Turkish Journal of Mathematics

Volume 48 | Number 2 Article 16

3-8-2024

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MERAL SÜER
meral.suer@batman.edu.tr

MEHMET YEŞİL
mehmet-yesil@outlook.com

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2024) 48: 346 – 359

doi:10.55730/1300-0098.3510

Special subdiagrams of Young diagrams and numerical semigroups

Meral SÜER, Mehmet YEŞİL*
Department of Mathematics, Faculty of Science and Letters, Batman University, Batman, Turkiye

Received: 18.11.2023 • **Accepted/Published Online:** 14.02.2024 • **Final Version:** 08.03.2024

Abstract: In this study, Young diagrams and their corresponding numerical sets are considered, and a new notion called special subdiagrams is described. Characterizations of special subdiagrams and their corresponding numerical sets, as well as the conditions when they are numerical semigroups, are provided. Young diagrams of symmetric, almost symmetric and Arf numerical semigroups are also considered and properties of their special subdiagrams are given.

Key words: Numerical sets, partitions, Young diagrams, special subdiagrams, numerical semigroups

1. Introduction

Let $\mathbb{Z}$ denote the set of integers, $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A subset S of $\mathbb{N}_0$ that contains zero and has finite complement in $\mathbb{N}_0$ is called a numerical set. Note that $\mathbb{N}_0$ is a numerical set having empty complement. A numerical set S is a numerical semigroup, if it satisfies that $x, y \in S \implies x + y \in S$. Numerical semigroups are important mathematical structures with wide-ranging applications in combinatorics, commutative algebra, and algebraic geometry. For understanding the structural properties of numerical semigroups, there are many tools such as the Frobenius number, the conductor, small elements etc. The use of Young diagrams is one of the newest tool which is still being developed.

A Young diagram is a collection of left-aligned rows of boxes where each row contains at least as many boxes as the row immediately below it. The notion of Young diagram is one of the fundamental combinatorial structures with applications in various branches of mathematics, including but not limited to representation theory. Particularly, they are instrumental for studying symmetric polynomials and representations of symmetric groups. Young diagrams are also used for visualising partitions of positive integers and numerical sets. There are bijective correspondences between the set of Young diagrams, the set of partitions and the set of numerical sets. The connection between Young diagrams, partitions, and numerical sets is given by Keith and Nath in [7], and Constantin, Houston-Edwards and Kaplan in [2]. As an important application of the correspondences between Young diagrams, partitions, and numerical sets, Arf numerical semigroups are characterized and many nice properties of them are given via their Young diagrams and corresponding partitions by Tutaş and her collaborators in a sequel of papers [4–6, 10, 11]. Young diagrams are also used by the authors to give new decompositions of symmetric and pseudosymmetric numerical semigroups in [9].

In this paper, for a given Young diagram Y, we define special diagrams of Y (see Definition 3.1) and show that they are subdiagrams of Y (see Proposition 3.3). Then for a given numerical set S and its corresponding

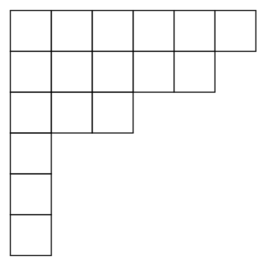
*Correspondence: mehmet-yesil@outlook.com
2010 *AMS Mathematics Subject Classification:* 20M14, 20M20, 05A17

Young diagram Y_S, we find the elements and gaps of the corresponding numerical sets to special subdiagrams of Y_S using the elements and gaps of S (see Propositions 3.8 and 3.9). We also give the exact conditions when the corresponding numerical set to a special subdiagram is a numerical semigroup (see Theorem 3.10). Then we consider symmetric, almost symmetric, and Arf numerical semigroups, and investigate the special subdiagrams of their corresponding Young diagrams.

2. Young diagrams, numerical sets, and partitions

Consider a Young diagram denoted as Y with n columns and g rows. The count of boxes in a column (or row) is referred to as the length of that respective column (or row).

Example 1 *The picture depicted below is a Young diagram with* 6 *columns and* 6 *rows.*



A numerical set S is considered proper if it is not equal to the set of nonnegative integers. Let us assume that S is indeed a proper numerical set. We represent the complement of S within $\mathbb{N}_0$ as $\mathrm{G}(S)$. The elements of $\mathrm{G}(S)$ are referred to as the gaps of S. The count of gaps in S is designated as its genus, denoted by $\mathrm{g}(S)$. The largest gap present in S is termed the Frobenius number and symbolized as $\mathrm{F}(S)$. Furthermore, $\mathrm{F}(S)+1$ is known as the conductor of S and represented as $\mathrm{C}(S)$. In particular, $\mathrm{C}(S)$ is the smallest element of S that satisfies the condition: for any nonnegative integer n, if $n \geq \mathrm{C}(S)$, then n belongs to S. It is important to note that $\mathrm{F}(\mathbb{N}_0) = -1$ and $\mathrm{C}(\mathbb{N}_0) = 0$.

The elements in a proper numerical set S that are less than $\mathrm{C}(S)$ are referred to as the small elements of S. If S contains n such small elements, they are arranged in ascending order and listed as $0 = s_0 < s_1 < \cdots < s_{n-1}$. This allows us to represent S as follows:

$$S = \{0, s_1, \ldots, s_{n-1}, s_n = \mathrm{C}(S), \rightarrow\}.$$

Here, the arrow at the end signifies that all integers greater than $\mathrm{C}(S)$ are considered part of the set S.

For an element $s \in S$, we define the set difference $S - s$ as follows: $S - s = \{x - s \mid x \in S \text{ and } x \geq s\}$. If $s \geq \mathrm{C}(S)$, then $S - s = \mathbb{N}$. However, if s is equal to a small element $s_i \in S$, then $S - s$ can be expressed as:

$$S - s = \{0 = s_i - s, s_{i+1} - s, \ldots, s_n - s, \rightarrow\}.$$

In this case, $S - s$ forms a numerical set with gaps, and its gap set $\mathrm{G}(S - s) = \{g - s \mid g \in \mathrm{G}(S) \text{ and } g > s\}$. Consequently, the Frobenius number of $S - s$ is given by $\mathrm{F}(S - s) = \mathrm{F}(S) - s$.

Example 2 *The set* $S = \{0, 4, 5, 7, 8, 10, 12, \rightarrow\}$ *is a numerical set which possesses a complement* $\mathrm{G}(S) = \{1, 2, 3, 6, 9, 11\}$. *Its genus is calculated as* $\mathrm{g}(S) = 6$. *Notably, the Frobenius number of* S *is* $\mathrm{F}(S) = 11$, *and the conductor of* S *is* $\mathrm{C}(S) = 12$.

For a numerical set S, we can create a Young diagram, labelled as Y_S, that corresponds to S by sketching a continuous polygonal path originating from the origin in $\mathbb{Z}^2$. We begin with $s = 0$, and for each subsequent s:

1. if $s \in S$, then we draw a unit-length line to the right.
2. if $s \notin S$, then we draw a unit-length line upwards.

We continue this process until we reach $s = \mathrm{F}(S)$. The region above this polygonal path and the horizontal line positioned $\mathrm{g}(S)$ units above the origin collectively define the corresponding Young diagram Y_S.

It is evident that every Young diagram corresponds to a unique proper numerical set. Consequently, the mapping from S to Y_S establishes a one-to-one correspondence, i.e. a bijection, between the set of proper numerical sets and the set of Young diagrams. For instance, the numerical set given in Example 2 corresponds to the Young diagram presented in Example 1. When we have a numerical set $S = \{0, s_1, \ldots, s_{n-1}, s_n = \mathrm{C}(S), \rightarrow\}$ with the corresponding Young diagram Y_S, by the construction process, it is evident that Y_S has $\mathrm{g}(S)$ rows and n columns.

For a positive integer N, a partition λ is a nonincreasing finite sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = N$, denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. For each $i = 1, 2, \ldots, n$, the number λ_i is called a part of the partition and the number n of parts called the length of the partition. When two partitions are being compared, it can be considered that they have the same length because it is convenient to tap a partition with zeros to the length we need.

For a Young diagram, listing all the lengths of each column gives a partition. Conversely, every partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ corresponds to a Young diagram with λ_1 rows and n columns where the lengths of columns are $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. It is evident that this correspondence is a bijection between the set of partitions and the set of Young diagrams. For example, the Young diagram in Example 1 corresponds to the partition $(6, 3, 3, 2, 2, 1)$ of 17.

Let Young diagrams Y_λ and Y_μ correspond to the partitions $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $(\mu_1, \mu_2, \ldots, \mu_n)$, respectively. We say that Y_μ is a subdiagram of Y_λ and we write $Y_\mu \subseteq Y_\lambda$ if $\mu_i \leq \lambda_i$ for each $i = 1, 2, \ldots, n$. This gives a partial order on the set of Young diagrams.

3. Special subdiagrams and numerical sets

In this section, we will introduce special subdiagrams of a Young diagram. We will explore the relationship between Young diagrams and numerical sets to identify the elements within the corresponding numerical sets of these special subdiagrams. Additionally, we will examine Young diagrams of numerical semigroups and establish conditions under which the corresponding numerical sets for special subdiagrams also qualify as numerical semigroups.

Definition 3.1 *Let* $g, n \in \mathbb{N}$. *Let* Y *represent a Young diagram with* g *rows and* n *columns. The following process gives a new Young diagram which has less rows and columns than* Y.

1. *Add a single unit box immediately to the right of each row in* Y, *except for the rightmost column.*

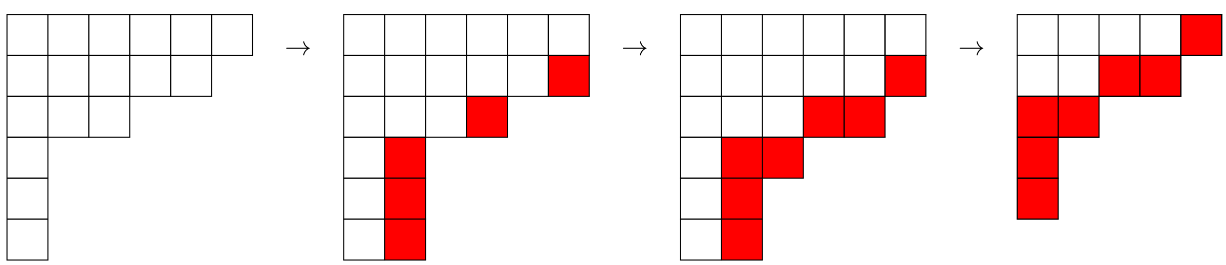
2. *Consider each row of Y, excluding the bottom row, and add one unit-box just beneath each box that did not receive a box under in the previous step.*

3. *Delete the columns on the left of the box added to the far left and most bottom, and delete the rows above the box added to the rightmost and very top.*

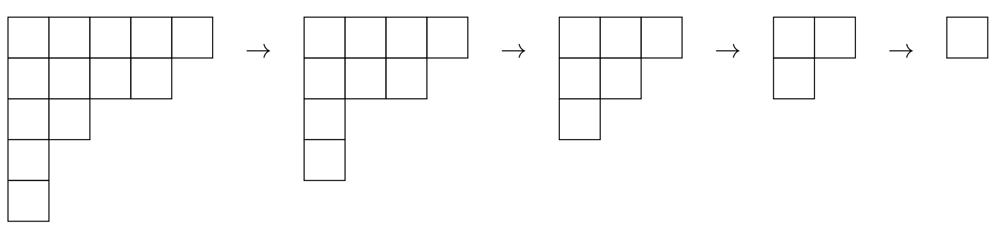
This Young diagram is called the first special diagram of Y.

By following the procedure outlined above on the initial special diagram of Y, we obtain another Young diagram, which we will refer to as the second special diagram of Y. Through induction, when the described process is applied to Y m times, it generates the mth special diagram of Y, provided it exists. It is essential to emphasize that, for a Young diagram with n columns, the maximum number of special diagrams can only be $n-1$. The final special diagram is achieved when we reach a Young diagram with either a single row or a single column. This implies that a Young diagram featuring only one row or one column does not possess any special diagram.

Example 3 *Let Y be the Young diagram having 6 rows and 6 columns as in Example 1. We get its first special diagram as follows;*



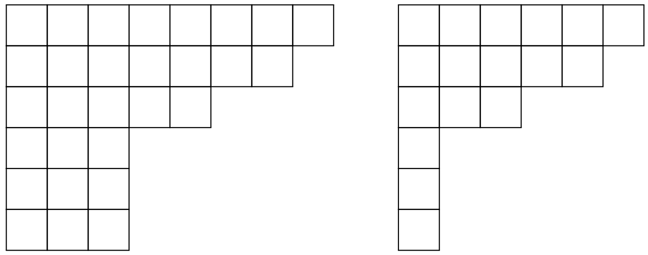
Then we get consecutive special diagrams of Y as follows;



Definition 3.2 *A Young diagram whose bottom row has length one is called a reduced Young diagram.*

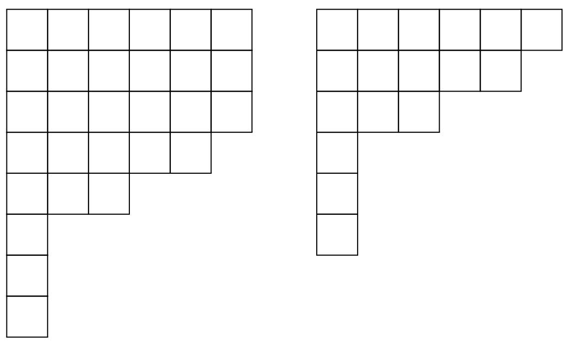
Let Y be a Young diagram whose bottom row has length a, and let $Y^\star$ denote the Young diagram obtained by deleting the first $a-1$ columns of Y. The way we define $Y^\star$ shows that if Y is reduced, then $Y^\star = Y$. It is also easy to see that special diagrams of Y and $Y^\star$ are completely the same. Therefore, to find the special diagrams of Y, we can consider $Y^\star$.

Example 4 *Let us consider the following two Young diagrams. Their special diagrams are all the same. Thus, to find the special diagrams of the one on the left, we can consider the one on the right.*



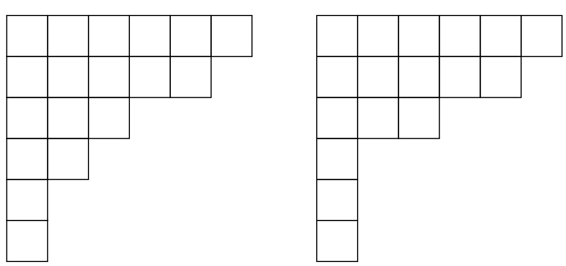
Likewise, if the rightmost column of a Young diagram Y has length b, to find its special diagrams, we can consider the Young diagram obtained by deleting the top $b-1$ rows of Y. It is evident that special diagrams in this scenario remain identical.

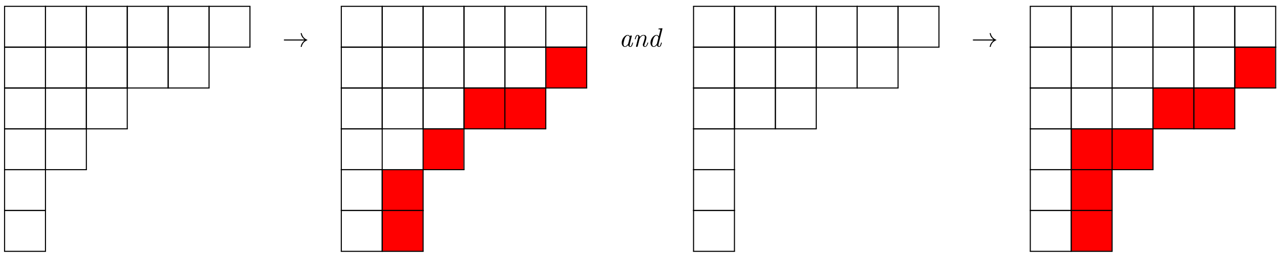
Example 5 *Let us consider the following Young diagrams. We can easily find out that their special diagrams are all the same.*



We also have some Young diagrams whose special diagrams are exactly the same even if their bottom rows and rightmost columns have length one. See the example below.

Example 6 *Let us consider the following Young diagrams. If we construct their first special diagrams, we can see that they are identical. Since they have the same first special diagrams, all the other consecutive special diagrams are completely the same.*





Next, we show that the first special diagram of a Young diagram is actually a subdiagram, and so special diagrams gives a sequence of subdiagrams.

Proposition 3.3 *The first special diagram of a Young diagram Y with n columns is a subdiagram of Y, and all special diagrams give a sequence of subdiagrams of Y with length smaller than n.*

Proof Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be the corresponding partition of Y. If Y is not reduced, then the bottom row of Y has length $a > 1$, i.e. $\lambda_1 = \lambda_2 = \cdots = \lambda_a > \lambda_{a+1}$, and $Y^\star$ corresponds to the partition $\lambda^\star = (\lambda_a, \lambda_{a+1}, \dots, \lambda_n)$. This means that $Y^\star$ is a subdiagram of Y. Since Y and $Y^\star$ have the same special diagrams, if the first special diagram of $Y^\star$ is a subdiagram of $Y^\star$, it is also a subdiagram of Y. Similarly, the length of the rightmost column of Y will not affect the special diagrams of Y. Thus, without loss of generality, we can assume that Y is reduced and $\lambda_n = 1$.

Let Y_T be the first special diagram of Y. Since Y is reduced, Y_T has $n-1$ columns. Now let $\mu = (\mu_1, \mu_2, \dots, \mu_{n-1})$ be the corresponding partition of Y_T. By the definition of Y_T, the ith column of Y_T is constructed by tapping $\lambda_i - \lambda_{i+1}$ unit boxes to the bottom of the $i+1$th column of Y and deleting one box from the top of the $i+1$th column of Y if $\lambda_i > \lambda_{i+1}$. And the ith column of Y_T is constructed by tapping one unit box to the bottom of the $i+1$th column of Y and deleting one box from the $i+1$th column of Y if $\lambda_i = \lambda_{i+1}$. This means that for $i = 1, 2, \dots, n-1$,

$$\mu_i = \begin{cases} \lambda_i - 1 & \text{if }, \lambda_i > \lambda_{i+1} \\ \lambda_{i+1} & \text{if } \lambda_i = \lambda_{i+1}. \end{cases}$$

Therefore, for each $i = 1, 2, \dots, n-1$, we get $\lambda_i \geq \mu_i$, i.e. Y_T is a subdiagram of Y. If we consider all the consecutive special diagrams of Y, the second special diagram of Y is a subdiagram of Y_T and so on. Hence, we get a sequence of subdiagrams of Y by induction. Since the maximum number of special diagrams is $n-1$, the length of this sequence must be smaller than n.

Henceforth, in this paper, we will refer to special diagrams of a Young diagram as special subdiagrams.

Definition 3.4 *A numerical set S is called reduced if its corresponding Young diagram Y_S is reduced.*

Proposition 3.5 *A numerical set S is reduced if and only if $1 \notin S$.*

Proof Let Y_S be the corresponding Young diagram to S. If Y_S is reduced, then the bottom row of Y_S has length one. Then by the construction of Y_S, we get $0 \in S$ and $1 \notin S$. The converse is true since the corresponding numerical set to Y_S is S.

Corollary 3.6 *All numerical semigroups except $\mathbb{N}_0$ are reduced.*

Proposition 3.7 *Let $S = \{0, s_1, \ldots, s_n, \rightarrow\}$ be a numerical set and s_r be the smallest element of S where $s_{r+1} - s_r \neq 1$. Then the numerical set $S^\star$ corresponding to $(Y_S)^\star$ is $S - r$.*

Proof If S is reduced, then $Y_S = (Y_S)^\star$ and $1 \notin S$. Since $1 \notin S$, the smallest element s_r of S such that $s_{r+1} - s_r \neq 1$ is $s_0 = 0$, and so $S - 0 = \{0 = s_0 - 0, s_1 - 0, s_2 - 0, \ldots, s_n - 0, \rightarrow\} = S$. And since $Y_S = (Y_S)^\star$, the corresponding numerical set to $(Y_S)^\star$ is S itself. This means that if S is reduced, then $S^\star = S$.

Suppose now that S is not reduced. Then $1 \in S$. This means that the smallest element s_r of S such that $s_{r+1} - s_r \neq 1$ is greater than or equal to 1. Thus, $s_0 = 0, s_1 = 1, \ldots, s_r = r$ and $s_{r+1} > r+1$. Then the bottom row of the corresponding Young diagram Y_S has length $r+1$. Deleting the first r columns of Y_S gives the Young diagram $(Y_S)^\star$, and the numerical set corresponding to $(Y_S)^\star$ is $\{s_r - r = 0, s_{r+1} - r > 1, s_{r+2} - r, \ldots, s_n - r, \rightarrow\}$. Hence, $S^\star = S - r$.

Next we consider a reduced numerical set S and its Young diagram Y_S to characterize elements and gaps of numerical sets corresponding to special subdiagrams of Y_S.

Proposition 3.8 *Let $S = \{0 = s_0, s_1, \ldots, s_n, \rightarrow\}$ be a reduced numerical set and Y_S be its Young diagram. If Y_T is the first special subdiagram of Y_S, then the corresponding numerical set to Y_T is $T = \{0 = t_0, t_1, \ldots, t_{n-1}, \rightarrow\}$ where each t_i is characterized as follows.*

1. $t_0 = s_0 = 0$.
2. *For* $i = 1, 2, \ldots, n-2$, $t_i = \begin{cases} s_i & \text{if } s_{i+1} - s_i \neq 1, \\ s_i - 1 & \text{if } s_{i+1} - s_i = 1. \end{cases}$
3. $t_{n-1} = s_{n-1}$.

Proof By the construction of Y_S, the number of columns of Y_S is the number of small elements of S, which is n, and the number of rows of Y_S is the number of gaps of S, which is the genus $g = \mathrm{g}(S)$ of S. Since S is reduced, the length of the bottom row of Y_S is 1. This means when we construct Y_T, we only get one column on the left of the box added to the far left and bottommost. Thus, we only delete one column from the left and we get $n-1$ columns for Y_T, i.e. the corresponding numerical set T has $n-1$ small elements. Also, by the correspondence between Y_T and T, we get $t_0 = 0$. Hence, T is of the form $\{0 = t_0, t_1, \ldots, t_{n-1}, \rightarrow\}$.

The length of the rightmost column of Y_S gives the number of the gaps between s_{n-1} and s_n. When we construct Y_T, the rightmost column of Y_S gets at least a box underneath. Then we delete rows over the one added to the rightmost and topmost, meaning we delete $s_n - s_{n-1} - 1$ rows. Therefore, the number of rows of Y_T is $g - (s_n - s_{n-1} - 1)$, which is the genus of T, i.e. $\mathrm{g}(T) = g - s_n + s_{n-1} + 1$. On the other hand, for every numerical set S, each small element s_i is the sum of the number of gaps smaller than s_i and the number of small elements smaller than s_i. In particular, the conductor s_n is the sum of the number of small elements and the number of gaps of S, i.e. $s_n = n + g$. Therefore, we get

$$t_{n-1} = (n-1) + g - (s_n - s_{n-1} - 1) = n - 1 + g - s_n + s_{n-1} + 1 = s_n - s_n + s_{n-1} = s_{n-1}.$$

By the correspondence between Y_S and S, each small element s_i of S is represented by the bottommost box of the $i+1$th column of Y_S, and each gap is represented by the rightmost box in rows of Y_S. Let the box representing s_i be denoted by B_i. Notice that the number of gaps smaller than s_i is the number of rows under B_i, and the number of small elements smaller than s_i is the number of columns which are on the left of B_i.

If s_i and s_{i+1} are consecutive numbers, i.e. $s_{i+1} - s_i = 1$, then the boxes B_i and B_{i+1} are in the same row. Then by the construction of Y_T, we get a box under B_{i+1} which represents a small element of T. Since S is reduced, this small element is the $i+1$th element, i.e. it is $t_i \in T$. Then since t_i is represented by the box added under B_{i+1}, t_i is the sum of the number of rows under B_{i+1} minus 1 and the number of columns which are on the left of B_{i+1} minus 1 in Y_S. Therefore, $t_i = s_{i+1} - 2 = (s_i + 1) - 2 = s_i - 1$.

If there is a gap between s_i and s_{i+1}, i.e. $s_{i+1} - s_i \neq 1$, then there are $s_{i+1} - s_i - 1$ gaps between s_i and s_{i+1}. Then we get $s_{i+1} - s_i - 1$ boxes under B_{i+1} when we construct Y_T. Since S is reduced, the column containing B_{i+1} is the $i+1$th column of Y_T, and the bottom box in this column represents the ith small element t_i of T, denote it by B_i'. Then B_i and B_i' are consecutive boxes in the same row of Y_T. Since t_i is the sum of the number of rows under B_i' and the number of columns on the left of B_i' in Y_T, which are exactly the number of rows under B_i and the number of columns on left of B_i in Y_S, respectively, $t_i = s_i$.

Proposition 3.9 *Let $S = \{0 = s_0, s_1, \dots, s_n, \rightarrow\}$ be a reduced numerical set and Y_S be its Young diagram where the gap set of S is $\mathrm{G}(S) = \{g_1, \dots, g_t\}$. If T is the numerical set corresponding to the first special subdiagram of Y_S, then for some $k \in \{1, 2, \dots, t-1\}$,*

1. $g_{k+1} - g_k \leq 2 \implies g_k \in \mathrm{G}(T)$,
2. $g_{k+1} - g_k > 2 \implies g_k \notin \mathrm{G}(T)$ *and* $g_{k+1} - 2 \in \mathrm{G}(T)$,
3. $g_{k+1} > s_{n-1} \implies g_r \notin \mathrm{G}(T)$ *for all* $r \geq k$,

where $\mathrm{G}(T)$ is the gap set of T.

Proof Let $T = \{0 = t_0, t_1, \dots, t_{n-1}, \rightarrow\}$ and $g_{k+1}, g_k \in \mathrm{G}(S)$ for some $k \in \{1, 2, \dots, t-1\}$. If $g_{k+1} - g_k = 1$, then $g_{k+1} = g_k + 1$. Therefore, if s_i is the largest element of S which is less than g_k, then s_{i+1} is the least element of S bigger than g_{k+1}. In this case, we get $s_{i+1} - s_i \neq 1$. By Proposition 3.8, $t_i = s_i \in T$, and $t_{i+1} = s_{i+1}$ or $t_{i+1} = s_{i+1} - 1$. If $t_{i+1} = s_{i+1}$, then we get $t_i < g_k < g_{k+1} < t_{i+1}$. If $t_{i+1} = s_{i+1} - 1$, then we also get $t_i < g_k < g_{k+1} \leq t_{i+1}$. This means $g_k \notin T$, i.e. $g_k \in \mathrm{G}(T)$.

Now suppose $g_{k+1} - g_k = 2$. If s_i is the largest element of S which is less than g_k, then we get $s_{i+1} = g_k + 1 = g_{k+1} - 1$ and $g_k < s_{i+1} < g_{k+1}$. Then s_{i+2} is the least element of S bigger than g_{k+1}. This means that $s_{i+1} - s_i \neq 1$ and $s_{i+2} - s_{i+1} \neq 1$. Therefore, by Proposition 3.8, we have $t_i = s_i \in T$ and $t_{i+1} = s_{i+1} \in T$, and so $g_k \notin T$. Hence, $g_k \in \mathrm{G}(T)$. This finishes the proof of *1*.

If $g_{k+1} - g_k > 2$, then S has at least two elements between g_k and g_{k+1}. In this case, if s_i is the largest element of S which is less than g_k, we have $s_i < g_k < s_{i+1} < s_{i+2} < \cdots < s_{i+n} < g_{k+1} < s_{i+n+1}$ where $n \geq 2$. Then $s_{i+1} - s_i \neq 1$ and $s_{i+2} - s_{i+1} = 1$. Also, by Proposition 3.8, we get $t_i = s_i \in T$ and $t_{i+1} = s_{i+1} - 1 = g_k \in T$. Hence, $g_k \notin G(T)$. On the other hand, since $s_{i+n+1} - s_{i+n} \neq 1$ and $s_{i+n} - s_{i+n-1} = 1$, by Proposition 3.8 again, we get $t_{i+n} = s_{i+n} \in T$ and $t_{i+n-1} = s_{i+n-1} - 1 \in T$. This means that $t_{i+n-1} = s_{i+n-1} - 1 < s_{i+n} = g_{k+1} - 2 < t_{i+n} = s_{i+n}$. Therefore, $g_{k+1} - 2 \notin T$, i.e. $g_{k+1} - 2 \in \mathrm{G}(T)$. This finishes the proof of *2*.

The proof of *3* follows from Definition 3.1.

When S is a numerical semigroup, the following theorem provides precise conditions for when corresponding numerical sets of special subdiagrams of Y_S become numerical semigroups.

Theorem 3.10 *Let $S = \{0 = s_0, s_1, \ldots, s_n, \rightarrow\}$ be a numerical semigroup and Y_S be its Young diagram. Let Y_T be the first special subdiagram of Y_S, and T be the corresponding numerical set to Y_T. Then T is a numerical semigroup if and only if for any nonzero elements $x, y \in T$ we have $x + y \geq t_{n-1}$, or $x + y < t_{n-1}$ and there exists an element $s_k \in S$ for some $k \in \{1, 2, \ldots, n-2\}$ where one of the following conditions hold*

1. $x + y = s_k$ *and* $s_k + 1 \notin S$,
2. $x + y = s_k$ *and* $s_k + 1, s_k + 2 \in S$,
3. $x + y = s_k - 1$ *and* $s_k + 1 \in S$,
4. $x + y = s_k - 2$ *and* $s_k - 1 \in S$,
5. $x + y = s_k - 2$ *and* $s_k - 2 \in S$.

Proof Let $T = \{0 = t_0, t_1, \ldots, t_{n-1}, \rightarrow\}$. By Proposition 3.8, we know how to get the elements of T from S. To determine when T is a numerical semigroup, let us take two nonzero elements $x, y \in T$. If $x+y \geq t_{n-1}$, then $x+y \in T$ and there is nothing to consider. Thus, we suppose that $x = t_i$, $y = t_j$ for some $i, j \in \{1, 2, \ldots, n-2\}$ and $t_i + t_j < t_{n-1}$. By Proposition 3.8, $t_i = s_i$ if $s_{i+1} - s_i \neq 1$, i.e. $s_i + 1 \notin S$, and $t_i = s_i - 1$ if $s_{i+1} - s_i = 1$, i.e. $s_i + 1 \in S$. Similarly, $t_j = s_j$ if $s_{j+1} - s_i \neq 1$, i.e. $s_j + 1 \notin S$, and $t_j = s_j - 1$ if $s_{j+1} - s_j = 1$, i.e. $s_j + 1 \in S$. This means that we have four cases to investigate:

Case 1: Assume that $s_{i+1} - s_i \neq 1$ and $s_{j+1} - s_j \neq 1$. This means that $s_i + 1, s_j + 1 \notin S$ and $t_i = s_i$, $t_j = s_j$. Then $t_i + t_j = s_i + s_j \in S$. Since $t_i + t_j < t_{n-1}$, we have $s_i + s_j < s_n$ and an element $s_k \in S$ for some $k \in \{1, 2, \ldots, n-2\}$ such that $s_i + s_j = s_k$, i.e. $t_i + t_j = s_k$. On the other hand, since $s_k \in S$,

$$\begin{aligned} t_i + t_j \in T \Longleftrightarrow & s_k \in T \\ \Longleftrightarrow & s_{k+1} - s_k \neq 1 \text{ or } s_k + 1 = s_{k+1}, s_k + 2 = s_{k+2} \\ \Longleftrightarrow & s_k + 1 \notin S \text{ or } s_k + 1, s_k + 2 \in S. \end{aligned}$$

This covers *1* and *2*.

Case 2: Assume that $s_{i+1} - s_i = 1$ and $s_{j+1} - s_j \neq 1$. This implies that $s_i + 1 \in S$, $s_j + 1 \notin S$ and $t_i = s_i - 1$, $t_j = s_j$. Since $t_i + t_j < t_{n-1}$, we get $s_i + s_j < s_n$ and an element $s_k \in S$ for some $k \in \{1, 2, \ldots, n-2\}$ such that $s_i + s_j = s_k$, i.e. $t_i + t_j = s_i + s_j - 1 = s_k - 1$. Since $s_k \in S$, in this case

$$t_i + t_j \in T \Longleftrightarrow s_k - 1 \in T \Longleftrightarrow s_k + 1 \in S.$$

Case 3: Suppose that $s_{i+1} - s_i \neq 1$ and $s_{j+1} - s_j = 1$. This is similar to Case 2. These cover *3*.

Case 4: Suppose now that $s_{i+1} - s_i = 1$ and $s_{j+1} - s_j = 1$. This implies that $s_i + 1, s_j + 1 \in S$ and $t_i = s_i - 1$, $t_j = s_j - 1$. Since $t_i + t_j < t_{n-1}$ and $t_{n-1} \leq s_n - 2$, we get $s_i + s_j - 2 < s_n$ and an element $s_k \in S$ for some $k \in \{1, 2, \ldots, n-2\}$ such that $s_i + s_j = s_k$, i.e. $t_i + t_j = s_i + s_j - 2 = s_k - 2$. In this case, since $s_k \in S$,

$$t_i + t_j \in T \Longleftrightarrow s_k - 2 \in T \Longleftrightarrow s_{k-1} = s_k - 1 \text{ or } s_{k-1} = s_k - 2.$$

This covers *4* and *5*. The converse of the proof is clear by definitions.

Example 7 *Let us consider the numerical semigroup* $S = \{0, 6, 8, 9, 10, 12, \rightarrow\}$ *and its Young diagram* Y_S *where the gap set of* S *is* $\mathrm{G}(S) = \{1, 2, 3, 4, 5, 7, 11\}$*. If the first special subdiagram of* Y_S *is* Y_T *where* T *is its corresponding numerical set. We use Proposition 3.8 and find* $T = \{0, 6, 7, 8, 10, \rightarrow\}$*. Furthermore, by using Proposition 3.9, we see that* $1, 2, 3, 4, 5 \in \mathrm{G}(T)$ *and* $7, 11 \notin G(T)$*. However, we actually have* $G(T) = \{1, 2, 3, 4, 5, 9\}$*. It is also easy to see that* T *is a numerical semigroup and it satisfies the conditions of Theorem 3.10.*

4. Special subdiagrams of symmetric and almost symmetric numerical semigroups

In this section, we first consider symmetric numerical sets and we prove all numerical sets corresponding to special subdiagrams of the Young diagram of a symmetric numerical set are symmetric as well. Then we consider almost symmetric numerical semigroups and explain with an example that numerical sets corresponding to special subdiagrams of the Young diagram of an almost symmetric numerical semigroup do not have to be almost symmetric even if they are numerical semigroups.

Remember that a numerical set S is symmetric if and only if for each $k \in \{0, 1, 2, \ldots, \mathrm{F}(S)\}$ exactly one of k and $\mathrm{F}(S) - k$ is an element of S, and that a Young diagram is called symmetric if its corresponding numerical set is symmetric.

Proposition 4.1 *Let* S *be a numerical set and* Y_S *be its Young diagram. Let* Y_T *be the first special subdiagram of* Y_S *and* T *be the corresponding numerical set to* Y_T*. If* S *is symmetric, then* T *is also symmetric.*

Proof Let $S = \{0 = s_0, s_1, \ldots, s_n, \rightarrow\}$ and s_r be the smallest element of S which satisfies that $s_{r+1} - s_r \neq 1$. Then $s_0 = 0, s_1 = 1, \ldots, s_r = r$ and $s_{r+1} > r + 1$. Since S is symmetric, $1, 2, \ldots, r \in S$ implies that $\mathrm{F}(S) - 1, \mathrm{F}(S) - 2, \ldots, \mathrm{F}(S) - r \notin S$ and $r + 1 \notin S$ implies that $\mathrm{F}(S) - r - 1 \in S$, and so $s_{n-1} = \mathrm{F}(S) - r - 1$. Therefore, the lengths of the bottom row and the rightmost column of Y_S are equal and $r + 1$. By Examples 4 and 5, if we delete the first r columns and rows of Y_S, it will not affect the special subdiagrams and gives us a Young diagram whose corresponding numerical set is $S' = \{0, s_{r+1} - r, \ldots, s_{n-1} - r, s_{n-1} - r + 2, \rightarrow\}$. Namely, Y_S and $Y_{S'}$ have identical special subdiagrams. Since $s_{r+1} - r > 1$, S' is reduced. Therefore, to find T, we can now apply Proposition 3.8 to S'. On the other hand, take $k \in \{0, 1, 2, \ldots, \mathrm{F}(S') = s_{n-1} - r + 1\}$ and assume that $k \in S'$. Then $k = s_j - r$ for some $j \in \{r+1, \ldots, n-1\}$, and $\mathrm{F}(S') - k = s_{n-1} - r + 1 - (s_j - r) = s_{n-1} + 1 - s_j$. If $s_{n-1} + 1 - s_j \in S'$, then $s_{n-1} + r + 1 - s_j = \mathrm{F}(S) - s_j \in S$. However, this is impossible because S is symmetric. Therefore, $\mathrm{F}(S') - k \notin S'$, and so S' is symmetric. Hence, without loss of generality we can assume that S is reduced.

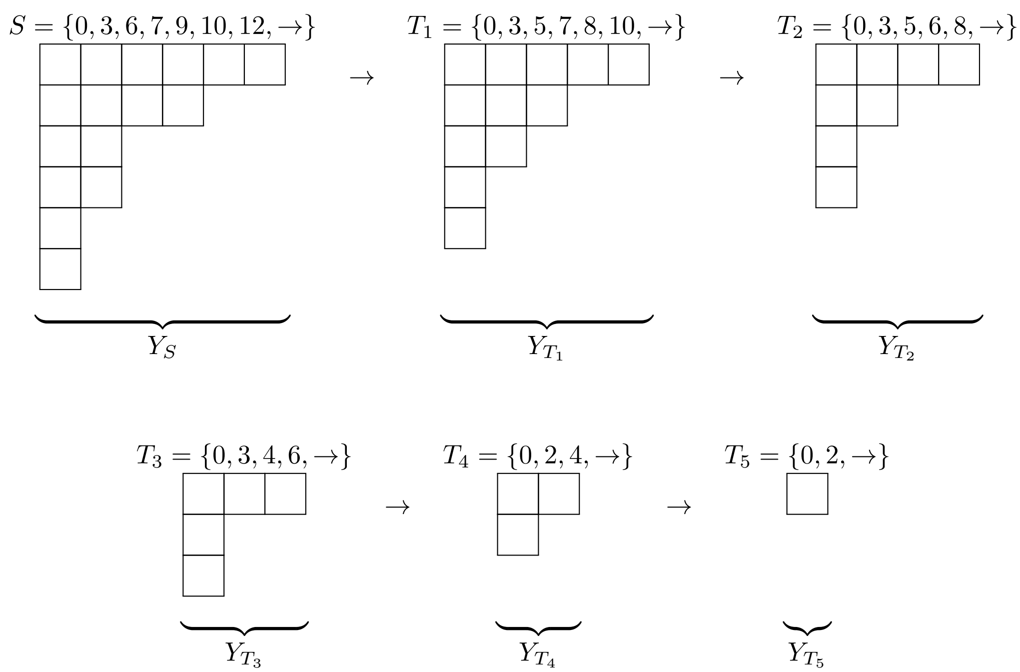
Assume now that $S = \{0 = s_0, s_1, \ldots, s_n, \rightarrow\}$ is reduced and $T = \{0 = t_0, t_1, \ldots, t_{n-1}, \rightarrow\}$. Since S is reduced, $1 \notin S$. And since S is symmetric, $1 \notin S \implies \mathrm{F}(S) - 1 \in S$. Therefore, $s_{n-1} = \mathrm{F}(S) - 1 = s_n - 2$, and so $s_n - s_{n-1} \neq 1$. This means that $t_{n-1} = s_{n-1}$ and $\mathrm{F}(T) = t_{n-1} - 1 = s_{n-1} - 1 = s_n - 3 = \mathrm{F}(S) - 2$. Now take $k \in \{1, 2, \ldots, \mathrm{F}(T)\}$ and suppose that $k \in T$. Then $k = t_i$ for some $i \in \{1, 2, \ldots, n-2\}$ and we have two cases.

Case 1: If $k = t_i = s_i$, then $s_{i+1} - s_i \neq 1$, i.e. $s_i + 1 \notin S$. Therefore, since S is symmetric, we have $\mathrm{F}(S) - s_i = s_n - s_i - 1 \notin S$, and $\mathrm{F}(S) - s_i - 1 = s_n - s_i - 2 \in S$. By Proposition 3.8, $s_n - s_i - 3$ cannot be an element of T. That is, $\mathrm{F}(T) - k = s_n - s_i - 3 \notin T$. Thus, in this case T is symmetric.

Case 2: If $k = t_i = s_i - 1$, then $s_{i+1} - s_i = 1$, meaning $s_{i+1} = s_i + 1$. Since S is symmetric, we have $\mathrm{F}(S) - s_{i+1} = s_n - 1 - s_i - 1 = s_n - s_i - 2 \notin S$ and $\mathrm{F}(S) - s_i = s_n - s_i - 1 \notin S$. Then by Proposition 3.9, $s_n - s_i - 2 \notin T$. On the other hand, $\mathrm{F}(T) - k = s_n - 3 - s_i + 1 = s_n - s_i - 2$. Thus, in this case as well, T is symmetric.

By Proposition 4.1, for a symmetric numerical semigroup S, all the numerical sets corresponding to special subdiagrams of Y_S are symmetric. However, it is not guaranteed that they are also semigroups. The next example illustrates special subdiagrams of a symmetric numerical semigroup.

Example 8 *Let us find special subdiagrams of Y_S when $S = \{0, 3, 6, 7, 9, 10, 12, \rightarrow\}$.*



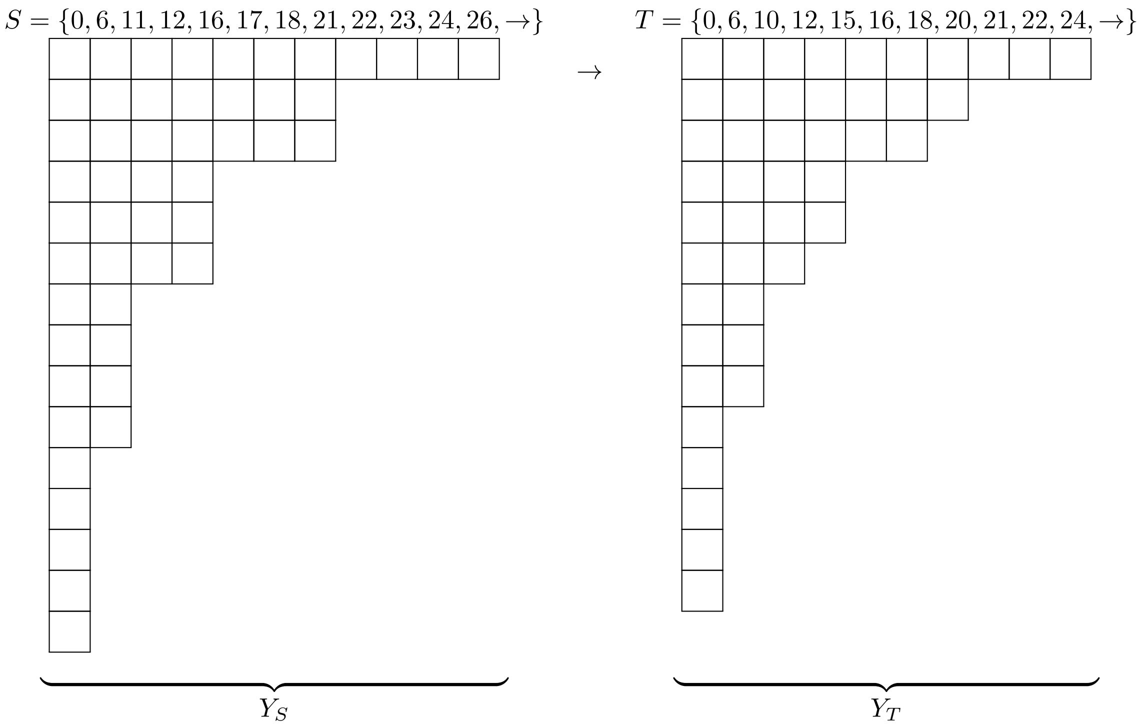
Notice that each T_i is symmetric, but T_1 is not a numerical semigroup.

Let S be numerical semigroup and $z \in \mathbb{Z}$. We say that z is a pseudo-Frobenius number of S if $z \notin S$ and $z + s \in S$ for all nonzero element $s \in S$. The set of pseudo-Frobenius numbers of S is denoted by $\mathrm{PF}(S)$, and the cardinality of $\mathrm{PF}(S)$ is called the type of S, denoted by $\mathrm{t}(S)$.

It is well-known that $2\,\mathrm{g}(S) \geq \mathrm{F}(S) + \mathrm{t}(S)$ is valid for any numerical semigroup S. When the equality holds, i.e. $2\,\mathrm{g}(S) = \mathrm{F}(S) + \mathrm{t}(S)$, S is called almost symmetric. The notion of almost symmetric numerical semigroups is one of the mostly studied concepts in numerical semigroup theory which were introduced by Barucci and Fröberg in [1]. They are the natural generalizations of symmetric numerical semigroups used for studying generalizations of one-dimensional Gorenstein rings. Remember that a numerical semigroup is almost symmetric of type 1 if and only if it is symmetric, and that a numerical semigroup is almost symmetric of type 2 if and only if it is pseudosymmetric.

Next example shows that given an almost symmetric numerical semigroup the corresponding numerical set to the first special subdiagram of its Young diagram does not have to be almost symmetric even if it is a numerical semigroup.

Example 9 *Let* $S = \{0, 6, 11, 12, 16, 17, 18, 21, 22, 23, 24, 26, \rightarrow\}$ *be an almost symmetric numerical semigroup with* $\mathrm{PF}(S) = \{5, 10, 15, 20, 25\}$*. Let* Y_S *be the Young diagram of* S*. We find the first special subdiagram* Y_T *of* Y_S *and its corresponding numerical set* $T = \{0, 6, 10, 12, 15, 16, 18, 20, 21, 22, 24, \rightarrow\}$ *depicted below. It is easy to find that* T *is a numerical semigroup with* $\mathrm{G}(T) = \{1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 14, 17, 19, 23\}$*,* $\mathrm{g}(T) = 14$ *and* $\mathrm{PF}(T) = \{14, 19, 23\}$*. This means* $2\,\mathrm{g}(T) \neq \mathrm{F}(T) + \mathrm{t}(T)$*, i.e.* T *is not almost symmetric.*



5. Special subdiagrams of Arf numerical semigroups

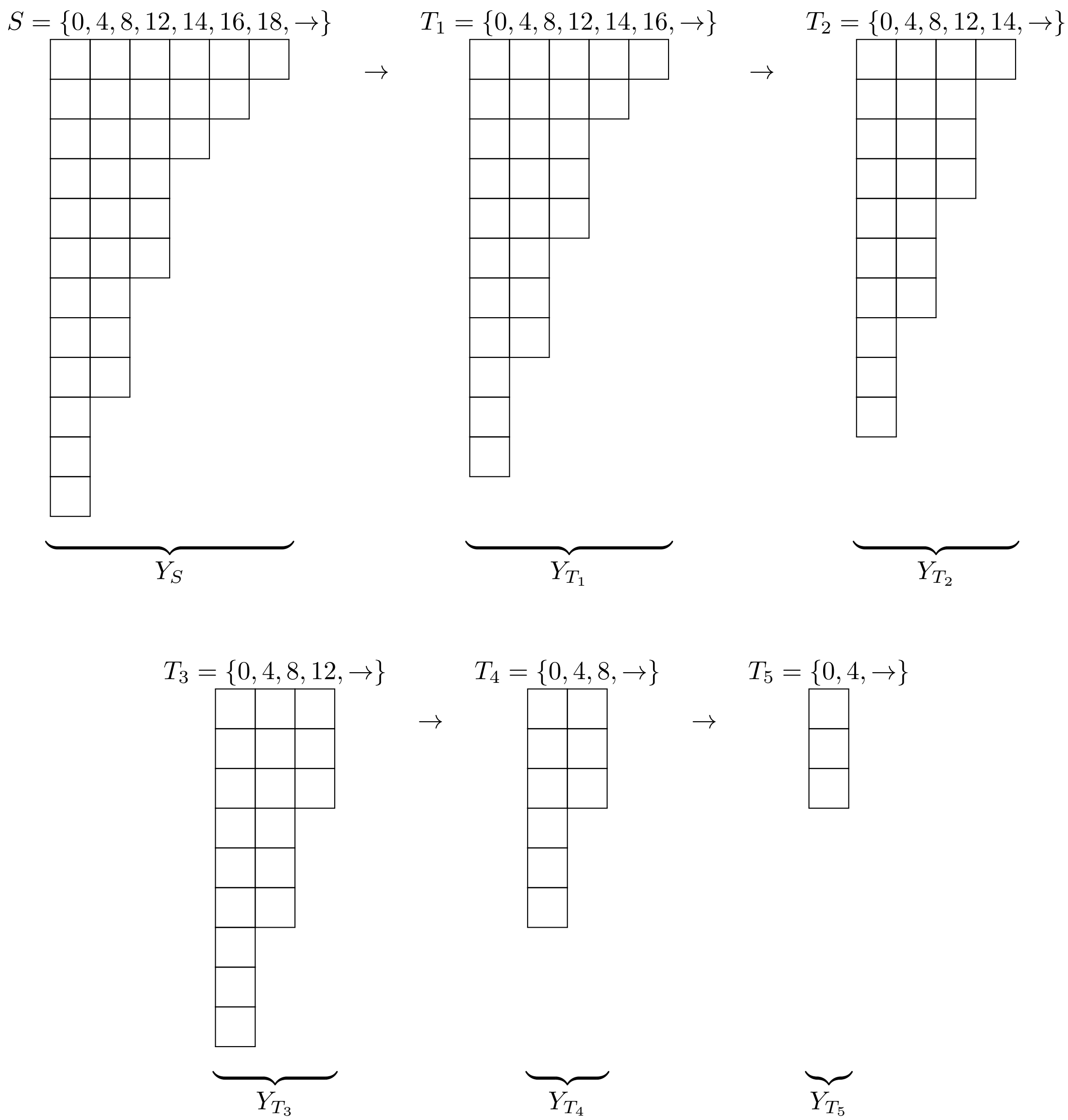
In this section, we focus on Arf numerical semigroups and demonstrate that the numerical sets corresponding to special subdiagrams of the Young diagram of an Arf numerical semigroup also satisfies the Arf property. Let us recall that a numerical semigroup S is Arf if for every $s_1, s_2, s_3 \in S$ with $s_1 \leq s_2 \leq s_3$, we have the property that $s_2 + s_3 - s_1 \in S$. This is equivalent that for every $s_1, s_2 \in S$ with $s_1 \leq s_2$ we have $2s_2 - s_1 \in S$.

Proposition 5.1 *Let* S *be a numerical set and* Y_S *be its Young diagram. Let* Y_T *be the first special subdiagram of* Y_S *and* T *be its corresponding numerical set. If* S *is an Arf numerical semigroup, then* T *is also an Arf numerical semigroup.*

Proof Let $S = \{0 = s_0, s_1, \ldots, s_n, \rightarrow\}$ and $T = \{0 = t_0, t_1, \ldots, t_{n-1}, \rightarrow\}$. We first note that $s_n - s_{n-1} \neq 1$, since s_n is the conductor. Recall that when S is an Arf numerical semigroup, we have $s_i - s_{i-1} \neq 1$ for $i \leq n$. To prove this property, assume on the contrary that $s_i - s_{i-1} = 1$ for some $i < n$. By the definition of Arf numerical semigroups, we have $2s_i - s_{i-1} \in S$. In this case, we get $2s_i - s_{i-1} = 2(s_{i-1} + 1) - s_{i-1} = s_{i-1} + 2 = s_i + 1 \in S$. Therefore, we have $s_{i+1} = s_i + 1$, i.e. $s_{i+1} - s_i = 1$. Similarly, we get $s_i + 2 \in S$ and $s_{i+2} = s_{i+1} + 1$. Inductively, we get $s_i + k \in S$ for all $k \in \mathbb{N}$. This contradicts with s_n being the conductor. Therefore, when S is an Arf numerical semigroup, $s_i - s_{i-1} \neq 1$ for $i \leq n$. Now by Proposition 3.8, $T = \{0 = t_0 = s_0, t_1 = s_1, \ldots, t_{n-1} = s_{n-1}, \rightarrow\}$ which is clearly an Arf numerical semigroup.

By Proposition 5.1, for an Arf numerical semigroup $S = \{0 = s_0, s_1, \ldots, s_n, \rightarrow\}$, all the numerical sets corresponding to special subdiagrams of Y_S are Arf numerical semigroups. If T_i is the corresponding Arf numerical semigroup to the ith special subdiagram of Y_S, then we can easily describe it by using induction as $T_i = \{0 = s_0, s_1, \ldots, s_{n-i}, \rightarrow\}$.

Example 10 *Consider the Arf numerical semigroup* $S = \{0, 4, 8, 12, 14, 16, 18, \rightarrow\}$*. By Proposition 5.1, we list the corresponding Arf numerical semigroups and the special subdiagrams of* Y_S *as follows:*



Acknowledgements

This work was supported by The Scientific and Technological Research Council of Türkiye (TÜBİTAK) (project number: 122F475). We thank the anonymous reviewers for their careful reading and their many insightful comments and suggestions.

References

[1] Barucci V, Fröberg R. One-dimensional almost Gorenstein rings. Journal of Algebra 1997; 188: 418-442. https://doi.org/10.1006/jabr.1996.6837

[2] Constantin H, Houston-Edwards B, Kaplan B. Numerical sets, core partitions, and integer points in polytopes. In: Proceedings of Combinatorial and Additive Number Theory II; 2017; New York, NY, USA. pp. 99-127.

[3] Fulton W. Young Tableaux: With Applications to Representation Theory and Geometry. New York, NY, USA: Cambridge University Press, 1997.

[4] Gümüşbaş N, Tutaş N. A Decomposition of Arf Semigroups Filomat 2020; 34 (2): 491-498. https://doi.org/10.2298/FIL2002491G

[5] Gümüşbaş N, Tutaş N, Er N. Almost symmetric Arf partitions. Turkish Journal of Mathematics 2020; 44 (6): 2185-2198. https://doi.org/10.3906/mat-2003-74

[6] Karakaş Hİ, Tutaş N. A decomposition of partitions and numerical sets. Semigroup Forum 2020; 101: 704-715. https://doi.org/10.1007/s00233-019-10080-7

[7] Keith W, Nath R. Partitions with prescribed hooksets. Journal of Combinatorics and Number Theory 2011; 3 (1): 39-50.

[8] Rosales JC, García-Sánchez PA. Numerical Semigroups. New York NY, USA: Springer, 2009.

[9] Süer M, Yeşil M. Symmetric and pseudo-symmetric numerical semigroups via Young diagrams and their semigroup rings. Journal of Korean Mathematical Society 2021; 58 (6): 1367-1383 https://doi.org/10.4134/JKMS.j210007

[10] Tutaş N, On Partitions and Arf Semigroups. Open Mathematics 2019; 17 (1): 343-355. https://doi.org/10.1515/math-2019-0025

[11] Tutaş N, Karakaş Hİ, Gümüşbaş N. Young tableaux and Arf partitions. Turkish Journal of Mathematics 2019; 43 (1): 448-459. https://doi.org/10.3906/mat-1807-181