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Lightcone framed curves in the Lorentz-Minkowski 3-space

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Abstract: For a nonlightlike nondegenerate regular curve, we have the arc-length parameter and the Frenet-Serret type formula by using a moving frame like a regular space curve in the Euclidean space. If a point of the curve moves between spacelike and timelike regions, then there is a lightlike point. In this paper, we consider mixed types of not only regular curves but also curves with singular points. In order to consider mixed type of curves with singular points, we introduce a frame, so-called the lightcone frame, and lightcone framed curves. We investigate differential geometric properties of lightcone framed curves.

Key words: Mixed type, lightcone framed curve, curvature, singular point

1. Introduction

For a nonlightlike nondegenerate regular curve, we have the arc-length parameter and the Frenet-Serret type formula by using a moving frame like a regular space curve in the Euclidean space. It follows that we have the curvature of a nonlightlike regular curve.

If a point of the curve moves between spacelike and timelike regions, then there is a lightlike point. However, at a lightlike point, there is no arc-length parameter. Also at a singular point, the situation is the same. In this paper, we consider mixed types of not only regular curves but also curves with singular points. In order to consider mixed type of curves with singular points, we introduce a frame, so-called the lightcone frame and lightcone framed curves. The idea is that the framed curve is in the Euclidean space [6] and lightcone frame is in Lorentz-Minkowski plane [8]. For curves in Lorentz-Minkowski plane, some authors also studied the differential geometric properties in [12–14]. However, we focus on mixed type of curves in Lorentz-Minkowski 3-space by using the lightcone frame. As results, we consider the uniqueness and existence theorems for the lightcone frame curves. We also give the necessary and sufficient condition that a lightcone framed curve is contained in a plane. Compared with other work, such as [9], our research is quite different. In [9], the authors had also defined a kind of lightcone frame (called lightcone circle frame in our case) for studying the mixed type of curves. However, not all of the mixed type of curve has the lightcone circle frame. We give a necessary and sufficient condition that a lightcone framed curve has a lightcone circle frame (cf. Theorem 3).

The organization of this paper is as follows. We present a brief review on regular curves without type changing in the Lorentz-Minkowski space and mixed type of curves in the Lorentz-Minkowski plane in §2. We also recall basic notions about smooth curves (with singular points) in Euclidean space, including framed curves

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in the Euclidean 3-space and Legendre curves in the Euclidean plane in this section. The definitions of lightcone framed curves are given in §3 and the existence and uniqueness theorems of the lightcone framed curves are discussed. As a special case of a lightcone frame, we consider a lightcone circle frame which was introduced in [9]. We investigate the relationship between a lightcone framed base curve with a lightcone circle frame and a frontal in the Euclidean plane. At the end of this section, we consider a condition that the trace of a curve is contained in some plane, including both regular curves and singular curves. Some examples are given for explaining our results.

All maps and manifolds considered here are differentiable of class C^∞ without unless stated.

2. Preliminaries

2.1. Lorentz-Minkowski 3-space

The *Lorentz-Minkowski space* $\mathbb{R}^3_1$ is the space $\mathbb{R}^3$ endowed with the metric induced by the pseudoscalar product $\langle \mathbf{x}, \mathbf{y}\rangle = -x_1y_1 + x_2y_2 + x_3y_3$, where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. For more details, refer to [11].

We say that a nonzero vector $\mathbf{x} \in \mathbb{R}^3_1$ is *spacelike* if $\langle \mathbf{x}, \mathbf{x}\rangle > 0$, *lightlike* if $\langle \mathbf{x}, \mathbf{x}\rangle = 0$, and *timelike* if $\langle \mathbf{x}, \mathbf{x}\rangle < 0$ respectively. The *signature* of a vector $\mathbf{x}$ is defined by $\mathrm{sign}(\mathbf{x}) = 1, 0$ or -1 if $\mathbf{x}$ is spacelike, lightlike or timelike, respectively. The *norm* of a vector $\mathbf{x} \in \mathbb{R}^3_1$ is defined by $||\mathbf{x}|| = \sqrt{|\langle \mathbf{x}, \mathbf{x}\rangle|}$.

Given a vector $\mathbf{v} \in \mathbb{R}^3_1$ and a real number $c \in \mathbb{R}$, the plane with pseudonormal $\mathbf{v}$ is given by

$$PL(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}^3_1 \mid \langle \mathbf{x}, \mathbf{v}\rangle = c\}.$$

We say that $PL(\mathbf{v}, c)$ is a spacelike, timelike, or lightlike plane if $\mathbf{v}$ is timelike, spacelike, or lightlike, respectively.

For any $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3_1$, we define a vector $\mathbf{x} \wedge \mathbf{y}$ by

$$\mathbf{x} \wedge \mathbf{y} = \det \begin{pmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}, \tag{1}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of $\mathbb{R}^3_1$. For any $\mathbf{w} \in \mathbb{R}^3_1$, we can easily check that

$$\langle \mathbf{w}, \mathbf{x} \wedge \mathbf{y}\rangle = \det(\mathbf{w}, \mathbf{x}, \mathbf{y}),$$

so that $\mathbf{x} \wedge \mathbf{y}$ is pseudoorthogonal to both $\mathbf{x}$ and $\mathbf{y}$. Moreover, if $\mathbf{x}$ is a unit timelike vector, $\mathbf{y}$ is a unit spacelike vector and $\langle \mathbf{x}, \mathbf{y}\rangle = 0$, $\mathbf{x} \wedge \mathbf{y} = \mathbf{z}$, then by a straightforward calculation, we have

$$\mathbf{z} \wedge \mathbf{x} = \mathbf{y}, \ \mathbf{y} \wedge \mathbf{z} = -\mathbf{x}.$$

We define *hyperbolic* 2*-space* by

$$H^2(-1) = \{\mathbf{x} \in \mathbb{R}^3_1 \mid \langle \mathbf{x}, \mathbf{x}\rangle = -1\},$$

de Sitter 2*-space* by

$$S^2_1 = \{\mathbf{x} \in \mathbb{R}^3_1 \mid \langle \mathbf{x}, \mathbf{x}\rangle = 1\},$$

lightcone at the origin by

$$LC^* = \{\mathbf{x} \in \mathbb{R}^3_1 \mid \langle \mathbf{x}, \mathbf{x}\rangle = 0\}.$$

If $\mathbf{x} = (x_0, x_1, x_2)$ is a nonzero lightlike vector, then $x_0 \neq 0$. Therefore, we have

$$\widetilde{\mathbf{x}} = \left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \in S^1_+ = \{\mathbf{x} = (x_0, x_1, x_2) \in LC^* |\ x_0 = 1\}.$$

We call S^1_+ the *lightcone circle*.

In this paper, we consider two double Legendrian fibrations (cf. [7]):

(1) $\Delta_1 = \{(\mathbf{v}, \mathbf{w}) \in H^2(-1) \times S^2_1 \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0\}$,

$\pi_{11} : \Delta_1 \to H^2(-1), \pi_{11}(\mathbf{v}, \mathbf{w}) = \mathbf{v},\ \pi_{12} : \Delta_1 \to S^2_1, \pi_{12}(\mathbf{v}, \mathbf{w}) = \mathbf{w}$,

$\theta_{11} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_1}, \theta_{12} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_1}$.

(2) $\Delta_4 = \{(\mathbf{v}, \mathbf{w}) \in LC^* \times LC^* \mid \langle \mathbf{v}, \mathbf{w} \rangle = -2\}$,

$\pi_{41} : \Delta_1 \to LC^*, \pi_{41}(\mathbf{v}, \mathbf{w}) = \mathbf{v},\ \pi_{42} : \Delta_1 \to LC^*, \pi_{42}(\mathbf{v}, \mathbf{w}) = \mathbf{w}$,

$\theta_{41} = \langle d\mathbf{v}, \mathbf{w} \rangle|_{\Delta_4}, \theta_{42} = \langle \mathbf{v}, d\mathbf{w} \rangle|_{\Delta_4}$.

Note that $\Phi : \Delta_4 \to \Delta_1, \Phi(\mathbf{v}, \mathbf{w}) = ((\mathbf{v} + \mathbf{w})/2, (\mathbf{v} - \mathbf{w})/2)$ is a contact diffeomorphsim.

2.2. Regular curves without type changing in the Lorentz-Minkowski 3-space

In this subsection, we review notions on regular curves without type changing in $\mathbb{R}^3_1$. For more details refer to [3, 4] and arXiv: 1905.03367. Let I be an interval of $\mathbb{R}$ and $\gamma : I \to \mathbb{R}^3_1$ be a smooth curve. We say that γ is *spacelike* (respectively, *lightlike, timelike*) if $\dot{\gamma}(t) = (d\gamma/dt)(t)$ is a spacelike (respectively, lightlike, timelike) vector for any $t \in I$. We call γ a *nonlightlike* curve if γ is spacelike or timelike. Moreover, a point t (or, $\gamma(t)$) is called a *spacelike* (respectively, *lightlike*, *timelike*) point if $\dot{\gamma}(t)$ is a spacelike (respectively, lightlike, timelike) vector.

Let $\gamma : I \to \mathbb{R}^3_1$ be a nonlightlike regular curve. In this case, we may take the arc-length parameter s of γ. It follows that $||\gamma'(s)|| = 1$ for all $s \in I$, where $\gamma'(s) = (d\gamma/ds)(s)$. We denote $\boldsymbol{t}\,(s) = \gamma'(s)$ and call it the *unit tangent vector field*. We define the *curvature function* of γ as $\kappa : I \to \mathbb{R},\ \kappa(s) = \sqrt{|\langle \gamma''(s), \gamma''(s) \rangle|}$. Although γ is a nonlightlike curve, the curvature function $\kappa(s)$ may be zero. In the case when $\kappa(s) \neq 0$, for any $s \in I$, we define two signs $\varepsilon_\gamma \in \{-1, +1\}$ and $\sigma_\gamma \in \{-1, +1\}$ as $\varepsilon_\gamma = 1, \sigma_\gamma = -1$ if γ is a spacelike curve of type S; $\varepsilon_\gamma = \sigma_\gamma = +1$ if γ is a spacelike curve of type T; $\varepsilon_\gamma = \sigma_\gamma = -1$ if γ is a timelike curve. Here, we call a nonlightlike curve a *spacelike curve of type* S (or *of type* T) if $\boldsymbol{t}\,(s)$ is a spacelike vector and $\boldsymbol{t}\,'(s)$ is a spacelike vector (or timelike vector) for any $s \in I$. For details about the spacelike curve of type S or of type T, see the research results of Honda (see arXiv: 1905.03367).

We now define two vector fields by $\mathbf{n}_1(s) = \gamma''(s)/\kappa(s)$ and $\mathbf{n}_2(s) = \varepsilon_\gamma \sigma_\gamma \mathbf{t}(s) \wedge \mathbf{n}_1(s)$. We call them the *principal normal vector field* and *binormal vector field*, respectively, which satisfy $\det(\mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s)) = 1$. According to a straightforward calculation, we get the following Frenet-Serret type formula.

$$\begin{pmatrix} \mathbf{t}'(s) \\ \mathbf{n}_1'(s) \\ \mathbf{n}_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ \varepsilon_\gamma \sigma_\gamma \kappa(s) & 0 & \sigma_\gamma \tau(s) \\ 0 & \varepsilon_\gamma \sigma_\gamma \tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}_1(s) \\ \mathbf{n}_2(s) \end{pmatrix}, \tag{2}$$

where $\tau(s) = \varepsilon_\gamma \langle \mathbf{n}_1'(s), \mathbf{n}_2(s)\rangle$ is called the *torsion function* of γ. Moreover, in the case when $\gamma''(s) \neq 0$ and $\kappa(s) = 0$, namely $\gamma''(s)$ is a lightlike vector field. The curve γ is called a *spacelike curve of type L* by Honda. We denote $\gamma''(s) = \mathbf{N}(s)$. It follows that there exists an unique lightlike vector field $\mathbf{B}(s)$ such that $\langle \mathbf{t}(s), \mathbf{B}(s)\rangle = 0$ and $\langle \mathbf{N}(s), \mathbf{B}(s)\rangle = -2$. Then, $\{\mathbf{t}, \mathbf{N}, \mathbf{B}\}$ is a frame of $\mathbb{R}^3_1$ along γ and the Frenet-Serret type formula is as follows.

$$\begin{pmatrix} \mathbf{t}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\mu(s) & 0 \\ 2 & 0 & \mu(s) \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}, \tag{3}$$

where $\mu(s) = \frac{1}{2}\langle \mathbf{N}'(s), \mathbf{B}(s)\rangle$ is called the *pseudotorsion function* of γ.

Let $\gamma : I \to \mathbb{R}^3_1$ be a nondegenerate lightlike curve, namely, $\boldsymbol{\alpha}(t) := \gamma'(t) \in LC^*$ and $\mathbf{N}(t) := \gamma''(t)$ is nonlightlike and nonzero vector field, for any $t \in I$. In this case, we can choose a parametrization $s = s(t)$ such that $\|\gamma''(s)\| = 1$. We call s the *pseudo arc-length* parameter of γ. Without loss of generality, we take s as pseudoarc-length parameter of a nondegenerate lightlike curve γ in the following equations. Then, there exists a unique lightlike vector field $\boldsymbol{\beta}(s)$ such that $\langle \boldsymbol{\alpha}(s), \boldsymbol{\beta}(s)\rangle = -2$ and $\langle \mathbf{N}(s), \boldsymbol{\beta}(s)\rangle = 0$. Then, $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{N}\}$ is a *Cartan frame* of $\mathbb{R}^3_1$ along γ and the Frenet-Serret type formula is as follows (cf. [3, 4]).

$$\begin{pmatrix} \boldsymbol{\alpha}'(s) \\ \boldsymbol{\beta}'(s) \\ \mathbf{N}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \nu(s) \\ \frac{1}{2}\nu(s) & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}(s) \\ \boldsymbol{\beta}(s) \\ \mathbf{N}(s) \end{pmatrix}, \tag{4}$$

where $\nu(s) = \langle \mathbf{N}(s), \boldsymbol{\beta}'(s)\rangle$ is called the *Cartan torsion function* of γ.

2.3. Mixed types of curves in the Lorentz-Minkowski plane

In this subsection, we prepare some notions on mixed types of curves in $\mathbb{R}^2_1$. For more details, refer to [8]. The *Lorentz-Minkowski plane* $\mathbb{R}^2_1$ is the plane $\mathbb{R}^2$ endowed with the metric induced by the pseudoscalar product $\langle \mathbf{u}, \mathbf{v}\rangle = -u_0v_0 + u_1v_1$, where $\mathbf{u} = (u_0, u_1)$ and $\mathbf{v} = (v_0, v_1)$.

We say that a nonzero vector $\mathbf{u} \in \mathbb{R}^2_1$ is *spacelike* if $\langle \mathbf{u}, \mathbf{u}\rangle > 0$, *lightlike* if $\langle \mathbf{u}, \mathbf{u}\rangle = 0$, and *timelike* if $\langle \mathbf{u}, \mathbf{u}\rangle < 0$ respectively. The *norm* of a vector $\mathbf{u} = (u_0, u_1) \in \mathbb{R}^2_1$ is defined by $||\mathbf{u}|| = \sqrt{|\langle \mathbf{u}, \mathbf{u}\rangle|}$.

We denote $\mathbf{L}^+ = (1, 1)$ and $\mathbf{L}^- = (1, -1)$. By definition, $\mathbf{L}^+$ and $\mathbf{L}^-$ are independent lightlike vectors and $\langle \mathbf{L}^+, \mathbf{L}^-\rangle = -2$. We call $\{\mathbf{L}^+, \mathbf{L}^-\}$ a *lightcone frame* on $\mathbb{R}^2_1$.

Let $\gamma : I \to \mathbb{R}^2_1$ be a smooth curve (with lightlike points). There exists a smooth function $(\alpha, \beta) : I \to \mathbb{R}^2 \setminus \{0\}$ such that

$$\dot\gamma(t) = \alpha(t)\mathbf{L}^+ + \beta(t)\mathbf{L}^- \tag{5}$$

for all $t \in I$. We say that a *curve* γ *with the lightlike tangential data* (α, β) if the condition (5) holds.

Since $\langle \dot\gamma(t), \dot\gamma(t)\rangle = -4\alpha(t)\beta(t)$, $\gamma(t)$ is a spacelike (respectively, lightlike or timelike) point if and only if $\alpha(t)\beta(t) < 0$ (respectively, $\alpha(t)\beta(t) = 0$ or $\alpha(t)\beta(t) > 0$).

2.4. Framed curves in the Euclidean space

We give a quick review on curves with moving frame in $\mathbb{R}^3$. For more details, refer to [5, 6]. Let $\mathbb{R}^3$ be the 3-dimensional Euclidean space equipped with the inner product $\mathbf{a}\cdot\mathbf{b} = a_1b_1+a_2b_2+a_3b_3$, where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. We denote $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$. We define the vector product,

$$\mathbf{a}\times\mathbf{b} = \det\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}, \tag{6}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the canonical basis on $\mathbb{R}^3$. We denote

$$Z := \mathrm{diag}(-1, 1, 1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

(refer to arXiv: 1905.03367). Then $Z(\mathbf{a}) = Z(a_1, a_2, a_3) = (-a_1, a_2, a_3)$. By definition, we have

- $Z(\mathbf{a}\wedge\mathbf{b}) = \mathbf{a}\times\mathbf{b}$,
- $\langle Z(\mathbf{a}), \mathbf{b}\rangle = \langle \mathbf{a}, Z(\mathbf{b})\rangle = \mathbf{a}\cdot\mathbf{b}$,
- $Z(\mathbf{a})\cdot\mathbf{b} = \mathbf{a}\cdot Z(\mathbf{b}) = \langle \mathbf{a}, \mathbf{b}\rangle$,
- $Z(\mathbf{a})\cdot Z(\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}$,
- $\langle Z(\mathbf{a}), Z(\mathbf{b})\rangle = \langle \mathbf{a}, \mathbf{b}\rangle$.

We denote the set Δ by

$$\{(\nu_1, \nu_2) \in \mathbb{R}^3\times\mathbb{R}^3 \mid \nu_1\cdot\nu_1 = 1, \nu_2\cdot\nu_2 = 1, \nu_1\cdot\nu_2 = 0\}$$

Then Δ is a 3-dimensional smooth manifold. If $(\nu_1, \nu_2) \in \Delta$, we define a unit vector $\boldsymbol{\mu} = \nu_1\times\nu_2$ of $\mathbb{R}^3$. A framed curve in the Euclidean space is a curve with a moving frame.

Definition 1 We say that $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3\times\Delta$ is a *framed curve* if $\dot\gamma(t)\cdot\nu_1(t) = \dot\gamma(t)\cdot\nu_2(t) = 0$ for all $t \in I$. We also say that $\gamma : I \to \mathbb{R}^3$ is a *framed base curve* if there exists $(\nu_1, \nu_2) : I \to \Delta$ such that (γ, ν_1, ν_2) is a framed curve.

Note that an analytic curve germ is always framed base curve at least locally, see [5].

Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3\times\Delta$ be a framed curve and $\boldsymbol{\mu}(t) = \nu_1(t)\times\nu_2(t)$. The Frenet-Serret type formula is given by

$$\begin{pmatrix} \dot\nu_1(t) \\ \dot\nu_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) & m(t) \\ -\ell(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix}\begin{pmatrix} \nu_1(t) \\ \nu_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \dot\gamma(t) = \alpha(t)\boldsymbol{\mu}(t), \tag{8}$$

where $\ell(t) = \dot\nu_1(t)\cdot\nu_2(t), m(t) = \dot\nu_1(t)\cdot\boldsymbol{\mu}(t), n(t) = \dot\nu_2(t)\cdot\boldsymbol{\mu}(t), \alpha(t) = \dot\gamma(t)\cdot\boldsymbol{\mu}(t)$. We call the mapping (ℓ, m, n, α) *the (framed) curvature* of the framed curve.

We denote a plane by

$$PE(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \cdot \mathbf{v} = c\},$$

where $\mathbf{v} \in S^2$ and $c \in \mathbb{R}$. Note that $PE(\mathbf{v}, c) = PL(Z(\mathbf{v}), c)$ and $PE(Z(\mathbf{v}), c) = PL(\mathbf{v}, c)$. If $\gamma(t) \in PE(\mathbf{v}, c)$, then we have $\det(\dot\gamma(t), \ddot\gamma(t), \dddot\gamma(t)) = 0$ for all $t \in I$. It follows that

$$\alpha^3(t)\left(m(t)\dot n(t) - \dot m(t)n(t) + (m^2(t) + n^2(t))\ell(t)\right) = 0$$

for all $t \in I$. Conversely, we have the following result.

Proposition 1 ([6]) *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with curvature (ℓ, m, n, α).*

(1) *If $m^2(t) + n^2(t) \neq 0$ and $m(t)\dot n(t) - \dot m(t)n(t) + (m^2(t) + n^2(t))\ell(t) = 0$ for all $t \in I$, then there exist a vector $\mathbf{v} \in S^2$ and a constant $c \in \mathbb{R}$ such that $\gamma(t) \in PE(\mathbf{v}, c)$.*

(2) *If (γ, ν_1, ν_2) is an analytic framed curve and $m(t)\dot n(t) - \dot m(t)n(t) + (m^2(t) + n^2(t))\ell(t) = 0$ for all $t \in I$, then there exist a vector $\mathbf{v} \in S^2$ and a constant $c \in \mathbb{R}$ such that $\gamma(t) \in PE(\mathbf{v}, c)$.*

For regular cases, the following result is well-known.

Proposition 2 *Let $\gamma : I \to \mathbb{R}^3$ be a regular curve and nondegenerate, that is, $\dot\gamma(t) \times \ddot\gamma(t) \neq 0$ for all $t \in I$. Then the trace of γ is contained in a plane if and only if the torsion $\tau(t) = 0$ for all $t \in I$.*

Note that the torsion is given by

$$\tau(t) = \frac{\det(\dot\gamma(t), \ddot\gamma(t), \dddot\gamma(t))}{|\dot\gamma(t) \times \ddot\gamma(t)|^2}.$$

In Proposition 2, the nondegenerate condition is needed. If $\gamma : I \to \mathbb{R}^3$ is a regular curve and $\det(\dot\gamma(t), \ddot\gamma(t), \dddot\gamma(t)) = 0$ for all $t \in I$, then it does not follow that the trace of γ is contained in a plane. That is, there are examples that the trace of γ is not contained in a plane even if $\dot\gamma(t) \neq 0$ and $\det(\dot\gamma(t), \ddot\gamma(t), \dddot\gamma(t)) = 0$ for all $t \in I$.

2.5. Legendre curves in the Euclidean plane

We quickly review the theory of Legendre curves in the unit tangent bundle over $\mathbb{R}^2$, for details see [5]. Let $\mathbb{R}^2$ be the Euclidean plane equipped with the inner product $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2$, where $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$. We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma, \nu)^*\theta = 0$ for all $t \in I$, where θ is a canonical contact form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ over $\mathbb{R}^2$ (cf. [1, 2]). This condition is equivalent to $\dot\gamma(t) \cdot \nu(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^2$ is a frontal if there exists $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve. We have the Frenet formula of a frontal γ as follows. We put on $\boldsymbol{\mu}(t) = J(\nu(t))$. Then we call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ a *moving frame of a frontal* $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet formula of a frontal (or, Legendre curve),

$$\begin{pmatrix} \dot\nu(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \ \dot\gamma(t) = \beta(t)\boldsymbol{\mu}(t), \tag{9}$$

where $\ell(t) = \dot\nu(t) \cdot \boldsymbol{\mu}(t)$ and $\beta(t) = \dot\gamma(t) \cdot \boldsymbol{\mu}(t)$. We call the pair (ℓ, β) *the (Legendre) curvature* of the Legendre curve.

3. Lightcone framed curves

In order to investigate mixed type of curves in the Lorentz-Minkowski 3-space, we introduce the lightcone frame.

Definition 2 Let $(\gamma, \ell^+, \ell^-) : I \to \mathbb{R}^3_1 \times \Delta_4$ be a smooth mapping. We say that (γ, ℓ^+, ℓ^-) is a *lightcone framed curve* if there exist smooth functions $\alpha, \beta : I \to \mathbb{R}$ such that $\dot\gamma(t) = \alpha(t)\ell^+(t) + \beta(t)\ell^-(t)$ for all $t \in I$. We also say that γ is a *lightcone framed base curve* if there exists a smooth mapping $(\ell^+, \ell^-) : I \to \Delta_4$ such that $(\gamma, \ell^+, \ell^-) : I \to \mathbb{R}^3_1 \times \Delta_4$ is a lightcone framed curve.

Since $(\ell^+, \ell^-) : I \to \Delta_4$, we have

$$\langle \ell^+(t), \ell^+(t)\rangle = 0,\ \langle \ell^-(t), \ell^-(t)\rangle = 0,\ \langle \ell^+(t), \ell^-(t)\rangle = -2$$

for all $t \in I$. By $\dot\gamma(t) = \alpha(t)\ell^+(t) + \beta(t)\ell^-(t)$, $\langle\dot\gamma(t), \dot\gamma(t)\rangle = -4\alpha(t)\beta(t)$. Therefore, γ is spacelike, lightlike, or timelike at t if $\alpha(t)\beta(t) < 0$, $\alpha(t)\beta(t) = 0$ or $\alpha(t)\beta(t) < 0$, respectively. Note that if t is a lightlike, then $\alpha(t)\beta(t) = 0$ with $\alpha(t) \neq 0$ or $\beta(t) \neq 0$. Moreover, t is a singular point of γ if and only if $\alpha(t) = \beta(t) = 0$. Although the relation among $\dot\gamma$, ℓ^+, and ℓ^- looks similar to the case in Lorentz-Minkwoski plane [8], our situation is quite different. In our case, ℓ^+ and ℓ^- are not fixed lightlike vectors.

We denote $(\mathbf{n}^T, \mathbf{n}^S) : I \to \Delta_1$,

$$\mathbf{n}^T(t) = \frac{\ell^+(t) + \ell^-(t)}{2},\ \mathbf{n}^S(t) = \frac{\ell^+(t) - \ell^-(t)}{2}.$$

We define $\mathbf{n} : I \to S^2_1, \mathbf{n}(t) = \mathbf{n}^T(t) \wedge \mathbf{n}^S(t) = -(1/2)\ell^+(t) \wedge \ell^-(t)$. Then $\{\mathbf{n}^T(t), \mathbf{n}^S(t), \mathbf{n}(t)\}$ is a pseudoorthonormal frame of $\gamma(t)$. We say that $\{\ell^+(t), \ell^-(t), \mathbf{n}(t)\}$ is a *lightcone frame* of $\gamma(t)$. Note that the lightcone frame is not a pseudoorthonormal frame. By a direct calculation, we have

$$\begin{pmatrix} \dot{\ell^+}(t) \\ \dot{\ell^-}(t) \\ \dot{\mathbf{n}}(t) \end{pmatrix} = \begin{pmatrix} \kappa_1(t) & 0 & 2\kappa_3(t) \\ 0 & -\kappa_1(t) & 2\kappa_2(t) \\ \kappa_2(t) & \kappa_3(t) & 0 \end{pmatrix} \begin{pmatrix} \ell^+(t) \\ \ell^-(t) \\ \mathbf{n}(t) \end{pmatrix}, \tag{10}$$

$$\dot\gamma(t) = \alpha(t)\ell^+(t) + \beta(t)\ell^-(t), \tag{11}$$

where

$$\kappa_1(t) = -\frac{1}{2}\langle \dot{\ell^+}(t), \ell^-(t)\rangle,\ \kappa_2(t) = -\frac{1}{2}\langle \dot{\mathbf{n}}(t), \ell^-(t)\rangle,\ \kappa_3(t) = -\frac{1}{2}\langle \dot{\mathbf{n}}(t), \ell^+(t)\rangle,$$

$$\alpha(t) = -\frac{1}{2}\langle\dot\gamma(t), \ell^-(t)\rangle,\ \beta(t) = -\frac{1}{2}\langle\dot\gamma(t), \ell^+(t)\rangle.$$

We call $(\kappa_1, \kappa_2, \kappa_3, \alpha, \beta)$ a *(lightcone) curvature* of the lightcone framed curve $(\gamma, \ell^+, \ell^-) : I \to \mathbb{R}^3_1 \times \Delta_4$.

On the other hand, we have

$$\begin{pmatrix} \dot{\mathbf{n}}^T(t) \\ \dot{\mathbf{n}}^S(t) \\ \dot{\mathbf{n}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(t) & \kappa_2(t) + \kappa_3(t) \\ \kappa_1(t) & 0 & -\kappa_2(t) + \kappa_3(t) \\ \kappa_2(t) + \kappa_3(t) & \kappa_2(t) - \kappa_3(t) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{n}^T(t) \\ \mathbf{n}^S(t) \\ \mathbf{n}(t) \end{pmatrix}, \tag{12}$$

$$\dot\gamma(t) = (\alpha(t) + \beta(s))\mathbf{n}^T(t) + (\alpha(t) - \beta(t))\mathbf{n}^S(t). \tag{13}$$

Example 1 Let $k>1$ be a natural number. We define $(\gamma,\ell^+,\ell^-):\mathbb{R}\to\mathbb{R}^3_1\times\Delta_4$ by

$$\begin{aligned}\gamma(t) &= \left(\frac{1}{k}t^k,\cos t,\sin t\right),\\ \ell^+(t) &= (1,\sin t,-\cos t),\\ \ell^-(t) &= (1,-\sin t,\cos t).\end{aligned}$$

Then $(\gamma,\ell^+,\ell^-):\mathbb{R}\to\mathbb{R}^3_1\times\Delta_4$ is a regular lightcone framed curve. By a direct calculation,

$$\mathbf{n}(t)=(0,\cos t,\sin t)$$

and the curvature $(\kappa_1(t),\kappa_2(t),\kappa_3(t),\alpha(t),\beta(t))$ of (γ,ℓ^+,ℓ^-) is given by

$$\left(0,\ -\frac{1}{2},\ \frac{1}{2},\ \frac{t^{k-1}-1}{2},\ \frac{t^{k-1}+1}{2}\right).$$

Since

$$\alpha(t)\beta(t)=\frac{1}{4}(t^{2(k-1)}-1),$$

γ is spacelike (respectively, lightlike or timelike) if $t^{2(k-1)}-1<0$ (respectively, $t^{2(k-1)}-1=0$ or $t^{2(k-1)}-1>0$).

We remark that if $k=1$, then $\gamma(t)=(t,\cos t,\sin t)$ is a nondegenerate lightlike curve with the Cartan torsion function $\nu(t)=-1$.

Example 2 Let $\gamma:I\to\mathbb{R}^3_1$ be a regular spacelike curve of type S (or T) with the curvature function κ and the torsion function τ. Suppose $\{\mathbf{t},\mathbf{n}_1,\mathbf{n}_2\}$ is the frame of $\mathbb{R}^3_1$ along γ (see §2.2). Regular-curves-without We denote that

$$\ell^\pm(t)=\frac{1+\sigma_\gamma}{2}\mathbf{n}_1(t)+\frac{1-\sigma_\gamma}{2}\mathbf{n}_2(t)\pm\mathbf{t}(t),$$

$$\mathbf{n}(t)=\frac{1-\sigma_\gamma}{2}\mathbf{n}_1(t)-\frac{1+\sigma_\gamma}{2}\mathbf{n}_2(t).$$

By a straightforward calculation, the curvature $(\kappa_1(t),\kappa_2(t),\kappa_3(t),\alpha(t),\beta(t))$ of (γ,ℓ^+,ℓ^-) is given by

$$\left(\frac{1+\sigma_\gamma}{2}\kappa(t),-\frac{\frac{1-\sigma_\gamma}{2}\kappa(t)+\tau(t)}{2},\frac{\frac{1-\sigma_\gamma}{2}\kappa(t)-\tau(t)}{2},\frac{1}{2},-\frac{1}{2}\right).$$

Example 3 Let $\gamma:I\to\mathbb{R}^3_1$ be a regular spacelike curve of type L with the pseudotorsion function μ. Suppose $\{\mathbf{t},\mathbf{N},\mathbf{B}\}$ is the frame of $\mathbb{R}^3_1$ along γ (see §2.2). We denote that

$$\ell^\pm(t)=\frac{1}{2}(\mathbf{N}(t)+\mathbf{B}(t))\pm\mathbf{t}(t),$$

$$\mathbf{n}(t)=-\frac{1}{2}(\mathbf{N}(t)-\mathbf{B}(t)).$$

By a straightforward calculation, the curvature $(\kappa_1(t),\kappa_2(t),\kappa_3(t),\alpha(t),\beta(t))$ of (γ,ℓ^+,ℓ^-) is given by

$$\left(1,\frac{1+\mu(t)}{2},-\frac{1-\mu(t)}{2},\frac{1}{2},-\frac{1}{2}\right).$$

Example 4 Let $\gamma : I \to \mathbb{R}^3_1$ be a regular timelike curve with the curvature function κ and the torsion function τ. Suppose $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$ is the frame of $\mathbb{R}^3_1$ along γ (see §2.2). Firstly, we denote that

$$\ell^{\pm}(t) = \mathbf{t}(t) \pm \mathbf{n}_1(t), \mathbf{n}(t) = \mathbf{n}_2(t).$$

By a straightforward calculation, the curvature $(\kappa_1(t), \kappa_2(t), \kappa_3(t), \alpha(t), \beta(t))$ of (γ, ℓ^+, ℓ^-) is given by

$$\left(\kappa(t), \frac{\tau(t)}{2}, -\frac{\tau(t)}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

Moreover, if we denote that

$$\ell^{\pm}(t) = \mathbf{t}(t) \pm \mathbf{n}_2(t), \mathbf{n}(t) = -\mathbf{n}_1(t).$$

By a straightforward calculation, the curvature $(\kappa_1(t), \kappa_2(t), \kappa_3(t), \alpha(t), \beta(t))$ of (γ, ℓ^+, ℓ^-) is given by

$$\left(0, -\frac{\kappa(t) - \tau(t)}{2}, -\frac{\kappa(t) + \tau(t)}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

Example 5 Let $\gamma : I \to \mathbb{R}^3_1$ be a regular nondegenerate lightlike curve with the Cartan torsion function ν. Suppose $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{N}\}$ is the frame of $\mathbb{R}^3_1$ along γ (see §2.2). We denote that

$$\ell^+(t) = \boldsymbol{\alpha}(t), \ell^-(t) = \boldsymbol{\beta}(t), \mathbf{n}(t) = -\mathbf{N}(t).$$

By a straightforward calculation, the curvature $(\kappa_1(t), \kappa_2(t), \kappa_3(t), \alpha(t), \beta(t))$ of (γ, ℓ^+, ℓ^-) is given by

$$\left(0, -\frac{\nu(t)}{2}, -\frac{1}{2}, 1, 0\right).$$

We consider a new lightcone frame. We denote $\overline{\ell}^+(t) = (1/c(t))\ell^+(t)$ and $\overline{\ell}^-(t) = c(t)\ell^-(t)$, where $c : I \to \mathbb{R}$ is a nonzero function. By a direct calculation, we have

$$\overline{\ell}^+(t) \wedge \overline{\ell}^-(t) = \ell^+(t) \wedge \ell^-(t) = -2\mathbf{n}(t).$$

Then $\{\overline{\ell}^+(t), \overline{\ell}^-(t), \mathbf{n}(t)\}$ is also a lightcone frame of $\gamma(t)$. By a direct calculation, we have

$$\begin{pmatrix} \dot{\overline{\ell}}^+(t) \\ \dot{\overline{\ell}}^-(t) \\ \dot{\mathbf{n}}(t) \end{pmatrix} = \begin{pmatrix} \overline{\kappa}_1(t) & 0 & 2\overline{\kappa}_3(t) \\ 0 & -\overline{\kappa}_1(t) & 2\overline{\kappa}_2(t) \\ \overline{\kappa}_2(t) & \overline{\kappa}_3(t) & 0 \end{pmatrix} \begin{pmatrix} \overline{\ell}^+(t) \\ \overline{\ell}^-(t) \\ \mathbf{n}(t) \end{pmatrix}, \tag{14}$$

$$\dot{\gamma}(t) = \overline{\alpha}(t)\overline{\ell}^+(t) + \overline{\beta}(t)\overline{\ell}^-(t), \tag{15}$$

where

$$\overline{\kappa}_1(t) = -\frac{\dot{c}(t) - c(t)\kappa_1(t)}{c(t)},\ \overline{\kappa}_2(t) = c(t)\kappa_2(t),\ \overline{\kappa}_3(t) = \frac{\kappa_3(t)}{c(t)},\ \overline{\alpha}(t) = c(t)\alpha(t),\ \overline{\beta}(t) = \frac{\beta(t)}{c(t)}.$$

If we take $\dot{c}(t) = c(t)\kappa_1(t)$, that is, $c(t) = Ae^{\int \kappa_1(t)dt}$, where A is a constant, then $\overline{\kappa}_1(t) = 0$. Hence, we can always take $\overline{\kappa}_1(t) = 0$. We say that the lightcone frame $\{\overline{\ell}^+(t), \overline{\ell}^-(t), \mathbf{n}(t)\}$ with $\overline{\kappa}_1(t) = 0$ is an *adapted frame*.

From now on, we only consider an adapted frame. Then we rewire $(\gamma,\ell^+,\ell^-):I\to\mathbb{R}^3_1\times\Delta_4$ with curvature (m,n,α,β), that is,

$$\begin{pmatrix}\dot{\ell^+}(t)\\ \dot{\ell^-}(t)\\ \dot{\mathbf{n}}(t)\end{pmatrix} = \begin{pmatrix}0 & 0 & 2n(t)\\ 0 & 0 & 2m(t)\\ m(t) & n(t) & 0\end{pmatrix}\begin{pmatrix}\ell^+(t)\\ \ell^-(t)\\ \mathbf{n}(t)\end{pmatrix}, \tag{16}$$

$$\dot{\gamma}(t) = \alpha(t)\ell^+(t)+\beta(t)\ell^-(t). \tag{17}$$

Remark 1 *Let (γ,ℓ^+,ℓ^-) be a lightcone framed curve with an adapted frame and curvature (m,n,α,β). If c is a nonzero constant, then $(\gamma,(1/c)\ell^+,c\ell^-)$ is also a lightcone framed curve with an adapted frame and curvature $(cm,n/c,c\alpha,\beta/c)$. Hence, an adapted frame is not unique in this sense.*

By a direct calculation, it is easy to show the following result.

Proposition 3 *Let $(\gamma,\ell^+,\ell^-):I\to\mathbb{R}^3_1\times\Delta_4$ be a lightcone framed curve with curvature (m,n,α,β). We have the following:*

(1) *$(\gamma,-\ell^+,-\ell^-):I\to\mathbb{R}^3_1\times\Delta_4$ is a lightcone framed curve with curvature $(-m,-n,-\alpha,-\beta)$.*

(2) *$(-\gamma,\ell^+,\ell^-):I\to\mathbb{R}^3_1\times\Delta_4$ is a lightcone framed curve with curvature $(m,n,-\alpha,-\beta)$.*

(3) *$(-\gamma,-\ell^+,-\ell^-):I\to\mathbb{R}^3_1\times\Delta_4$ is a lightcone framed curve with curvature $(-m,-n,\alpha,\beta)$.*

(4) *$(\gamma,\ell^-,\ell^+):I\to\mathbb{R}^3_1\times\Delta_4$ is a lightcone framed curve with curvature $(-n,-m,\beta,\alpha)$.*

Theorem 1 (Existence theorem for lightcone framed curves) *Let $(m,n,\alpha,\beta):I\to\mathbb{R}^4$ be a smooth mapping. There exists a lightcone framed curve $(\gamma,\ell^+,\ell^-):I\to\mathbb{R}^3_1\times\Delta_4$ such that the curvature is (m,n,α,β).*

Proof Fix a point $t_0\in I$. We consider the following linear ordinary differential equation with the initial value,

$$\begin{pmatrix}\dot{\ell^+}(t)\\ \dot{\ell^-}(t)\\ \dot{\mathbf{n}}(t)\end{pmatrix} = \begin{pmatrix}0 & 0 & 2n(t)\\ 0 & 0 & 2m(t)\\ m(t) & n(t) & 0\end{pmatrix}\begin{pmatrix}\ell^+(t)\\ \ell^-(t)\\ \mathbf{n}(t)\end{pmatrix}, \tag{18}$$

$$\langle\ell^+(t_0),\ell^+(t_0)\rangle=\langle\ell^-(t_0),\ell^-(t_0)\rangle=\langle\ell^+(t_0),\mathbf{n}(t_0)\rangle=\langle\ell^-(t_0),\mathbf{n}(t_0)\rangle=0, \tag{19}$$

$$\langle\ell^+(t_0),\ell^-(t_0)\rangle=-2,\ \langle\mathbf{n}(t_0),\mathbf{n}(t_0)\rangle=1. \tag{20}$$

Then we have a solution $(\ell^+,\ell^-,\mathbf{n})$ of the linear differential equation. We define functions $a_1,\dots,a_6:I\to\mathbb{R}$ by

$$a_1(t)=\langle\ell^+(t),\ell^+(t)\rangle,\ a_2(t)=\langle\ell^+(t),\ell^-(t)\rangle,\ a_3(t)=\langle\ell^-(t),\ell^-(t)\rangle,$$
$$a_4(t)=\langle\ell^+(t),\mathbf{n}(t)\rangle,\ a_5(t)=\langle\ell^-(t),\mathbf{n}(t)\rangle,\ a_6(t)=\langle\mathbf{n}(t),\mathbf{n}(t)\rangle.$$

We also consider the following linear ordinary differential equation with the initial value,

$$\begin{pmatrix}\dot{a_1}(t)\\ \dot{a_2}(t)\\ \dot{a_3}(t)\\ \dot{a_4}(t)\\ \dot{a_5}(t)\\ \dot{a_6}(t)\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 4n(t) & 0 & 0\\ 0 & 0 & 0 & 2m(t) & 2n(t) & 0\\ 0 & 0 & 0 & 0 & 4m(t) & 0\\ m(t) & n(t) & 0 & 0 & 0 & 2n(t)\\ 0 & m(t) & n(t) & 0 & 0 & 2m(t)\\ 0 & 0 & 0 & 2m(t) & 2n(t) & 0\end{pmatrix}\begin{pmatrix}a_1(t)\\ a_2(t)\\ a_3(t)\\ a_4(t)\\ a_5(t)\\ a_6(t)\end{pmatrix}, \tag{21}$$

$$a_1(t_0) = 0, a_2(t_0) = -2, a_3(t_0) = 0, a_4(t_0) = 0, a_5(t_0) = 0, a_6(t_0) = 1. \quad (22)$$

By the uniqueness of the linear ordinary differential equation with the initial value, we have

$$a_1(t) = 0, a_2(t) = -2, a_3(t) = 0, a_4(t) = 0, a_5(t) = 0, a_6(t) = 1$$

for all $t \in I$. It follows that $(\ell^+, \ell^-) : I \to \Delta_4$. We define $\gamma(t) = \int(\alpha(t)\ell^+(t) + \beta(t)\ell^-(t))dt$. Then $(\gamma, \ell^+, \ell^-) : I \to \mathbb{R}^3_1 \times \Delta_4$ is a lightcone framed curve with the curvature (m, n, α, β). □

Let $O(1, 2)$ be the Lorentz group which consists of square matrices A of order 3 such that ${}^tAZA = Z$. We set $SO(1, 2)$ as

$$SO(1, 2) := \{A \in O(1, 2) \mid \det A = 1\}.$$

For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3_1$ and $A \in SO(1, 2)$, we have

$$\langle \mathbf{a}, \mathbf{b} \rangle = \langle A(\mathbf{a}), A(\mathbf{b}) \rangle, \ A(\mathbf{a} \wedge \mathbf{b}) = A(\mathbf{a}) \wedge A(\mathbf{b}).$$

Let (γ, ℓ^+, ℓ^-) and $(\overline{\gamma}, \overline{\ell}^+, \overline{\ell}^-) : I \to \mathbb{R}^3_1 \times \Delta_4$ be lightcone framed curves.

Definition 3 We say that (γ, ℓ^+, ℓ^-) and $(\overline{\gamma}, \overline{\ell}^+, \overline{\ell}^-) : I \to \mathbb{R}^3_1 \times \Delta_4$ are *congruent as lightcone framed curves* if there exist $A \in SO(1, 2)$ and $\mathbf{a} \in \mathbb{R}^3_1$ such that

$$\overline{\gamma}(t) = A(\gamma(t)) + \mathbf{a}, \ \overline{\ell}^+(t) = A(\ell^+(t)), \ \overline{\ell}^-(t) = A(\ell^-(t))$$

for all $t \in I$.

Theorem 2 (Uniqueness theorem for lightcone framed curves) *Let (γ, ℓ^+, ℓ^-) and $(\overline{\gamma}, \overline{\ell}^+, \overline{\ell}^-) : I \to \mathbb{R}^3_1 \times \Delta_4$ be two lightcone framed curves with curvatures (m, n, α, β) and $(\overline{m}, \overline{n}, \overline{\alpha}, \overline{\beta})$, respectively. (γ, ℓ^+, ℓ^-) and $(\overline{\gamma}, \overline{\ell}^+, \overline{\ell}^-)$ are congruent as lightcone framed curves if and only if $(m, n, \alpha, \beta) = (\overline{m}, \overline{n}, \overline{\alpha}, \overline{\beta})$.*

Proof Suppose that (γ, ℓ^+, ℓ^-) and $(\overline{\gamma}, \overline{\ell}^+, \overline{\ell}^-)$ are congruent as lightcone framed curves. There exist $A \in SO(1, 2)$ and $\mathbf{a} \in \mathbb{R}^3_1$ such that

$$\overline{\gamma}(t) = A(\gamma(t)) + \mathbf{a}, \ \overline{\ell}^+(t) = A(\ell^+(t)), \ \overline{\ell}^-(t) = A(\ell^-(t))$$

for all $t \in I$. Then $\overline{\mathbf{n}}(t) = A(\mathbf{n}(t))$. By a direct calculation, we have $(\overline{m}, \overline{n}, \overline{\alpha}, \overline{\beta}) = (m, n, \alpha, \beta)$.

Conversely, suppose that $(m, n, \alpha, \beta) = (\overline{m}, \overline{n}, \overline{\alpha}, \overline{\beta})$. For any fixed $t_0 \in I$, we can choose a matrix $A \in SO(1, 2)$ and a constant vector $\mathbf{a} \in \mathbb{R}^3_1$ such that

$$\overline{\gamma}(t_0) = A(\gamma(t_0)) + \mathbf{a}, \ \overline{\ell}^+(t_0) = A(\ell^+(t_0)), \ \overline{\ell}^-(t_0) = A(\ell^-(t_0))$$

(cf. [10]). By the necessaries, the curvature of the lightcone framed curves $(\gamma(t), \ell^+(t), \ell^-(t))$ and $(A(\gamma(t)) + \mathbf{a}, A(\ell^+(t)), A(\ell^-(t)))$ are equal. Therefore, the curvature of the lightcone framed curves $(\overline{\gamma}(t), \overline{\ell}^+(t), \overline{\ell}^-(t))$ and

$(A(\gamma(t))+\mathbf{a}, A(\ell^+(t)), A(\ell^-(t)))$ are also equal. Then they satisfy the same ODE system with the same initial value condition in Theorem 1. According to the uniqueness of ODE system, we have

$$\overline{\gamma}(t)=A(\gamma(t))+\mathbf{a},\ \overline{\ell}^+(t)=A(\ell^+(t)),\ \overline{\ell}^-(t)=A(\ell^-(t))$$

for any $t\in I$. It follows that (γ,ℓ^+,ℓ^-) and $(\overline{\gamma},\overline{\ell}^+,\overline{\ell}^-)$ are congruent as lightcone framed curves. □

We say that $t:\overline{I}\to I$ is a *(positive) parameter change* if t is surjective and $t'(u)>0$ for all $u\in\overline{I}$. By a direct calculation, we have the following result.

Proposition 4 *Let $(\gamma,\ell^+,\ell^-):I\to\mathbb{R}^3_1\times\Delta_4$ be a lightcone framed curve with curvature (m,n,α,β). Suppose that $t:\overline{I}\to I$ is a parameter change. Then $(\overline{\gamma},\overline{\ell}^+,\overline{\ell}^-)=(\gamma\circ t,\ell^+\circ t,\ell^-\circ t):\overline{I}\to\mathbb{R}^3_1\times\Delta_4$ is a lightcone framed curve with curvature $(t'm\circ t,t'n\circ t,t'\alpha\circ t,t'\beta\circ t)$.*

We consider a special lightcone frame (cf. [9]). Let $(\gamma,\ell^+,\ell^-):I\to\mathbb{R}^3_1\times\Delta_4$ be a lightcone framed curve, where ℓ^+ and $\ell^-:I\to S^1_+$. Then there exists a smooth function $\theta:I\to\mathbb{R}$ such that

$$\ell^+(t)=(1,\cos\theta(t),\sin\theta(t)),\ \ell^-(t)=(1,-\cos\theta(t),-\sin\theta(t)).$$

Therefore,

$$\mathbf{n}(t)=-\frac{1}{2}\ell^+(t)\wedge\ell^-(t)=(0,-\sin\theta(t),\cos\theta(t)).$$

We call the above lightcone frame $\{\ell^+(t),\ell^-(t),\mathbf{n}(t)\}$ a *lightcone circle frame* of $\gamma(t)$. By a direct calculation, we have

$$\begin{pmatrix}\dot{\ell^+}(t)\\ \dot{\ell^-}(t)\\ \dot{\mathbf{n}}(t)\end{pmatrix}=\begin{pmatrix}0&0&\dot\theta(t)\\0&0&-\dot\theta(t)\\-\dot\theta(t)/2&\dot\theta(t)/2&0\end{pmatrix}\begin{pmatrix}\ell^+(t)\\ \ell^-(t)\\ \mathbf{n}(t)\end{pmatrix}, \tag{23}$$

$$\dot\gamma(t)=\alpha(t)\ell^+(t)+\beta(t)\ell^-(t). \tag{24}$$

In this case, the curvature is given by $(-\dot\theta(t)/2,\dot\theta(t)/2,\alpha(t),\beta(t))$. We denote briefly $(\dot\theta,\alpha,\beta)$ as the (lightcone) curvature of the lightcone framed curve $(\gamma,\ell^+,\ell^-):I\to\mathbb{R}^3_1\times\Delta_4$. In [9], (α,β,θ) is called a *lightcone semipolar coordinate*. Moreover, evolutes of a mixed type curve are investigated by using lightcone circle frame of γ in [9]. However, not any mixed type of curve has the lightcone circle frame, namely our lightcone frame is a more general frame for the mixed type of curve. We give an existence condition that a lightcone framed base curve has a lightcone circle frame. A canonical projection on the Euclidean plane will be denoted by $\pi:\mathbb{R}^3_1\to\mathbb{R}^2,\pi(x,y,z)=(y,z)$.

Theorem 3 *Let $\gamma:I\to\mathbb{R}^3_1,\gamma(t)=(x(t),y(t),z(t))$ be a smooth curve. Then $\gamma:I\to\mathbb{R}^3_1$ is a lightcone framed base curve with a lightcone circle frame if and only if $\pi\circ\gamma:I\to\mathbb{R}^2$ is a frontal.*

Proof Suppose that there exists $\{\ell^+(t),\ell^-(t),\mathbf{n}(t)\}$ a lightcone circle frame such that $(\gamma,\ell^+,\ell^-):I\to\mathbb{R}^3_1\times\Delta_4$ is a lightcone framed curve. Then there exists a smooth function $\theta:I\to\mathbb{R}$ such that

$$\ell^+(t)=(1,\cos\theta(t),\sin\theta(t)),\ \ell^-(t)=(1,-\cos\theta(t),-\sin\theta(t)).$$

By definition,

$$\dot{x}(t) = \alpha(t) + \beta(t),\ \dot{y}(t) = (\alpha(t) - \beta(t)) \cos\theta(t),\ \dot{z}(t) = (\alpha(t) - \beta(t)) \sin\theta(t),$$

where $(\dot{\theta}, \alpha, \beta)$ is the curvature of (γ, ℓ^+, ℓ^-). We define $\nu : I \to S^1$ by $\nu(t) = (\sin\theta(t), -\cos\theta(t))$. Since $\dot{y}(t)\sin\theta(t) - \dot{z}(t)\cos\theta(t) = 0$ for all $t \in I$, $(\pi \circ \gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre curve, see §2.5. Hence, $\pi \circ \gamma : I \to \mathbb{R}^2$ is a frontal. Note that the Legendre curvature of $(\pi \circ \gamma, \nu)$ is given by $(\dot{\theta}, \alpha - \beta)$, see §2.5.

Conversely, there exists a smooth mapping $\nu : I \to S^1$ such that $(\pi \circ \gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre curve. We may denote $\nu(t) = (\sin\theta(t), -\cos\theta(t))$, where $\theta : I \to \mathbb{R}$ is a smooth function. We denote the Legendre curvature by $(\dot{\theta}, \widetilde{\beta})$. Then we define $(\ell^+, \ell^-) : I \to \Delta_4$ by

$$\ell^+(t) = (1, \cos\theta(t), \sin\theta(t)),\ \ell^-(t) = (1, -\cos\theta(t), -\sin\theta(t)).$$

It follows that $\dot{\gamma}(t) = \alpha(t)\ell^+(t) + \beta(t)\ell^-(t)$, where $\alpha(t) = (1/2)(\dot{x}(t) + \widetilde{\beta}(t))$ and $\beta(t) = (1/2)(\dot{x}(t) - \widetilde{\beta}(t))$. Therefore, $(\gamma, \ell^+, \ell^-) : I \to \mathbb{R}^3_1 \times \Delta_4$ is a lightcone framed curve with a lightcone circle frame. Hence, $\gamma : I \to \mathbb{R}^3_1$ is a lightcone framed base curve with a lightcone circle frame. □

Example 6 Let n_1, n_2, n_3, k_1, k_2 be natural numbers with $n_1 > 1$, $n_2 = n_1 + k_1$ and $n_3 = n_2 + k_2$. We define $(\gamma, \ell^+, \ell^-) : \mathbb{R} \to \mathbb{R}^3_1 \times \Delta_4$ by

$$\begin{aligned}
\gamma(t) &= \left(\frac{1}{n_1}t^{n_1}, \frac{1}{n_2}t^{n_2}, \frac{1}{n_3}t^{n_3}\right),\\
\ell^+(t) &= \left(1, \frac{1}{\sqrt{1+t^{2k_2}}}, \frac{t^{k_2}}{\sqrt{1+t^{2k_2}}}\right),\\
\ell^-(t) &= \left(1, -\frac{1}{\sqrt{1+t^{2k_2}}}, -\frac{t^{k_2}}{\sqrt{1+t^{2k_2}}}\right).
\end{aligned}$$

Then $(\gamma, \ell^+, \ell^-) : \mathbb{R} \to \mathbb{R}^3_1 \times \Delta_4$ is a lightcone framed curve with lightcone circle frame. By a direct calculation,

$$\mathbf{n}(t) = \left(0, -\frac{t^{k_2}}{\sqrt{1+t^{2k_2}}}, \frac{1}{\sqrt{1+t^{2k_2}}}\right)$$

and the curvature $(\dot{\theta}(t), \alpha(t), \beta(t))$ of (γ, ℓ^+, ℓ^-) is given by

$$\left(\frac{k_2 t^{k_2-1}}{1+t^{2k_2}},\ \frac{1}{2}\left(t^{n_1-1} + t^{n_2-1}\sqrt{1+t^{2k_2}}\right),\ \frac{1}{2}\left(t^{n_1-1} - t^{n_2-1}\sqrt{1+t^{2k_2}}\right)\right).$$

Then $t = 0$ is a singular point of γ. Since

$$\alpha(t)\beta(t) = \frac{1}{4}t^{2(n_1-1)}\left(1 - t^{2k_1}(1+t^{2k_2})\right),$$

γ at $t \neq 0$ is spacelike (respectively, lightlike or timelike) if $1 - t^{2k_1}(1+t^{2k_2}) < 0$ (respectively, $1 - t^{2k_1}(1+t^{2k_2}) = 0$ or $1 - t^{2k_1}(1+t^{2k_2}) > 0$).

Example 7 Let k be a natural number. We define $(\gamma, \ell^+, \ell^-) : \mathbb{R} \to \mathbb{R}^3_1 \times \Delta_4$ by

$$\begin{aligned}\gamma(t) &= \left(\frac{1}{k}t^k, \cos^3 t, \sin^3 t\right),\\ \ell^+(t) &= (1, \cos t, -\sin t),\\ \ell^-(t) &= (1, -\cos t, \sin t).\end{aligned}$$

Then $(\gamma, \ell^+, \ell^-) : \mathbb{R} \to \mathbb{R}^3_1 \times \Delta_4$ is a lightcone framed curve with lightcone circle frame. By a direct calculation,

$$\mathbf{n}(t) = (0, \sin t, \cos t)$$

and the curvature $(\dot\theta(t), \alpha(t), \beta(t))$ of (γ, ℓ^+, ℓ^-) is given by

$$\left(-1,\ \frac{1}{2}(t^{k-1} - 3\cos t \sin t),\ \frac{1}{2}(t^{k-1} + 3\cos t\sin t)\right).$$

Then in the case when $k > 1$, we have $t = 0$ is a singular point of γ. Since

$$\alpha(t)\beta(t) = \frac{1}{4}(t^{2(k-1)} - 9\cos^2 t \sin^2 t),$$

γ at $t \neq 0$ is spacelike (respectively, lightlike or timelike) if $t^{2(k-1)} - 9\cos^2 t\sin^2 t < 0$ (respectively, $t^{2(k-1)} - 9\cos^2 t\sin^2 t = 0$ or $t^{2(k-1)} - 9\cos^2 t \sin^2 t > 0$).

We draw the pictures of lightcone framed curves with lightcone circle frames below (see Figures 1 and 2 when $k = 1$ and 2 in Examples 7 and 8).

Example 8 Let k be a positive natural number. We define $(\gamma, \ell^+, \ell^-) : [0, 2\pi) \to \mathbb{R}^3_1 \times \Delta_4$ by

$$\begin{aligned}\gamma(t) &= \left(\frac{1}{k}\cos^k t, \cos^3 t, \sin^3 t\right),\\ \ell^+(t) &= (1, \cos t, -\sin t),\\ \ell^-(t) &= (1, -\cos t, \sin t).\end{aligned}$$

Then $(\gamma, \ell^+, \ell^-) : [0, 2\pi) \to \mathbb{R}^3_1 \times \Delta_4$ is a lightcone framed curve with lightcone circle frame. By a direct calculation,

$$\mathbf{n}(t) = (0, \sin t, \cos t)$$

and the curvature $(\dot\theta(t), \alpha(t), \beta(t))$ of (γ, ℓ^+, ℓ^-) is given by

$$\left(-1,\ \frac{1}{2}(-\cos^{k-1} t \sin t - 3\cos t\sin t),\ \frac{1}{2}(-\cos^{k-1} t\sin t + 3\cos t \sin t)\right).$$

In the case when $k > 1$, we have $t = 0, \pi/2, \pi$ and $3\pi/2$ are singular points of γ. By a direct calculation, we have

$$\alpha(t)\beta(t) = \frac{1}{4}(\cos^{2(k-1)} t\sin^2 t - 9\cos^2 t\sin^2 t).$$

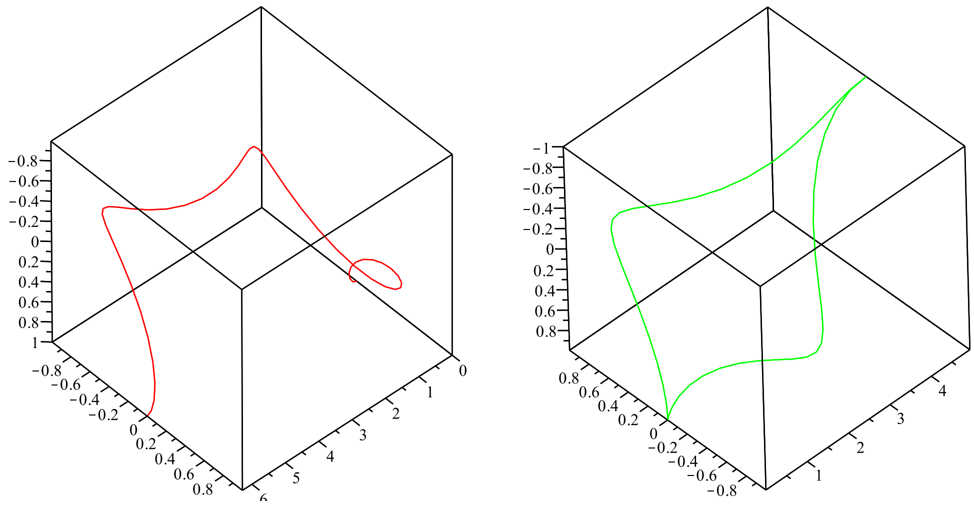

Figure 1. γ in Example 7 when $k=1$ and $k=2$.

Thus, γ at $t\neq 0$ or π is spacelike (respectively, lightlike) if $\cos^{2(k-1)} t - 9\cos^2 t < 0$ (respectively, $\cos^{2(k-1)} t - 9\cos^2 t = 0$). γ is never timelike. Moreover, if $k=1$, since

$$\alpha(t)\beta(t) = \frac{1}{4}(1-9\cos^2 t)\sin^2 t,$$

γ at $t\neq 0$ or π is spacelike (respectively, lightlike or timelike) if $1-9\cos^2 t < 0$ (respectively, $1-9\cos^2 t = 0$ or $1-9\cos^2 t > 0$).

Suppose that $\gamma : (I,t_0) \to \mathbb{R}^3_1$ is an analytic curve germ. Then $\pi\circ\gamma$ is also an analytic curve germ. It follows that $\pi\circ\gamma$ is a frontal (cf. [6]). By Theorem 3, γ is a lightcone framed base curve with a lightcone circle frame. Therefore, we have the following corollary.

Corollary 1 *If $\gamma : (I,t_0) \to \mathbb{R}^3_1$ is an analytic curve germ, then γ is a lightcone framed base curve with a lightcone circle frame.*

We also denote canonical projections on the Lorentz-Minkowski plane by $\pi_1 : \mathbb{R}^3_1 \to \mathbb{R}^2_1, \pi_1(x,y,z) = (x,y)$ and $\pi_2 : \mathbb{R}^3_1 \to \mathbb{R}^2_1, \pi_1(x,y,z) = (x,z)$.

Proposition 5 *Suppose that $(\gamma,\ell^+,\ell^-) : I \to \mathbb{R}^3_1 \times \Delta_4$ is a lightcone framed curve with the curvature (m,n,α,β), where $\ell^+ = (\ell^+_0,\ell^+_1,\ell^+_2)$ and $\ell^- = (\ell^-_0,\ell^-_1,\ell^-_2)$. Then*

(1) $\pi_1\circ\gamma : I \to \mathbb{R}^2_1$ *with the lightlike tangential data*

$$(\widetilde{\alpha},\widetilde{\beta}) = \left(\frac{1}{2}(\alpha\ell^+_0 + \beta\ell^-_0 + \alpha\ell^+_1 + \beta\ell^-_1), \frac{1}{2}(\alpha\ell^+_0 + \beta\ell^-_0 - \alpha\ell^+_1 - \beta\ell^-_1)\right).$$

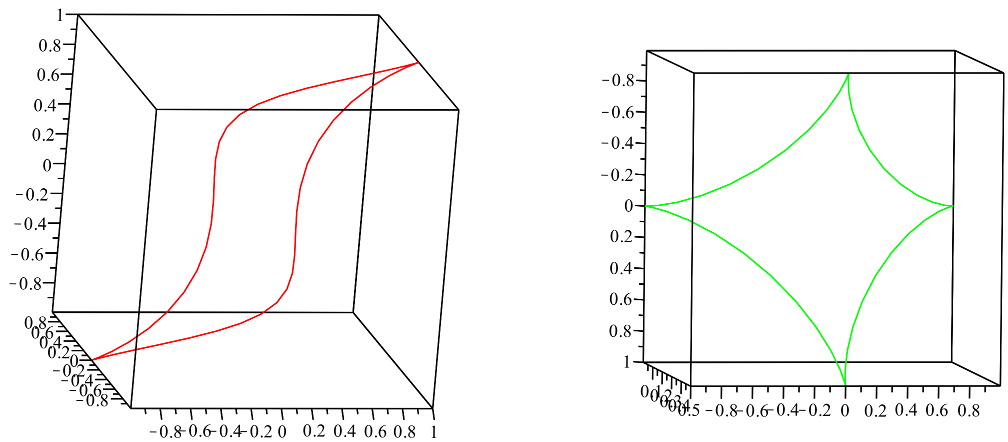

Figure 2. γ in Example 8 when $k=1$ and $k=2$.

(2) $\pi_2 \circ \gamma : I \to \mathbb{R}^2_1$ *with the lightlike tangential data*

$$(\widetilde{\alpha}, \widetilde{\beta}) = \left(\frac{1}{2}(\alpha \ell_0^+ + \beta \ell_0^- + \alpha \ell_2^+ + \beta \ell_2^-), \frac{1}{2}(\alpha \ell_0^+ + \beta \ell_0^- - \alpha \ell_2^+ - \beta \ell_2^-) \right).$$

Proof (1) Since $\dot{\gamma}(t) = \alpha(t)\ell^+(t) + \beta(t)\ell^-(t)$,

$$\begin{aligned} \frac{d}{dt}(\pi_1 \circ \gamma)(t) &= (\alpha(t)\ell_0^+(t) + \beta(t)\ell_0^-(t), \alpha(t)\ell_1^+(t) + \beta(t)\ell_1^-(t)) \\ &= \widetilde{\alpha}(t)\mathbf{L}^+ + \widetilde{\beta}(t)\mathbf{L}^- = (\widetilde{\alpha}(t) + \widetilde{\beta}(t), \widetilde{\alpha}(t) - \widetilde{\beta}(t)). \end{aligned}$$

We have the result. (2) is similar to the calculation. $\square$

4. Planar lightcone framed curves

We now consider a condition that the trace of a curve is contained in some plane. Firstly, consider the case of regular curves (cf. Proposition 2). We use the fact that planes in the sense of Lorentz-Minkowski space and of Euclidean space are the same. In fact, we have $PE(\mathbf{v}, c) = PL(Z(\mathbf{v}), c)$ and $PE(Z(\mathbf{v}), c) = PL(\mathbf{v}, c)$, see §2.4.

Proposition 6 *Let* $(\gamma, \ell^+, \ell^-) : I \to \mathbb{R}^3_1 \times \Delta_4$ *be a lightcone framed curve with curvature* (m, n, α, β). *Suppose that*

$$(\alpha(t)\dot{\beta}(t) - \dot{\alpha}(t)\beta(t))^2 + (\alpha(t)^2 + \beta(t)^2)(\alpha(t)n(t) + \beta(t)m(t))^2 \neq 0 \tag{25}$$

for all $t \in I$. *Then the trace of* γ *is contained in a plane if and only if*

$$\begin{aligned} &(\alpha(t)\dot{\beta}(t) - \dot{\alpha}(t)\beta(t))(2\dot{\alpha}(t)n(t) + \alpha(t)\dot{n}(t) + 2\dot{\beta}(t)m(t) + \beta(t)\dot{m}(t)) \\ &-\alpha(t)(\alpha(t)n(t) + \beta(t)m(t))(\ddot{\beta}(t) + 2(\alpha(t)n(t) + \beta(t)m(t))n(t)) \\ &-\beta(t)(\alpha(t)n(t) + \beta(t)m(t))(\ddot{\alpha}(t) + 2(\alpha(t)n(t) + \beta(t)m(t))m(t)) = 0 \end{aligned} \tag{26}$$

for all $t \in I$*.*

Proof By a direct calculation, we have

$$\begin{aligned}
\dot{\gamma}(t) &= \alpha(t)\ell^+(t) + \beta(t)\ell^-(t),\\
\ddot{\gamma}(t) &= \dot{\alpha}(t)\ell^+(t) + \dot{\beta}(t)\ell^-(t) + 2(\alpha(t)n(t) + \beta(t)m(t))\mathbf{n}(t),\\
\dddot{\gamma}(t) &= (\ddot{\alpha}(t) + 2(\alpha(t)n(t) + \beta(t)m(t))m(t))\ell^+(t) + (\ddot{\beta}(t) + 2(\alpha(t)n(t) + \beta(t)m(t))n(t))\ell^-(t)\\
&\quad +2(2\dot{\alpha}(t)n(t) + \alpha(t)\dot{n}(t) + 2\dot{\beta}(t)m(t) + \beta(t)\dot{m}(t))\mathbf{n}(t).
\end{aligned}$$

By the relation $\dot{\gamma}(t) \times \ddot{\gamma}(t) = Z(\dot{\gamma}(t) \wedge \ddot{\gamma}(t))$, we have $\dot{\gamma}(t) \times \ddot{\gamma}(t) \neq 0$ if and only if $\dot{\gamma}(t) \wedge \ddot{\gamma}(t) \neq 0$. Since

$$\dot{\gamma}(t) \wedge \ddot{\gamma}(t) = \Big(\alpha(t)\dot{\beta}(t) - \dot{\alpha}(t)\beta(t), \alpha(t)(\alpha(t)n(t) + \beta(t)m(t)), \beta(t)(\alpha(t)n(t) + \beta(t)m(t))\Big)$$

and (25), the nondegenerate condition $\dot{\gamma}(t) \times \ddot{\gamma}(t) \neq 0$ is satisfied. Moreover,

$$\begin{aligned}
&\det(\dot{\gamma}(t), \ddot{\gamma}(t), \dddot{\gamma}(t)) = \langle \dot{\gamma}(t) \wedge \ddot{\gamma}(t), \dddot{\gamma}(t)\rangle\\
&\quad = -4((\alpha(t)\dot{\beta}(t) - \dot{\alpha}(t)\beta(t))(2\dot{\alpha}(t)n(t) + \alpha(t)\dot{n}(t) + 2\dot{\beta}(t)m(t) + \beta(t)\dot{m}(t))\\
&\quad -\alpha(t)(\alpha(t)n(t) + \beta(t)m(t))(\ddot{\beta}(t) + 2(\alpha(t)n(t) + \beta(t)m(t))n(t))\\
&\quad -\beta(t)(\alpha(t)n(t) + \beta(t)m(t))(\ddot{\alpha}(t) + 2(\alpha(t)n(t) + \beta(t)m(t))m(t))).
\end{aligned}$$

The trace of γ is contained in a plane if and only if (26) holds for all $t \in I$. □

For lightcone circle frame, we have the following result.

Corollary 2 *Let* $(\gamma, \ell^+, \ell^-) : I \to \mathbb{R}^3_1 \times \Delta_4$ *be a lightcone framed curve with lightcone circle frame and curvature* $(\dot{\theta}, \alpha, \beta)$*. Suppose that*

$$(\alpha(t)\dot{\beta}(t) - \dot{\alpha}(t)\beta(t))^2 + \frac{1}{4}\dot{\theta}(t)^2(\alpha(t) - \beta(t))^2(\alpha(t)^2 + \beta(t)^2) \neq 0$$

for all $t \in I$*. Then the trace of* γ *is contained in a plane if and only if*

$$\begin{aligned}
&(\alpha(t)\dot{\beta}(t) - \dot{\alpha}(t)\beta(t))\left(\frac{\ddot{\theta}(t)}{2}(\alpha(t) - \beta(t)) + \dot{\theta}(t)(\dot{\alpha}(t) - \dot{\beta}(t))\right)\\
&+\alpha(t)(\alpha(t) - \beta(t))\dot{\theta}(t)\left(\ddot{\beta}(t) + \frac{1}{2}(\alpha(t) - \beta(t))\dot{\theta}(t)^2\right)\\
&+\beta(t)(\alpha(t) - \beta(t))\dot{\theta}(t)\left(\ddot{\alpha}(t) + \frac{1}{2}(\alpha(t) - \beta(t))\dot{\theta}(t)^2\right) = 0
\end{aligned}$$

for all $t \in I$*.*

Example 9 *Let* $(\gamma, \ell^+, \ell^-) : I \to \mathbb{R}^3_1 \times \Delta_4$ *be a lightcone framed curve with lightcone circle frame and curvature* $(\dot{\theta}, \alpha, \beta)$*. Suppose that* θ *is constant and* $\alpha(t)\dot{\beta}(t) - \dot{\alpha}(t)\beta(t) \neq 0$ *for all* $t \in I$*. By Corollary 2, the trace of* γ *is contained in a plane. By the assumption,* $\mathbf{n}$ *is a constant spacelike vector. There exists a constant* $c \in \mathbb{R}$ *such that* $\langle \gamma(t), \mathbf{n}\rangle = c$*. Therefore, the trace of* γ *is contained in a timelike plane.*

Remark 2 *If t is a singular point of γ, then $\alpha(t)=\beta(t)=0$ and condition (25) does not hold. Therefore, it follows from condition (4.1) that γ must be a regular curve.*

Next, we consider the case of curves with singular points under a condition. Let $(\gamma,\ell^+,\ell^-):I\to\mathbb{R}^3_1\times\Delta_4$ be a lightcone framed curve with lightcone circle frame and curvature $(\dot\theta,\alpha,\beta)$. We consider γ as a framed base curve. Suppose that α and $\beta:I\to\mathbb{R}$ are linearly dependent, that is, there exists a smooth mapping $(k_1,k_2):I\to\mathbb{R}^2$ with $(k_1(t),k_2(t))\neq(0,0)$ such that $k_1(t)\alpha(t)+k_2(t)\beta(t)=0$ for all $t\in I$. It is equivalent to the condition that there exists a smooth function $\omega:I\to\mathbb{R}$ such that

$$(\alpha(t)+\beta(t))\cos\omega(t)+(\alpha(t)-\beta(t))\sin\omega(t)=0 \tag{27}$$

for all $t\in I$. Since

$$\begin{aligned}\dot\gamma(t) &= \alpha(t)\ell^+(t)+\beta(t)\ell^-(t)\\ &= (\alpha(t)+\beta(t),(\alpha(t)-\beta(t))\cos\theta(t),(\alpha(t)-\beta(t))\sin\theta(t)),\end{aligned}$$

$\langle\dot\gamma(t),\mathbf{n}(t)\rangle=0$ and $\langle\dot\gamma(t),-\cos\omega(t)\mathbf{n}^T(t)+\sin\omega(t)\mathbf{n}^S(t)\rangle=0$. By definition of Z (see §2.4), we have $Z(\mathbf{n}(t))=\mathbf{n}(t)$, $Z(\mathbf{n}^T(t))=-\mathbf{n}^T(t)$ and $Z(\mathbf{n}^S(t))=\mathbf{n}^S(t)$. Hence, $\dot\gamma(t)\cdot\mathbf{n}(t)=0$ and $\dot\gamma(t)\cdot(\cos\omega(t)\mathbf{n}^T(t)+\sin\omega(t)\mathbf{n}^S(t))=0$ for all $t\in I$. It follows that $(\gamma,\nu_1,\nu_2):I\to\mathbb{R}^3\times\Delta$ is a framed curve, where $\nu_1(t)=\mathbf{n}(t)$ and $\nu_2(t)=\cos\omega(t)\mathbf{n}^T(t)+\sin\omega(t)\mathbf{n}^S(t)$. Since

$$\begin{aligned}\mathbf{n}(t)\times\mathbf{n}^T(t) &= Z(\mathbf{n}(t)\wedge\mathbf{n}^T(t))=Z(\mathbf{n}^S(t))=\mathbf{n}^S(t),\\ \mathbf{n}(t)\times\mathbf{n}^S(t) &= Z(\mathbf{n}(t)\wedge\mathbf{n}^S(t))=Z(\mathbf{n}^T(t))=-\mathbf{n}^T(t),\end{aligned}$$

$\mu(t)=\nu_1(t)\times\nu_2(t)=\cos\omega(t)\mathbf{n}^S(t)-\sin\omega(t)\mathbf{n}^T(t)$. By a direct calculation, the framed curvature $(\bar\ell,\overline{m},\overline{n},\overline{\alpha})$ of the framed curve (γ,ν_1,ν_2) is given by

$$(-\dot\theta(t)\sin\omega(t),-\dot\theta(t)\cos\omega(t),\dot\omega(t),-(\alpha(t)+\beta(t))\sin\omega(t)+(\alpha(t)-\beta(t))\cos\omega(t)).$$

By Proposition 1, we have the following results.

Proposition 7 *Let $(\gamma,\ell^+,\ell^-):I\to\mathbb{R}^3_1\times\Delta_4$ be a lightcone framed curve with lightcone circle frame and curvature $(\dot\theta,\alpha,\beta)$. Suppose that α and $\beta:I\to\mathbb{R}$ are linearly dependent, that is, satisfies the condition (27). Under the above notations, we have the following.*

(1) If $(\dot\theta(t)\cos\omega(t))^2+\dot\omega(t)^2\neq0$ and $\cos\omega(t)(\dot\theta(t)\ddot\omega(t)+\ddot\theta(t)\dot\omega(t)+\dot\theta(t)^3\cos\omega(t)\sin\omega(t))=0$ for all $t\in I$, then the trace of γ is contained in a plane.

(2) If (γ,ℓ^+,ℓ^-) is an analytic lightcone framed curve and

$$\cos\omega(t)(\dot\theta(t)\ddot\omega(t)+\ddot\theta(t)\dot\omega(t)+\dot\theta(t)^3\cos\omega(t)\sin\omega(t))=0$$

for all $t\in I$, then the trace of γ is contained in a plane.

Proof By a direct calculation, we have $\overline{m}^2(t)+\overline{n}^2(t)=(\dot\theta(t)\cos\omega(t))^2+\dot\omega(t)^2$ and

$$\overline{m}(t)\dot{\overline{n}}(t)-\dot{\overline{m}}(t)\overline{n}(t)+(\overline{m}^2(t)+\overline{n}^2(t))\bar\ell(t)=-\cos\omega(t)(\dot\theta(t)\ddot\omega(t)+\ddot\theta(t)\dot\omega(t)+\dot\theta(t)^3\cos\omega(t)\sin\omega(t)).$$

By Proposition 1, we have the results. □

Example 10 *Let* $(\gamma, \ell^+, \ell^-) : [0, 2\pi) \to \mathbb{R}^3_1 \times \Delta_4$,

$$\gamma(t) = (1, \cos^3 t, -\sin^3 t), \ \ell^+(t) = (1, \cos t, \sin t), \ \ell^-(t) = (1, -\cos t, -\sin t).$$

Then (γ, ℓ^+, ℓ^-) *is a lightcone framed curve with lightcone circle frame and curvature is given by*

$$(\dot{\theta}(t), \alpha(t), \beta(t)) = \left(1, -\frac{3}{2}\cos t \sin t, \frac{3}{2}\cos t \sin t\right).$$

Since $\alpha(t) + \beta(t) = 0$*, we have* $\omega(t) = 0$*.*

5. Conclusion

An effective frame, so-called the lightcone frame, is introduced for studying the mixed type of curves, namely lightcone framed curves. We investigate the differential geometric properties of the mixed type of not only regular curves but also curves with singularities. Existence and Uniqueness theorems for the lightcone framed curves are shown as results. We also give the conditions that the trace of a lightcone framed curve is contained in some plane in the last section.

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