Turkish Journal of Mathematics

Volume 48 | Number 2 Article 12

3-8-2024

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## Recommended Citation

ÖZÇAĞ, EMİN (2024) "An extension of the definition on the compositions of the singular distributions," *Turkish Journal of Mathematics*: Vol. 48: No. 2, Article 12. https://doi.org/10.55730/1300-0098.3506
Available at: https://journals.tubitak.gov.tr/math/vol48/iss2/12

Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2024) 48: 279 – 295

doi:10.55730/1300-0098.3506

# An extension of the definition on the compositions of the singular distributions

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**Received:** 02.12.2023 • **Accepted/Published Online:** 22.12.2023 • **Final Version:** 08.03.2024

**Abstract:** Gelfand and Shilov give the definition of the composition $\delta(g(x))$ for an infinitely differentiable function $g(x)$ having any number of simple roots. In the paper, we consider their definition for an infinitely differentiable function having any number of multiple roots by using the method of the discarding of unwanted infinite quantities from asymptotic expansions and give some examples. Further, we define the compositions $\delta(g_+)$ and $\delta(g_-)$ for a locally summable function $g(x)$.

**Key words:** Regular sequence, divergent integral, Hadamard's finite part, distribution, neutrices, Dirac-delta function

## 1. Introduction

The symbol $\delta^2$ often appears in quantum mechanics which leads to reasonable results and further the symbol $\delta^{-1}$ appeared in cosmological models [16, 17] in which the following distributional identities

$$\big[f(\omega)+C\delta(\omega)\big]^{-1}=f^{-1}(\omega),\qquad \frac{d}{d\omega}\big[f(\omega)+C\delta(\omega)\big]^{-1}=\frac{d}{d\omega}f^{-1}(\omega)$$

were used. In the theory of Schwartz distributions, there is no reasonable way of introducing neither the square $\delta^2$ nor $\delta^{-1}$, generally the composition $G(g(x))$ of distributions $G$ and $g$. Nevertheless, Fisher, in numerous papers [7–10, 14], has approached the problems of distributional operations namely composition, multiplication and convolution. In addition, the meaning has recently been given to the power of the composition $\delta(g(x))$ in [19] and to the symbol $\delta^{-k}$ in [21].

Now recall that $\mathcal{D}$ denotes the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ denotes the space of distributions defined on $\mathcal{D}$. For a locally summable function (absolutely integrable in every bounded region of $\mathbb{R}^n$) $g(x)$ and for every $\varphi\in\mathcal{D}$ the equation $\langle g,\varphi\rangle=\int_{\mathbb{R}^n}g(x)\varphi(x)\,dx$ defines the distribution. So the distribution given by this equation is called regular, and all others (including the delta function) are called singular.

Explicit sequences that approach the Dirac delta function and its derivatives are often helpful and used in presenting distributions. In fact, this idea based on the important property of the space $\mathcal{D}'$ is that every distribution is the limit of a regular sequence of infinitely differentiable functions with compact support [13]. In the following, we construct a regular sequence of infinitely differentiable functions which converges to $\delta(x)$.

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2020 *AMS Mathematics Subject Classification:* 41A30, 46F10, 40A10.

Let $\tau$ be a fixed infinitely differentiable function having the properties:

(i) $\tau(x) = 0$ for $|x| \geq 1$, (ii) $\tau(x) \geq 0$,

(iii) $\tau(x) = \tau(-x)$, (iv) $\int_{-1}^{1} \tau(x)\, dx = 1$,

and put $\delta_n(x) = n\tau(nx)$ for $n = 1, 2, \ldots$. Then it follows that $\delta_n(x)$ is a regular sequence [22] of infinitely differentiable functions converging to Dirac delta-function $\delta(x)$.

Further, if $h$ is a distribution in $\mathcal{D}'$ and $h_n(x) = \langle h(t), \delta_n(x-t)\rangle$, then $\{h_n(x)\}$ is a regular sequence of infinitely differentiable functions and converges to $h(x)$.

Antosik [1] defines the composition $G(g(x))$ of distributions $G$ and $g$ as the limit of the sequence of composition $\{G_n(g_n)\}$ on $\mathbb{R}$ proving that the limit exists and converges to a distribution $F(x)$. By this definition, he obtained

$$(i)\sqrt{\delta} = 0, \quad (ii)\sqrt{\delta^2+1} = 1+\delta, \quad (iii)\ log(1+\delta) = 0, \quad (iv)\sin\delta = 0, \quad (v)\tfrac{1}{1+\delta} = 1.$$

However, it is not possible to define the composition for many pairs of distributions, by using Antosik's definition.

## 2. Extension of the definition of the composition $\delta(g(x))$

Gelfand and Shilov defined the composition $\delta(g(x))$ [13] for an infinitely differentiable function $g$ having simple zeros at $x_1, x_2, \ldots, x_n$ by

$$\delta(g(x)) = \sum_n \frac{1}{|g'(x_n)|}\delta(x - x_n).$$

It will be valuable as a side to give the definition of $\delta(g(x))$ for an infinitely differentiable $g(x)$ having any number of multiple roots. We will need the following definition given by Fisher [7] for larger class of distributions, but, first of all, let $N$ denote the neutrix having domain $N' = \{1, 2, \ldots, n, \ldots\}$, range the real numbers with negligible functions which are finite linear sums of the functions $n^\lambda \ln^{r-1} n, \ln^r n \quad (\lambda > 0, r = 1, 2, \ldots)$ and all functions which converge to zero in the usual sense as $n$ tends to infinity [3].

**Definition 1** *Let $G$ be distribution in $\mathcal{D}'$ and let $g$ be infinitely differentiable function. We say that the distribution $G(g(x))$, the neutrix composition of $G$ and $g$, exists and is equal to $F(x)$ on the interval $(a, b)$ if the neutrix limit*

$$\mathrm{N}\!-\!\lim_{n\to\infty}\left[\int_{-\infty}^{\infty} G_n(g(x))\varphi(x)\, dx\right] = \langle F(x), \varphi(x)\rangle$$

*for all $\varphi$ in $\mathcal{D}$ with support contained in the interval $(a, b)$, where $G_n(x) = (G * \delta_n)(x)$, and $N$ is the neutrix defined above.*

The reader can find some examples of the neutrix limit and some examples of compositions and some applications of the neutrix limit in conjunction with special functions in [9, 10, 14, 18, 20]. The essential use of the neutrix limit is to extract an appropriate finite part from a divergent quantity as one has usually done to subtract the divergent terms via rather complicated procedures in the renormalization theory. In fact, the technique of neglecting appropriately defined infinite quantities given by Hadamard and the resulting finite value extracted from divergent integral usually known as the Hadamard finite part [5, 12, 15] can be regarded as a particular application of the neutrix calculus developed by J.G. van der Corput [3]. In connection with this, Fisher gave a general principle via the theory of neutrices for the discarding of unwanted infinite quantities from asymptotic expansions [8, 9].

It is of interest to emphasize that the novel approach to the computation of the Hadamard Finite Part of the singular integral $\int_a^b g(x)\,dx$, where $g(x) = \frac{f(x)}{(x-t)^m}$, $m = 1, 2, \ldots$, $a < t < b$ and $f \in C^\infty[a, b]$ are given in [23].

The following proposition will be an intermediary to present our results.

**Proposition 1** *Let $g(x)$ be an infinitely differentiable function. Assume that $g$ does not have any root in the interval $(a, b)$. Then the distribution $\delta(g(x))$ exists on the interval $(a, b)$ and*

$$\delta(g(x)) = 0. \tag{1}$$

**Proof** Let $\varphi(x)$ be an arbitrary test function with support contained in the interval $(a, b)$. Since the equation $g(x) = 0$ has no root in the interval $(a, b)$, then we have either $g(x) > 0$ or $g(x) < 0$ on the interval $(a, b)$ which implies that at least $\inf\{f(x) : a < x < b\}$ or $\sup\{g(x) : a < x < b\}$ exists.

Now suppose that $\inf g$ exists and let $\inf\{g(x) : a < x < b\} = c > 0$. Choosing $K > c^{-1}$, we have $ng(x) \geq 1$ for $n > K$ and all $x$. Thus, $\delta_n(g(x)) = n\tau(ng(x)) = 0$ for $n > K$. So it follows that

$$\langle \delta(g(x)), \varphi(x) \rangle = \mathrm{N-}\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(g(x))\varphi(x)\,dx = 0. \tag{2}$$

If $c < 0$, then by choosing $K > -c^{-1}$ we have for $n > K$ and all $x$, $ng(x) < -1$ which gives $\tau(ng(x)) = 0$.

Similarly, we have for $\varphi \in \mathcal{D}$

$$\langle \delta(g(x)), \varphi(x) \rangle = \mathrm{N-}\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(g(x))\varphi(x)\,dx = 0. \tag{3}$$

If $\sup\{g(x) : a < x < b\}$ exists, then we define $f(x) = -g(x)$ and it follows from Equations 2 and 3 that

$$\begin{aligned} \langle \delta(f(x)), \varphi(x) \rangle &= \mathrm{N-}\lim_{n\to\infty} \langle \delta_n(-g(x)), \varphi(x) \rangle = \mathrm{N-}\lim_{n\to\infty} \langle \delta_n(g(x)), \varphi(x) \rangle \\ &= 0. \end{aligned}$$

□

Now let us assume that $g(x)$ is an infinitely differentiable real-valued function having multiple roots at $x_0 \in \mathbb{R}$ with a multiplicity $s \in \mathbb{Z}^+$ and let us define the function $h(x)$ as follows:

$$h(x) = \begin{cases} \frac{g(x)}{(x-x_0)^s}, & x \neq x_0, \\ \frac{g^{(s)}(x_0)}{s!}, & x = x_0 \end{cases}$$

so that it is continuous and infinitely differentiable on the real line and $h^{(n)}(x_0) = \frac{g^{(s+n)}(x_0)}{(s+n)!}$ holds for all $n \in \mathbb{Z}^+$. Thus, we may write

$$g(x) = (x - x_0)^s h(x). \tag{4}$$

Further, the similar argument works for the function $g(x)$ having distinct multiple roots at the points $x_1, x_2, \ldots, x_n$ of $\mathbb{R}$ with the multiplicities $r_1, r_2, \ldots, r_n \in \mathbb{Z}^+$ respectively. This time, we can define infinitely differentiable function $h(x)$ on the real line by

$$h(x) = \begin{cases} \frac{g(x)}{(x-x_1)^{r_1}(x-x_2)^{r_2}\ldots(x-x_n)^{r_n}}, & x \neq x_i,\ i = 1, 2, \ldots, n \\ \frac{g^{(r_i)}(x_i)}{r_i! \prod_{i \neq k}(x_i - x_k)^{r_k}}, & x = x_i,\ i = 1, 2, \ldots, n, \end{cases}$$

and consequently, we have

$$g(x) = (x - x_1)^{r_1}(x - x_2)^{r_2} \ldots (x - x_n)^{r_n} h(x). \tag{5}$$

Equations 4 and 5 will be used in the following section to present the main results. Before presenting one of the main contributions, which is the definition of the composition $\delta(g(x))$ for the function $g$ with multiple roots, we first provide the Fa$\acute{a}$ di Bruno's formulae [4], defining the n$th$ derivative of a composite function. The explicit formula for the derivative of a smooth composite function $\varphi \circ g$, of arbitrary order, in terms of derivatives of $\varphi$ and $g$ is as follows:

$$\Big[\varphi(g(x))\Big]^{(n)} = \sum_{r=1}^{n} \varphi^{(r)}(g) B_{n,r}(g'(x), g''(x), \ldots, g^{(n-r+1)}(x)), \tag{6}$$

where $B_{n,r}(g'(x), g''(x), \ldots, g^{(n-r+1)}(x))$ are exponential Bell polynomials [2] defined by the equation

$$B_{n,r}\Big(g'(x), g''(x), \ldots, g^{(n-r+1)}(x)\Big) = \\ = \sum \frac{n!}{b_1! b_2! \ldots b_{n-r+1}!}\Big(\frac{g'(x)}{1!}\Big)^{b_1}\Big(\frac{g''(x)}{2!}\Big)^{b_2} \ldots \Big(\frac{g^{(n-r+1)}(x)}{(n-r+1)!}\Big)^{b_{n-r+1}} \tag{7}$$

where the sum is over all possible combinations of nonnegative integers $b_1, b_2, \ldots, b_{n-r+1}$ such that two conditions $b_1 + b_2 + \ldots + b_{n-r+1} = r$ and $b_1 + 2b_2 + 3b - 3 \ldots + (n - r + 1)b_{n-r+1} = n$ are satisfied.

**Theorem 1** *Let $g$ be an infinitely differentiable function having multiple roots at $x_0$ with multiplicity $s \in \mathbb{Z}^+$ on the open interval $(a, b)$. Then the composition $\delta(g(x))$ of Dirac delta function and $g$ exists on the open interval $(a, b)$ for all $s$ and defined by*

$$\delta(g(x)) = 0 \tag{8}$$

*for even* $s$ *and*

$$\langle \delta(g(x)), \varphi(x) \rangle = \sum_{k=0}^{s-1} \binom{s-1}{k} \frac{1}{s!|g^{(s-k)}(x_0)|} \times$$

$$\times \sum_{r=1}^{k} B_{k,r}\Big(\frac{1}{|g'(x_0)|}, \frac{1}{|g''(x_0)|}, \dots, \frac{1}{|g^{(k-r+1)}(x_0)|}\Big)\varphi^{(r)}(x_0)$$

$$= \sum_{k=0}^{s-1} \binom{s-1}{k} \frac{1}{s!|g^{(s-k)}(x_0)|} \times$$

$$\times \sum_{r=1}^{k} (-1)^r B_{k,r}\Big(\frac{1}{|g'(x_0)|}, \frac{1}{|g''(x_0)|}, \dots, \frac{1}{|g^{(k-r+1)}(x_0)|}\Big)\langle \delta^{(r)}(x - x_0), \varphi(x) \rangle \tag{9}$$

*for* $s = 1, 3, \dots$. *In particular, we have*

$$\delta(g(x)) = \frac{1}{|g'(x_0)|}\delta(x - x_0)$$

*for* $s = 1$ *and further*

$$\frac{d}{dx}\delta(g(x)) = g'(x)\delta'(g(x)). \tag{10}$$

**Proof** We prove the theorem for $x_0 = 0$, then the case $x_0 \neq 0$ follows by translation. Now we may write $g(x)$ from equation (4) as $g(x) = x^s h(x)$ where $h$ is defined by

$$h(x) = \begin{cases} \frac{g(x)}{x^s}, & x \neq 0, \\ \frac{g^{(s)}(0)}{s!}, & x = 0. \end{cases}$$

Let us put $g_1(x) = x h^{1/s}(x)$ and assume that the interval $(a, b)$ is bounded and $g_1'(x) \neq 0$ on $(a, b)$. Then the equation $y = g_1(x)$ will have inverse $x = f_1(y) \in C^\infty$ on the interval $(a, b)$. Let $\varphi$ be an infinitely differentiable function with $\sup(\varphi) \subset (a, b)$. Then

$$\int_{-\infty}^{\infty} \delta_n(g(x))\varphi(x)\, dx = \int_0^{\infty} \delta_n(g(x))\varphi(x)\, dx + \int_0^{\infty} \delta_n(g(-x))\varphi(-x)\, dx \tag{11}$$

in which we make the substitution $t^{1/s} = g_1(x)$ or $x = f_1(t^{1/s})$ for the first integral on the right-hand side of equation 11. Then we have

$$\int_0^{\infty} \delta_n(g(x))\varphi(x)\, dx = \frac{1}{s}\int_0^{\infty} \delta_n(t)\varphi(f_1(t^{1/s})|f_1'(t^{1/s})|t^{1/s-1}\, dt.$$

The function $\Phi(y) = \varphi(f_1(y))|f_1'(y)|$ is infinitely differentiable function and from Taylor's formula

$$\Phi(y) = \sum_{i=0}^{s-1} \frac{\Phi^{(i)}(0)}{i!}y^i + \frac{\Phi^{(s)}(\xi y)}{s!}y^s \quad (0 < \xi < 1).$$

Thus,

$$s\int_0^\infty \delta_n(g(x))\varphi(x)\,dx = \sum_{i=0}^{s-1}\frac{\Phi^{(i)}(0)}{i!}\int_0^\infty \delta_n(t)t^{\frac{i+1}{s}-1}\,dt + \int_0^\infty \frac{\Phi^{(s)}(\xi t^{1/s})}{s!}\delta_n(t)t^{1/s}\,dt$$
$$= \sum_{i=0}^{s-2}\frac{\Phi^{(i)}(0)}{i!}\int_0^1 \tau(u)(\frac{u}{n})^{\frac{i+1}{s}-1}\,du + \frac{\Phi^{(s-1)}(0)}{(s-1)!}\int_0^1 \tau(u)\,du +$$
$$+\int_0^1 \frac{\Phi^{(s)}(\xi(u/n)^{1/s})}{s!}\tau(u)(u/n)^s\,du$$

on making the substitution $nt = u$. It now follows that the neutrix limit of $\int_0^\infty \delta_n(g(x))\varphi(x)\,dx$ exists and is equal to

$$\underset{n\to\infty}{\text{N}-\lim}\int_0^\infty \delta_n(g(x))\varphi(x)\,dx = \frac{\Phi^{(s-1)}(0)}{2s!} \tag{12}$$

where $\int_0^1 \tau(u)\,du = \frac{1}{2}$. Next consider the integral $\int_0^\infty \delta_n(g(-x))\varphi(-x)\,dx$. Similarly, by making the substitution $-t^{1/s} = g_1(-x)$ where $t^{1/s} \geq 0$, we have that

$$s\int_0^\infty \delta_n(g(-x))\varphi(-x)\,dx = \int_0^\infty \delta_n((-1)^s t)\varphi(f_1(-t^{1/s}))|f_1'(-t^{1/s})|t^{1/s-1}\,dt$$
$$= \int_0^\infty \delta_n(t)\Phi(-t^{1/s})t^{1/s-1}\,dt$$

where $\Phi$ is defined above. Thus,

$$\underset{n\to\infty}{\text{N}-\lim}\int_0^\infty \delta_n(g(-x))\varphi(-x)\,dx = \frac{(-1)^{s-1}\Phi^{(s-1)}(0)}{s!}\int_0^1 \tau(u)\,du$$
$$= \frac{(-1)^{s-1}\Phi^{(s-1)}(0)}{2s!}. \tag{13}$$

It now follows from Equations 11–13 that

$$\underset{n\to\infty}{\text{N}-\lim}\int_{-\infty}^\infty \delta_n(g(x))\varphi(x)\,dx = \begin{cases} 0, & s = 2,4,\dots \\ \frac{\Phi^{(s-1)}(0)}{s!}, & s = 1,3,\dots, \end{cases}$$

proving the existence of $\delta_n(g(x))$ on the interval $(a,b)$ for $s \in \mathbb{N}$. It is obvious that $\delta_n(g(x)) = 0$ for $s = 2,4,\dots$.

Furthermore, $\delta_n(g(x)) = \frac{\Phi^{(s-1)}(0)}{s!}$ for $s = 1, 3, \ldots$. Let us evaluate $\Phi^{(s-1)}(0)$

$$\begin{aligned}
\Phi^{(s-1)}(0) &= \left\{\Phi^{(s-1)}(y)\right\}_{y=0} = \{\varphi(f_1(y))|f_1'(y)|\}^{(s-1)}\Big]_{y=0} \\
&= \sum_{k=0}^{s-1}\binom{s-1}{k}\{\varphi(f_1(y))\}^{(k)}\,|f_1^{(s-k)}(y)|\Big]_{y=0} \\
&= \sum_{k=0}^{s-1}\binom{s-1}{k}\sum_{r=1}^{k}\varphi^{(r)}(x_0)B_{k,r}\Big(f_1', f_1'', \ldots, f_1^{(k-r+1)}\Big)|f_1^{(s-k)}(y)|\Big]_{y=0} \\
&= \sum_{k=0}^{s-1}\binom{s-1}{k}\frac{1}{|g^{(s-k)}(x_0)|}\times \\
&\qquad \times\sum_{r=1}^{k}B_{k,r}\Big(\frac{1}{|g'(x_0)|}, \frac{1}{|g''(x_0)|}, \ldots, \frac{1}{|g^{(k-r+1)}(x_0)|}\Big)\varphi^{(r)}(x_0)
\end{aligned}$$

where $B_{k,r}$ is the incomplete exponential Bell polynomial. Thus, we have proven so far that the composition $\delta(g(x))$ exists and equals

$$\begin{aligned}
\langle\delta(g(x)), \varphi(x)\rangle = \sum_{k=0}^{s-1}\binom{s-1}{k}\frac{1}{s!|g^{(s-k)}(x_0)|}\times \\
\times\sum_{r=1}^{k}(-1)^r B_{k,r}\Big(\frac{1}{|g'(x_0)|}, \frac{1}{|g''(x_0)|}, \ldots, \frac{1}{|g^{(k-r+1)}(x_0)|}\Big)\langle\delta^{(r)}(x-x_0), \varphi(x)\rangle.
\end{aligned}$$

It obviously follows from the last equation that for $s = 1$, we have $\delta(g(x)) = \frac{1}{|g'(x_0)|}\delta(x - x_0)$. Finally, we have

$$\begin{aligned}
\langle(\delta(g(x)))', \varphi(x)\rangle &= -\langle\delta(g(x)), \varphi'(x)\rangle \\
&= -\,\mathrm{N-lim}\int_{-\infty}^{\infty}\delta_n(g(x))\varphi'(x)\,dx \\
&= \mathrm{N-lim}\int_{-\infty}^{\infty}\delta_n'(g(x))\varphi(x)g'(x)\,dx
\end{aligned}$$

on integrating by parts and thus

$$\langle(\delta(g(x)))', \varphi(x)\rangle = \langle\delta'(g(x)), \varphi(x)g'(x)\rangle = \langle g'(x)\delta'(g(x)), \varphi(x)\rangle.$$

□

**Remark 1** *The reader can see that if $g(x) = x^s$ and $s$ is an odd in Theorem 1, then the function $\Phi$ is identical to an arbitrary function $\varphi$ so that*

$$\begin{aligned}
\langle\delta(g(x)), \varphi(x)\rangle &= \frac{\Phi^{(s-1)}(0)}{s!} \\
&= \frac{(-1)^{s-1}}{s!}\langle\delta^{(s-1)}(x), \varphi(x)\rangle
\end{aligned}$$

*and so*

$$\delta(x^s) = \frac{(-1)^{s-1}}{s!}\delta^{(s-1)}(x)$$

*and*

$$\delta(x^s) = 0$$

for even $s$ on the real line which are in agreement with the results given in [19].

The other main contribution of this paper is as follows.

**Theorem 2** *Assume that $g(x)$ is an infinitely differentiable function having distinct multiple roots at $x_1, x_2, \ldots, x_n$ with multiplicities $r_1, r_2, r_3, \ldots, r_n$, $(n, r_i \in \mathbb{Z}^+)$ respectively on the open interval $(a,b)$. Then the distribution $\delta(g(x))$ exists on the interval $(a,b)$ and*

$$\begin{aligned}
\langle \delta(g(x)), \varphi(x)\rangle = \sum_{i=1}^{n}\sum_{k=0}^{r_i-1}\binom{r_i-1}{k}\frac{1}{r_i!|g^{(r_i-k)}(x_i)|} \times \\
\times \sum_{r=1}^{k} B_{k,r}\Big(\frac{1}{|g'(x_i)|}, \frac{1}{|g''(x_i)|}, \ldots, \frac{1}{|g^{(k-r+1)}(x_i)|}\Big)\varphi^{(r)}(x_i) \\
= \sum_{i=1}^{n}\sum_{k=0}^{r_i-1}\binom{r_i-1}{k}\frac{1}{r_i!|g^{(r_i-k)}(x_i)|} \times \\
\times \sum_{r=1}^{k}(-1)^r B_{k,r}\Big(\frac{1}{|g'(x_i)|}, \frac{1}{|g''(x_i)|}, \ldots, \frac{1}{|g^{(k-r+1)}(x_i)|}\Big)\langle \delta^{(r)}(x-x_i), \varphi(x)\rangle
\end{aligned} \tag{14}$$

*for all $i = 1, 2, \ldots, n$, $r_i \in \mathbb{Z}^+$. In particular, if $r_i = 1$ for all $i$ then*

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x - x_i)$$

which was given by Gelfand and Shilov in [13].

**Proof** It follows from equation 5 that $g(x) = (x-x_1)^{r_1}(x-x_2)^{r_2}(x-x_3)^{r_3}\ldots(x-x_n)^{r_n}h(x)$ where $h(x)$ is infinitely differentiable function defined by

$$h(x) = \begin{cases} \frac{g(x)}{(x-x_1)^{r_1}(x-x_2)^{r_2}\ldots(x-x_n)^{r_n}}, & x \neq x_i, \\ \frac{g^{(r_i)}(x_i)}{r_i!\prod_{i\neq k}(x_i-x_k)^{r_k}}, & x = x_i, \end{cases}$$

for $i = 1, 2, \ldots, n$.

Let $(\lambda_i, \upsilon_i)$ be disjoint open subintervals of $(a,b)$ containing $x_i$ such that $A = \cup_{i=1}^{n}(\lambda_i, \upsilon_i)$ for $i = 1, 2, \ldots, n$. Let us write $g_i(x) = \{(x-x_1)^{r_1}(x-x_2)^{r_2}(x-x_3)^{r_3}\ldots(x-x_n)^{r_n}h(x)\}^{1/r_i}$ and assume that the interval $(\lambda_i, \upsilon_i)$ is bounded and since $x_i$ is a simple root, we have $g_i'(x) \neq 0$ on $(\lambda_i, \upsilon_i)$ and also assume that

$g_i(x)$ is increasing. Then the equation $y = g_i(x)$ will have inverse $x = f_i(y) \in C^\infty$ on the interval $(\lambda_i, \upsilon_i)$. Now let $\varphi(x) \in \mathcal{D}$ with $\operatorname{supp}(\varphi) \subset (a,b)$, then we have

$$\begin{aligned}\int_{-\infty}^{\infty} \delta_n(g(x))\varphi(x)\,dx &= \int_{\lambda_1}^{\upsilon_1} \delta_n(g(x))\varphi(x)\,dx + \int_{\lambda_2}^{\upsilon_2} \delta_n(g(x))\varphi(x)\,dx + \\ &\quad + \ldots + \int_{\lambda_n}^{\upsilon_n} \delta_n(g(x))\varphi(x)\,dx + \int_{\mathbb{R}\setminus A} \delta_n(g(x))\varphi(x)\,dx \end{aligned} \tag{15}$$

where the last integral on the right-hand side of Equation 15 equals zero.

Next, for each $i$, we consider

$$\int_{\lambda_i}^{\upsilon_i} \delta_n(g(x))\varphi(x)\,dx = \int_{\lambda_i}^{x_i} \delta_n(g(x))\varphi(x)\,dx + \int_{x_i}^{\upsilon_i} \delta_n(g(x))\varphi(x)\,dx. \tag{16}$$

Making the substitution $t^{1/r_i} = g_i(x)$ or $x = f_i(t^{1/r_i})$ for the second integral on the right-hand side of equation 16, we have

$$\int_{x_i}^{\upsilon_i} \delta_n(g(x))\varphi(x)\,dx = \frac{1}{r_i}\int_0^{\alpha_i} \delta_n(t)\varphi(f_i(t^{1/r_i}))|f_i'(t^{1/r_i})|t^{1/r_i-1}\,dt$$

where $\alpha_i = g(\upsilon_i)$. The function $\Phi(y) = \varphi(f_i(y))|f_i'(y)|$ is infinitely differentiable and so by Taylor's theorem

$$\Phi(y) = \sum_{i=0}^{r_i-1} \frac{\Phi^{(i)}(0)}{i!}y^i + \frac{\Phi^{(r_i)}(\xi y)}{r_i!}y^{r_i} \quad (0<\xi<1).$$

Thus,

$$\begin{aligned} r_i\int_{x_i}^{\upsilon_i} \delta_n(g(x))\varphi(x)\,dx &= \sum_{j=0}^{r_i-1} \frac{\Phi^{(j)}(0)}{j!}\int_0^{\alpha_i} \delta_n(t)t^{\frac{j+1}{r_i}-1}\,dt + \int_0^{\alpha_i} \frac{\Phi^{(r_i)}(\xi t^{1/r_i})}{r_i!}\delta_n(t)t^{1/r_i}\,dt \\ &= \sum_{j=0}^{r_i-2} \frac{\Phi^{(j)}(0)}{j!}\int_0^1 \tau(u)(\frac{u}{n})^{\frac{i+1}{r_i}-1}\,du + \frac{\Phi^{(r_i-1)}(0)}{(r_i-1)!}\int_0^1 \tau(u)\,du + \\ &\quad + \int_0^1 \frac{\Phi^{(r_i)}(\xi(u/n)^{1/r_i})}{r_i!}\tau(u)(u/n)^{r_i}\,du \end{aligned}$$

on making the substitution $nt = u$ for $n^{-1} < \alpha_i$. Taking the neutrix limit of $\int_{x_i}^{\upsilon_i} \delta_n(g(x))\varphi(x)\,dx$, we get

$$\operatorname*{N-lim}_{n\to\infty} \int_{x_i}^{\upsilon_i} \delta_n(g(x))\varphi(x)\,dx = \frac{\Phi^{(r_i-1)}(0)}{2r_i!} \tag{17}$$

where $\int_0^1 \tau(u)\,du = \frac{1}{2}$.

Next, consider the integral $\int_{\lambda_i}^{x_i} \delta_n(g(x))\varphi(x)\,dx$. Similarly,by making the substitution $-t^{1/r_i} = g_i(-x)$ or $-$

$x = f_i(-t^{1/r_i})$ where $t^{1/r_i} \geq 0$, we have that

$$\begin{aligned} r_i \int_{\lambda_i}^{x_i} \delta_n(g(x))\varphi(x)\,dx &= \int_0^{\mu_i} \delta_n(g(-x))\varphi(-x)\,dx \\ &= \int_0^{\mu_i} \delta_n((-1)^s t)\varphi(f_i(-t^{1/s}))|f_i'(-t^{1/s})|t^{1/s-1}\,dt \\ &= \int_0^{\mu_i} \delta_n(t)\Phi(-t^{1/s})t^{1/s-1}\,dt \end{aligned}$$

where $\mu_i = -g(\lambda_i)$ and $\Phi$ is defined above.

Thus, with $n^{-1} < \mu_i$

$$\begin{aligned} \mathrm{N}\!-\!\lim_{n\to\infty} \int_{\lambda_i}^{x_i} \delta_n(g(x))\varphi(x)\,dx &= \frac{(-1)^{r_i-1}\Phi^{(r_i-1)}(0)}{r_i!}\int_0^1 \tau(u)\,du \\ &= \frac{(-1)^{r_i-1}\Phi^{(r_i-1)}(0)}{2r_i!}. \end{aligned} \tag{18}$$

It now follows from Equations 15–18 that

$$\mathrm{N}\!-\!\lim_{n\to\infty} \int_{\lambda_i}^{\upsilon_i} \delta_n(g(x))\varphi(x)\,dx = \begin{cases} 0, & r_i = 2,4,\dots \\ \frac{\Phi^{(r_i-1)}(0)}{r_i!}, & r_i = 1,3,\dots, \end{cases}$$

proving the existence of the composition $\delta(g(x))$ on the interval $(\lambda_i, \upsilon_i)$ for $i = 1,2,\dots,n,$ and consequently, it follows from what we have proven that the composition $\delta(g(x))$ exists on the interval $(a,b)$ and equals

$$\begin{aligned} \langle \delta(g(x)), \varphi(x)\rangle &= \sum_{i=1}^{n} \left\{\Phi^{(r_i-1)}(y)\right\}_{y=0} = \sum_{i=1}^{n} \{\varphi(f_i(y))|f_i'(y)|\}^{(r_i-1)}\Big|_{y=0} \\ &= \sum_{i=1}^{n}\sum_{k=0}^{r_i-1}\binom{r_i-1}{k}\{\varphi(f_i(y))\}^{(k)}\,|f_i^{(r_i-k)}(y)|\Big|_{y=0} \\ &= \sum_{i=1}^{n}\sum_{k=0}^{r_i-1}\binom{r_i-1}{k}\sum_{r=1}^{k}\varphi^{(r)}(f_i(y))B_{k,r}\Big(f_i', f_i'', \dots, h_i^{(k-r+1)}\Big)|f_i^{(r_i-k)}(y)|\Big|_{y=0} \\ &= \sum_{i=1}^{n}\sum_{k=0}^{r_i-1}\binom{r_i-1}{k}\frac{1}{|g^{(r_i-k)}(x_i)|}\times \\ &\qquad \times \sum_{r=1}^{k} B_{k,r}\Big(\frac{1}{|g'(x_i)|}, \frac{1}{|g''(x_i)|}, \dots, \frac{1}{|g^{(k-r+1)}(x_i)|}\Big)\varphi^{(r)}(x_i) \\ &= \sum_{i=1}^{n}\sum_{k=0}^{r_i-1}\binom{r_i-1}{k}\frac{1}{r_i!|g^{(r_i-k)}(x_i)|}\times \\ &\qquad \times \sum_{r=1}^{k}(-1)^r B_{k,r}\Big(\frac{1}{|g'(x_i)|}, \frac{1}{|g''(x_i)|}, \dots, \frac{1}{|g^{(k-r+1)}(x_i)|}\Big)\langle \delta^{(r)}(x-x_i), \varphi(x)\rangle \end{aligned}$$

for all $i = 1, 2, \dots, n,\ r_i \in \mathbb{Z}^+$ where $B_{k,r}(x_1, x_2, \dots, x_{k-r+1})$ are again the exponential Bell polynomials. This completes the proof. □

**Example 1** *Let us consider the function* $g(x) = \tan^3 x$. *Using the notation of the proof of Theorem 1,* $g_1(x) = \tan x$ *has simple roots at the points* $x = 0, \pm\pi, \pm 2\pi, \dots$ *and we have*

$$\begin{aligned} g_1(y) &= \tan^{-1} y = y - \frac{1}{3}y^3 + \frac{1}{5}y^5 - \frac{1}{7}y^7 + \dots, \\ f_1'(y) &= \frac{1}{1+y^2} = 1 - y^2 + y^4 - y^6 + \dots \end{aligned}$$

*on the open interval* $(-\infty, \infty)$. *Thus,* $\Phi(y) = \varphi(\tan^{-1} y)(1+y^2)^{-1}$ *and it can be shown that* $\Phi''(0) = 2\varphi(0) + \varphi''(0)$. *It follows from the proof of Theorem 1 that*

$$\langle \delta(\tan^3 x), \varphi(x) \rangle = \frac{1}{6}\Phi''(0) = \frac{1}{3}\varphi(0) + \frac{1}{6}\varphi''(0)$$

*and so*

$$\delta(\tan^3 x) = \frac{1}{3}\delta(x) + \frac{1}{6}\delta''(x)$$

on the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

**Example 2** *As an example of Theorem 2, let us consider the* $g(x) = \sin^3 x$. *The equation* $g(x) = \sin x$ *has simple roots at the points* $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots$ *and*

$$f_1(y) = \arcsin y = y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \dots \quad \textit{and} \quad |f_1'(y)| = (1-y^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}y^2 + \frac{3}{8}y^4 + \dots$$

*on the open interval* $(-1, 1)$. *Thus,* $\Phi(y) = \varphi(\arcsin y)(1-y^2)^{-\frac{1}{2}}$ *and it can be shown as* $\Phi''(0) = \varphi(0) + \varphi''(0)$. *It follows from the proof of the theorem that*

$$\langle \delta(\sin^3 x), \varphi(x) \rangle = \frac{1}{6}\Phi''(0) = \frac{1}{6}\Phi(0) + \frac{1}{6}\Phi''(0).$$

*on the open interval* $(-\pi, \pi)$. *Since* $\delta$ *is an even distribution* $\delta(-\sin^3 x) = \delta(\sin^3 x)$ *it follows by translation*

$$\delta(\sin^3 x) = \sum_{n=-\infty}^{\infty} \frac{1}{6}[\delta(x - n\pi) + \delta''(x - n\pi)].$$

## 3. More on the composition $\delta(g(x))$

We now extend Definition 1 with the following definition given in [6].

**Definition 2** *Let $G$ be a distribution in $\mathcal{D}'$, and let $g$ be a locally summable function. We say that the distribution $G(g(x))$, the neutrix composition of $G$ and $g$, exists and is equal to $F(x)$ on the interval $(a,b)$ if the neutrix limit*

$$\mathop{\mathrm{N-lim}}_{n\to\infty}\left[\int_{-\infty}^{\infty} G_n(g(x))\varphi(x)\,dx\right] = \langle F(x),\varphi(x)\rangle$$

*for all $\varphi$ in $\mathcal{D}$ with support contained in the interval $(a,b)$, where $G_n(x)=(G*\delta_n)(x)$, and $N$ is the neutrix taken as in Definition 1.*

For a summable function $g(x)$, the functions $g_+$ and $g_-$ are defined by

$$g_+(x)=\begin{cases} g(x), & x\geq 0,\\ 0, & x<0,\end{cases} \quad \text{and} \quad g_-(x)=\begin{cases} g(x), & x\leq 0,\\ 0, & x>0.\end{cases}$$

In accordance with the usual practice, we define the summable functions $x_+^r$ and $x_-^r$ by

$$x_+^r=\begin{cases} x^r, & x\geq 0,\\ 0, & x<0,\end{cases} \quad \text{and} \quad x_-^r=\begin{cases} |x|^r, & x\leq 0,\\ 0, & x>0.\end{cases}$$

If the term infinitely differentiable function is replaced by summable function, then Proposition 1 turns out to be as follows:

**Proposition 2** *Let $g$ be a summable function and suppose that $g$ is continuous on $[a,b]$ and $g(x)\neq 0$ on this interval, where $a<0<b$. Then the composition $\delta(g_+(x))$ exists and*

$$\delta(g_+(x))=0$$

*on the interval $(-\infty,b)$, in particular $\delta(H(x))=0$ on the interval $(-\infty,\infty)$, where $H$ denotes Heaviside's function.*

**Proof** Let $\varphi\in\mathcal{D}'$ with compact support contained in the interval $(-\infty,b)$. Then

$$\begin{aligned}\int_{-\infty}^{\infty}\delta_n(g_+(x))\varphi(x)\,dx &= \int_{-\infty}^{0}\delta_n(0)\varphi(x)\,dx+\int_{0}^{\infty}\delta_n(g(x))\varphi(x)\,dx\\ &= n\tau(0)\int_{-\infty}^{0}\varphi(x)\,dx+n\int_{0}^{b}\tau(ng(x))\varphi(x)\,dx\end{aligned}$$

where $n\tau(0)\int_{-\infty}^{0}\varphi(x)\,dx$ is either negligible or zero. Further, since $g$ is continuous and nonzero on $[0,b]$, we can find an integer $N$ such that $|ng(x)|\geq 1$ for $n>N$. It follows that we have $\tau(ng(x))=0$ for $n>N$. Thus,

$$\mathop{\mathrm{N-lim}}_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(g_+(x))\varphi(x)\,dx=$$

$$=\mathop{\mathrm{N-lim}}_{n\to\infty} n\tau(0)\int_{-\infty}^{0}\varphi(x)\,dx+\lim_{n\to\infty} n\int_{0}^{b}\tau(ng(x))\varphi(x)\,dx=\langle 0,\varphi(x)\rangle$$

and so $\delta(g_+(x)) = 0$. □

Let $g(x)$ be as in the proposition 2 and by writing $h(x) = g(-x)$, then $h(x)$ is continuous and nonzero for $-b \leq x \leq -a$ and so by the proposition 2.2, we have $\delta(h_+(x)) = 0$ on the interval $(-\infty, -a)$. Now replacing $x$ by $-x$ we see that $\delta(h_+(-x)) = \delta(g_-(x)) = 0$. Thus, we reach the following result.

**Corollary 1** *Let $g$ be a summable function and suppose that $g$ is continuous and nonzero on the interval $[a, b]$, where $a < 0 < b$. Then $\delta(g_-(x))$ exists and*

$$\delta(g_-(x)) = 0$$

*on the interval $(a, \infty)$ and the result of the corollary follows.*

**Proof** It is similar to the proof of Proposition 2. □

**Corollary 2** *Let $g$ be a summable function and suppose that $g$ is continuous and non-zero on the interval $[a, b]$, where $a < 0 < b$. Then the composition $\delta(g_+(x) - g_-(x))$ exists and*

$$\delta(g_+(x)) = \delta(g_+(x) - g_-(x)) = 0$$

*on the interval $(a, b)$.*

**Proof** It is evident. □

**Example 3** *Consider the function $g(x) = \cos x$. It follows immediately from Gelfand and Shilov's definition and Proposition 2 that $\delta(\cos_+ x) = \sum_{n=0}^{\infty} \delta(x - n\pi - \frac{\pi}{2})$ and it follows from corollary 3.4 that*

$$\delta(\cos_+ x - \cos_- x) = \sum_{n=0}^{\infty} \delta(x - n\pi - \frac{\pi}{2}) + \sum_{n=1}^{\infty} \delta(x + n\pi - \frac{\pi}{2}).$$

*We have from Theorem 1 and Proposition 2 that $\delta(\cos_+^2 x) = 0$. It was proved in Example 2 that $\delta(\sin^3 x) = \sum_{n=-\infty}^{\infty} \frac{1}{6}[\delta(x - n\pi) + \delta''(x - n\pi)]$ and it follows by translation and Proposition 2 that*

$$\delta(\cos^3 x) = \sum_{n=-\infty}^{\infty} \frac{1}{6}[\delta(x - n\pi - \frac{\pi}{2}) + \delta''(x - n\pi - \frac{\pi}{2})]$$

*and*

$$\delta(\cos_+^3 x) = \sum_{n=0}^{\infty} \frac{1}{6}[\delta(x - n\pi - \frac{\pi}{2}) + \delta''(x - n\pi - \frac{\pi}{2})].$$

*Of course these results hold on the real line. In the following, we consider Theorem 1 for summable function $g(x)$.*

**Theorem 3** *Let $G$ be a summable function which is $k+1$ times continuously differentiable on the interval $[a,b]$, where $a<0<b$. Suppose that the equation $G(x)=0$ has a single simple root at the point $x=0$ in the interval $[a,b]$. Then if $g=G^k$, the composition $\delta(g_+(x))$ exists on the interval $(-\infty,b)$ and the composition $\delta(g_-(x))$ exists on the interval $(a,\infty)$ for $k=1,2,\ldots$. In particular*

$$\delta(x_+^k)=\frac{(-1)^{k-1}}{2k!}\delta^{(k-1)}(x) \quad and \qquad \delta(x_-^k)=\frac{1}{2k!}\delta^{(k-1)}(x)$$

*on the interval $(-\infty,\infty)$ for $k=1,2,\ldots$.*

**Proof** The proof is slightly similar to the proof of Theorem 1. Since $x=0$ is a simple root of the equation $G(x)=0$, this implies that $G'(x)\neq 0$ on the interval $[o,c]$, where $0<c\leq b$. The equation $G(x)=y$ will therefore have inverse $x=h(y)$ on the interval $[0,c]$ and the function $h$ will be $k+1$ times continuously differentiable. Let $\varphi\in\mathcal{D}'$ with $\operatorname{supp}(\varphi)\subset(-\infty,c)$. Then

$$\begin{aligned}\int_{-\infty}^{\infty}\delta_n(g_+(x))\varphi(x)\,dx &= \int_{-\infty}^{0}\delta_n(0)\varphi(x)\,dx+\int_0^{\infty}\delta_n(g(x))\varphi(x)\,dx=\\ &= n\tau(0)\int_{-\infty}^{0}\varphi(x)\,dx+\int_0^{\infty}\delta_n(g(x))\varphi(x)\,dx\end{aligned}$$

where again $n\tau(0)\int_{-\infty}^0\varphi(x)\,dx$ is either negligible or zero. By making the substitution $t^{1/k}=G(x)$ or $x=h(t^{1/k})$, we have

$$\int_0^{\infty}\delta_n(g(x))\varphi(x)\,dx=\frac{1}{k}\int_0^{\infty}\delta_n(t)\varphi(h(t^{1/k}))|h'(t^{1/k})|t^{1/k-1}\,dt.$$

The function $\Psi(y)=\varphi(h(y))|h'(y)|$ is $k$ times continuously differentiable and so we have by Taylor's theorem

$$\Psi(y)=\sum_{i=0}^{k-1}\frac{\Psi^{(i)}(0)}{i!}y^i+\frac{\Psi^{(k)}(\xi y)}{k!}y^k,\ (0\leq\xi\leq 1).$$

Thus,

$$\begin{aligned}k\int_0^{\infty}\delta_n(g(x))\varphi(x)\,dx=\sum_{i=0}^{k-2}\frac{\Psi^{(i)}(0)}{i!}\int_0^1 n\tau(u)(u/n)^{(i+1)/k-1}\,du+\\ +\frac{\Psi^{(k-1)}(0)}{(k-1)!}\int_0^1\tau(u)u\,du+\int_0^1\frac{\Psi^{(k)}(\xi(u/n)^{1/k})}{k!}n\tau(u)(u/n)^{1/k}\,du\end{aligned}$$

where the substitution $nt=u$ has been made. It follows that

$$\begin{aligned}\underset{n\to\infty}{\text{N}-\lim}\int_{-\infty}^{\infty}\delta_n(g_+(x))\varphi(x)\,dx &= \underset{n\to\infty}{\text{N}-\lim}\int_0^{\infty}\delta_n(g(x))\varphi(x)\,dx\\ &= \frac{\Psi^{(k-1)}(0)}{k!}\int_0^1\tau(u)u\,du\\ &= \frac{(-1)^k\Psi^{(k-1)}(0)}{2k!}.\end{aligned}$$

This proves the existence of $\delta(g_+(x))$ on the interval $(-\infty, c)$. We of course have $\delta(g_+(x)) = 0$ on the interval $(\frac{c}{2}, b)$ by Corollary 2. □

**Remark 2** *The reader may realize that the value $\Psi^{(k-1)}(0)$ is exactly the same as in the proof of Theorem 2.3. Thus, consider the particular case when $G = x$, $G$ is infinitely differentiable, $g_+ = x^k$ and $\Psi$ is identical to $\varphi$, it follows that*

$$\langle \delta(x_+^k), \varphi(x) \rangle = \frac{\varphi^{(k-1)}(0)}{2k!} = \frac{(-1)^{k-1}}{2k!} \langle \delta^{(k-1)}(x), \varphi(x) \rangle.$$

*Note that $\delta(g_-(x)) = \delta(g_+(-x))$ the right-hand side existing above and in particular replacing $x$ by $-x$ we have*

$$\delta(x_-^k) = \frac{(-1)^{(k-1)}}{2k!} \delta^{(k-1)}(-x) = \frac{1}{2k!} \delta^{(k-1)}(x). \quad \square$$

**Corollary 3** *Let $G$ be a summable function which is $k+1$ times continuously differentiable on the interval $[a, b]$, where $a < 0 < b$. Suppose that the equation $G(x) = 0$ has a single simple root at the point $x = 0$ in the interval $[a, b]$. Then if $g = G^k$, the compositions $\delta(g(x))$ and $\delta(g_+(x) - g_-(x))$ exist on the interval $(a, b)$ for $k = 1, 2, \ldots$. In particular*

$$\delta(\text{sgn}(x)|x|^k) = \delta(|x|^k) = 0 \tag{19}$$

*for $k = 2, 4, 6 \ldots$. and*

$$\delta(\text{sgn}(x)|x|^k) = \delta(|x|^k) = \frac{(-1)^k}{k!} \delta^{(k-1)}(x) \tag{20}$$

*for $k = 1, 3, 5 \ldots$, on the interval $(-\infty, \infty)$.*

**Proof** Writing

$$\begin{aligned} \int_{-\infty}^{\infty} \delta_n(g(x))\varphi(x)\,dx &= \int_{-\infty}^{0} \delta_n(g(x))\varphi(x)\,dx + \int_{0}^{\infty} \delta_n(g(x))\varphi(x)\,dx \\ &= \int_{-\infty}^{\infty} \delta_n(g_-(x))\varphi(x)\,dx + \int_{-\infty}^{\infty} \delta_n(g_+(x))\varphi(x)\,dx - \int_{-\infty}^{\infty} \delta_n(0)\varphi(x)\,dx, \end{aligned}$$

where the last term is either negligible or zero, it follows that

$$\text{N}-\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(g(x))\varphi(x)\,dx = \langle \delta(g_+(x)) + \delta(g_-(x)), \varphi(x) \rangle$$

implies

$$\delta(g(x)) = \delta(g_+(x)) + \delta(g_-(x)).$$

Similarly, we can simply prove that

$$\begin{aligned} \delta(g_+(x) - g_-(x)) &= \delta(g_+(x)) + \delta(-g_-(x)) \\ &= \delta(g_+(x)) + \delta(g_-(x)). \end{aligned}$$

Equations 19 and 20 now follow from these results on using Theorem 3. □

It follows on using the fact that $x^k = x_+^k + (-1)^k x_-^k$ in the last equation above, we have

$$\delta(x^k) = \delta(x_+^k) + (-1)^k \delta(x_-^k) = \begin{cases} \frac{(-1)^k}{k!}\delta^{(k-1)}(x), & k \quad \text{odd}, \\ 0, & k \quad \text{even}, \end{cases}$$

which is in agreement with the result given in the note and also results given in [19]. We leave the reader as an exercise to prove that the composition $\delta(x_+^\mu) = 0$ for $\mu > 0$, $\mu \neq 1, 2, \ldots$

**Example 4** *As a final example, consider* $g(x) = \sin^2 x$. *It follows from Theorem 1 that* $\delta(\sin^2 x) = 0$ *on the interval* $(-\infty, \infty)$. *Further, for arbitrary test function* $\varphi$, *we have* $\Phi(y) = \varphi(\sin^{-1}(y)(1-y^2)^{-1/2}$ *and it can be shown that* $\Phi'(0) = \varphi'(0)$, *and* $\Phi'''(0) = 4\varphi'(0) + \varphi'''(0)$. *It follows from Theorem 3.6 and its corollaries that*

$$\begin{aligned} \langle \delta(\sin_+^2 x), \varphi(x) \rangle &= \frac{1}{4}\Phi'(0) = \frac{1}{4}\varphi'(0) \\ \langle \delta'(\sin_+^2 x), \varphi(x) \rangle &= -\frac{1}{24}\Phi'''(9) = -\frac{1}{6}\varphi'(0) - \frac{1}{24}\varphi'''(0) \end{aligned}$$

*and so*

$$\begin{aligned} \langle \delta(\sin_+^2 x), \varphi(x) \rangle &= -\frac{1}{4}\delta'(x), \\ \langle \delta(\sin_-^2 x), \varphi(x) \rangle &= \frac{1}{4}\delta'(x), \\ \langle \delta'(\sin_+^2 x), \varphi(x) \rangle &= \frac{1}{6}\delta'(x) + \frac{1}{24}\delta'''(x), \\ \langle \delta'(\sin_-^2 x), \varphi(x) \rangle &= -\frac{1}{6}\delta'(x) - \frac{1}{24}\delta'''(x), \\ \langle \delta(\operatorname{sgn} x. \sin^2 x), \varphi(x) \rangle &= -\frac{1}{2}\delta'(x), \\ \langle \delta'(\operatorname{sgn} x. \sin^2 x), \varphi(x) \rangle &= \frac{1}{3}\delta'(x) + \frac{1}{24}\delta'''(x) \end{aligned}$$

*on the interval* $(-\infty, \infty)$. □

**Conclusion.** The composition $\delta(g(x))$ is defined for infinitely differentiable function with simple roots at $x_1, x_2, \ldots, x_n$ by Gelfand and Shilov. In this study, we extend their definition for an infinitely differentiable function having distinct multiple roots via neutrices theory and also the composition $\delta(g(x))$ is considered for summable $g(x)$ and defined as the neutrix limit of the regular sequence $\{\delta_n(g(x))\}$, due to van der Corput. The compositions $\delta(x^s)$, $\delta(x_+^s)$, $\delta(x_-^s)$, $\delta(\operatorname{sgn} x|x|^s)$, $\delta(\tan^3 x)$, and $\delta(\cos^3 x)$ etc. are evaluated as the applications of the definitions.

**Acknowledgment**

The author is grateful for the financial support provided by the Scientific Research Projects Coordination Unit of Hacettepe University, Project Number FHD2024-21165.

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