(x,y)| \leq \|z_x\|\,T_{m,n}(|t-x|\,;x,y) + \|z_y\|\,T_{m,n}(|s-y|\,;x,y)\,.$$

From the Cauchy-Schwarz inequality, we find that

$$\begin{aligned} |T_{m,n}(z;x,y)-z(x,y)| &\leq \|z_x\| \left[T_{m,n}\left((t-x)^2;x,y\right)\right]^{\frac{1}{2}} \left[T_{m,n}(e_0;x,y)\right]^{\frac{1}{2}} \\ &\quad + \|z_y\| \left[T_{m,n}\left((s-y)^2;x,y\right)\right]^{\frac{1}{2}} \left[T_{m,n}(e_0;x,y)\right]^{\frac{1}{2}} \\ &\leq \|z_x\| \sqrt{\delta_m(x)} + \|z_y\| \sqrt{\delta_n(y)}. \end{aligned}$$

Thus, we achieve the desired result.

Now, we will consider the Voronovskaja-type theorem for the operators $T_{m,n}$. For this theorem, let the space $G(I^2)$ is the space of all functions on I^2 satisfying $|z(x,y)| \leq M_z\left(1+x^2+y^2\right)$, $H(I^2)$ is the subspace of all continuous functions of the space $G(I^2)$ and $S(I^2)$ is the subspace of all $z \in H(I^2)$ satisfying that $\lim_{(x,y)\to\infty} \frac{z(x,y)}{1+x^2+y^2}$ is finite.

Theorem 2.12 *If $z \in S\left(I^2\right)$ such that $z', z'' \in S\left(I^2\right)$ and (x,y) be in each compact subset I_{ab} of I^2, then we get*

$$\begin{aligned} \lim_{n\to\infty} n\left(T_{n,n}(z;x,y)-z(x,y)\right) &= z_x(x,y)\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right) + z_y(x,y)\frac{1}{2}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right) \\ &\quad + \frac{1}{2}x\ z_{xx}(x,y) + \frac{1}{2}y\ z_{yy}(x,y). \end{aligned}$$

Proof Let $(x,y) \in I_{ab}$. From the Taylor's expansion formula, we obtain

$$\begin{aligned} z(t,s) &= z(x,y) + z_x(x,y)(t-x) + z_y(x,y)(s-y) \\ &\quad + \frac{1}{2}\left\{z_{xx}(x,y)(t-x)^2 + 2z_{xy}(x,y)(t-x)(s-y) + z_{yy}(x,y)(s-y)^2\right\} \\ &\quad + \rho(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4} \end{aligned}$$

where $(t,s) \in I^2$, $\rho(t,s;x,y) \in C\left(I^2\right)$ and $\rho(t,s;x,y) \to 0$ as $(t,s) \to (x,y)$. Applying $T_{n,n}(z;x,y)$ on above equality, we have

$$\begin{aligned} T_{n,n}(z;x,y) &= z(x,y) + z_x(x,y)T_{n,n}(t-x;x,y) + z_y(x,y)T_{n,n}(s-y;x,y) \\ &\quad + \frac{1}{2}\left[z_{xx}(x,y)T_{n,n}\left((t-x)^2;x,y\right) + 2z_{xy}(x,y)T_{n,n}\left((t-x)(s-y);x,y\right)\right. \\ &\quad \left. + z_{yy}(x,y)T_{n,n}\left((s-y)^2;x,y\right)\right] + T_{n,n}\left(\rho(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4};x,y\right). \end{aligned}$$

If we multiply with n, take limit as $n \to \infty$ and then consider Lemma 2.3, then we find

$$\begin{aligned} \lim_{n\to\infty} n\left(T_{n,n}(z;x,y)-z(x,y)\right) &= z_x(x,y)\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right) + z_y(x,y)\frac{1}{2}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right) \\ &\quad + \frac{1}{2}x\ z_{xx}(x,y) + z_{xy}(x,y)\lim_{n\to\infty} nT_{n,n}\left((t-x)(s-y);x,y\right) \\ &\quad + \frac{1}{2}y\ z_{yy}(x,y) + \lim_{n\to\infty} nT_{n,n}\left(\rho(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4};x,y\right). \quad (2.25) \end{aligned}$$

In addition, we have from Lemma 1.6,

$$\lim_{n\to\infty} nT_{n,n}\left((t-x)(s-y);x,y\right) = \lim_{n\to\infty} n\left(T_n(t-x;x)T_n(s-y;y)\right) = 0.$$

From the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & nT_{n,n}\left(\rho\left(t,s;x,y\right)\sqrt{(t-x)^4+(s-y)^4};x,y\right) \\ \leq\ & \left(T_{n,n}\left(\rho^2\left(t,s;x,y\right);x,y\right)\right)^{\frac{1}{2}}\left(n^2T_{n,n}\left((t-x)^4+(s-y)^4;x,y\right)\right)^{\frac{1}{2}} \\ =\ & \left(T_{n,n}\left(\rho^2\left(t,s;x,y\right);x,y\right)\right)^{\frac{1}{2}}\left(n^2T_{n,n}\left((t-x)^4;x,y\right)+n^2T_{n,n}\left((s-y)^4;x,y\right)\right)^{\frac{1}{2}}. \end{aligned}$$

Here, $\rho^2\left(t,s;x,y\right)\to 0$ as $(t,s)\to(x,y)$. Thus, by Theorem 2.4, we obtain that

$$\lim_{n\to\infty} T_{n,n}\left(\rho^2\left(t,s;x,y\right)\right) = 0$$

uniformly with respect to $(x,y)\in I_{ab}$. With the help of Lemma 2.3, we find

$$\begin{aligned} \lim_{n\to\infty} n^2T_{n,n}\left((t-x)^4;x,y\right) &= 3x^2, \\ \lim_{n\to\infty} n^2T_{n,n}\left((s-y)^4;x,y\right) &= 3y^2. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} nT_{n,n}\left(\rho\left(t,s;x,y\right)\sqrt{(t-x)^4+(s-y)^4};x,y\right) = 0.$$

Finally, from (2.25), we obtain the desired result as follows

$$\begin{aligned} \lim_{n\to\infty} n\left(T_{n,n}\left(z;x,y\right)-z(x,y)\right) &= z_x(x,y)\left(\frac{D'(1;h)}{D(1;h)}(1+h)+2y\right)+z_y(x,y)\frac{1}{2}\left(\frac{D'(1;h)}{D(1;h)}(1+h)+x\right) \\ &\quad +\frac{1}{2}x\ z_{xx}(x,y)+\frac{1}{2}y\ z_{yy}(x,y). \end{aligned}$$

3. Kantorovich form of The Δ_h-Gould-Hopper Appell polynomials

In this section, we consider the Kantorovich modification of the operators including Δ_h-Gould-Hopper Appell polynomials as follows

$$\begin{aligned} T^*_{m,n}(z;x,y) &= \frac{mn}{\left(D(1;h)\right)^2(1+h)^{\frac{(m+1)x+(n+1)y}{h}}} \\ &\quad\times\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{\mathcal{A}_u(mx,y;h)}{u!}\frac{\mathcal{A}_v(x,ny;h)}{v!}\int_{\frac{u}{m}}^{\frac{u+1}{m}}\int_{\frac{v}{n}}^{\frac{v+1}{n}} z\left(k(1+h),\frac{l(1+h)}{2}\right)dl\ dk \qquad (3.1) \end{aligned}$$

where $m,n\in\mathbb{N},(x,y)\in I^2, z(x,y)\in C\left(I^2\right)$. Now, we obtain the approximation properties.

Lemma 3.1 *For $e_{i,j}=t^i s^j$; $i,j=0,1,2,3,4$ and $x,y\in I$, the operators $T^*_{m,n}(e_{i,j};x,y)$ have the equalities*

$$\begin{aligned}
T^*_{m,n}(e_{0,0};x,y) &= T_{m,n}(e_{0,0};x,y),\\
T^*_{m,n}(e_{1,0};x,y) &= T_{m,n}(e_{1,0};x,y)+\frac{1+h}{2m}T_{m,n}(e_{0,0};x,y),\\
T^*_{m,n}(e_{0,1};x,y) &= T_{m,n}(e_{0,1};x,y)+\frac{1+h}{4n}T_{m,n}(e_{0,0};x,y),\\
T^*_{m,n}(e_{2,0};x,y) &= T_{m,n}(e_{2,0};x,y)+\frac{1+h}{m}T_{m,n}(e_{1,0};x,y)+\frac{(1+h)^2}{3m^2}T_{m,n}(e_{0,0};x,y),\\
T^*_{m,n}(e_{0,2};x,y) &= T_{m,n}(e_{0,2};x,y)+\frac{1+h}{2n}T_{m,n}(e_{0,1};x,y)+\frac{(1+h)^2}{12n^2}T_{m,n}(e_{0,0};x,y),\\
T^*_{m,n}(e_{3,0};x,y) &= T_{m,n}(e_{3,0};x,y)+\frac{3(1+h)}{2m}T_{m,n}(e_{2,0};x,y)+\frac{(1+h)^2}{m^2}T_{m,n}(e_{1,0};x,y)\\
&\quad+\frac{(1+h)^3}{4m^3}T_{m,n}(e_{0,0};x,y),\\
T^*_{m,n}(e_{0,3};x,y) &= T_{m,n}(e_{0,3};x,y)+\frac{3(1+h)}{4n}T_{m,n}(e_{0,2};x,y)+\frac{(1+h)^2}{4n^2}T_{m,n}(e_{0,1};x,y)\\
&\quad+\frac{(1+h)^3}{32n^3}T_{m,n}(e_{0,0};x,y),\\
T^*_{m,n}(e_{4,0};x,y) &= T_{m,n}(e_{4,0};x,y)+\frac{2(1+h)}{m}T_{m,n}(e_{3,0};x,y)+\frac{2(1+h)^2}{m^2}T_{m,n}(e_{2,0};x,y)\\
&\quad+\frac{(1+h)^3}{m^3}T_{m,n}(e_{1,0};x,y)+\frac{(1+h)^4}{5m^4}T_{m,n}(e_{0,0};x,y),\\
T^*_{m,n}(e_{0,4};x,y) &= T_{m,n}(e_{0,4};x,y)+\frac{1+h}{n}T_{m,n}(e_{0,3};x,y)+\frac{(1+h)^2}{2n^2}T_{m,n}(e_{0,2};x,y)\\
&\quad+\frac{(1+h)^3}{8n^3}T_{m,n}(e_{0,1};x,y)+\frac{(1+h)^4}{80n^4}T_{m,n}(e_{0,0};x,y).
\end{aligned}$$

Lemma 3.2 *For the operators $T^*_{m,n}$, the following equalities are satisfied*

$$\begin{aligned}
T^*_{m,n}(t-x;x,y) &= \frac{1}{m}\left(\left(\frac{1}{2}+\frac{D'(1;h)}{D(1;h)}\right)(1+h)+2y\right),\\
T^*_{m,n}(s-y;x,y) &= \frac{1}{2n}\left(\left(\frac{1}{2}+\frac{D'(1;h)}{D(1;h)}\right)(1+h)+x\right),\\
T^*_{m,n}\left((t-x)^2;x,y\right) &= \frac{x}{m}+\frac{1}{m^2}\left[\left(\frac{D''(1;h)+2D'(1;h)}{D(1;h)}+\frac{1}{3}\right)(1+h)^2\right.\\
&\quad\left.+2y\left(\frac{2D'(1;h)}{D(1;h)}+1\right)(1+h)+4y(y+1)\right],\\
T^*_{m,n}\left((s-y)^2;x,y\right) &= \frac{y}{n}+\frac{1}{4n^2}\left[\left(\frac{D''(1;h)+2D'(1;h)}{D(1;h)}+\frac{1}{3}\right)(1+h)^2\right.\\
&\quad\left.+x\left(\frac{2D'(1;h)}{D(1;h)}+1\right)(1+h)+x(x+1)\right]
\end{aligned}$$

where $x, y \in I$.

When we look at the approximation properties of the operators $T^*_{m,n}$ closely, we see that the operators are associated with the approximation properties of the operators $T_{m,n}$. Because of this, we will mention only theorems for $T^*_{m,n}$.

Theorem 3.3 *Let* $z \in C\left(I^2\right) \cap N$. *Then,*

$$\lim_{m,n\to\infty} T^*_{m,n}(z;x,y) = z(x,y).$$

uniformly on each compact I_{ab}.

Theorem 3.4 *Let* $z \in C\left(I_{ab}\right)$ *and* $(x,y) \in I_{ab}$. *Then, for* $\varphi_m(x) = T^*_{m,n}\left((t-x)^2; x, y\right)$ *and* $\varphi_n(x) = T^*_{m,n}\left((s-y)^2; x, y\right)$, *we obtain*

$$\left|T^*_{m,n}(z;x,y) - z(x,y)\right| \leq 4\omega(z; \sqrt{\varphi_m(x)}, \sqrt{\varphi_n(y)}).$$

Theorem 3.5 *For* $z \in C\left(I_{ab}\right)$, $\varphi_m(x) = T^*_{m,n}\left((t-x)^2; x, y\right)$ *and* $\varphi_n(x) = T^*_{m,n}\left((s-y)^2; x, y\right)$; *the operators* $T^*_{m,n}$ *satisfy the following inequality*

$$\left|T^*_{m,n}(z;x,y) - z(x,y)\right| \leq 2\left[\omega^{(1)}\left(z; \sqrt{\varphi_m(x)}\right) + \omega^{(2)}\left(z; \sqrt{\varphi_n(y)}\right)\right].$$

Theorem 3.6 *Let* $z \in C\left(I_{ab}\right)$. *Then, for the operators* $T^*_{m,n}$, *we get*

$$\begin{aligned}\left|T^*_{m,n}(z;x,y) - z(x,y)\right| &\leq M\Bigg[\omega_2\left(z; \frac{1}{2}\sqrt{\varphi_m(x) + \varphi_n(y) + \gamma_m^2 + \gamma_n^2}\right) \\ &\quad + \min\left\{1, \varphi_m(x) + \varphi_n(y) + \gamma_m^2 + \gamma_n^2\right\} \|z\|_{C(I_{ab})}\Bigg] + \omega\left(z; \gamma_m, \gamma_n\right)\end{aligned}$$

where C *is a positive constant,* $\varphi_m(x) = T^*_{m,n}\left((t-x)^2; x, y\right)$, $\varphi_n(x) = T^*_{m,n}\left((s-y)^2; x, y\right)$ *and* ω_2 *is the second order modulus of continuity. In addition,* γ_m *and* γ_n *is defined by*

$$\begin{aligned}\gamma_m &= \frac{1}{m}\left(\left(\frac{1}{2} + \frac{D'(1;h)}{D(1;h)}\right)(1+h) + 2y\right), \\ \gamma_n &= \frac{1}{2n}\left(\left(\frac{1}{2} + \frac{D'(1;h)}{D(1;h)}\right)(1+h) + x\right).\end{aligned}$$

Theorem 3.7 *For* $z \in Lip_M(b_1, b_2)$, *the following inequality is satisfied*

$$\left|T^*_{m,n}(z;x,y) - z(x,y)\right| \leq L\left(\theta_m(x)\right)^{\frac{b_1}{2}} \left(\theta_n(y)\right)^{\frac{b_2}{2}}$$

where $L > 0$, $\theta_m(x)$ *and* $\theta_n(y)$ *are defined by* $T^*_m((t-x)^2; x)$ *and* $T^*_n((s-y)^2; y)$ *respectively. Here, the operator* T^*_m *are Kantorovich form of the operators (1.14) (see [12]).*

Theorem 3.8 *For $z \in C^1\left(I^2\right)$, we have*

$$\left\|T^*_{m,n}(z) - z\right\| \leq \|z_x\| \sqrt{\varphi_m(x)} + \|z_y\| \sqrt{\varphi_n(y)}$$

where $\varphi_m(x) = T^*_{m,n}\left((t-x)^2; x, y\right)$, $\varphi_n(x) = T^*_{m,n}\left((s-y)^2; x, y\right)$.

Theorem 3.9 *For $z, z', z'' \in S\left(I^2\right)$, we get*

$$\begin{aligned}\lim_{n\to\infty} n\left(T^*_{n,n}(z; x, y) - z(x, y)\right) &= z_x(x,y)\left(\left(\frac{1}{2} + \frac{D'(1;h)}{D(1;h)}\right)(1+h) + 2y\right)(1+h) \\ &\quad + z_y(x,y)\frac{1}{2}\left(\left(\frac{1}{2} + \frac{D'(1;h)}{D(1;h)}\right)(1+h) + x\right)(1+h) \\ &\quad + \frac{1}{2}x\ z_{xx}(x,y) + \frac{1}{2}y\ z_{yy}(x,y)\end{aligned}$$

where $(x, y) \in I_{ab}$ and I_{ab} is compact subset of I^2.

4. Conclusion

In this paper, we introduced the approximation properties of the linear positive operators including Δ_h-Gould-Hopper Appell polynomials. In recent years, there have been many studies on Appell polynomials, which have very practical areas in pure and applied mathematics. From these studies and the information given in the introduction, we obtained more general impressions and we predict that this paper will contribute to many mathematical processes. Besides, we obtained two variable versions of the operators including degenerate Appell polynomials. After this study, the authors may research multi-variable versions of the operators including degenerate Appell polynomials and other operators.

Acknowledgment

The authors would like to thank the Scientific and Technological Research Council of Türkiye (TÜBİTAK) for the graduate scholarship that supported the first author.

References

[1] Altomare F, Campiti M. Korovkin-type approximation theory and its applications. de Gruyter Studies in Mathematics, Berlin: Walter de Gruyter &Co, 1994.

[2] Anastassiou GA, Gal SG. Approximation Theory: Moduli of Continuity and Global Smoothness Preservation. Birkhauser: Boston, MA, USA, 2000.

[3] Appell PE. Sur une classe de polynômes. Annales scientifiques de l'École Normale Supérieure 1880; 2 (9): 119-144. https://doi.org/10.24033/asens.186 (in French).

[4] Biricik B, Çekim B, Özarslan MA. Sequences of twice-iterated Δ_w-Gould–Hopper Appell polynomials, Journal of Taibah University for Science 2024; 18 (1). https://doi.org/10.1080/16583655.2023.2286714

[5] Costabile FA, Longo, E. Δ_h-Appell sequences and related interpolation problem, Numerical Algorithms 2013; 63 (1): 165-186.

[6] Devore RA, Lorentz GG. Constructive Approximation. Berlin: Springer-Verlag, 1993.

[7] Ismail MEH. On a generalization of Szász operators. Mathematica (Cluj) 1974; 39: pp. 259-267.

[8] İçöz G, Varma S, Sucu S. Approximation by operators including generalized Appell polynomials. Filomat 2016; 30 (2): 429-440. https://www.doi.org/10.2298/FIL1602429I

[9] Jakimovski A, Leviatan D. Generalized Szász operators for the approximation in the infinite interval. Mathematica (Cluj) 1969; 11 (34): pp. 97-103.

[10] Lebesgue H. Sur les intégrales singulières. Annales de la Faculté des sciences de l'Université de Toulouse pour les sciences mathématiques et les sciences physiques 1909; 3 (1): pp. 25-117 (in French).

[11] Özarslan MA, Yılmaz Yaşar B. Δ_h-Gould-Hopper Appell Polynomials. Acta Mathematica Scientia 2021; 41: 1196-1222. https://doi.org/10.1007/s10473-021-0411-y

[12] Sergi BZ, İçöz G, Çekim B. Some approximation properties of operators including degenerate Appell polynomials. Mathematica Slovaca 2023; 73 (6): 1545-1558. https://doi.org/10.1515/ms-2023-0111

[13] Szász O. Generalization of Bernstein's polynomials to the infinite interval. Journal of Research of the National Bureau of Standards (U. S) 1950; 45 (3): 239-245. https://www.doi.org/10.6028/JRES.045.024

[14] Varma S, Sucu S, İçöz G. Generalization of Szász operators involving Brenke type polynomials. Computers & Mathematics with Applications 2012; 64 (2): 121-127. https://www.doi.org/10.1016/j.camwa.2012.01.025

[15] Volkov VI. On the convergence of sequence of linear positive operators in the space of continuous functions of two variables. (Russian) Doklady Akademii Nauk SSSR (N.S) 1957; 115 (1): pp. 17-19.

[16] Weierstrass KG. Uber die analytische Darstellbarkeit sogenannter willkurlicher Functionen einer reellen Veranderlichen Sitzungsberichteder. Koniglich Preussischen Akademie der Wissenschcaften zu Berlin 1885; pp. 633-639 & pp. 789-805 (in German).