Turkish Journal of Mathematics

Volume 48 | Number 2 Article 15

3-8-2024

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Recommended Citation

BENCHEVA, VICTORIA and MILOUSHEVA, VELICHKA (2024) "Timelike surfaces with parallel normalized mean curvature vector field in the Minkowski 4-space," *Turkish Journal of Mathematics*: Vol. 48: No. 2, Article 15. https://doi.org/10.55730/1300-0098.3509
Available at: https://journals.tubitak.gov.tr/math/vol48/iss2/15

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2024) 48: 327 – 345
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doi:10.55730/1300-0098.3509

Timelike surfaces with parallel normalized mean curvature vector field in the Minkowski 4-space

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Received: 10.07.2023 • **Accepted/Published Online:** 16.02.2024 • **Final Version:** 08.03.2024

Abstract: In the present paper, we study timelike surfaces with parallel normalized mean curvature vector field in the four-dimensional Minkowski space. We introduce special isotropic parameters on each such surface, which we call canonical parameters, and prove a fundamental existence and uniqueness theorem stating that each timelike surface with parallel normalized mean curvature vector field is determined up to a rigid motion in the Minkowski space by three geometric functions satisfying a system of three partial differential equations. In this way, we minimize the number of functions and the number of partial differential equations determining the surface, thus solving the Lund-Regge problem for this class of surfaces.

Key words: Parallel normalized mean curvature vector field, canonical parameters, fundamental theorem

1. Introduction

In the local theory of surfaces, both in Euclidean and pseudo-Euclidean spaces, one of the basic problems is to find a minimal number of invariant functions, satisfying some natural conditions, that determine the surface up to a motion. This problem is known as the Lund-Regge problem [19]. It is solved for minimal (or maximal) surfaces of codimension two in the Euclidean 4-space $\mathbb{R}^4$, the Minkoswki space $\mathbb{R}^4_1$, and the pseudo-Euclidean space $\mathbb{R}^4_2$. The surfaces with zero mean curvature in these spaces admit locally geometrically determined special isothermal parameters, called *canonical*, such that the two main invariants (the Gaussian curvature and the normal curvature) of the surface satisfy a system of two partial differential equations called a *system of natural PDEs*. The number of the invariant functions determining the surfaces and the number of the differential equations are reduced to two. Moreover, the geometry of the corresponding zero mean curvature surface (minimal or maximal) is determined by the solutions of this system of natural PDEs.

Special geometric parameters on minimal surfaces in $\mathbb{R}^4$ were introduced by T. Itoh in [14], and further, these parameters were used to prove that a minimal surface in $\mathbb{R}^4$ is determined up to a motion by two invariant functions satisfying a system of two PDEs [23]. Based on the canonical parameters, the system of natural PDEs was solved explicitly in terms of two holomorphic functions [9]. The same problem was solved for maximal spacelike surfaces and minimal timelike surfaces in the Minkowksi space $\mathbb{R}^4_1$. Special isothermal parameters on maximal spacelike surfaces in $\mathbb{R}^4_1$ were introduced in [2] and it was proved that the local geometry of these surfaces is determined by two invariant functions satisfying two PDEs. On the base of these canonical

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2010 *AMS Mathematics Subject Classification:* Primary 53B30, Secondary 53A35, 53B25

parameters the system of natural PDEs of maximal spacelike surfaces was solved explicitly in [10]. Minimal timelike surfaces in $\mathbb{R}^4_1$ were studied by G. Ganchev and the second author in [12] and it was proved that they admit locally canonical parameters and their geometry is determined by two invariant functions, satisfying the following system of natural PDEs:

$$\begin{array}{lcl} \sqrt[4]{K^2+\varkappa^2}\ \Delta^h \ln \sqrt[4]{K^2+\varkappa^2} & = & 2K; \\ \sqrt[4]{K^2+\varkappa^2}\ \Delta^h \arctan \dfrac{\varkappa}{K} & = & 2\varkappa; \end{array} \qquad K^2+\varkappa^2 \neq 0,$$

where K is the Gaussian curvature, $\varkappa$ is the curvature of the normal connection (the normal curvature), and Δ^h is the hyperbolic Laplace operator.

Similar results were obtained for minimal Lorentz surfaces in the pseudo-Euclidean space with neutral metric $\mathbb{R}^4_2$ in [1], [11], and [15].

Thus, the following natural question arises: *How to introduce canonical parameters and obtain natural equations for other classes of surfaces in 4-dimensional spaces?*

This problem can be solved for the surfaces with parallel normalized mean curvature vector field—another important class of surfaces both in Riemannian and pseudo-Riemannian geometry, since being a natural extension of the surfaces with parallel mean curvature vector field, they play an important role in differential geometry and physics.

Surfaces with parallel normalized mean curvature vector field in the Euclidean 4-space $\mathbb{R}^4$ and spacelike surfaces with parallel normalized mean curvature vector field in the Minkowski 4-space $\mathbb{R}^4_1$ were studied by G. Ganchev and the second author in [13]. These classes of surfaces are described in terms of the so-called canonical parameters. Each surface with parallel normalized mean curvature vector field in $\mathbb{R}^4$ is determined up to a motion by three functions $\lambda(u,v)$, $\mu(u,v)$ and $\nu(u,v)$ satisfying the following system of partial differential equations

$$\begin{aligned} &\nu_u = \lambda_v - \lambda(\ln|\mu|)_v; \\ &\nu_v = \lambda_u - \lambda(\ln|\mu|)_u; \\ &\nu^2 - (\lambda^2+\mu^2) = \tfrac{1}{2}|\mu|\Delta \ln|\mu|, \end{aligned}$$

where Δ denotes the Laplace operator.

The class of spacelike surfaces with parallel normalized mean curvature vector field in the Minkowski space $\mathbb{R}^4_1$ is described by three functions $\lambda(u,v)$, $\mu(u,v)$, and $\nu(u,v)$, satisfying the following system of PDEs

$$\begin{aligned} &\nu_u = \lambda_v - \lambda(\ln|\mu|)_v; \\ &\nu_v = \lambda_u - \lambda(\ln|\mu|)_u; \\ &\varepsilon(\nu^2 - \lambda^2+\mu^2) = \tfrac{1}{2}|\mu|\Delta \ln|\mu|, \end{aligned}$$

where $\varepsilon = 1$ corresponds to the case where the mean curvature vector field is spacelike, and $\varepsilon = -1$ corresponds to the case where the mean curvature vector field is timelike.

In the present paper, we focus our attention on the class of timelike surfaces with parallel normalized mean curvature vector field in the Minkowski 4-space $\mathbb{R}^4_1$. On each such surface, we introduce special isotropic parameters (u,v), which we call *canonical*, that allow us to prove the fundamental existence and uniqueness theorem in terms of three geometrically determined functions. With respect to these parameters, the metric function and all invariants of the surface are expressed by these geometric functions. The timelike surfaces with parallel normalized mean curvature vector field in $\mathbb{R}^4_1$ can be divided into three subclasses:

- surfaces satisfying $K - H^2 > 0$;
- surfaces satisfying $K - H^2 < 0$;
- surfaces satisfying $K - H^2 = 0$,

where K is the Gauss curvature and H is the mean curvature vector field.

The timelike surfaces with parallel normalized mean curvature vector field in $\mathbb{R}^4_1$ for which $K - H^2 > 0$ are determined up to a rigid motion in $\mathbb{R}^4_1$ by three functions $\lambda(u,v)$, $\mu(u,v)$, and $\nu(u,v)$ satisfying the following system of partial differential equations:

$$\begin{aligned} &\nu_u + \lambda_v = \lambda(\ln|\mu|)_v; \\ &\lambda_u - \nu_v = \lambda(\ln|\mu|)_u; \\ &|\mu|(\ln|\mu|)_{uv} = -\nu^2 - (\lambda^2 + \mu^2). \end{aligned} \tag{1}$$

The surfaces from the second subclass (characterized by the inequality $K - H^2 < 0$) are determined up to a rigid motion in $\mathbb{R}^4_1$ by three functions $\lambda(u,v)$, $\mu(u,v)$, and $\nu(u,v)$ satisfying the system of PDEs:

$$\begin{aligned} &\nu_u + \lambda_v = \lambda(\ln|\mu|)_v; \\ &\lambda_u + \nu_v = \lambda(\ln|\mu|)_u; \\ &|\mu|(\ln|\mu|)_{uv} = -\nu^2 + (\lambda^2 + \mu^2). \end{aligned} \tag{2}$$

The surfaces from the third subclass (characterized by $K - H^2 = 0$) are determined up to a rigid motion by three functions $\lambda(u,v)$, $\mu(u,v)$, and $\nu(u)$ satisfying:

$$\begin{aligned} &\nu_u + \lambda_v = \lambda(\ln|\mu|)_v; \\ &|\mu|(\ln|\mu|)_{uv} = -\nu^2. \end{aligned} \tag{3}$$

The above systems (1), (2), and (3) are the background systems of natural partial differential equations describing the three subclasses of timelike surfaces with parallel normalized mean curvature vector field in $\mathbb{R}^4_1$. In this way, we solve the Lund-Regge problem for this class of surfaces in $\mathbb{R}^4_1$.

2. Preliminaries

Let $\mathbb{R}^4_1$ be the four-dimensional Minkowski space endowed with the metric $\langle .,. \rangle$ of signature $(3,1)$. The standard flat metric is given in local coordinates by $dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$.

Let $\mathcal{M} = (\mathcal{D}, z)$ be a surface in $\mathbb{R}^4_1$, where $\mathcal{D} \subset \mathbb{R}^2$ and $z : \mathcal{D} \to \mathbb{R}^4_1$ is an immersion, i.e. $\mathcal{M}$ is locally parametrized by $\mathcal{M} : z = z(u,v)$, $(u,v) \in \mathcal{D}$. The surface $\mathcal{M}$ is said to be *spacelike* (resp. *timelike*), if $\langle .,. \rangle$ induces a Riemannian (resp. Lorentzian) metric g on $\mathcal{M}$.

We use the notations $\widetilde{\nabla}$ and ∇ for the Levi Civita connections on $\mathbb{R}^4_1$ and $\mathcal{M}$, respectively. Thus, if x and y are vector fields tangent to $\mathcal{M}$ and ξ is a normal vector field, then we have the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_x y = \nabla_x y + \sigma(x, y);$$
$$\widetilde{\nabla}_x \xi = -A_\xi x + D_x \xi,$$

which define the second fundamental tensor σ, the normal connection D, and the shape operator A_ξ with respect to ξ. In general, A_ξ is not diagonalizable.

The mean curvature vector field H of $\mathcal{M}$ is defined as

$$H = \frac{1}{2}\,\mathrm{tr}\,\sigma.$$

A surface $\mathcal{M}$ is called *totally geodesic* if its second fundamental form vanishes identically. The surface is called *minimal* if its mean curvature vector vanishes identically, i.e. $H = 0$.

A normal vector field ξ on a surface $\mathcal{M}$ is called *parallel in the normal bundle* (or simply *parallel*) if $D\xi = 0$ [7]. The surface $\mathcal{M}$ is said to have *parallel mean curvature vector field* if its mean curvature vector H is parallel, i.e. $DH = 0$. In the early 1970s, the surfaces with parallel mean curvature vector field in Riemannian space forms were classified by Chen [3] and Yau [24]. In 2009, Chen classified spacelike surfaces with parallel mean curvature vector field in pseudo-Euclidean spaces with arbitrary codimension and later, Lorentz surfaces with parallel mean curvature vector field in arbitrary pseudo-Euclidean space $\mathbb{R}^m_s$ were studied in [5] and [8]. Some classical and recent results on submanifolds with parallel mean curvature vector in Riemannian manifolds as well as in pseudo-Riemannian manifolds are presented in the survey [6].

The class of surfaces with parallel mean curvature vector field is naturally extended to the class of surfaces with parallel normalized mean curvature vector field as follows: a surface is said to have *parallel normalized mean curvature vector field* if H is nonzero and there exists a unit vector field in the direction of H which is parallel in the normal bundle [4]. It is proved that every analytic surface with parallel normalized mean curvature vector in the Euclidean m-space $\mathbb{R}^m$ must either lie in a 4-dimensional space $\mathbb{R}^4$ or in a hypersphere of $\mathbb{R}^m$ as a minimal surface [4].

Complete classification of biconservative surfaces with parallel normalized mean curvature vector field in $\mathbb{R}^4$ is given in [21] and biconservative m-dimensional submanifolds with parallel normalized mean curvature vector field in $\mathbb{R}^{n+2}$ are studied in [20]. Recently, 3-dimensional biconservative and biharmonic submanifolds with parallel normalized mean curvature vector field in the Euclidean 5-space $\mathbb{R}^5$ have been studied in [22].

Let $\mathcal{M} : z = z(u,v),\ (u,v) \in \mathcal{D}\ (\mathcal{D} \subset \mathbb{R}^2)$ be a local parametrization on a timelike surface in $\mathbb{R}^4_1$. The tangent space $T_p\mathcal{M}$ at an arbitrary point $p = z(u,v)$ of $\mathcal{M}$ is spanned by z_u and z_v. We use the standard denotations $E(u,v) = \langle z_u, z_u\rangle,\ F(u,v) = \langle z_u, z_v\rangle,\ G(u,v) = \langle z_v, z_v\rangle$ for the coefficients of the first fundamental form and denote $W = \sqrt{|EG - F^2|}$. Without loss of generality, we assume that $E < 0$ and $G > 0$. Choosing an orthonormal frame field $\{n_1, n_2\}$ of the normal bundle, i.e. $\langle n_1, n_1\rangle = 1,\ \langle n_2, n_2\rangle = 1,\ \langle n_1, n_2\rangle = 0$, we can write the following derivative formulas:

$$\begin{aligned}
\widetilde{\nabla}_{z_u} z_u &= z_{uu} = -\Gamma_{11}^1\, z_u + \Gamma_{11}^2\, z_v + c_{11}^1\, n_1 + c_{11}^2\, n_2;\\
\widetilde{\nabla}_{z_u} z_v &= z_{uv} = -\Gamma_{12}^1\, z_u + \Gamma_{12}^2\, z_v + c_{12}^1\, n_1 + c_{12}^2\, n_2;\\
\widetilde{\nabla}_{z_v} z_v &= z_{vv} = -\Gamma_{22}^1\, z_u + \Gamma_{22}^2\, z_v + c_{22}^1\, n_1 + c_{22}^2\, n_2;
\end{aligned} \tag{4}$$

where Γ_{ij}^k are the Christoffel's symbols and the functions $c_{ij}^k,\ i,j,k = 1,2$ are defined by

$$\begin{array}{lll}
c_{11}^1 = \langle z_{uu}, n_1\rangle; & c_{12}^1 = \langle z_{uv}, n_1\rangle; & c_{22}^1 = \langle z_{vv}, n_1\rangle;\\
c_{11}^2 = \langle z_{uu}, n_2\rangle; & c_{12}^2 = \langle z_{uv}, n_2\rangle; & c_{22}^2 = \langle z_{vv}, n_2\rangle.
\end{array}$$

It is obvious that the surface $\mathcal{M}$ lies in a two-dimensional plane if and only if it is totally geodesic, i.e. $c_{ij}^k = 0$ for all $i,j,k = 1,2$. Thus, further we assume that at least one of the coefficients c_{ij}^k is not zero.

Let us consider the following determinants:

$$\Delta_1 = \begin{vmatrix} c_{11}^1 & c_{12}^1 \\ c_{11}^2 & c_{12}^2 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} c_{12}^1 & c_{22}^1 \\ c_{12}^2 & c_{22}^2 \end{vmatrix}.$$

At a given point $p \in \mathcal{M}$, the *first normal space* of $\mathcal{M}$ in $\mathbb{R}^4_1$, denoted by $\operatorname{Im}\sigma_{\rm p}$, is the subspace given by

$$\operatorname{Im}\sigma_p = \operatorname{span}\{\sigma(x,y) : x, y \in T_p\mathcal{M}\}.$$

It is obvious that the condition $\Delta_1 = \Delta_2 = \Delta_3 = 0$ characterizes points at which the first normal space $\operatorname{Im}\sigma_p$ is one-dimensional. Such points are called *flat* or *inflection* points of the surface [16, 18]. Lane proved in [16] that every point of a surface in a 4-dimensional affine space is an inflection point if and only if the surface is developable or lies in a 3-dimensional space. Thus, further we consider timelike surfaces in $\mathbb{R}^4_1$ that are free of inflection points, i.e. we assume that $(\Delta_1, \Delta_2, \Delta_3) \neq (0,0,0)$.

3. Canonical parameters on timelike surfaces with parallel normalized mean curvature vector field

For a timelike surface $\mathcal{M}$ in $\mathbb{R}^4_1$, locally there exists a coordinate system (u,v) such that the metric tensor g of $\mathcal{M}$ has the form [17]:

$$g = -f^2(u,v)(du \otimes dv + dv \otimes du)$$

for some positive function $f(u,v)$. Let $z = z(u,v), (u,v) \in \mathcal{D}\,(\mathcal{D} \subset \mathbb{R}^2)$ be such a local parametrization on $\mathcal{M}$. Then, the coefficients of the first fundamental form are

$$E = \langle z_u, z_u\rangle = 0; \quad F = \langle z_u, z_v\rangle = -f^2(u,v); \quad G = \langle z_v, z_v\rangle = 0.$$

We consider the pseudoorthonormal tangent frame field given by $x = \dfrac{z_u}{f}$, $y = \dfrac{z_v}{f}$. Obviously, $\langle x, x\rangle = 0$, $\langle x, y\rangle = -1$, $\langle y, y\rangle = 0$. Then, the mean curvature vector field H of $\mathcal{M}$ is given by

$$H = -\sigma(x,y).$$

In the case $H \neq 0$ (nonminimal surface), we can choose a unit normal vector field n_1 which is collinear with the mean curvature vector field H, i.e. $H = \nu n_1$ for a smooth function $\nu = ||H||$. Then, $\sigma(x,y) = -\nu n_1$. We choose a unit normal vector field n_2 such that $\{n_1, n_2\}$ is an orthonormal frame field of the normal bundle (n_2 is determined up to orientation). Then we have the following formulas:

$$\begin{aligned}
\sigma(x,x) &= \lambda_1 n_1 + \mu_1 n_2;\\
\sigma(x,y) &= -\nu n_1;\\
\sigma(y,y) &= \lambda_2 n_1 + \mu_2 n_2,
\end{aligned}$$

where $\nu \neq 0$, $\lambda_1, \mu_1, \lambda_2, \mu_2$ are smooth functions determined by:

$$\begin{aligned}
\lambda_1 &= \langle \widetilde{\nabla}_x x, n_1\rangle; & \mu_1 &= \langle \widetilde{\nabla}_x x, n_2\rangle;\\
\lambda_2 &= \langle \widetilde{\nabla}_y y, n_1\rangle; & \mu_2 &= \langle \widetilde{\nabla}_y y, n_2\rangle.
\end{aligned}$$

Using that $\langle z_u, z_u\rangle = 0$, $\langle z_u, z_v\rangle = -f^2(u,v)$, $\langle z_v, z_v\rangle = 0$, after differentiation we calculate the coefficients Γ_{ij}^k, $\quad i,j,k = 1,2$:

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{2f_u}{f}; & \Gamma_{11}^2 &= 0;\\
\Gamma_{12}^1 &= 0; & \Gamma_{12}^2 &= 0;\\
\Gamma_{22}^1 &= 0; & \Gamma_{22}^2 &= \frac{2f_v}{f}.
\end{aligned} \tag{5}$$

Having in mind that $x = \dfrac{z_u}{f}$, $y = \dfrac{z_v}{f}$, from (4) and (5), after calculations, we obtain:

$$\begin{aligned}
\nabla_x x &= \frac{f_u}{f^2}\, x\\
\nabla_x y &= \qquad -\frac{f_u}{f^2}\, y\\
\nabla_y x &= -\frac{f_v}{f^2}\, x\\
\nabla_y y &= \qquad \frac{f_v}{f^2}\, y
\end{aligned} \tag{6}$$

We denote $\gamma_1 = \dfrac{f_u}{f^2} = x(\ln f)$, $\gamma_2 = \dfrac{f_v}{f^2} = y(\ln f)$. Thus, using equalities (4) and (6), we obtain the following derivative formulas:

$$\begin{aligned}
\widetilde{\nabla}_x x &= \gamma_1 x \qquad + \lambda_1 n_1 + \mu_1 n_2\\
\widetilde{\nabla}_x y &= \qquad -\gamma_1 y - \nu n_1\\
\widetilde{\nabla}_y x &= -\gamma_2 x \qquad - \nu n_1\\
\widetilde{\nabla}_y y &= \qquad \gamma_2 y \; + \lambda_2 n_1 + \mu_2 n_2
\end{aligned} \tag{7}$$

Remark: The pseudoorthonormal frame field $\{x, y, n_1, n_2\}$ is geometrically determined: x, y are the two lightlike directions in the tangent space; n_1 is the unit normal vector field collinear with the mean curvature vector field H; n_2 is determined by the condition that $\{n_1, n_2\}$ is an orthonormal frame field of the normal bundle (n_2 is determined up to a sign). We call this pseudoorthonormal frame field $\{x, y, n_1, n_2\}$ a *geometric frame field* of the surface.

Using (7), we can easily derive the following derivative formulas for the normal frame field $\{n_1, n_2\}$:

$$\begin{aligned} \widetilde{\nabla}_x n_1 &= -\nu x + \lambda_1 y \qquad + \beta_1 n_2 \\ \widetilde{\nabla}_y n_1 &= \lambda_2 x - \nu y \qquad + \beta_2 n_2 \\ \widetilde{\nabla}_x n_2 &= \qquad + \mu_1 y - \beta_1 n_1 \\ \widetilde{\nabla}_y n_2 &= \mu_2 x \qquad - \beta_2 n_1 \end{aligned} \tag{8}$$

where $\beta_1 = \langle \widetilde{\nabla}_x n_1, n_2 \rangle$ and $\beta_2 = \langle \widetilde{\nabla}_y n_1, n_2 \rangle$. Formulas (7) and (8) are the derivative formulas of the surface with respect to the pseudoorthonormal frame field $\{x, y, n_1, n_2\}$ which is geometrically determined as explained above.

The geometric meaning of the functions β_1 and β_2 is revealed by the next two propositions.

Proposition 1 *Let $\mathcal{M}$ be a timelike surface in the Minkowski space $\mathbb{R}^4_1$. Then, $\mathcal{M}$ has parallel mean curvature vector field if and only if $\beta_1 = \beta_2 = 0$ and $\nu = const$.*

Proof Let $\mathcal{M}$ be a timelike surface in $\mathbb{R}^4_1$ with geometric pseudoorthonormal frame field $\{x, y, n_1, n_2\}$. It follows from (8) that for the normal mean curvature vector field $H = \nu n_1$, we have the formulas:

$$\begin{aligned} D_x H &= x(\nu) n_1 + \nu \beta_1 n_2; \\ D_y H &= y(\nu) n_1 + \nu \beta_2 n_2, \end{aligned}$$

which imply that H is parallel in the normal bundle if and only if $\beta_1 = \beta_2 = 0$ and $\nu = const$.

Proposition 2 *Let $\mathcal{M}$ be a timelike surface in the Minkowski space $\mathbb{R}^4_1$. Then, $\mathcal{M}$ has parallel normalized mean curvature vector field if and only if $\beta_1 = \beta_2 = 0$ and $\nu \neq const$.*

Proof Recall that $\mathcal{M}$ is a surface with parallel normalized mean curvature vector field if H is nonzero (and nonparallel) and there exists a unit vector field in the direction of H which is parallel in the normal bundle. Since n_1 is collinear with H and

$$\begin{aligned} D_x n_1 &= \beta_1 n_2; \\ D_y n_1 &= \beta_2 n_2, \end{aligned}$$

we conclude that $\mathcal{M}$ is a surface with parallel normalized mean curvature vector field if and only if $\beta_1 = \beta_2 = 0$ and $\nu \neq const$.

Further, we consider timelike surfaces with parallel normalized mean curvature vector field, i.e. we assume that $\beta_1 = \beta_2 = 0$ and $\nu \neq const$. For this class of surfaces we will introduce special, so-called canonical parameters, which we will prove to exist locally on each such surface.

Using that $\beta_1 = \beta_2 = 0$, from (7) and (8), we derive the following derivative formulas for the class of surfaces with parallel normalized mean curvature vector field:

$$\begin{array}{ll}
\widetilde{\nabla}_x x = \gamma_1 x \qquad + \lambda_1 n_1 + \mu_1 n_2; & \widetilde{\nabla}_x n_1 = -\nu x + \lambda_1 y;\\
\widetilde{\nabla}_x y = \qquad -\gamma_1 y - \nu n_1; & \widetilde{\nabla}_y n_1 = \lambda_2 x - \nu y;\\
\widetilde{\nabla}_y x = -\gamma_2 x \qquad - \nu n_1; & \widetilde{\nabla}_x n_2 = \qquad +\mu_1 y;\\
\widetilde{\nabla}_y y = \qquad \gamma_2 y \ + \lambda_2 n_1 + \mu_2 n_2; & \widetilde{\nabla}_y n_2 = \mu_2 x.
\end{array} \tag{9}$$

Further, we calculate the integrability conditions for this class of surfaces. Since the Levi Civita connection $\widetilde{\nabla}$ of $\mathbb{R}^4_1$ is flat, we have

$$\widetilde{R}(x,y,x) = 0; \quad \widetilde{R}(x,y,y) = 0; \quad \widetilde{R}(x,y,n_1) = 0; \quad \widetilde{R}(x,y,n_2) = 0, \tag{10}$$

where

$$\widetilde{R}(x,y,z) = \widetilde{\nabla}_x \widetilde{\nabla}_y z - \widetilde{\nabla}_y \widetilde{\nabla}_x z - \widetilde{\nabla}_{[x,y]} z$$

for arbitrary vector fields x, y, z. It follows from (9) that the commutator $[x,y]$ is expressed as follows

$$[x,y] = \widetilde{\nabla}_x y - \widetilde{\nabla}_y x = \gamma_2 x - \gamma_1 y.$$

Then, by use of formulas (9), we calculate:

$$\begin{array}{rl}
\widetilde{R}(x,y,x) = & \big(-x(\gamma_2) - y(\gamma_1) - 2\gamma_1\gamma_2 + \nu^2 - \lambda_1\lambda_2 - \mu_1\mu_2\big)\ x - \\
& - \left(x(\nu) + y(\lambda_1) + 2\gamma_2\lambda_1\right)\ n_1 - \left(y(\mu_1) + 2\gamma_2\mu_1\right)\ n_2;\\
\widetilde{R}(x,y,y) = & \big(x(\gamma_2) + y(\gamma_1) + 2\gamma_1\gamma_2 - \nu^2 + \lambda_1\lambda_2 + \mu_1\mu_2\big)\ y + \\
& + \left(x(\lambda_2) + y(\nu) + 2\gamma_1\lambda_2\right)\ n_1 + \left(x(\mu_2) + 2\gamma_1\mu_2\right)\ n_2;\\
\widetilde{R}(x,y,n_1) = & \left(x(\lambda_2) + y(\nu) + 2\gamma_1\lambda_2\right)\ x - \left(x(\nu) + y(\lambda_1) + 2\gamma_2\lambda_1\right)\ y + \\
& + \left(\mu_1\lambda_2 - \lambda_1\mu_2\right)\ n_2;\\
\widetilde{R}(x,y,n_2) = & \left(x(\mu_2) + 2\gamma_1\mu_2\right)\ x - \left(y(\mu_1) + 2\gamma_2\mu_1\right)\ y + \\
& + \left(\lambda_1\mu_2 - \mu_1\lambda_2\right)\ n_1.
\end{array} \tag{11}$$

Now, taking into consideration (10) and (11), we obtain the following integrability conditions:

$$\begin{array}{l}
x(\lambda_2) + y(\nu) + 2\gamma_1\lambda_2 = 0;\\
x(\nu) + y(\lambda_1) + 2\gamma_2\lambda_1 = 0;\\
x(\mu_2) + 2\gamma_1\mu_2 = 0;\\
y(\mu_1) + 2\gamma_2\mu_1 = 0;\\
x(\gamma_2) + y(\gamma_1) + 2\gamma_1\gamma_2 - \nu^2 + \lambda_1\lambda_2 + \mu_1\mu_2 = 0;\\
\mu_1\lambda_2 - \lambda_1\mu_2 = 0.
\end{array} \tag{12}$$

Remark: If we assume that both μ_1 and μ_2 are zero functions, i.e. $\mu_1(u,v) = 0$ and $\mu_2(u,v) = 0$ for all $(u,v) \in \mathcal{D}$, then from (9) we obtain that $\Delta_1 = \Delta_2 = \Delta_3 = 0$, which means that the surface consists of inflection points. Moreover, from $\widetilde{\nabla}_x n_2 = 0$ and $\widetilde{\nabla}_y n_2 = 0$, we get that the normal vector field n_2 is constant, which implies that the surface $\mathcal{M}$ lies in the three-dimensional Minkowski space $\mathbb{R}^3_1 = \text{span}\{x, y, n_1\}$.

Thus, further we assume that $\mu_1^2 + \mu_2^2 \neq 0$ at least in a subdomain $\mathcal{D}_0$ of $\mathcal{D}$. Without loss of generality, we may assume that $\mu_1 \neq 0$. Then, from the last equality of (12) we obtain that $\mu_1 \lambda_2 = \lambda_1 \mu_2$, which implies $\lambda_2 = \dfrac{\mu_2}{\mu_1}\lambda_1$.

The Gauss curvature of the surface is defined by the following formula:

$$K = \frac{\langle R(x,y,y), x\rangle}{\langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2}.$$

Now, using that $R(x,y,y) = \nabla_x \nabla_y y - \nabla_y \nabla_x y - \nabla_{[x,y]} y$, from formulas (9), we obtain

$$R(x,y,y) = (x(\gamma_2) + y(\gamma_1) + 2\gamma_1\gamma_2)\; y,$$

and hence, the Gauss curvature K is given by

$$K = x(\gamma_2) + y(\gamma_1) + 2\gamma_1\gamma_2.$$

Having in mind the fifth equality of (9), we obtain that the Gauss curvature of the surface $\mathcal{M}$ is expressed in terms of the functions ν, λ_1, λ_2, μ_1, μ_2 as follows:

$$K = \nu^2 - \lambda_1\lambda_2 - \mu_1\mu_2.$$

The last equality together with $\nu^2 = H^2$ (for simplicity we denote $H^2 = \langle H, H\rangle$) implies that $K - H^2 = -(\lambda_1\lambda_2 + \mu_1\mu_2)$. Using that $\lambda_2 = \dfrac{\mu_2}{\mu_1}\lambda_1$, we get

$$K - H^2 = -\frac{\mu_2}{\mu_1}(\lambda_1^2 + \mu_1^2).$$

Hence, the surfaces with parallel normalized mean curvature vector field can be divided into two main classes:

- $K - H^2 \neq 0$ (which is equivalent to $\mu_1\mu_2 \neq 0$) in a subdomain;
- $K - H^2 = 0$ (which is equivalent to $\mu_1\mu_2 = 0$) in a subdomain.

3.1. Surfaces satisfying $K - H^2 \neq 0$

First, we shall consider the case $K - H^2 \neq 0$, i.e. $\mu_1\mu_2 \neq 0$. In this case, from the third and forth equalities of (12), we get:

$$\begin{aligned} x(\ln|\mu_2|) &= -2\gamma_1;\\ y(\ln|\mu_1|) &= -2\gamma_2. \end{aligned}$$

On the other hand, the functions γ_1 and γ_2 are expressed by the metric function f as follows: $\gamma_1 = x(\ln f)$, $\gamma_2 = y(\ln f)$. Hence, we obtain:

$$\begin{aligned} x(\ln f^2|\mu_2|) &= 0;\\ y(\ln f^2|\mu_1|) &= 0. \end{aligned} \tag{13}$$

It follows from (13) that the function $f^2|\mu_1|$ depends only on the parameter u, and the function $f^2|\mu_2|$ depends only on v. Therefore, there exist smooth functions $\varphi(u) > 0$ and $\psi(v) > 0$ such that:

$$f^2|\mu_1| = \varphi(u); \quad f^2|\mu_2| = \psi(v).$$

We consider the following change of the parameters:

$$\overline{u} = \int_{u_0}^{u} \sqrt{\varphi(u)}\, du + \overline{u}_0, \quad \overline{u}_0 = const$$
$$\overline{v} = \int_{v_0}^{v} \sqrt{\psi(v)}\, dv + \overline{v}_0, \quad \overline{v}_0 = const$$

Under this change of the parameters, we obtain:

$$z_{\overline{u}} = \frac{z_u}{\sqrt{\varphi(u)}} = \frac{z_u}{f\sqrt{|\mu_1|}},$$

$$z_{\overline{v}} = \frac{z_v}{\sqrt{\psi(v)}} = \frac{z_v}{f\sqrt{|\mu_2|}},$$

which imply that

$$\langle z_{\overline{u}}, z_{\overline{u}} \rangle = 0; \quad \langle z_{\overline{u}}, z_{\overline{v}} \rangle = -\frac{1}{\sqrt{|\mu_1||\mu_2|}}, \quad \langle z_{\overline{v}}, z_{\overline{v}} \rangle = 0.$$

Therefore, $(\overline{u}, \overline{v})$ are special isotropic parameters with respect to which the metric tensor of the surface is given by

$$g = -\overline{f}^2(\overline{u}, \overline{v})(d\overline{u} \otimes d\overline{v} + d\overline{v} \otimes d\overline{u}),$$

where the metric function $\overline{f}$ is expressed in terms of μ_1 and μ_2 as follows:

$$\overline{f}(\overline{u}, \overline{v}) = \frac{1}{\sqrt[4]{|\mu_1||\mu_2|}}.$$

With respect to the isotropic directions:

$$\overline{x} = \frac{z_{\overline{u}}}{\overline{f}} = \frac{\sqrt[4]{|\mu_1||\mu_2|}}{\sqrt{|\mu_1|}} x,$$
$$\overline{y} = \frac{z_{\overline{v}}}{\overline{f}} = \frac{\sqrt[4]{|\mu_1||\mu_2|}}{\sqrt{|\mu_2|}} y,$$

we have the flowing expressions for the second fundamental tensor σ:

$$\sigma(\overline{x}, \overline{x}) = \frac{\sqrt{|\mu_1||\mu_2|}}{|\mu_1|} \sigma(x, x) = \lambda_1 \frac{\sqrt{|\mu_1||\mu_2|}}{|\mu_1|} n_1 + \mu_1 \frac{\sqrt{|\mu_1||\mu_2|}}{|\mu_1|} n_2;$$
$$\sigma(\overline{x}, \overline{y}) = \sigma(x, y) = -\nu\, n_1;$$
$$\sigma(\overline{y}, \overline{y}) = \frac{\sqrt{|\mu_1||\mu_2|}}{|\mu_2|} \sigma(y, y) = \lambda_2 \frac{\sqrt{|\mu_1||\mu_2|}}{|\mu_2|} n_1 + \mu_2 \frac{\sqrt{|\mu_1||\mu_2|}}{|\mu_2|} n_2.$$

Since μ_1 and μ_2 are smooth functions and we consider a local theory, we may assume that $sign(\mu_1) = \varepsilon_1$, $\varepsilon_1 = \pm 1$ and $sign(\mu_2) = \varepsilon_2$, $\varepsilon_2 = \pm 1$ in some subdomain. Now, using that $\dfrac{\lambda_2}{\lambda_1} = \dfrac{\mu_2}{\mu_1} = \dfrac{|\mu_2|}{|\mu_1|}\dfrac{\varepsilon_2}{\varepsilon_1}$, we get the formulas:

$$\sigma(\overline{x}, \overline{x}) = \lambda_1 \frac{\sqrt{|\mu_1||\mu_2|}}{|\mu_1|}\, n_1 + \varepsilon_1 \sqrt{|\mu_1||\mu_2|}\, n_2;$$
$$\sigma(\overline{y}, \overline{y}) = \lambda_1 \frac{\varepsilon_2}{\varepsilon_1} \frac{\sqrt{|\mu_1||\mu_2|}}{|\mu_1|}\, n_1 + \varepsilon_2 \sqrt{|\mu_1||\mu_2|}\, n_2.$$

Denoting $\overline{\lambda} = \lambda_1 \dfrac{\sqrt{|\mu_1||\mu_2|}}{|\mu_1|}$, $\overline{\mu} = \varepsilon_1 \sqrt{|\mu_1||\mu_2|}$, we obtain:

$$\sigma(\overline{x}, \overline{x}) = \overline{\lambda}\, n_1 + \overline{\mu}\, n_2;$$
$$\sigma(\overline{y}, \overline{y}) = \frac{\varepsilon_2}{\varepsilon_1} \overline{\lambda}\, n_1 + \frac{\varepsilon_2}{\varepsilon_1} \overline{\mu}\, n_2.$$

Thus, we conclude that there exist two subcases:

1. μ_1 and μ_2 have one and the same sign in the considered subdomain, i.e. $\varepsilon_1 \varepsilon_2 = 1$; hence, we have $\sigma(\overline{x}, \overline{x}) = \sigma(\overline{y}, \overline{y})$.

2. μ_1 and μ_2 have opposite signs in the considered subdomain, i.e. $\varepsilon_1 \varepsilon_2 = -1$; hence, we have $\sigma(\overline{x}, \overline{x}) = -\sigma(\overline{y}, \overline{y})$.

Having in mind that $K - H^2 = -\dfrac{\mu_2}{\mu_1}(\lambda_1^2 + \mu_1^2)$, we get that the first subcase corresponds to $K - H^2 < 0$, the second subcase corresponds to $K - H^2 > 0$.

Hence, after the change of the parameters, we have the formulas:

$$\begin{array}{l} \sigma(\overline{x}, \overline{x}) = \overline{\lambda}\, n_1 + \overline{\mu}\, n_2 \\ \sigma(\overline{x}, \overline{y}) = -\nu\, n_1 \\ \sigma(\overline{y}, \overline{y}) = -\overline{\lambda}\, n_1 - \overline{\mu}\, n_2 \end{array} \quad , \quad \text{if } K - H^2 > 0;$$

or

$$\begin{array}{l} \sigma(\overline{x}, \overline{x}) = \overline{\lambda}\, n_1 + \overline{\mu}\, n_2 \\ \sigma(\overline{x}, \overline{y}) = -\nu\, n_1 \\ \sigma(\overline{y}, \overline{y}) = \overline{\lambda}\, n_1 + \overline{\mu}\, n_2 \end{array} \quad , \quad \text{if } K - H^2 < 0.$$

In both cases ($K - H^2 > 0$ or $K - H^2 < 0$), the metric function $\overline{f}$ is expressed by:

$$\overline{f} = \frac{1}{\sqrt{|\overline{\mu}|}}.$$

We introduce the notion of canonical isotropic parameters on a timelike surface with parallel normalized mean curvature vector field by the following definition.

Definition 1 *Let $\mathcal{M}$ be a timelike surface with parallel normalized mean curvature vector field in $\mathbb{R}^4_1$ and $K - H^2 \neq 0$. The isotropic parameters (u, v) are said to be canonical if the metric function f is expressed by:*

$$f(u, v) = \frac{1}{\sqrt{|\mu|}}, \quad \mu \neq 0.$$

With the above considerations, we have proved that:

Proposition 3 *Each timelike surface with parallel normalized mean curvature vector field satisfying $K-H^2 \neq 0$ locally admits canonical parameters.*

Let $\mathcal{M} : z = z(u, v),\ (u, v) \in \mathcal{D}$ be a timelike surface with parallel normalized mean curvature vector field satisfying $K - H^2 \neq 0$ and parametrized by isotropic canonical parameters (u, v). With respect to canonical isotropic parametrization the derivative formulas of $\mathcal{M}$ take the form:

$$\begin{array}{ll}
\widetilde{\nabla}_x x = \gamma_1 x \qquad\quad + \lambda n_1 + \mu n_2; & \widetilde{\nabla}_x n_1 = -\nu x + \lambda y; \\
\widetilde{\nabla}_x y = \qquad -\gamma_1 y - \nu n_1; & \widetilde{\nabla}_y n_1 = -\varepsilon\lambda x - \nu y; \\
\widetilde{\nabla}_y x = -\gamma_2 x \qquad - \nu n_1; & \widetilde{\nabla}_x n_2 = \qquad +\mu y; \\
\widetilde{\nabla}_y y = \qquad\ \ \gamma_2 y\ - \varepsilon\lambda n_1 - \varepsilon\mu n_2; & \widetilde{\nabla}_y n_2 = -\varepsilon\mu x,
\end{array}$$

where $\varepsilon = 1$ in the case $K - H^2 > 0$, and $\varepsilon = -1$ in the case $K - H^2 < 0$.

The geometric meaning of the canonical parametrization can be explained as follows: if (u, v) are canonical isotropic parameters, then the canonical directions $x = \dfrac{z_u}{f}$ and $y = \dfrac{z_v}{f}$ satisfy the relation:

$$\begin{array}{ll}
\sigma(x, x) = -\sigma(y, y), & \text{in the case } K - H^2 > 0; \\
\sigma(x, x) = \sigma(y, y), & \text{in the case } K - H^2 < 0.
\end{array}$$

Moreover, with respect to canonical isotropic parameters (u, v), the functions γ_1 and γ_2 are expressed by:

$$\gamma_1 = -\frac{|\mu|_u}{2\sqrt{|\mu|}}, \quad \gamma_2 = -\frac{|\mu|_v}{2\sqrt{|\mu|}}. \tag{14}$$

From integrability conditions (12), in the case $\lambda_1 = \lambda,\ \lambda_2 = -\varepsilon\lambda,\ \mu_1 = \mu,\ \mu_2 = -\varepsilon\mu$, we get

$$\begin{array}{l}
x(\nu) + y(\lambda) + 2\gamma_2\lambda = 0; \\
-\varepsilon x(\lambda) + y(\nu) - \varepsilon 2\gamma_1\lambda = 0; \\
x(\gamma_2) + y(\gamma_1) + 2\gamma_1\gamma_2 - \nu^2 - \varepsilon(\lambda^2 + \mu^2) = 0.
\end{array}$$

Then, having in mind (14), from the equalities above, we obtain

$$\begin{array}{l}
\nu_u + \lambda_v = \lambda(\ln|\mu|)_v; \\
\lambda_u - \varepsilon\nu_v = \lambda(\ln|\mu|)_u; \\
|\mu|\,(\ln|\mu|)_{uv} = -\nu^2 - \varepsilon(\lambda^2 + \mu^2).
\end{array}$$

Thus, by introducing canonical parameters on a surface with parallel normalized mean curvature vector field, we manage to reduce the number of functions and the number of partial differential equations up to three. In the next section, we shall prove that these three functions, λ, μ, and ν, determine the surface up to a motion.

3.2. Surfaces satisfying $K - H^2 = 0$

Now we shall consider the case $K - H^2 = 0$, i.e. $\mu_1\mu_2 = 0$, $\mu_1^2 + \mu_2^2 \neq 0$. Without loss of generality, we assume that $\mu_1 \neq 0$ and $\mu_2 = 0$ in a subdomain $\mathcal{D}_0$. From $\mu_1\lambda_2 - \lambda_1\mu_2 = 0$, it follows that $\lambda_2 = 0$, which implies that $K = \nu^2$. In this case, the derivative formulas take the following form:

$$\begin{array}{ll}
\widetilde{\nabla}_x x = \gamma_1 x \qquad + \lambda_1 n_1 + \mu_1 n_2; & \widetilde{\nabla}_x n_1 = -\nu x + \lambda_1 y; \\
\widetilde{\nabla}_x y = \qquad -\gamma_1 y - \nu n_1; & \widetilde{\nabla}_y n_1 = \qquad -\nu y; \\
\widetilde{\nabla}_y x = -\gamma_2 x \qquad - \nu n_1; & \widetilde{\nabla}_x n_2 = \qquad \mu_1 y; \\
\widetilde{\nabla}_y y = \qquad \gamma_2 y; & \widetilde{\nabla}_y n_2 = 0.
\end{array} \tag{15}$$

From integrability conditions (12), we get:

$$\begin{aligned}
& y(\nu) = 0; \\
& x(\nu) + y(\lambda_1) + 2\gamma_2\lambda_1 = 0; \\
& y(\mu_1) + 2\gamma_2\mu_1 = 0; \\
& x(\gamma_2) + y(\gamma_1) + 2\gamma_1\gamma_2 = \nu^2.
\end{aligned} \tag{16}$$

Thus, the first equality of (16) implies that:

$$\nu = \nu(u),$$

and from the third one, we get:

$$y(\ln|\mu_1|) = -2\gamma_2.$$

Having in mind that $\gamma_2 = y(\ln f)$, we obtain

$$y(\ln(f^2|\mu_1|)) = 0,$$

which implies that there exists a function $\varphi(u) > 0$ such that $f^2|\mu_1| = \varphi(u)$.

Consider the following change of the parameters:

$$\begin{aligned}
& \overline{u} = \int_{u_0}^{u} \varphi(u)du + \overline{u}_0, \quad \overline{u}_0 = const \\
& \overline{v} = v + \overline{v}_0, \quad \overline{v}_0 = const
\end{aligned}$$

Under this change of the parameters, we obtain

$$\begin{aligned}
& z_{\overline{u}} = \frac{z_u}{f^2|\mu_1|}; \\
& z_{\overline{v}} = z_v,
\end{aligned}$$

which implies that

$$\langle z_{\overline{u}}, z_{\overline{u}} \rangle = 0; \quad \langle z_{\overline{u}}, z_{\overline{v}} \rangle = -\frac{1}{|\mu_1|}; \quad \langle z_{\overline{v}}, z_{\overline{v}} \rangle = 0.$$

Hence, $(\overline{u}, \overline{v})$ are isotropic parameters with respect to which the new metric function is:

$$\overline{f} = \frac{1}{\sqrt{|\mu_1|}}.$$

We consider the isotropic directions

$$\overline{x} = \frac{z_{\overline{u}}}{\overline{f}} = \frac{x}{f\sqrt{|\mu_1|}};$$
$$\overline{y} = \frac{z_{\overline{v}}}{\overline{f}} = f\sqrt{|\mu_1|}y.$$

Then, the second fundamental tensor is expressed as follows:

$$\sigma(\overline{x}, \overline{x}) = \frac{\lambda_1}{f^2|\mu_1|}n_1 + \frac{\mu_1}{f^2|\mu_1|}n_2;$$
$$\sigma(\overline{x}, \overline{y}) = -\nu n_1;$$
$$\sigma(\overline{y}, \overline{y}) = 0.$$

Denoting $\overline{\lambda} = \dfrac{\lambda_1}{f^2|\mu_1|}$ and $\overline{\mu} = \dfrac{\mu_1}{f^2|\mu_1|}$, we get

$$\sigma(\overline{x}, \overline{x}) = \overline{\lambda}n_1 + \overline{\mu}n_2;$$
$$\sigma(\overline{x}, \overline{y}) = -\nu n_1;$$
$$\sigma(\overline{y}, \overline{y}) = 0.$$

Note that $\overline{\mu} = \dfrac{\varepsilon}{f^2}$, where $\varepsilon = sign(\mu_1)$. Obviously, $\dfrac{\overline{\lambda}}{\overline{\mu}} = \dfrac{\lambda_1}{\mu_1}$.

Thus, in the case $K - H^2 = 0$, we can also introduce canonical isotropic parameters by the next definition.

Definition 2 *Let $\mathcal{M}$ be a timelike surface with parallel normalized mean curvature vector field in $\mathbb{R}^4_1$ and $K - H^2 = 0$. The isotropic parameters (u, v) are said to be canonical if the metric function f is expressed by:*

$$f(u, v) = \frac{1}{\sqrt{|\mu|}}, \quad \mu \neq 0.$$

With the above considerations, we have proved that:

Proposition 4 *Each timelike surface with parallel normalized mean curvature vector field satisfying $K - H^2 = 0$ locally admits canonical parameters.*

With respect to canonical isotropic parameters, in the case $K - H^2 = 0$, we have derivative formulas (15). From integrability conditions (12), in the case $\lambda_1 = \lambda,\ \lambda_2 = 0,\ \mu_1 = \mu,\ \mu_2 = 0$, we get

$$\begin{aligned} &x(\nu) + y(\lambda) + 2\gamma_2\lambda = 0; \\ &x(\gamma_2) + y(\gamma_1) + 2\gamma_1\gamma_2 = \nu^2, \end{aligned}$$

which in view of (14) imply

$$\begin{aligned} &\nu_u + \lambda_v = \lambda(\ln|\mu|)_v; \\ &|\mu|(\ln|\mu|)_{uv} = -\nu^2. \end{aligned}$$

Hence, in the case $K - H^2 = 0$, by introducing canonical parameters on a surface with parallel normalized mean curvature vector field, we manage to reduce the number of functions and the number of partial differential equations determining the surface.

4. Fundamental theorems

Now we shall prove fundamental existence and uniqueness theorems for the class of timelike surfaces with parallel normalized mean curvature vector field in terms of canonical parameters.

Theorem 1 *Let $\lambda(u,v)$, $\mu(u,v)$ and $\nu(u,v)$ be smooth functions, $\mu \neq 0$, $\nu \neq const$, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions*

$$\begin{aligned} &\nu_u + \lambda_v = \lambda(\ln|\mu|)_v; \\ &\lambda_u - \varepsilon\nu_v = \lambda(\ln|\mu|)_u; \\ &|\mu|\,(\ln|\mu|)_{uv} = -\nu^2 - \varepsilon(\lambda^2 + \mu^2), \end{aligned} \tag{17}$$

where $\varepsilon = \pm 1$. If $\{x_0,\, y_0,\, (n_1)_0,\, (n_2)_0\}$ is a pseudoorthonormal frame at a point $p_0 \in \mathbb{R}^4_1$, then there exists a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique timelike surface $\mathcal{M} : z = z(u,v),\ (u,v) \in \mathcal{D}_0$ with parallel normalized mean curvature vector field, such that $\mathcal{M}$ passes through p_0, $\{x_0,\, y_0,\, (n_1)_0,\, (n_2)_0\}$ is the geometric frame of $\mathcal{M}$ at the point p_0, the functions $\lambda(u,v)$, $\mu(u,v)$, $\nu(u,v)$ are the geometric functions of the surface, and $K - H^2 > 0$ in the case $\varepsilon = 1$, resp. $K - H^2 < 0$ in the case $\varepsilon = -1$. Furthermore, (u,v) are canonical isotropic parameters of $\mathcal{M}$.

Proof Let us denote $\gamma_1 = -(\sqrt{|\mu|})_u$, $\gamma_2 = -(\sqrt{|\mu|})_v$ and consider the following system of partial differential equations for the unknown vector functions $x = x(u,v),\ y = y(u,v),\ n_1 = n_1(u,v),\ n_2 = n_2(u,v)$ in $\mathbb{R}^4_1$:

$$\begin{aligned} x_u &= \frac{1}{\sqrt{|\mu|}}\,(\gamma_1\, x + \lambda\, n_1 + \mu\, n_2) & x_v &= \frac{1}{\sqrt{|\mu|}}\,(-\gamma_2\, x - \nu\, n_1) \\ y_u &= \frac{1}{\sqrt{|\mu|}}\,(-\gamma_1\, y - \nu\, n_1) & y_v &= \frac{1}{\sqrt{|\mu|}}\,(\gamma_2\, y - \varepsilon\lambda\, n_1 - \varepsilon\mu\, n_2) \\ (n_1)_u &= \frac{1}{\sqrt{|\mu|}}\,(-\nu\, x + \lambda\, y) & (n_1)_v &= \frac{1}{\sqrt{|\mu|}}\,(-\varepsilon\lambda\, x - \nu\, y) \\ (n_2)_u &= \frac{1}{\sqrt{|\mu|}}\,(\mu\, y) & (n_2)_v &= \frac{1}{\sqrt{|\mu|}}\,(-\varepsilon\mu\, x) \end{aligned} \tag{18}$$

We denote

$$\mathcal{F} = \begin{pmatrix} x \\ y \\ n_1 \\ n_2 \end{pmatrix}; \ \mathcal{A} = \frac{1}{\sqrt{|\mu|}} \begin{pmatrix} \gamma_1 & 0 & \lambda & \mu \\ 0 & -\gamma_1 & -\nu & 0 \\ -\nu & \lambda & 0 & 0 \\ 0 & \mu & 0 & 0 \end{pmatrix}; \ \mathcal{B} = \frac{1}{\sqrt{|\mu|}} \begin{pmatrix} -\gamma_2 & 0 & -\nu & 0 \\ 0 & \gamma_2 & -\varepsilon\lambda & -\varepsilon\mu \\ -\varepsilon\lambda & -\nu & 0 & 0 \\ -\varepsilon\mu & 0 & 0 & 0 \end{pmatrix}.$$

Then, system (18) can be written in matrix form as follows:

$$\begin{aligned} \mathcal{F}_u &= \mathcal{A}\,\mathcal{F}, \\ \mathcal{F}_v &= \mathcal{B}\,\mathcal{F}. \end{aligned} \tag{19}$$

The integrability conditions of system (19) are $\mathcal{F}_{uv} = \mathcal{F}_{vu}$, i.e.

$$\frac{\partial a_i^k}{\partial v} - \frac{\partial b_i^k}{\partial u} + \sum_{j=1}^{4} (a_i^j\, b_j^k - b_i^j\, a_j^k) = 0, \quad i,k = 1, \dots, 4, \tag{20}$$

where, by a_i^j and b_i^j, we denote the elements of the matrices $\mathcal{A}$ and $\mathcal{B}$. Using (17), one can check that equalities (20) are fulfilled. Hence, there exists a subdomain $\mathcal{D}_1 \subset \mathcal{D}$ and unique vector functions $x = x(u,v)$, $y = y(u,v)$, $n_1 = n_1(u,v)$, $n_2 = n_2(u,v)$, $(u,v) \in \mathcal{D}_1$, which satisfy system (18) and the conditions

$$x(u_0,v_0) = x_0, \quad y(u_0,v_0) = y_0, \quad n_1(u_0,v_0) = (n_1)_0, \quad n_2(u_0,v_0) = (n_2)_0.$$

It can be proved that $x(u,v)$, $y(u,v)$, $n_1(u,v)$, $n_2(u,v)$ form a pseudoorthonormal frame in $\mathbb{R}^4_1$ for each $(u,v) \in \mathcal{D}_1$. Indeed, let us consider the following functions:

$$\begin{array}{lll} \varphi_1 = \langle x, x \rangle; & \varphi_5 = \langle x, y \rangle + 1; & \varphi_8 = \langle y, n_1 \rangle; \\ \varphi_2 = \langle y, y \rangle; & \varphi_6 = \langle x, n_1 \rangle; & \varphi_9 = \langle y, n_2 \rangle; \\ \varphi_3 = \langle n_1, n_1 \rangle - 1; & \varphi_7 = \langle x, n_2 \rangle; & \varphi_{10} = \langle n_1, n_2 \rangle; \\ \varphi_4 = \langle n_2, n_2 \rangle - 1; & & \end{array}$$

defined for $(u,v) \in \mathcal{D}_1$. Having in mind that $x(u,v)$, $y(u,v)$, $n_1(u,v)$, $n_2(u,v)$ satisfy (18), we obtain the system

$$\begin{aligned} \frac{\partial \varphi_i}{\partial u} &= p_i^j\, \varphi_j, \\ \frac{\partial \varphi_i}{\partial v} &= q_i^j\, \varphi_j; \end{aligned} \qquad i = 1, \dots, 10, \tag{21}$$

where p_i^j, q_i^j, $i,j = 1, \dots, 10$ are functions of $(u,v) \in \mathcal{D}_1$. System (21) is a linear system of partial differential equations for the functions $\varphi_i(u,v)$, satisfying the conditions $\varphi_i(u_0,v_0) = 0$ for all $i = 1, \dots, 10$, since $\{x_0,\, y_0,\, (n_1)_0,\, (n_2)_0\}$ is a pseudoorthonormal frame. Therefore, $\varphi_i(u,v) = 0$, $i = 1, \dots, 10$ for each $(u,v) \in \mathcal{D}_1$. Hence, the vector functions $x(u,v)$, $y(u,v)$, $n_1(u,v)$, $n_2(u,v)$ form a pseudoorthonormal frame in $\mathbb{E}^4_1$ for each $(u,v) \in \mathcal{D}_1$.

Finally, we consider the following system of partial differential equations for the vector function $z(u,v)$:

$$\begin{aligned} z_u &= \frac{1}{\sqrt{|\mu|}}\, x \\ z_v &= \frac{1}{\sqrt{|\mu|}}\, y \end{aligned} \tag{22}$$

It follows from equalities (17) and (18) that the integrability conditions $z_{uv} = z_{vu}$ of system (22) are fulfilled. Hence, there exists a subdomain $\mathcal{D}_0 \subset \mathcal{D}_1$ and a unique vector function $z = z(u,v)$, defined for $(u,v) \in \mathcal{D}_0$ and satisfying $z(u_0, v_0) = p_0$.

Now, we consider the surface $\mathcal{M} : z = z(u,v),\ (u,v) \in \mathcal{D}_0$. Obviously, $\mathcal{M}$ is a timelike surface in $\mathbb{R}^4_1$. It follows from (18) that $\mathcal{M}$ has parallel normalized mean curvature vector field, since $H = \nu\, n_1$; $D_x n_1 = 0$ and $D_y n_1 = 0$. Moreover, (u,v) are canonical isotropic parameters of $\mathcal{M}$, since $\langle z_u, z_v\rangle = -\dfrac{1}{|\mu|}$, and the metric function is $f = \dfrac{1}{\sqrt{|\mu|}}$.

Theorem 2 *Let $\lambda(u,v)$, $\mu(u,v)$, and $\nu(u)$ be smooth functions, $\mu \neq 0$, $\nu \neq const$, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions*

$$\begin{aligned} &\nu_u + \lambda_v = \lambda(\ln|\mu|)_v; \\ &|\mu|\,(\ln|\mu|)_{uv} = -\nu^2. \end{aligned} \tag{23}$$

If $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is a pseudoorthonormal frame at a point $p_0 \in \mathbb{R}^4_1$, then there exists a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique timelike surface $\mathcal{M} : z = z(u,v),\ (u,v) \in \mathcal{D}_0$ with parallel normalized mean curvature vector field, such that $\mathcal{M}$ passes through p_0, $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is the geometric frame of $\mathcal{M}$ at the point p_0, the functions $\lambda(u,v)$, $\mu(u,v)$, $\nu(u)$ are the geometric functions of the surface, and $K - H^2 = 0$. Furthermore, (u,v) are canonical isotropic parameters of $\mathcal{M}$.

Proof Let us consider the following system of partial differential equations for the unknown vector functions $x = x(u,v)$, $y = y(u,v)$, $n_1 = n_1(u,v)$, $n_2 = n_2(u,v)$ in $\mathbb{R}^4_1$:

$$\begin{aligned} x_u &= \frac{1}{\sqrt{|\mu|}}\,(\gamma_1\, x + \lambda\, n_1 + \mu\, n_2) & x_v &= \frac{1}{\sqrt{|\mu|}}\,(-\gamma_2\, x - \nu\, n_1) \\ y_u &= \frac{1}{\sqrt{|\mu|}}\,(-\gamma_1\, y - \nu\, n_1) & y_v &= \frac{1}{\sqrt{|\mu|}}\,(\gamma_2\, y) \\ (n_1)_u &= \frac{1}{\sqrt{|\mu|}}\,(-\nu\, x + \lambda\, y) & (n_1)_v &= \frac{1}{\sqrt{|\mu|}}\,(-\nu\, y) \\ (n_2)_u &= \frac{1}{\sqrt{|\mu|}}\,(\mu\, y) & (n_2)_v &= 0 \end{aligned} \tag{24}$$

where $\gamma_1 = -(\sqrt{|\mu|})_u$ and $\gamma_2 = -(\sqrt{|\mu|})_v$. It follows from equalities (23) that the integrability conditions of system (24) are fulfilled.

Further, the proof follows the steps in the proof of Theorem 1; therefore, we are not going to give the details.

Remark: We can also introduce canonical nonisotropic parameters which in the case where $K - H^2 > 0$ have the same geometric meaning as the canonical parameters of spacelike surfaces with parallel normalized mean curvature vector field in $\mathbb{R}^4_1$ and $\mathbb{R}^4$.

Acknowledgment

The authors are partially supported by the National Science Fund, Ministry of Education and Science of Bulgaria under contract KP-06-N52/3.

References

[1] Aleksieva Y, Milousheva V. Minimal Lorentz surfaces in pseudo-Euclidean 4-space with neutral metric. Journal of Geometry and Physics 2019; 142: 240-253. https://doi.org/10.1016/j.geomphys.2019.04.008

[2] Alías L, Palmer B. Curvature properties of zero mean curvature surfaces in four dimensional Lorenzian space forms. Mathematical Proceedings of the Cambridge Philosophical Society 1998; 124 (2): 315-327. https://doi.org/10.1017/S0305004198002618

[3] Chen BY. Geometry of Submanifolds. Marcel Dekker 1973; New York.

[4] Chen BY. Surfaces with parallel normalized mean curvature vector. Monatshefte für Mathematik 1980; 90 (3): 185-194. https://doi.org/10.1007/BF01295363

[5] Chen BY. Complete classification of Lorentz surfaces with parallel mean curvature vector in arbitrary pseudo-Euclidean space. Kyushu Journal of Mathematics 2010; 64 (2): 261-279. doi: 10.2206/kyushujm.64.261

[6] Chen BY. Submanifolds with parallel mean curvature vector in Riemannian and indefinite space forms. Arab Journal of Mathematical Sciences 2010; 16 (1): 1-46. doi: 10.48550/arXiv.1307.0430

[7] Chen BY. Pseudo-Riemannian geometry, δ-invariants and applications. World Scientific Publishing 2011; Singapore.

[8] Fu Y, Hou ZH. Classification of Lorentzian surfaces with parallel mean curvature vector in pseudo-Euclidean spaces. Journal of Mathematical Analysis and Applications 2010; 371 (1): 25-40. https://doi.org/10.1016/j.jmaa.2010.04.022

[9] Ganchev G, Kanchev K. Explicit solving of the system of natural PDE's of minimal surfaces in the four-dimensional Euclidean space. Comptes rendus de l'Académie bulgare des Sciences 2014; 67 (5): 623-628.

[10] Ganchev G, Kanchev K. Explicit solving of the system of natural PDE's of minimal space-like surfaces in Minkowski space-time. Comptes rendus de l'Académie bulgare des Sciences 2017; 70 (6): 761-768.

[11] Ganchev G, Kanchev K. Canonical coordinates and natural equations for minimal time-like surfaces in $\mathbb{R}^4_2$. Kodai Mathematical Journal 2020; 43 (3): 524-572. doi: 10.2996/kmj/1605063628

[12] Ganchev G, Milousheva V. Timelike surfaces with zero mean curvature in Minkowski 4-space. Israel Journal of Mathematics 2013; 196: 413-433. https://doi.org/10.1007/s11856-012-0169-y

[13] Ganchev G, Milousheva V. Surfaces with parallel normalized mean curvature vector field in Euclidean or Minkowski 4-space. Filomat 2019; 33 (4): 1135-1145. https://doi.org/10.2298/FIL1904135G

[14] Itoh T. Minimal surfaces in 4-dimensional Riemannian manifolds of constant curvature. Kodai Mathematical Seminar Reports 1971; 23: 451-458. https://doi.org/10.2996/kmj/1138846772

[15] Kassabov O, Milousheva V. Weierstrass representations of Lorentzian minimal surfaces in $\mathbb{R}^4_2$. Mediterranean Journal of Mathematics 2020; 17 (6, 199). https://doi.org/10.1007/s00009-020-01636-x

[16] Lane E. Projective differential geometry of curves and surfaces. University of Chicago Press 1932; Chicago.

[17] Larsen JC. Complex analysis, maximal immersions and metric singularities. Monatshefte für Mathematik 1996; 122: 105-156. https://doi.org/10.1007/BF01297766

[18] Little J. On singularities of submanifolds of higher dimensional Euclidean spaces. Annali di Matematica Pura ed Applicata. Series IV 1969; 83: 261-335. https://doi.org/10.1007/BF02411172

[19] Lund F, Regge T. Unified approach to strings and vortices with soliton solutions. Physical Review D 1976; 14 (6): 1524-1536. https://doi.org/10.1103/PhysRevD.14.1524

[20] Sen R. Biconservative submanifolds with parallel normalized mean curvature vector field in Euclidean spaces. Bulletin of the Iranian Mathematical Society 2022; 48: 3185-3194. https://doi.org/10.1007/s41980-022-00691-2

[21] Sen R, Turgay NC. On biconservative surfaces in 4-dimensional Euclidean space. Journal of Mathematical Analysis and Applications 2018; 460: 565-581. https://doi.org/10.1016/j.jmaa.2017.12.009

[22] Sen R, Turgay NC. Biharmonic PNMCV submanifolds in Euclidean 5-space. Turkish Journal of Mathematics 2023; 47: 296-316. doi: 10.55730/1300-0098.3361

[23] Tribuzy R, Guadalupe I. Minimal immersions of surfaces into 4-dimensional space forms. Rendiconti del Seminario Matematico della Università di Padova 1985; 73: 1-13.

[24] Yau S. Submanifolds with constant mean curvature. American Journal of Mathematics 1974; 96 (2): 346-366. https://doi.org/10.2307/2373638