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Existence of solutions by coincidence degree theory for Hadamard fractional differential equations at resonance

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Abstract: Using the coincidence degree theory of Mawhin and constructing appropriate operators, we investigate the existence of solutions to Hadamard fractional differential equations (FRDEs) at resonance

$$\begin{cases} -\left({}^{\mathrm{H}}D^{\gamma}u\right)(t) = f(t, u(t)), \quad t \in (1, e), \\ u(1) = 0, \quad u(e) = \int_1^e u(t)\mathrm{d}A(t), \end{cases}$$

where $1 < \gamma < 2$, $f : [1, e] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory conditions, $\int_1^e u(t)\mathrm{d}A(t)$ is the Riemann–Stieltjes integration, and $\left({}^{\mathrm{H}}D^{\gamma}u\right)$ is the Hadamard fractional derivation of u of order γ. An example is included to illustrate our result.

Key words: Fractional integral, fractional derivative, Hadamard derivative, boundary value problem, existence of solutions, coincidence degree theory

1. Introduction

Fractional differential equations (FRDEs) are gaining popularity as a modeling tool for complex systems in a variety of fields of science and engineering. Unlike traditional differential equations, FRDEs involve derivatives of noninteger order, which allows for the modeling of systems with long-range dependencies and memory effects. A new field of research has emerged, focusing on the analysis and solution of FRDEs. There are various types of fractional derivatives that have been defined which are being used to construct various factional boundary value problems. One of them is the Hadamard fractional derivative that was first introduced by French mathematician Jacques Hadamard in the early 20th century. Over the past few years, there has been renewed interest in the Hadamard fractional derivative, particularly in the field of signal processing, logarithmic decay, fracture analysis in material mechanics, probability, and geology. More information on these applications can be found in references [3, 11, 12, 19]. Let us note some of the recent work in direction of FRDEs with Hadamard derivatives. Existence and uniqueness of solutions was studied by Agarwal et al. [1] for Hadamard fractional functional differential equations. Ma et al. [20] obtained results concerning Lyapunov-type inequalities

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for FRDEs

$$\begin{cases} \left({}^{\mathrm{H}}D^{\gamma}u\right)(t) - q(t)u(t) = 0, \quad t \in (1,e), \\ u(1) = 0, \quad u(e) = 0. \end{cases}$$

In [5], Benchohra et al., using coincidence degree theory, studied existence of solutions considering Hadamard FRDEs with periodic condition

$$\begin{cases} \left({}^{\mathrm{H}}D^{\gamma}u\right)(t) = f\left(t, u(t), \left({}^{\mathrm{H}}D^{\gamma}u\right)\right), \ 0 < \gamma \leq 1, \ t \in (1,T), \\ u(1) = u(T), \end{cases}$$

where $T > 1$. One can find some other analysis work in Aibout et al. [2] which contains study of Caputo-Hadamard FRDEs. Hristova et al. [14] studied variable-order FRDEs, Jiang et al. [15] studied nonlinear Hadamard FRDEs with coupled integral boundary conditions. In [13], Graef et al. obtained results regarding the asymptotic behavior of solutions of higher-order Caputo–Hadamard FRDEs. Recently, Bohner et al. [6] applied a Vallée-Poussin theorem and obtained explicit inequality tests for fractional functional differential equations

$$\begin{cases} \left({}^{\mathrm{H}}D^{\gamma}u\right)(t) + (\Upsilon u)(t) = f(t), \\ u(1) = u(e) = 0, \end{cases}$$

where the operator $\Upsilon : C \to L_{\infty}$ can be an operator with deviation (of delayed or advanced type), an integral operator, or various linear combinations and superpositions. The exploration of equations involving Hadamard fractional derivatives is a rare occurrence in the existing literature.

Motivated by the above mentioned applications and theoretical investigations, here we consider the fractional boundary value problem

$$\begin{cases} -\left({}^{\mathrm{H}}D^{\gamma}u\right)(t) = f(t, u(t)), \quad t \in (1,e), \\ u(1) = 0, \quad u(e) = \int_1^e u(t)\mathrm{d}A(t), \end{cases} \tag{1}$$

where ${}^{\mathrm{H}}D^{\gamma}$ is the Hadamard fractional derivative of order $1 < \gamma < 2$, $f : [1,e] \times \mathbb{R}^2 \to \mathbb{R}$, and $\int_1^e u(t)\mathrm{d}A(t)$ is the Riemann–Stieltjes integral. The problem stated in (1) is in resonance, as there exists a nontrivial solution to the linear equation $-\left({}^{\mathrm{H}}D^{\gamma}u\right)(t) = 0$ for $t \in (1,e)$ under the given integral boundary condition. Throughout this article, we assume that the following holds:

(H1) $\int_1^e (\ln t)^{\gamma-1} dA(t) = 1$, $\int_1^e (\ln t)^{\gamma} \mathrm{d}A(t) \neq 1$;

(H2) $f : [1,e] : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ fulfills Carathéodory conditions, which means that $f(\cdot, u)$ is measurable for each fixed $u \in \mathbb{R}$, $f(t, \cdot)$ is continuous for almost every $t \in [1,e]$, and for each $l > 0$, there exists $\omega_l \in L^{\infty}[1,e]$ such that $|f(t,u)| \leq \omega_l(t)$ holds for all $|u| \leq l$ and $t \in [1,e]$.

One may observe from the literature [4, 5, 8, 10, 16, 21, 27] that it is required to choose the upper bounds on the function f as $u_1(t) + u_2(t)u + u_3(t)y$, where f is a function of u and y, where y includes the derivative terms, such as $D^{\gamma}u$, $D^{\gamma-1}u$, $D^{\gamma-2}u$, u', u'' and so on. Further, the authors in [4, 8, 10, 16, 21, 27] assumed the additional condition that there exists a constant $M > 0$, where M is the bound for the derivative term present in the function, and used bounds on $K_p y$ in order to prove

$$\Omega_1 = \{u \in \mathrm{dom}(B) \setminus \mathrm{Ker}(B) : Bu = \mu N u \text{ for some } \mu \in [0,1]\}$$

is bounded, where the function was assumed either $u_1(t)+u_2(t)u$ or $u_1(t)+u_2(t)u+u_3(t)y$. Our condition avoids such situations and does not require bounds on $K_p y$. In fact, we have assumed a simple condition (see (H3) and (H4) below) which is completely new in the literature.

The paper is organized into five sections for clarity. Section 1 serves as the introduction, while Section 2 presents the basic principles of Hadamard fractional calculus and Mawhin's coincidence degree theory. Section 3 focuses on the main results of the study, and Section 4 provides an illustrative example to reinforce our findings. In the last section, we provide conclusions with several possible open problems.

2. Preliminaries

Definition 1 (see [17, 18]) *The Hadamard fractional integral of order* $\gamma > 0$ *is defined for a function* $f \in L^p[a,b]$, *where* $1 \le p \le \infty$ *and* $0 \le a \le t \le b \le \infty$, *as*

$$^{\mathrm{H}}I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \left(\ln \frac{t}{s}\right)^{\gamma-1} f(s) \frac{\mathrm{d}s}{s}.$$

Definition 2 (see [17, 18]) *Consider a finite interval* $[a,b]$ *such that* $-\infty < a < b < \infty$, *and let* $\mathrm{AC}[a,b]$ *be the space of absolutely continuous functions on* $[a,b]$. *Let* $\delta = f\frac{\mathrm{d}}{\mathrm{d}t}$ *and define the space* $\mathrm{AC}^n_\delta[a,b]$ *as*

$$\mathrm{AC}^n_\delta[a,b] = \{f : t \in [a,b] \to \mathbb{R} \text{ such that } (\delta^{n-1} f) \in \mathrm{AC}[a,b]\}$$

For $n=1$, it is clear that $\mathrm{AC}^1_\delta \equiv \mathrm{AC}[a,b]$.

Definition 3 (see [17, 18]) *The Hadamard fractional derivative of order* $\gamma > 0$ *for a function* $f \in \mathrm{AC}^n_\delta[a,b]$, $0 < a < b < \infty$, *is defined as*

$$^{\mathrm{H}}D^{\gamma} f(t) = \delta^n (I^{n-\gamma} f)(t) = \frac{1}{\Gamma(n-\gamma)} \left(t \frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \left(\ln \frac{t}{s}\right)^{n-\gamma-1} f(s) \frac{\mathrm{d}s}{s},$$

where $n-1 < \gamma < n$, $n = [\gamma]+1$, $[\gamma]$ *denotes the integer part of the real number* γ.

Lemma 1 (see [17, 18]) *Let* $n \in \mathbb{N}$, $\gamma \in (n, n-1)$ *and* $u \in \mathrm{AC}^n_\delta[a,b]$. *Then the solution of the Hadamard FRDE* $\left({}^{\mathrm{H}}D^{\gamma} u\right)(t) = 0$ *is represented as*

$$u(t) = \sum_{j=1}^{n} c_j (\ln t)^{\gamma-i},$$

and the formula

$$^{\mathrm{H}}I^{\gamma\,\mathrm{H}}D^{\gamma} u(t) = u(t) + \sum_{j=1}^{n} c_j (\ln t)^{\gamma-j}$$

holds, where $c_j \in \mathbb{R}$, $j = 1, 2 \ldots, n$.

Lemma 2 (see [1, 20]) *The problem*

$$\begin{cases} \left({}^{\mathrm{H}}D^{\gamma}u\right)(t) + h(t) = 0, \\ u(1) = u(e) = 0 \end{cases} \tag{2}$$

is equivalent to

$$u(t) = \int_1^e G(t,s)h(s)\mathrm{d}s,$$

where u is unique and $G(t,s)$ is Green's function of problem (2) represented as

$$G(t,s) = \frac{1}{\Gamma(\gamma)} \begin{cases} \left(\ln \frac{e}{s}\right)^{\gamma-1} (\ln t)^{\gamma-1}\frac{1}{s} - \left(\ln \frac{t}{s}\right)^{\gamma-1}\frac{1}{s}, & 1 \le s \le t \le e, \\ \left(\ln \frac{e}{s}\right)^{\gamma-1} (\ln t)^{\gamma-1}\frac{1}{s}, & 1 \le t \le s \le e. \end{cases} \tag{3}$$

Lemma 3 (see [1, 20]) *Green's function represented in (3) is positive for $(t,s) \in (1,e)$.*

Now, let us note some basic definitions and properties related to the technique of coincidence degree theory. Suppose U and V are real Banach spaces and $B : \mathrm{dom}(B) \subset U \to V$ is a Fredholm operator with index zero. If $T_1 : U \to U$ and $T_2 : V \to V$ are two continuous projectors such that $\mathrm{Im}(T_2) = \mathrm{Ker}(B)$, $\mathrm{Ker}(T_2) = \mathrm{Im}(B)$, $U = \mathrm{Ker}(B) \oplus \mathrm{Ker}(T_1)$ and $V = \mathrm{Im}(B) \oplus \mathrm{Im}(T_2)$, then it follows that $B|_{\mathrm{dom}(B)\cap \mathrm{Ker}(T_1)} : \mathrm{dom}(B) \cap \mathrm{Ker}(T_1) \to \mathrm{Im}(B)$ is invertible and referred as K_p. Assuming that Ω is an open bounded subset of U such that $\mathrm{dom}(B) \cap \Omega \neq 0$, we can define $N : U \to V$ as B-compact on $\bar{\Omega}$, provided that $T_2N(\overline{\Omega})$ is bounded and $K_p(I - T_2)N : \overline{\Omega} \to U$ is compact.

Theorem 1 (see [22]) *Let N be B-compact on $\overline{\Omega}$ and let B be a Fredholm operator of index zero. Assume*

1) *$Bu \neq \mu Nu$ for every $(u, \mu) \in [(\mathrm{dom}(B) \setminus \mathrm{Ker}(B)) \cap \partial\Omega] \times (0, 1)$;*

2) *$Nu \notin \mathrm{Im}(B)$ for every $u \in \mathrm{Ker}(B) \cap \partial\Omega$;*

3) *If $T_2 : V \to V$ is a projector as above with $\mathrm{Im}(B) = \mathrm{Ker}(T_2)$, then $\deg(T_2N|_{\mathrm{Ker}\,L}, \mathrm{Ker}\,L \cap \Omega, 0) \neq 0$.*

Then, at least one solution exists in $\mathrm{dom}(B) \cap \overline{\Omega}$ for the equation $Bu = Nu$.

To implement Theorem 1, we use the classical Banach space $V = \mathrm{C}[1,e]$ with the norm $\|u\|_\infty = \max_{1\le t\le e} |u(t)|$ and the Banach space

$$U = \{u : [1,e] \to \mathbb{R}|\ u \in C[1,e]\}$$

with the norm $||u||_U = \max_{t\in[1,e]} |u(t)|$.

Let $B : \mathrm{dom}(B) \subset U \to V$ and $N : U \to V$ be defined by $(Bu)(t) = -\left({}^{H}D^{\gamma}u\right)(t)$ and $(Nu)(t) = f(t, u(t))$, $t \in [1,e]$, where

$$\mathrm{dom}(B) = \left\{ u \in U | -\left({}^{\mathrm{H}}D^{\gamma}u\right) \in V,\ u(1) = 0,\ u(e) = \int_1^e u(t)\mathrm{d}A(t) \right\}.$$

Then, (1) can be expressed as $(Bu)(t) = (Nu)(t)$ for $u \in \mathrm{dom}(B)$.

3. Main result

We use the following assumptions to derive the results of this article:

(H3) Assume that there exists $M \in \left(0, \frac{\Gamma(\gamma+1)}{2}\right)$ such that $|f(t,u)| \leq M\|u\|_U$, $t \in [1,e]$.

(H4) If $|u(t)| > r$ for a positive value r and for all $t \in [1,e]$, then $T_2Nu \neq 0$.

(H5) For $|c| > W$, where $c \in \mathbb{R}$, a positive constant W exists such that either $cT_2N(c(\ln t)^{\gamma-1}) < 0$ or $cT_2N(c(\ln t)^{\gamma-1}) > 0$ for all $t \in [1,e]$.

Lemma 4 *B is a Fredholm operator with zero index, which maps from a subset* $\mathrm{dom}(B)$ *that is a subset of U, to V represented as* $B : \mathrm{dom}(B) \subset U \to V$.

Proof Lemma 1 and $Bu = 0$ imply that $u(t) = c(\ln t)^{\gamma-1} + c_1(\ln t)^{\gamma-2}$, where $c, c_1 \in \mathbb{R}$. The kernel of the operator B, as determined by the boundary conditions (1), is given by $\mathrm{Ker}(B) = \{c(\ln t)^{\gamma-1} : c \in \mathbb{R}\}$. Here, we note

$$\mathrm{Im}(B) = \left\{ y \in V : \int_1^e (1-\ln s)^{\gamma-1} y(s) \frac{\mathrm{d}s}{s} - \int_1^e \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} y(s) \frac{\mathrm{d}s}{s} \mathrm{d}A(t) = 0 \right\}.$$

Let $u \in \mathrm{dom}(B)$ and $Bu = y$. Then by Lemma 1

$$u(t) = c_1(\ln t)^{\gamma-1} + c_2(\ln t)^{\gamma-2} - \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} y(s) \frac{\mathrm{d}s}{s}. \tag{4}$$

Using the boundary conditions, we obtain $c_2 = 0$. Then,

$$u(e) = c_1 - \frac{1}{\Gamma(\gamma)} \int_1^e (1-\ln s)^{\gamma-1} y(s) \frac{\mathrm{d}s}{s},$$

$$\int_1^e u(t)\mathrm{d}A(t) = c_1 \int_1^e (\ln t)^{\gamma-1} \mathrm{d}A(t) - \frac{1}{\Gamma(\gamma)} \int_1^e \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} y(s) \frac{\mathrm{d}s}{s} \mathrm{d}A(t).$$

Since $u(e) = \int_1^e u(t)\mathrm{d}A(t)$, we get

$$\int_1^e (1-\ln s)^{\gamma-1} y(s) \frac{\mathrm{d}s}{s} = \int_1^e \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} y(s) \frac{\mathrm{d}s}{s} \mathrm{d}A(t).$$

On the contrary, if $y \in V$, then $\int_1^e (1-\ln s)^{\gamma-1} y(s) \frac{\mathrm{d}s}{s} = \int_1^e \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} y(s) \frac{\mathrm{d}s}{s} \mathrm{d}A(t)$. If

$$u(t) = \frac{(\ln t)^{\gamma-1}}{\Gamma(\gamma)} \int_1^e (1-\ln s)^{\gamma-1} y(s) \frac{\mathrm{d}s}{s} - \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} y(s) \frac{\mathrm{d}s}{s},$$

then $Bu = y$,

$$u(e) = \frac{1}{\Gamma(\gamma)} \int_1^e (1-\ln s)^{\gamma-1} y(s) \frac{\mathrm{d}s}{s} - \frac{1}{\Gamma(\gamma)} \int_1^e (1-\ln s)^{\gamma-1} y(s) \frac{\mathrm{d}s}{s} = 0,$$

and

$$\int_1^e u(t)\mathrm{d}A(t) = \frac{1}{\Gamma(\gamma)}\int_1^e (\ln t)^{\gamma-1}\mathrm{d}A(t)\int_1^t (1-\ln s)^{\gamma-1}y(s)\frac{\mathrm{d}s}{s} - \frac{1}{\Gamma(\gamma)}\int_1^e\int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1} y(s)\frac{\mathrm{d}s}{s} = 0.$$

Thus, $u \in \mathrm{dom}(B)$ implies that $y \in \mathrm{Im}(B)$ and $Bu = y$. So,

$$\mathrm{Im}(B) = \left\{ y \in V : \int_1^e (1-\ln s)^{\gamma-1}y(s)\frac{\mathrm{d}s}{s} - \int_1^e\int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1} y(s)\frac{\mathrm{d}s}{s}\mathrm{d}A(t) = 0 \right\}. \quad (5)$$

Consequently, $\dim \mathrm{Ker}(B) = 1$ and $\mathrm{Im}(B)$ is closed. We can define a linear mapping $T_1 : U \to U$ by $(T_1u)(t) = u(e)(\ln t)^{\gamma-1}$. It can be shown that T_1 is a projection operator since

$$(T_1^2u)(t) = T_1(T_1u)(t) = (\ln t)^{\gamma-1}[(\ln t)^{\gamma-1}u(e)]|_{t=e} = (T_1u)(t).$$

Also, $\mathrm{Ker}(T_1) = \{u \in U |\ u(e) = 0\}$ and $\mathrm{Im}(T_1) = \mathrm{Ker}(B)$. For any $u \in U$, together with $u = (u - T_1u) + T_1u$, we get $U = \mathrm{Ker}(T_1) \oplus \mathrm{Ker}(B)$. Clearly, $\mathrm{Ker}(B) \cap \mathrm{Ker}(T_1) = \{0\}$, which implies $U = \mathrm{Ker}(T_1) \oplus \mathrm{Ker}(B)$. Define a linear continuous projector $T_2 : V \to V$ by

$$(T_2y)(t) = \frac{\gamma}{1 - \int_1^e (\ln t)^{\gamma}\mathrm{d}A(t)}\left[\int_1^e (1-\ln s)^{\gamma-1}y(s)\frac{\mathrm{d}s}{s} - \int_1^e\int_1^t (\ln t - \ln s)^{\gamma-1}y(s)\frac{\mathrm{d}s}{s}\mathrm{d}A(t)\right]. \quad (6)$$

Since

$$\begin{aligned} T_2^2y(t) &= T_2(T_2y(t)) \\ &= \frac{\gamma}{1 - \int_1^e (\ln t)^{\gamma}\mathrm{d}A(t)}\left[\int_1^e (1-\ln s)^{\gamma-1}T_2y(s)\frac{\mathrm{d}s}{s} - \int_1^e\int_1^t (\ln t - \ln s)^{\gamma-1}T_2y(s)\frac{\mathrm{d}s}{s}\mathrm{d}A(t)\right] \\ &= T_2y(t)\left[\frac{\gamma}{1 - \int_1^e (\ln t)^{\gamma}\mathrm{d}A(t)}\right]\left[\frac{1 - \int_1^e (\ln t)^{\gamma}}{\gamma}\mathrm{d}A(t)\right] = T_2y(t), \end{aligned}$$

T_2 is a projection operator. Furthermore, $\mathrm{Ker}(T_2) = \mathrm{Im}(B)$. Also, for any $y \in V$, let $y_1 = y - T_2y$. It follows that $(T_2y_1)(t) = T_2(y - T_2(y))(t) = T_2y(t) - T_2^2y(t) = 0$. Thus, $y_1 \in \mathrm{Im}(B)$ and $V = \mathrm{Im}(B) + \mathrm{Im}(T_2)$. Additionally, we can easily verify that $\mathrm{Im}(T_2) \cap \mathrm{Im}(B) = \{0\}$. Therefore, $V = \mathrm{Im}(B) \oplus \mathrm{Im}(T_2)$. Since $\mathrm{Im}(B)$ is a closed subspace of V and $\dim\ \mathrm{Ker}(B) = \mathrm{codim}\ \mathrm{Im}(B) = 1$, B is a Fredholm operator of index zero. This completes the proof. □

Now, $K_p : \mathrm{Im}(B) \to \mathrm{dom}(B) \cap \mathrm{Ker}(T_1)$ is defined as a generalized operator

$$(K_py)(t) = \int_1^e G(t,s)y(s)\mathrm{d}s = \frac{(\ln t)^{\gamma-1}}{\Gamma(\gamma)}\int_1^e (1-\ln s)^{\gamma-1}\frac{y(s)}{s}\mathrm{d}s - \frac{1}{\Gamma(\gamma)}\int_1^t (\ln t - \ln s)^{\gamma-1}\frac{y(s)}{s}\mathrm{d}s. \quad (7)$$

Lemma 5 *K_p is the inverse of $B_{\mathrm{dom}(B)\cap\mathrm{Ker}(T_1)}$.*

Proof For $y \in \mathrm{Im}(B)$, we get

$$BK_py = -\left({}^{\mathrm{H}}D^{\gamma}\left(\frac{(\ln t)^{\gamma-1}}{\Gamma(\gamma)}\int_1^e (1-\ln s)^{\gamma-1}\frac{y(s)}{s}\mathrm{d}s - \frac{1}{\Gamma(\gamma)}\int_1^e (\ln t - \ln s)^{\gamma-1}\frac{y(s)}{s}\mathrm{d}s\right)\right) = y.$$

Considering $u \in \mathrm{dom}(B) \cap \mathrm{Ker}(T_1)$ and $Bu = y$, we have

$$-\left({}^{\mathrm{H}}D^{\gamma}u\right)(t) = y(t), \qquad t \in (1,e), \qquad u(1) = 0, \qquad u(e) = 0.$$

Moreover, for $u \in \mathrm{dom}(B) \cap \mathrm{Ker}(T_1)$, we obtain

$$(K_pBu)(t) = \int_1^e G(t,s)(-\left({}^{\mathrm{H}}D^{\gamma}u\right)(s))\mathrm{d}s = \int_1^e G(t,s)y(s)\mathrm{d}s = u(t),$$

which implies that $K_p = (B|_{\mathrm{dom}(B)\cap\mathrm{Ker}(T_1)})^{-1}$. The proof is complete. □

Now, considering (6) and (7), we derive

$$K_p(I-T_2)Ny(t) = \int_1^e G(t,s)Ny(s)\mathrm{d}s - \left[\frac{(\ln t)^{\gamma-1}}{\Gamma(\gamma)}\left(\frac{1}{\gamma}\right) - \frac{1}{\Gamma(\gamma)}\left(\frac{(\ln t)^{\gamma}}{\gamma}\right)\right]y(t)$$
$$\times\left(\frac{\gamma}{1-\int_1^e(\ln t)^{\gamma}\mathrm{d}A(t)}\right)\left[\int_1^e(1-\ln s)^{\gamma-1}Ny(s)\frac{\mathrm{d}s}{s} - \int_1^e\int_1^t(\ln t-\ln s)^{\gamma-1}Ny(s)\frac{\mathrm{d}s}{s}\mathrm{d}A(t)\right].$$

Then, the operator $K_p(I-T_2)Ny : V \to V$ is completely continuous.

Lemma 6 *$T_2N : U \to V$ is a bounded and continuous operator and $K_p(I-T_2)N : \overline{\Omega} \to U$ is compact, where Ω is a bounded subset of U.*

Proof By the boundness of $T_2N(\overline{\Omega})$, and $(I-T_2)N(\overline{\Omega})$, there exists a positive constant H such that $|(I-T_2)Nu(t)| \leq H$ for all $u \in \overline{\Omega}$ and $t \in [0,1]$. The remaining proof follows from Arzelà–Ascoli theorem. This concludes the proof. □

Lemma 7 *If the conditions (H1)–(H4) are satisfied, then*

$$\Omega_1 = \{u \in \mathrm{dom}(B)\backslash\mathrm{Ker}(B) : Bu = \mu Nu \text{ for some } \mu \in [0,1]\}$$

is bounded.

Proof Let $u \in \Omega_1$. Then $u \in \mathrm{dom}(B)\backslash\mathrm{Ker}(B)$ and $Nu \in \mathrm{Im}(B)$. By (5), we have

$$\int_1^e(1-\ln s)^{\gamma-1}y(s)\frac{\mathrm{d}s}{s} - \int_1^e\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}y(s)\frac{\mathrm{d}s}{s}\mathrm{d}A(t) = 0.$$

Based on assumption (H4), there exists $t_0 \in [1,e]$ such that $|u(t_0)| \leq r$. Since $Bu = \mu Nu$, $u(1) = 0$, we have

$$u(t) = c(\ln t)^{\gamma-1} - \frac{\mu}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}Nu(s)\frac{\mathrm{d}s}{s}.$$

Considering $|u(t_0)| \leq r$, we get

$$|c(\ln t)^{\gamma-1}| \leq r + \frac{\mu}{\Gamma(\gamma)}\int_1^{t_0}\left(\ln\frac{t_0}{s}\right)^{\gamma-1}|Nu(s)|\frac{\mathrm{d}s}{s},$$

and hence

$$
\begin{aligned}
|u(t)| &\leq \left| c(\ln t_0)^{\gamma-1} - \frac{\mu}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} Nu(s) \frac{\mathrm{d}s}{s} \right| \\
&\leq |c(\ln t)^{\gamma-1}| + \frac{\mu}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} |Nu(s)| \frac{\mathrm{d}s}{s} \\
&\leq r + \frac{\mu}{\Gamma(\gamma)} \int_1^{t_0} \left(\ln \frac{t_0}{s}\right)^{\gamma-1} |Nu(s)| \frac{\mathrm{d}s}{s} + \frac{\mu}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} |Nu(s)| \frac{\mathrm{d}s}{s} \\
&\leq r + \frac{2\mu}{\Gamma(\gamma)} \|Nu\|_U \int_1^e \left(\ln \frac{e}{s}\right)^{\gamma-1} \frac{\mathrm{d}s}{s} \\
&\leq r + \frac{2}{\Gamma(\gamma+1)} M \|u\|_U.
\end{aligned}
$$

Hence, $\|u(t)\|_U \leq \frac{\Gamma(\gamma+1)r}{\Gamma(\gamma+1)-2M}$. Consequently, Ω_1 is bounded. □

Lemma 8 *If (H1), (H2) and (H5) are fulfilled, then*

$$\Omega_2 = \{u : u \in \mathrm{Ker}(B), Nu \in \mathrm{Im}(B)\}$$

is bounded.

Proof Suppose $u \in \Omega_2$, $u(t) = c(\ln t)^{\gamma-1}$, $c \in \mathbb{R}$. Then $\mathrm{Im}(B) = \mathrm{Ker}(T_2)$, and consequently $T_2Nu(t) = 0$. Considering (H5), it follows that $|c| \leq W$. Therefore, we can conclude that Ω_2 is bounded. □

Our next result relies on the definition of an isomorphism $\Psi : \mathrm{Ker}(B) \to \mathrm{Im}(T_2)$, where Ψ is defined as $\Psi(c(\ln t)^{\gamma-1}) = c$.

Lemma 9 *If (H1), (H2), and (H5) are satisfied, then*

$$\Omega_3 = \{u : u \in \mathrm{Ker}(B),\ \mu\Psi u + \beta(1-\mu)T_2Nu = 0,\ \mu \in [0,1]\}$$

with

$$\beta = \begin{cases} -1, & \textit{if } \ cT_2N(c(\ln t)^{\gamma-1}) < 0, \\ \ \ 1, & \textit{if } \ cT_2N(c(\ln t)^{\gamma-1}) > 0, \end{cases}$$

is bounded.

Proof Consider $u \in \Omega_3$. Then $u(t) = c(\ln t)^{\gamma-1}$, where $c \in \mathbb{R}$. We know that $\mu c + \beta(1-\mu)T_2N(c(\ln t)^{\gamma-1}) = 0$, where μ is a parameter in the interval $[0,1]$, and β is a sign function that depends on the sign of $cT_2N(c(\ln t)^{(\gamma-1)}$. If $\mu = 1$, then subsequently $c = 0$. Provided that $\mu = 0$, using condition (H5), we have $|c| \leq W$. Now, assume that $\mu \in (0,1)$. We argue that $|c| \leq W$. Allowing $|c| \geq W$, we get $\mu c^2 = -\beta(1-\mu)cT_2N(ct^{\gamma-1}) < 0$, which contradicts $\mu c^2 > 0$. Therefore, our argument remains valid, that is, $|c| \leq W$. Thus, Ω_3 is bounded. □

Theorem 2 *If assumptions (H1)–(H5) are fulfilled, then there exists at least one solution in U to problem (1).*

Proof Let Ω be any bounded open subset of U such that $\overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3} \subset \Omega$. Lemma 6 guarantees that N is B-compact. Additionally, Lemmas 7, 8, and 9 establish that the conditions 1) and 2) required by Theorem 1 are satisfied. To complete the proof, it is necessary to verify condition 3) of Theorem 1. We define

$$Z(u,\mu) = \mu u + \beta(1-\mu)T_2 N u$$

and apply Lemma 9 to conclude that $Z(u,\mu) \neq 0$ for $u \in \mathrm{Ker}(B) \cap \partial\Omega$. Thus, by the homotopy property of degree,

$$\begin{aligned}\deg(T_2 N|_{\mathrm{Ker}(B)}, \Omega \cap \mathrm{Ker}(B), 0) &= \deg(Z(\cdot,0), \Omega \cap Ker(B), 0)\\ &= \deg(Z(\cdot,1), \Omega \cap \mathrm{Ker}(B), 0)\\ &= \deg(Z(\beta\Psi, \Omega \cap \mathrm{Ker}(B), 0) \neq 0.\end{aligned}$$

By Theorem 1, we conclude that (1) has at least one solution u in $\mathrm{dom}(B) \cap \overline{\Omega}$. □

4. Example

Consider

$$\begin{cases} -({}^{\mathrm{H}}D^{\frac{3}{2}}u)(t) = \frac{1}{14}(u + \frac{1}{2}\sin u), & 1 < \gamma \leq 2, \quad t \in (1,e),\\ u(1) = 0, \qquad u(e) = \int_0^1 u(t)\mathrm{d}A(t). \end{cases} \tag{8}$$

Here, $\gamma = \frac{3}{2}$. Then $\Gamma(\frac{5}{2}) = 1.32934$ and $f(t,u) = \frac{1}{14}(u + \frac{1}{2}\sin u)$. Set $A = \frac{3}{2}(\ln t)$ and $M = \frac{3}{28}$. Then $\frac{3}{28} < 0.664675 = \frac{\Gamma(\gamma+1)}{2}$. Let $r = 1$. If $u(t) > 1$ holds for any $t \in [1,e]$, then we have

$$f(t,u) = \frac{1}{14}\left(u(t) + \frac{1}{2}\sin u\right) > \frac{1}{14}\left(1 - \frac{1}{2}\right) = \frac{1}{28} > 0.$$

If $u(t) < -1$ holds for any $t \in [1,e]$, then we have

$$f(t,u) = \frac{1}{14}\left(u(t) + \frac{1}{2}\sin u\right) < \frac{1}{14}\left(-1 + \frac{1}{2}\right) = -\frac{1}{28} < 0.$$

Take $W = 1$. Now for $|c| > 1$, we obtain

$$cf(t,ct) = c\frac{1}{14}\left(c(\ln t)^{\gamma-1} + \frac{1}{2}\sin(c(\ln t)^{\gamma-1})\right) \neq 0$$

for $1 < t < e$. Thus, conditions (H1)–(H5) are satisfied. By Theorem 2, (8) has at least one solution.

5. Conclusion and open problems

In conclusion, we use the application of coincidence degree theory to the study of Hadamard FRDEs, which has proven to be a valuable tool for analyzing the existence of solutions. What makes the present work novel is that the nonlinearity of the function f is independent of both ordinary and fractional derivatives. This characteristic sets it apart from previous research in the field, which often assumed that the nonlinearity was dependent on ordinary or fractional types of derivatives. We believe that the present work will inspire future research to involve use of coincidence degree theory in conjunction with other mathematical tools for the analysis of more complex problems in the sense of Hadamard fractional derivative. For example, one can use coincidence degree with matrix spectral theory [25], graph theory [26] and the method of matrix diagonalization [23]. Researchers may work on Hadamard FRDEs with higher-order integral, derivative or mixed boundary conditions

$$u(a) = u'(a) = \ldots = u^{n-2}(a) = 0, \quad u^k(b) = \int_a^b u^k(t)\mathrm{d}A(t),$$

where k can be any integer between 0 and 1, or

$$u(a) = 0, \quad {}^{\mathrm{H}}D^{\gamma}u(b) = \sum_{i=1}^{m} a_i \, {}^{\mathrm{H}}D^{\gamma}u(\xi_i),$$

where $a < \xi_1 < \xi_2 < \ldots < \xi_m < b$ and $\sum_{i=1}^m a_i = b$, or

$$\lim_{t\to a_+} t^{2-\gamma}u(t) = \int_a^b u(t)\mathrm{d}A(t), \quad {}^{\mathrm{H}}D^{\gamma}({}^{\mathrm{H}}D^{\alpha}u)(b) = \sum_{i=1}^{m} a_i \, {}^{\mathrm{H}}D^{\gamma}u(\xi_i),$$

where γ and α can be in different intervals. The literature currently available does not include an examination of Hadamard FRDEs with the aforementioned boundary conditions. Other potential avenues involve exploring generalizations of our main results to include FRDEs with general functional operator Υ

$$-({}^{\mathrm{H}}D^{\gamma}u)(t) = f(t, (\Upsilon u)(t)), \quad t \in (1, e),$$

where Υ can take on various forms, including but not limited to $\Upsilon = q(t)u(t-\tau(t))$, $\Upsilon = \int_1^e Q(t,s)u(\theta(s))\mathrm{d}s$, or $\Upsilon = \int_1^e u(s)\mathrm{d}_sQ(t,s)$. Corresponding results for linear fractional functional differential equations were obtained for Riemann–Liouville fractional derivatives in [9, 24], for Caputo fractional derivatives in [7] and for Hadamard fractional derivatives in [6].

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