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Abstract: In this paper, we introduce in a braided setting the notion of left module for an invertible 1-cocycle and we prove some categorical equivalences between categories of modules associated to an invertible 1-cocycle and categories of modules associated to Hopf braces.

Key words: Braided monoidal category, Hopf algebra, Hopf brace, invertible 1-cocycles, module

1. Introduction

Hopf braces were introduced recently in [3] as the linearisation of skew braces given in [6]. A skew brace consists of two different group structures, denoted as $(T, \diamond)$ and $(T, \circ)$, defined on the same set T. They satisfy $\forall\, a, b, c \in T$ the compatibility condition

$$a \circ (b \diamond c) = (a \circ b) \diamond a^{\diamond} \diamond (a \circ c),$$

where $a^{\diamond}$ represents the inverse with respect to $\diamond$. Thus, a Hopf brace comprises two structures of Hopf algebras defined on the same object, sharing a common coalgebra structure and satisfying the same compatibility condition that generalizes the previous identity. The relevance of these structures comes from the fact that provide solutions of the Yang-Baxter equation. As pointed out in [3], they establish the right setting for considering left symmetric algebras as Lie-theoretical analogs of the notion of brace introduced by W. Rump in [9]. Moreover, it is noteworthy that there exists a profound connection between Hopf braces and invertible 1-cocycles. In fact, the category of Hopf braces is equivalent to the category of invertible 1-cocycles.

Thus, invertible 1-cocycles are nothing more than coalgebra isomorphisms between Hopf algebras that share the underlying coalgebra and that are related by a module algebra structure.

As long as we are dealing with Hopf-type structures, we are somehow forced to have a deep look into their categories of modules for a complete overview. For example, in [5], the author introduces the category of left modules for a Hopf brace in order to prove Fundamental Theorem of Hopf modules (see, for example

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[1, 11]) in the Hopf brace setting. This notion of module for a Hopf brace is weaker than that introduced by H. Zhu in [12] and in the cocommutative setting both notions are equivalent. The main difference between the two definitions is the following: The Hopf brace with the two associated products is an example of module in the sense of [5] while with the definition proposed by Zhu that property only holds when the underlying object of the Hopf brace endowed with a particular action is an object belonging to a class of cocommutativity in the sense of [2]. In other words, under certain circumstances, for example, the lack of cocommutativity, the category of left modules over a Hopf brace introduced by Zhu may not contain the obvious object as it happens always in the case of Hopf algebras.

Hence, the primary goal of this article is to identify the suitable notion of a module associated to an invertible 1-cocycle. This identification will enable the extension of the established categorical equivalence between Hopf braces and invertible 1-cocycles to their respective categories of modules. It's essential to note that, in our context, the definition of a module associated with a Hopf brace aligns with the one introduced in [5].

The paper is organized as follows: The second section presents the basic notions that we will need in the rest of the paper and the main results are provided in the third section. More concretely, working in a braided setting, in Section 3, we introduce the notion of module for an invertible 1-cocycle and the category of these objects (see Definition 10). Next, we prove some functorial results and show that under symmetry and cocommutativity conditions the category of modules associated to an invertible 1-cocycle is symmetric monoidal (see Theorem 5). Finally, in Theorems 6, 7 and Corollary 1, we obtain the desired categorical equivalences between categories of modules associated to an invertible 1-cocycle and categories of modules associated to a Hopf brace.

2. Preliminaries

Throughout this paper, C denotes a strict braided monoidal category with tensor product $\otimes$, unit object K, and braiding c. Recall that a monoidal category is a category C together with a functor $\otimes : \mathsf{C}\times\mathsf{C}\to\mathsf{C}$, called tensor product, an object K of C, called the unit object, and families of natural isomorphisms

$$a_{M,N,P} : (M\otimes N)\otimes P \to M\otimes(N\otimes P), \quad r_M : M\otimes K\to M, \quad l_M : K\otimes M\to M,$$

in C, called associativity, right unit and left unit constraints, respectively, satisfying the Pentagon Axiom and the Triangle Axiom, i.e.,

$$a_{M,N,P\otimes Q}\circ a_{M\otimes N,P,Q} = (id_M\otimes a_{N,P,Q})\circ a_{M,N\otimes P,Q}\circ(a_{M,N,P}\otimes id_Q),$$

$$(id_M\otimes l_N)\circ a_{M,K,N} = r_M\otimes id_N,$$

where for each object X in C, id_X denotes the identity morphism of X (see [8]). A monoidal category is called strict if the constraints of the previous paragraph are identities. It is a well-known fact (see for example [7]) that every nonstrict monoidal category is monoidal equivalent to a strict one. This lets us treat monoidal categories as if they were strict and, as a consequence, the results proved in a strict setting hold for every nonstrict monoidal category, for example the category $\mathbb{F}$-Vect of vector spaces over a field $\mathbb{F}$, or the category R-Mod of left modules over a commutative ring R. For simplicity of notation, given objects M, N, P in C and a morphism $f : M\to N$, in most cases, we will write $P\otimes f$ for $id_P\otimes f$ and $f\otimes P$ for $f\otimes id_P$.

A braiding for a strict monoidal category C is a natural family of isomorphisms $c_{M,N} : M \otimes N \to N \otimes M$ subject to the conditions

$$c_{M,N\otimes P} = (N \otimes c_{M,P}) \circ (c_{M,N} \otimes P), \quad c_{M\otimes N,P} = (c_{M,P} \otimes N) \circ (M \otimes c_{N,P}).$$

A strict braided monoidal category C is a strict monoidal category with a braiding. Note that, as a consequence of the definition, the equalities $c_{M,K} = c_{K,M} = id_M$ hold, for all object M of C. If the braiding satisfies that $c_{N,M} \circ c_{M,N} = id_{M\otimes N}$, for all M, N in C, we will say that C is symmetric. In this case, we call the braiding c a symmetry for the category C.

Definition 1 *An algebra in C is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in C and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in C such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, a morphism $f : A \to B$ in C is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$.*

If A, B are algebras in C, the tensor product $A \otimes B$ is also an algebra in C where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

Definition 2 *A coalgebra in C is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in C and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in C such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, a morphism $f : D \to E$ in C is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$.*

Given D, E coalgebras in C, the tensor product $D \otimes E$ is a coalgebra in C where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

Definition 3 *Let $D = (D, \varepsilon_D, \delta_D)$ be a coalgebra and let $A = (A, \eta_A, \mu_A)$ be an algebra. By $\mathcal{H}(D, A)$, we denote the set of morphisms $f : D \to A$ in C. With the convolution operation $f * g = \mu_A \circ (f \otimes g) \circ \delta_D$, $\mathcal{H}(D, A)$ is an algebra where the unit element is $\eta_A \circ \varepsilon_D = \varepsilon_D \otimes \eta_A$.*

Definition 4 *Let A be an algebra. The pair (M, φ_M) is a left A-module if M is an object in C and $\varphi_M : A \otimes M \to M$ is a morphism in C satisfying $\varphi_M \circ (\eta_A \otimes M) = id_M$, $\varphi_M \circ (A \otimes \varphi_M) = \varphi_M \circ (\mu_A \otimes M)$. Given two left A-modules (M, φ_M) and (N, φ_N), $f : M \to N$ is a morphism of left A-modules if $\varphi_N \circ (A \otimes f) = f \circ \varphi_M$.*

The composition of morphisms of left A-modules is a morphism of left A-modules. Then left A-modules form a category that we will denote by ${}_{\mathsf{A}}\mathsf{Mod}$.

Definition 5 *We say that H is a bialgebra in C if (H, η_H, μ_H) is an algebra, $(H, \varepsilon_H, \delta_H)$ is a coalgebra, and ε_H and δ_H are algebra morphisms (equivalently, η_H and μ_H are coalgebra morphisms). Moreover, if there exists a morphism $\lambda_H : H \to H$ in C, called the antipode of H, satisfying that λ_H is the inverse of id_H in $\mathcal{H}(H, H)$, i.e.,*

$$id_H * \lambda_H = \eta_H \circ \varepsilon_H = \lambda_H * id_H, \tag{1}$$

we say that H is a Hopf algebra.

If H is a Hopf algebra, the antipode is antimultiplicative and anticomultiplicative

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H, \tag{2}$$

and leaves the unit and counit invariant, i.e., $\lambda_H \circ \eta_H = \eta_H$, $\varepsilon_H \circ \lambda_H = \varepsilon_H$.

A morphism of Hopf algebras is an algebra-coalgebra morphism. Note that, if $f : H \to D$ is a Hopf algebra morphism, the following equality holds:

$$\lambda_D \circ f = f \circ \lambda_H.$$

With the composition of morphisms in C, we can define a category whose objects are Hopf algebras and whose morphisms are morphisms of Hopf algebras. We denote this category by Hopf.

A Hopf algebra is commutative if $\mu_H \circ c_{H,H} = \mu_H$ and cocommutative if $c_{H,H} \circ \delta_H = \delta_H$. It is easy to see that in both cases $\lambda_H \circ \lambda_H = id_H$.

Definition 6 *Let D be a Hopf algebra. An algebra B is said to be a left D-module algebra if (B, Φ_B) is a left D-module and η_B, μ_B are morphisms of left D-modules, i.e.,*

$$\Phi_B \circ (D \otimes \eta_B) = \varepsilon_D \otimes \eta_B, \;\; \Phi_B \circ (D \otimes \mu_B) = \mu_B \circ \Phi_{B\otimes B},$$

where $\Phi_{B\otimes B} = (\Phi_B \otimes \Phi_B) \circ (D \otimes c_{D,B} \otimes B) \circ (\delta_D \otimes B \otimes B)$ is the left action on $B \otimes B$.

3. Modules over invertible 1-cocycles and Hopf braces

We begin the main section of this paper by defining the notion of invertible 1-cocycle in the braided monoidal category C. This definition can be directly generalized from the one given for the symmetric setting, for example in the category of vector spaces over a field $\mathbb{F}$ (see [3]).

Definition 7 *Let A, H be Hopf algebras in C. Let's assume that H is a left A-module algebra with action Φ_H. Let $\pi : A \to H$ be a coalgebra morphism. We will say that π is an invertible 1-cocycle if it is an isomorphism such that*

$$\pi \circ \mu_A = \mu_H \circ (\pi \otimes \Phi_H) \circ (\delta_A \otimes \pi) \tag{3}$$

holds.

Let $\pi : A \to H$ and $\tau : B \to D$ be invertible 1-cocycles. A morphism between them is a pair

$$(f, g) \; : \quad \pi \downarrow \begin{matrix} A \\ \\ H \end{matrix} \quad \longrightarrow \quad \tau \downarrow \begin{matrix} B \\ \\ D \end{matrix}$$

where $f : A \to B$ and $g : H \to D$ are algebra-coalgebra morphisms satisfying the following identities:

$$g \circ \pi = \tau \circ f, \tag{4}$$

$$g \circ \Phi_H = \Phi_D \circ (f \otimes g). \tag{5}$$

Then, with these morphisms, invertible 1-cocycles form a category denoted by IC. Note that $\pi \circ \eta_A = \eta_H$ holds (see [3]).

Remark 1 *It is easy to see that there exists a functorial connection between the categories* Hopf *and* IC *given by the following: If* A *is a Hopf algebra,* $(A, t_A = \varepsilon_A \otimes A)$ *is a left* A*-module algebra. Then,* $id_A : A \to A$ *is an object in* IC*. On the other hand, if* $f : A \to B$ *is a morphism of Hopf algebras, we have that the pair* (f, f) *is a morphism in* IC *between* $id_A : A \to A$ *and* $id_B : B \to B$*. Therefore, there exists a functor*

$$\mathsf{H} : \mathsf{Hopf} \to \mathsf{IC}$$

defined on objects by

$$\mathsf{H}(A) = \quad id_A \begin{array}{c} A \\ \downarrow \\ A \end{array} \, ,$$

where the action is $t_A = \varepsilon_A \otimes A$ *(the trivial action), and on morphisms by* $\mathsf{H}(f) = (f, f)$*.*

As was pointed in [3], there exists a closed relation between the Hopf theoretical generalization of skew braces, called Hopf braces, and invertible 1-cocycles in the category of vector spaces over a field $\mathbb{F}$. In the braided setting, we have the same relation and the definition of Hopf brace is the following:

Definition 8 *Let* $H = (H, \varepsilon_H, \delta_H)$ *be a coalgebra in* C*. Let us assume that there are two algebra structures* (H, η_H^1, μ_H^1)*,* (H, η_H^2, μ_H^2) *defined on* H *and suppose that there exist two endomorphisms of* H *denoted by* λ_H^1 *and* λ_H^2*. We will say that*

$$(H, \eta_H^1, \mu_H^1, \eta_H^2, \mu_H^2, \varepsilon_H, \delta_H, \lambda_H^1, \lambda_H^2)$$

is a Hopf brace in C *if:*

(i) $H_1 = (H, \eta_H^1, \mu_H^1, \varepsilon_H, \delta_H, \lambda_H^1)$ *is a Hopf algebra in* C*.*

(ii) $H_2 = (H, \eta_H^2, \mu_H^2, \varepsilon_H, \delta_H, \lambda_H^2)$ *is Hopf algebra in* C*.*

(iii) The following equality holds:

$$\mu_H^2 \circ (H \otimes \mu_H^1) = \mu_H^1 \circ (\mu_H^2 \otimes \Gamma_{H_1}) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H),$$

where

$$\Gamma_{H_1} = \mu_H^1 \circ (\lambda_H^1 \otimes \mu_H^2) \circ (\delta_H \otimes H).$$

Following [5], a Hopf brace will be denoted by $\mathbb{H} = (H_1, H_2)$ *or in a simpler way by* $\mathbb{H}$*.*

If $\mathbb{H}$ *is a Hopf brace in* C*, we will say that* $\mathbb{H}$ *is cocommutative if*

$$\delta_H = c_{H,H} \circ \delta_H,$$

i.e., H_1 *and* H_2 *are cocommutative Hopf algebras in* C*. Note that by [10, Corollary 5], if* H *is a cocommutative Hopf algebra in the braided monoidal category* C*, the identity*

$$c_{H,H} \circ c_{H,H} = id_{H \otimes H} \tag{6}$$

holds.

The previous definition is the general notion of Hopf brace in a braided monoidal setting. If we restrict it to a category of Yetter-Drinfeld modules over a Hopf algebra in $\mathbb{F}$-Vect which antipode is an isomorphism we obtain the definition of braided Hopf brace introduced by H. Zhu and Z. Ying in [13, Definition 2.1].

Definition 9 *Given two Hopf braces $\mathbb{H}$ and $\mathbb{B}$ in C, a morphism x in C between the two underlying objects is called a morphism of Hopf braces if both $x : H_1 \to B_1$ and $x : H_2 \to B_2$ are algebra-coalgebra morphisms, i.e., Hopf algebra morphisms.*

Hopf braces together with morphisms of Hopf braces form a category which we denote by HBr. This category is a subcategory of the category of Hopf trusses introduced by T. Brzeźniski in [4].

Theorem 1 *There exists a functor between the categories Hopf and HBr.*

Proof If H is a Hopf algebra, $\mathbb{H}_{triv} = (H, \eta_H, \mu_H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda_H, \lambda_H)$ is an object in HBr. On the other hand, if $x : H \to B$ is a morphism of Hopf algebras, we have that the pair (x, x) is a morphism in HBr between $\mathbb{H}_{triv}$ and $\mathbb{B}_{triv}$. Therefore, there exists a functor

$$\mathsf{H}' : \mathsf{Hopf} \to \mathsf{HBr}$$

defined on objects by $\mathsf{H}'(H) = \mathbb{H}_{triv}$ and on morphisms by $\mathsf{H}'(x) = (x, x)$. □

Let $\mathbb{H}$ be a Hopf brace in C. Then

$$\eta_H^1 = \eta_H^2,$$

holds and, by [3, Lemma 1.7], in this braided setting the equality

$$\Gamma_{H_1} \circ (H \otimes \lambda_H^1) = \mu_H^1 \circ ((\lambda_H^1 \circ \mu_H^2) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) \tag{7}$$

also holds. Moreover, in our braided context [3, Lemma 1.8, Remark 1.9] hold and then we have that (H, η_H^1, μ_H^1) is a left H_2-module algebra with action Γ_{H_1} and μ_H^2 admits the following expression:

$$\mu_H^2 = \mu_H^1 \circ (H \otimes \Gamma_{H_1}) \circ (\delta_H \otimes H). \tag{8}$$

Now, taking into account that every Hopf brace is an example of Hopf truss, by [4, Theorem 6.4], we have that (H, η_H^1, μ_H^1) also is a left H_2-module algebra with action

$$\Gamma'_{H_1} = \mu_H^1 \circ (\mu_H^2 \otimes \lambda_H^1) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$

because the symmetry is not needed in the proof as in the case of Γ_{H_1}.

Finally, by the naturality of c and the coassociativity of δ_H, we obtain that

$$\mu_H^1 \circ (\mu_H^2 \otimes \Gamma_{H_1}) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H)$$

$$= \mu_H^1 \circ (\Gamma'_{H_1} \otimes \mu_H^2) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H)$$

and then (iii) of Definition 8 is equivalent to

$$\mu_H^2 \circ (H \otimes \mu_H^1) = \mu_H^1 \circ (\Gamma'_{H_1} \otimes \mu_H^2) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H).$$

Therefore, the equality

$$\mu_H^2 = \mu_H^1 \circ (\Gamma'_{H_1} \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) \tag{9}$$

holds.

Remark 2 *Note that if $\mathbb{H}$ is a cocommutative Hopf brace in C, we obtain that the morphisms Γ_{H_1} and Γ'_{H_1} are coalgebra morphisms. Indeed, first note that it is easy to show that $\varepsilon_H \circ \Gamma_{H_1} = \varepsilon_H \otimes \varepsilon_H$. Moreover,*

$$\delta_H \circ \Gamma_{H_1}$$

$$= \mu_{H_1 \otimes H_1} \circ (((\lambda_H^1 \otimes \lambda_H^1) \circ c_{H,H} \circ \delta_H) \otimes (\mu_{H_2 \otimes H_2} \circ (\delta_H \otimes \delta_H))) \circ (\delta_H \otimes H) \text{ (by the condition of coalgebra morphisms for } \mu_H^1 \text{ and } \mu_H^2 \text{ and (2))}$$

$$= (\Gamma_{H_1} \otimes \Gamma_{H_1}) \circ \delta_{H \otimes H} \text{ (by the naturality of } c \text{ and the cocommutativity and coassociativity conditions)}$$

On the other hand, as in the case of Γ_{H_1}, trivially $\varepsilon_H \circ \Gamma'_{H_1} = \varepsilon_H \otimes \varepsilon_H$. Finally,

$$\delta_H \circ \Gamma'_{H_1}$$

$$= \mu_{H_1 \otimes H_1} \circ ((\mu_{H_2 \otimes H_2} \circ (\delta_H \otimes \delta_H)) \otimes ((\lambda_H^1 \otimes \lambda_H^1) \circ \delta_H)) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) \text{ (by the condition of coalgebra morphisms for } \mu_H^1 \text{ and } \mu_H^2, \text{ (2) and cocommutativity of } \delta_H)$$

$$= ((\mu_H^1 \circ (\mu_H^2 \otimes \lambda_H^1)) \otimes (\mu_H^1 \circ (\mu_H^2 \otimes \lambda_H^1))) \circ (H \otimes H \otimes c_{H,H} \otimes H \otimes H) \circ (H \otimes c_{H,H} \otimes (c_{H,H} \circ c_{H,H}) \otimes H)$$
$$\circ (\delta_H \otimes c_{H,H} \otimes c_{H,H}) \circ (H \otimes \delta_{H \otimes H}) \circ (\delta_H \otimes H) \text{ (by the naturality of } c)$$

$$= ((\mu_H^1 \circ (\mu_H^2 \otimes \lambda_H^1)) \otimes (\mu_H^1 \circ (\mu_H^2 \otimes \lambda_H^1))) \circ (H \otimes H \otimes c_{H,H} \otimes H \otimes H) \circ (H \otimes c_{H,H} \otimes H \otimes H \otimes H)$$
$$\circ (\delta_H \otimes c_{H,H} \otimes c_{H,H}) \circ (H \otimes \delta_{H \otimes H}) \circ (\delta_H \otimes H) \text{ (by (6))}$$

$$= ((\mu_H^1 \circ (\mu_H^2 \otimes \lambda_H^1)) \otimes (\mu_H^1 \circ (\mu_H^2 \otimes \lambda_H^1))) \circ (H \otimes ((c_{H,H} \otimes H) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \circ \delta_H) \otimes H)) \otimes c_{H,H})$$
$$\circ (\delta_H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) \text{ (by the naturality of } c \text{ and the coassociativity condition)}$$

$$= (\Gamma'_{H_1} \otimes \Gamma'_{H_1}) \circ \delta_{H \otimes H} \text{ (by the naturality of } c \text{ and the cocommutativity and coassociativity conditions)}$$

As was proved in [3, Theorem 1.12], the category of invertible 1-cocycles associated to a fixed Hopf algebra is equivalent to the category of Hopf braces where the first Hopf algebra structure is the same one fixed before. This categorical equivalence remains valid for general invertible 1-cocycles and Hopf braces in braided monoidal categories.

Theorem 2 *The categories* IC *and* HBr *are equivalent.*

Proof The proof follows as in [3]. In the following lines, we give a brief summary of this proof to introduce some notation and for the convenience of the reader.

Let $\mathbb{H}$ be an object in HBr. Then, $id_H : H_2 \to H_1$ is an invertible 1-cocycle. Also, if $\mathbb{H}$ and $\mathbb{H}'$ are objects in HBr and $x : \mathbb{H} \to \mathbb{H}'$ is a morphism between them, the pair (x, x) is a morphism in IC between $id_H : H_2 \to H_1$ and $id_{H'} : H'_2 \to H'_1$. Therefore, there exists a functor $\mathsf{E} : \mathsf{HBr} \to \mathsf{IC}$ defined on objects by

$$\mathsf{E}(\mathbb{H}) = \begin{array}{c} H_2 \\ id_H \downarrow \\ H_1 \end{array},$$

where $\Phi_{H_1} = \Gamma_{H_1}$, and on morphisms by $\mathsf{E}(x) = (x, x)$.

Conversely, let $\pi : A \to H$ be an object in IC. Define $\mu_{H_\pi} := \pi \circ \mu_A \circ (\pi^{-1} \otimes \pi^{-1})$, $\eta_{H_\pi} := \eta_H$ and $\lambda_{H_\pi} = \pi \circ \lambda_A \circ \pi^{-1}$. Then, if we denote by H_π the algebra $(H, \eta_{H_\pi}, \mu_{H_\pi})$,

$$(H, \eta_H, \mu_H, \eta_{H_\pi}, \mu_{H_\pi}, \varepsilon_H, \delta_H, \lambda_H, \lambda_{H_\pi})$$

is an object in HBr that we will denote by $\mathbb{H}_\pi = (H, H_\pi)$.

Moreover, if (f, g) is a morphism in IC between $\pi : A \to H$ and $\pi' : A' \to H'$, the morphism g is a morphism in HBr between $\mathbb{H}_\pi$ and $\mathbb{H}'_{\pi'}$. As a consequence of these facts, we have a functor $\mathsf{Q} :\ \mathsf{IC} \to \mathsf{HBr}$ defined by

$$\mathsf{Q}\left(\begin{array}{c} A \\ \pi \downarrow \\ H \end{array} \right) = \mathbb{H}_\pi$$

on objects and by $\mathsf{Q}((f, g)) = g$ on morphisms.

The functors induce an equivalence between the two categories because, clearly, $\mathsf{QE} = \mathsf{id}_{\mathsf{HBr}}$ and, on the other hand, $\mathsf{EQ} \backsimeq \mathsf{id}_{\mathsf{IC}}$ because, if $\Gamma_H = \mu_H \circ (\lambda_H \otimes \mu_{H_\pi}) \circ (\delta_H \otimes H)$,

$$\Phi_H = \Gamma_H \circ (\pi \otimes H) \tag{10}$$

holds and

$$(\pi, id_H) : \begin{array}{c} A \\ \pi \downarrow \\ H \end{array} \longrightarrow \begin{array}{c} H_\pi \\ id_H \downarrow \\ H \end{array} = \mathsf{EQ}\left(\begin{array}{c} A \\ \pi \downarrow \\ H \end{array} \right)$$

is an isomorphism in IC. □

Lemma 1 *Let $\pi : A \to H$ be an object in IC with action Φ_H. Then*

$$\Phi_H \circ (A \otimes (\lambda_H \circ \pi))$$

$$= \mu_H \circ ((\lambda_H \circ \mu_H) \otimes H) \circ (\pi \otimes \Phi_H \otimes \pi) \circ (\delta_A \otimes H \otimes A) \circ (A \otimes c_{A,H}) \circ (\delta_A \otimes \pi).$$

Proof The proof is the following:

$$\mu_H \circ ((\lambda_H \circ \mu_H) \otimes H) \circ (\pi \otimes \Phi_H \otimes \pi) \circ (\delta_A \otimes H \otimes A) \circ (A \otimes c_{A,H}) \circ (\delta_A \otimes \pi)$$

$= \mu_H \circ ((\lambda_H \circ \pi \circ \mu_A) \otimes \pi) \circ (A \otimes c_{A,A}) \circ (\delta_A \otimes A)$ (by naturality of c and (3))

$= \mu_H \circ ((\lambda_H \circ \mu_H^\pi) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \pi) \otimes \pi)$ (by naturality of c, the condition of coalgebra isomorphism for π and the definition of μ_H^π)

$= \Gamma_H \circ (\pi \otimes (\lambda_H \circ \pi))$ (by (7) for $\mathbb{H}_\pi$)

$= \Phi_H \circ (A \otimes (\lambda_H \circ \pi))$ (by (10)).

□

Theorem 3 *Let A and H be Hopf algebras in C, let $\pi : A \to H$ be an isomorphism of coalgebras such that $\pi \circ \eta_A = \eta_H$, and let us assume that H is a left A-module algebra with action $\Phi_H : A \otimes H \to H$. Then the following are equivalent:*

(i) The morphism $\pi : A \to H$ is an invertible 1-cocycle.

(ii) The pair (H, Φ'_H) is a left A-module algebra, where

$$\Phi'_H = \mu_H \circ (\mu_H \otimes H) \circ (\pi \otimes \Phi_H \otimes (\lambda_H \circ \pi)) \circ (\delta_A \otimes c_{A,H}) \circ (\delta_A \otimes H),$$

and moreover

$$\pi \circ \mu_A = \mu_H \circ (\Phi'_H \otimes \pi) \circ (A \otimes c_{A,H}) \circ (\delta_A \otimes \pi) \tag{11}$$

holds.

Proof First, we will prove that (i) ⇒ (ii). Indeed, let $\mathbb{H}_\pi$ be the Hopf brace introduced in the proof of Theorem 2. Then (H, Γ_H) is a left H_π-module algebra and (10) holds. Then, if

$$\Phi'_H = \mu_H \circ (\mu_H \otimes H) \circ (\pi \otimes \Phi_H \otimes (\lambda_H \circ \pi)) \circ (\delta_A \otimes c_{A,H}) \circ (\delta_A \otimes H),$$

we have that

$$\Phi'_H = \Gamma'_H \circ (\pi \otimes H) \tag{12}$$

holds because

Φ'_H

$= \mu_H \circ ((\mu_H \circ (H \otimes (\Phi_H \circ (\pi^{-1} \otimes H))) \circ (\delta_H \otimes H)) \otimes \lambda_H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \pi) \otimes H)$ (by the condition of coalgebra isomorphism for π and the naturality of c)

$= \mu_H \circ ((\mu_H \circ (H \otimes \Gamma_H) \circ (\delta_H \otimes H)) \otimes \lambda_H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \pi) \otimes H)$ (by (10))

$= \mu_H \circ (\mu_{H_\pi} \otimes \lambda_H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \pi) \otimes H)$ (by (8))

$= \Gamma'_H \circ (\pi \otimes H)$ (by definition of Γ'_H).

Then, as a consequence of (12), (H, Φ'_H) is a left A-module algebra because (H, Γ'_H) is a left H_π-module algebra. Finally,

$\mu_H \circ (\Phi'_H \otimes \pi) \circ (A \otimes c_{A,H}) \circ (\delta_A \otimes \pi)$

$= \mu_H \circ (\Gamma'_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \pi) \otimes \pi)$ (by (12), the naturality of c and the condition of coalgebra morphism for π)

$= \mu_{H_\pi} \circ (\pi \otimes \pi)$ (by (9))

$= \pi \circ \mu_A$ (by definition of μ_{H_π})

and then (11) holds.

Conversely, to prove that (ii) $\Rightarrow$ (i), we only need to show that (3) holds. Indeed:

$\pi \circ \mu_A$

$= \mu_H \circ (\Phi'_H \otimes \pi) \circ (A \otimes c_{A,H}) \circ (\delta_A \otimes \pi)$ (by (11))

$= \mu_H \circ ((\mu_H \circ (\mu_H \otimes H) \circ (\pi \otimes \Phi_H \otimes (\lambda_H \circ \pi)) \circ (\delta_A \otimes c_{A,H}) \circ (\delta_A \otimes H)) \otimes \pi) \circ (A \otimes c_{A,H}) \circ (\delta_A \otimes \pi)$

(by the definition of Φ'_H)

$= \mu_H \circ ((\mu_H \circ (\pi \otimes \Phi_H) \circ (\delta_A \otimes H)) \otimes ((\lambda_H * id_H) \circ \pi)) \circ (A \otimes c_{A,H}) \circ (\delta_A \otimes \pi)$ (by the associativity of μ_H, the naturality of c, the coassociativity of δ_A and the condition of coalgebra morphism for π)

$= \mu_H \circ (\pi \otimes \Phi_H) \circ (\delta_A \otimes \pi)$ (by (1), naturality of c and the unit and counit properties).

□

Remark 3 *Observe that in the previous theorem, we can recover Φ_H from Φ'_H as*

$$\Phi_H = \mu_H \circ (\mu_H \otimes H) \circ ((\lambda_H \circ \pi) \otimes \Phi'_H \otimes \pi) \circ (\delta_A \otimes c_{A,H}) \circ (\delta_A \otimes H),$$

and Φ_H induces a left A-module algebra structure on H if, and only if, Φ'_H does. As a consequence, we can define an invertible 1-cocycle as a coalgebra isomorphism satisfying condition (11), where (H, Φ'_H) is an A-module algebra.

Lemma 2 *Let $\pi : A \to H$ be an object in IC with action Φ_H and such that H is cocommutative. Then Φ_H is a coalgebra morphism.*

Proof The proof follows directly from the equality (10) and Remark 2. □

Lemma 3 *Let $\pi : A \to H$ be an object in IC with action Φ_H and such that H is cocommutative. Then the action Φ'_H defined in Theorem 3 is a coalgebra morphism.*

Proof The proof follows directly from the equality (12) and Remark 2. □

Remark 4 *Let $\mathsf{H} : \mathsf{Hopf} \to \mathsf{IC}$ be the functor defined in Remark 1. If A is a Hopf algebra, its image by H is the invertible 1-cocycle $id_A : A \to A$ where the action is the trivial one, i.e., $\Phi_A = t_A = \varepsilon_A \otimes A$ and then Φ_A is a coalgebra morphism. Also, by (ii) of Theorem 3, we have that*

$$\Phi'_A = \mu_A \circ (\mu_A \otimes \lambda_A) \circ (A \otimes c_{A,A}) \circ (\delta_A \otimes A) = \varphi_A^{ad}.$$

Taking into account the previous considerations, we can introduce the notion of left module for an invertible 1-cocycle.

Definition 10 *Let $\pi : A \to H$ be an invertible 1-cocycle. A left module over the invertible 1-cocycle $\pi : A \to H$ is a 6-tuple $(M, N, \Phi_M, \varphi_M, \Phi_N, \gamma)$ where*

(i) (M, Φ_M) is a left A-module and (M, φ_M) is a left H-module such that

$$\Phi_M \circ (A \otimes \varphi_M) = \varphi_M \circ (\Phi_H \otimes \Phi_M) \circ (A \otimes c_{A,H} \otimes M) \circ (\delta_A \otimes H \otimes M). \tag{13}$$

(ii) (N, Φ_N) is a left A-module.

(iii) $\gamma : N \to M$ is an isomorphism in C such that

$$\gamma \circ \Phi_N = \varphi_M \circ (\pi \otimes \Phi_M) \circ (\delta_A \otimes \gamma). \tag{14}$$

Let $(M, N, \Phi_M, \varphi_M, \Phi_N, \gamma)$ and $(M', N', \Phi_{M'}, \varphi_{M'}, \Phi_{N'}, \gamma')$ be left modules over an invertible 1-cocycle $\pi : A \to H$. A morphism between them is a pair (h, l) such that $h : M \to M'$ is a morphism of left A-modules and left H-modules, $l : N \to N'$ is a morphism of left A-modules and the following identity holds:

$$h \circ \gamma = \gamma' \circ l. \tag{15}$$

Note that, by (15), the morphism l is determined by h because $l = (\gamma')^{-1} \circ h \circ \gamma$.

With the obvious composition of morphisms, left modules over an invertible 1-cocycle $\pi : A \to H$ with action Φ_H form a category that we will denote by ${}_{(\pi,\Phi_H)}\mathsf{Mod}$.

Remark 5 *If $(M, N, \Phi_M, \varphi_M, \Phi_N, \gamma)$ is a left module over the invertible 1-cocycle $\pi : A \to H$, by (14), we obtain that Φ_N is determined by Φ_M and φ_M because*

$$\Phi_N = \gamma^{-1} \circ \varphi_M \circ (\pi \otimes \Phi_M) \circ (\delta_A \otimes \gamma). \tag{16}$$

Also, composing in both sides of the equality (14) with $(((\lambda_H \circ \pi) \otimes A) \circ \delta_A) \otimes \gamma^{-1}$ on the right and with φ_M on the left we obtain the identity

$$\Phi_M = \varphi_M \circ ((\lambda_H \circ \pi) \otimes (\gamma \circ \Phi_N)) \circ (\delta_A \otimes \gamma^{-1}). \tag{17}$$

Example 1 *It is easy to see that if $\pi : A \to H$ is an invertible 1-cocycle, the 6-tuple $(H, A, \Phi_H, \mu_H, \mu_A, \pi)$ is an example of left module over the invertible cocycle $\pi : A \to H$.*

Also, the unit object K of C is an example of left module over the invertible 1-cocycle $\pi : A \to H$, where $\Phi_K = \varepsilon_A$, $\varphi_K = \varepsilon_H$ and $\gamma = id_K$ because by (10) we have that $\varepsilon_H \circ \Phi_H = \varepsilon_A \otimes \varepsilon_H$. We call $(K, K, \Phi_K, \varphi_K, \Phi_K, id_K)$ the trivial left module over the invertible 1-cocycle $\pi : A \to H$.

Note that, if (M, Φ_M) is an object in ${}_{\mathsf{H}}\mathsf{Mod}$, the 6-tuple $(M, M, \Phi_M, t_M = \varepsilon_H \otimes M, \Phi_M, id_M)$ is a left module over the invertible 1-cocycle $id_H : H \to H$ defined in Remark 1. Also, if f is a morphism between two left H-modules (M, Φ_M) and (P, Φ_P), the pair (f, f) is a morphism of left modules over the invertible 1-cocycle $id_H : H \to H$ between $(M, M, \Phi_M, t_M, \Phi_M, id_M)$ and $(P, P, \Phi_P, t_P, \Phi_P, id_P)$. Therefore, we have a functor

$$\mathsf{I}_H : {}_{\mathsf{H}}\mathsf{Mod} \to {}_{(id_H, t_H)}\mathsf{Mod}$$

defined on objects by

$$\mathsf{I}_H((M,\Phi_M)) = (M, M, \Phi_M, t_M, \Phi_M, id_M)$$

and on morphisms by $\mathsf{I}_H(f) = (f, f)$.

Theorem 4 *Assume that* (f,g) *is a morphism between the invertible 1-cocycles* $\pi : A \to H$ *and* $\tau : B \to D$. *Then, there exists a functor*

$$\mathsf{M}_{(f,g)} : \ {}_{(\tau,\Phi_D)}\mathsf{Mod} \ \to \ {}_{(\pi,\Phi_H)}\mathsf{Mod}$$

defined on objects by

$$\mathsf{M}_{(f,g)}((P,Q,\Phi_P,\varphi_P,\Phi_Q,\theta)) = (P,Q,\Phi_P^{\pi} = \Phi_P \circ (f \otimes P), \varphi_P^{\pi} = \varphi_P \circ (g \otimes P), \Phi_Q^{\pi} = \Phi_Q \circ (f \otimes Q), \theta)$$

and on morphisms by the identity.

Proof The existence of the functor $\mathsf{M}_{(f,g)}$ is a consequence of the following facts: Trivially (P,Φ_P^{π}), (Q,Φ_Q^{π}) are left A-modules and (P,φ_P^{π}) is a left H-module. Also,

$$\varphi_P^{\pi} \circ (\Phi_H \otimes \Phi_P^{\pi}) \circ (A \otimes c_{A,H} \otimes P) \circ (\delta_A \otimes H \otimes P)$$

$$= \varphi_P \circ ((g \circ \Phi_H) \otimes (\Phi_P \circ (f \otimes P))) \circ (A \otimes c_{A,H} \otimes P) \circ (\delta_A \otimes H \otimes P) \ \text{(by definition of } \Phi_P^{\pi} \text{ and } \varphi_P^{\pi})$$

$$= \varphi_P \circ (\Phi_D \otimes \Phi_P) \circ (B \otimes c_{B,D} \otimes P) \circ (((f \otimes f) \circ \delta_A) \otimes g \otimes P) \ \text{(by (5) and natirality of } c)$$

$$= \varphi_P \circ (\Phi_D \otimes \Phi_P) \circ (B \otimes c_{B,D} \otimes P) \circ ((\delta_B \circ f) \otimes g \otimes P) \ \text{(by the coalgebra morphism condition for } f)$$

$$= \Phi_P^{\pi} \circ (A \otimes \varphi_P^{\pi}) \text{(by (13))},$$

and

$$\varphi_P^{\pi} \circ (\pi \otimes \Phi_P^{\pi}) \circ (\delta_A \otimes \theta)$$

$$= \varphi_P \circ ((g \circ \pi) \otimes (\Phi_P \circ (f \otimes P))) \circ (\delta_A \otimes \theta) \ \text{(by definition of } \Phi_P^{\pi} \text{ and } \varphi_P^{\pi})$$

$$= \varphi_P \circ ((\tau \circ f) \otimes (\Phi_P \circ (f \otimes P))) \circ (\delta_A \otimes \theta) \ \text{(by (4))}$$

$$= \varphi_P \circ (\tau \otimes \Phi_P) \circ ((\delta_B \circ f) \otimes \theta) \ \text{(by the coalgebra morphism condition for } f)$$

$$= \theta \circ \Phi_Q^{\pi} \ \text{(by (14))}$$

Then $(P,Q,\Phi_P^{\pi},\varphi_P^{\pi},\Phi_Q^{\pi},\theta)$ is an object in ${}_{(\pi,\Phi_H)}\mathsf{Mod}$. Finally, it is obvious that if (h,l) is a morphism in ${}_{(\tau,\Phi_D)}\mathsf{Mod}$, (h,l) is a morphism in ${}_{(\pi,\Phi_H)}\mathsf{Mod}$.

□

Remark 6 *Let $f : H \to H'$ be a Hopf algebra morphisms. Then, by Example 1 and Theorem 4, we have the following commutative diagram*

$$\begin{array}{ccc} {}_{H'}\mathsf{Mod} & \xrightarrow{\ \mathsf{I}_{H'}\ } & {}_{(id_{H'},t_{H'})}\mathsf{Mod} \\ \mathsf{M}_f \downarrow & & \downarrow \mathsf{M}_{(f,f)} \\ {}_{H}\mathsf{Mod} & \xrightarrow[\ \mathsf{I}_{H}\ ]{} & {}_{(id_{H},t_{H})}\mathsf{Mod} \end{array}$$

where M_f is the restriction of scalars functor.

Remark 7 *If (f,g) is an isomorphism defined between the invertible 1-cocycles $\pi : A \to H$ and $\tau : B \to D$ with inverse (f^{-1}, g^{-1}), the functor $\mathsf{M}_{(f,g)}$ is an isomorphism of categories with inverse $\mathsf{M}_{(f^{-1},g^{-1})}$. For example, in the proof of Theorem 2, we proved that, for all invertible 1-cocycle $\pi : A \to H$, (π, id_H) is an isomorphism between the invertible 1-cocycles $\pi : A \to H$ and $id_H : H_\pi \to H$. Therefore, the functor*

$$\mathsf{M}_{(\pi,id_H)} : {}_{(id_H,\Gamma_H)}\mathsf{Mod} \to {}_{(\pi,\Phi_H)}\mathsf{Mod}$$

is an isomorphism of categories with inverse

$$\mathsf{M}_{(\pi^{-1},id_H)} : {}_{(\pi,\Phi_H)}\mathsf{Mod} \to {}_{(id_H,\Gamma_H)}\mathsf{Mod}.$$

Theorem 5 *Let us assume that C is symmetric with natural isomorphism of symmetry c. Let A and H be cocommutative Hopf algebras in C. Then the category of left modules over an invertible 1-cocycle $\pi : A \to H$ is symmetric monoidal with unit object the trivial left module over the invertible 1-cocycle $\pi : A \to H$.*

Proof Let $(M, N, \Phi_M, \varphi_M, \Phi_N, \gamma)$, $(P, Q, \Phi_P, \varphi_P, \Phi_Q, \theta)$ be objects in ${}_{(\pi,\Phi_H)}\mathsf{Mod}$. Then we will define their tensor product as

$$(M, N, \Phi_M, \varphi_M, \Phi_N, \gamma) \otimes (P, Q, \Phi_P, \varphi_P, \Phi_Q, \theta)$$

$$= (M \otimes P, N \otimes Q, \Phi_{M\otimes P}, \varphi_{M\otimes P}, \Phi_{N\otimes Q}, \gamma \otimes \theta)$$

where the left A-actions are defined by $\Phi_{M\otimes P} = (\Phi_M \otimes \Phi_P) \circ (A \otimes c_{A,M} \otimes P) \circ (\delta_A \otimes M \otimes P)$, $\Phi_{N\otimes Q} = (\Phi_N \otimes \Phi_Q) \circ (A \otimes c_{A,N} \otimes Q) \circ (\delta_A \otimes N \otimes Q)$ and the left H-action is $\varphi_{M\otimes P} = (\varphi_M \otimes \varphi_P) \circ (H \otimes c_{H,M} \otimes P) \circ (\delta_H \otimes M \otimes P)$. By the monoidal property of the category of modules over a Hopf algebra, we have that $(M \otimes P, \Phi_{M\otimes P})$ and $(N \otimes Q, \Phi_{N\otimes Q})$ are left A-modules and $(M \otimes P, \varphi_{M\otimes P})$ is a left H-module. Moreover, the equality (13) holds because

$$\varphi_{M\otimes P} \circ (\Phi_H \otimes \Phi_{M\otimes P}) \circ (A \otimes c_{A,H} \otimes M \otimes P) \circ (\delta_A \otimes H \otimes M \otimes P)$$

$$= (\varphi_M \otimes \varphi_P) \circ (H \otimes c_{H,M} \otimes P) \circ ((\delta_H \circ \Phi_H) \otimes \Phi_{M\otimes P}) \circ (A \otimes c_{A,H} \otimes M \otimes P) \circ (\delta_A \otimes H \otimes M \otimes P)$$

(by definition)

$$= ((\varphi_M \circ (H \otimes \Phi_M)) \otimes \varphi_P) \circ (H \otimes ((A \otimes c_{H,M}) \circ (c_{H,A} \otimes M)) \otimes A \otimes P) \circ (((\Phi_H \otimes \Phi_H) \circ \delta_{A\otimes H})$$

$\otimes A \otimes M \otimes \Phi_P) \circ (A \otimes ((c_{A,H} \otimes c_{A,M}) \circ (A \otimes c_{A,H} \otimes M)) \otimes P) \circ (((A \otimes \delta_A) \circ \delta_A) \otimes H \otimes M \otimes P)$

(by the naturality of c and the condition of coalgebra morphism for Φ_H (see Lemma 2))

$= ((\varphi_M \circ (\Phi_H \otimes \Phi_M)) \otimes (\varphi_P \circ (\Phi_H \otimes \Phi_P))) \circ (A \otimes ((H \otimes A \otimes c_{A,M} \otimes H \otimes A) \circ (H \otimes c_{A,A} \otimes c_{H,M} \otimes A)$

$\circ (c_{A,H} \otimes A \otimes H \otimes c_{A,M})) \otimes P) \circ (\delta_A \otimes ((c_{A,H} \otimes c_{A,H}) \circ \delta_{A\otimes H})) \otimes M \otimes P) \circ (\delta_A \otimes H \otimes M \otimes P)$

(by the naturality of c and $c_{H,A} \circ c_{A,H} = id_{A\otimes H}$)

$= ((\varphi_M \circ (\Phi_H \otimes \Phi_M)) \otimes (\varphi_P \circ (\Phi_H \otimes \Phi_P))) \circ (A \otimes ((c_{A,H} \otimes c_{A,M} \otimes H \otimes A) \circ (A \otimes c_{A,H} \otimes c_{H,M} \otimes A)$

$\circ ((c_{A,A} \circ \delta_A) \otimes H \otimes H \otimes c_{A,M}) \circ (A \otimes H \otimes c_{A,H} \otimes M) \circ (\delta_{A\otimes H} \otimes M)) \otimes P) \circ (\delta_A \otimes H \otimes M \otimes P)$ (by the coassociativity of δ_A and the naturality of c)

$= ((\varphi_M \circ (\Phi_H \otimes \Phi_M)) \otimes (\varphi_P \circ (\Phi_H \otimes \Phi_P))) \circ (A \otimes ((c_{A,H} \otimes c_{A,M} \otimes c_{A,H}) \circ (A \otimes c_{A,H} \otimes c_{A,M} \otimes H)$

$\circ (A \otimes A \otimes c_{A,H} \otimes c_{H,M})) \otimes P) \circ (((\delta_A \otimes \delta_A) \circ \delta_A) \otimes \delta_H \otimes M \otimes P)$ (by the cocommutativity, the coassociativity of δ_A and the naturality of c)

$= ((\varphi_M \circ (\Phi_H \otimes \Phi_M) \circ (A \otimes c_{A,H} \otimes M) \circ (\delta_A \otimes H \otimes M)) \otimes (\varphi_P \circ (\Phi_H \otimes \Phi_P) \circ (A \otimes c_{A,H} \otimes P)$

$\circ (\delta_A \otimes H \otimes P))) \circ (A \otimes H \otimes c_{A,M} \otimes H \otimes P) \circ (A \otimes c_{A,H} \otimes c_{H,M} \otimes P) \circ (\delta_A \otimes \delta_H \otimes M \otimes P)$

(by the naturality of c)

$= ((\Phi_M \circ (A \otimes \varphi_M)) \otimes (\Phi_P \circ (A \otimes \varphi_P)) \circ (A \otimes H \otimes c_{A,M} \otimes H \otimes P) \circ (A \otimes c_{A,H} \otimes c_{H,M} \otimes P)$

$\circ (\delta_A \otimes \delta_H \otimes M \otimes P)$ (by (13))

$= \Phi_{M\otimes P} \circ (A \otimes \varphi_{M\otimes P})$ (by the naturality of c)

and, on the other hand, (14) follows by

$\varphi_{M\otimes P} \circ (\pi \otimes \Phi_{M\otimes P}) \circ (\delta_A \otimes \gamma \otimes \theta)$

$= ((\varphi_M \circ (\pi \otimes (\Phi_M \circ (A \otimes \gamma)))) \otimes (\varphi_P \circ (\pi \otimes (\Phi_P \circ (A \otimes \theta))))) \circ (A \otimes ((A \otimes c_{A,N}) \circ ((c_{A,A} \circ \delta_A)$

$\otimes N)) \otimes A \otimes Q) \circ (\delta_A \otimes c_{A,N} \otimes Q) \circ (\delta_A \otimes N \otimes Q)$ (by the coalgebra morphism condition for π, the coassociativity of δ_A and the naturality of c)

$= ((\varphi_M \circ (\pi \otimes (\Phi_M \circ (A \otimes \gamma)))) \otimes (\varphi_P \circ (\pi \otimes (\Phi_P \circ (A \otimes \theta))))) \circ (\delta_A \otimes ((c_{A,N} \otimes A) \circ (A \otimes c_{A,N})$

$\circ (\delta_A \otimes N)) \otimes Q) \circ (\delta_A \otimes N \otimes Q)$ (by the cocommutativity and the coassociativity of δ_A)

$= ((\varphi_M \circ (\pi \otimes (\Phi_M \circ (A \otimes \gamma)) \circ (\delta_A \otimes N))) \otimes (\varphi_P \circ (\pi \otimes (\Phi_P \circ (A \otimes \theta)) \circ (\delta_A \otimes Q))))$

$\circ (A \otimes c_{A,N} \otimes Q) \circ (\delta_A \otimes N \otimes Q)$ (by the the naturality of c)

$= (\gamma \otimes \theta) \circ \Phi_{N\otimes Q}$ (by (14)).

Finally, by the cocommutativity condition and the symmetry condition, it is easy to prove that $(c_{M,P}, c_{N,Q})$ is a morphism in ${}_{(\pi,\Phi_H)}\mathsf{Mod}$ between $(M, N, \Phi_M, \varphi_M, \Phi_N, \gamma)\otimes(P, Q, \Phi_P, \varphi_P, \Phi_Q, \theta)$ and $(P, Q, \Phi_P, \varphi_P, \Phi_Q, \theta)\otimes(M, N, \Phi_M, \varphi_M, \Phi_N, \gamma)$. As a consequence, ${}_{(\pi,\Phi_H)}\mathsf{Mod}$ is symmetric. □

Following [5], we recall the notion of left module over a Hopf brace.

Definition 11 Let $\mathbb{H}$ be a Hopf brace. A left $\mathbb{H}$-module is a triple (M, ψ_M^1, ψ_M^2), where (M, ψ_M^1) is a left H_1-module, (M, ψ_M^2) is a left H_2-module and the following identity

$$\psi_M^2 \circ (H \otimes \psi_M^1) = \psi_M^1 \circ (\mu_H^2 \otimes \Gamma_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes H \otimes M) \tag{18}$$

holds, where

$$\Gamma_M = \psi_M^1 \circ (\lambda_H^1 \otimes \psi_M^2) \circ (\delta_H \otimes M).$$

Given two left $\mathbb{H}$-modules (M, ψ_M^1, ψ_M^2) and (N, ψ_N^1, ψ_N^2), a morphism $f : M \to N$ is called a morphism of left $\mathbb{H}$-modules if f is a morphism of left H_1-modules and left H_2-modules. Left $\mathbb{H}$-modules with morphisms of left $\mathbb{H}$-modules form a category which we denote by ${}_{\mathbb{H}}\mathsf{Mod}$.

Example 2 *Let $\mathbb{H}$ be a Hopf brace. The triple (H, μ_H^1, μ_H^2) is an example of left $\mathbb{H}$-module. Also, if K is the unit object of C, $(K, \psi_K^1 = \varepsilon_H, \psi_K^2 = \varepsilon_H)$ is a left $\mathbb{H}$-module called the trivial module.*

Let $H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda_H)$ be a Hopf algebra. Then (H, μ_H, μ_H) is an example of left $\mathbb{H}$-module for the Hopf brace $\mathbb{H}$ with $H_1 = H_2 = H$. Also, if (M, ψ_M) is a left H-module, the triple (M, ψ_M, ψ_M) is a left $\mathbb{H}$-module for the same Hopf brace. Then, there exists an obvious functor $J :_{\mathsf{H}}\mathsf{Mod} \to {}_{\mathbb{H}}\mathsf{Mod}$ defined on objects by $J((M, \psi_M)) = (M, \psi_M, \psi_M)$ and by the identity on morphisms. Also, there exists a functor $L :_{\mathbb{H}}\mathsf{Mod} \to {}_{\mathsf{H}}\mathsf{Mod}$ defined on objects by $L((M, \psi_M^1, \psi_M^2)) = (M, \psi_M^1)$ and by the identity on morphisms. Obviously, $L \circ J = \mathsf{id}_{{}_H\mathsf{Mod}}$.

Remark 8 *As was pointed in [5], Definition 11 is weaker than the one introduced by H. Zhu in [12]. For this author, if $\mathbb{H}$ is a Hopf brace, a left $\mathbb{H}$-module is a triple (M, ψ_M^1, ψ_M^2), where (M, ψ_M^1) is a left H_1-module, (M, ψ_M^2) is a left H_2-module, and the equalities (18) and*

$$(\psi_M^2 \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M) = (\psi_M^1 \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes \Gamma_M) \circ (\delta_H \otimes M) \tag{19}$$

hold (see [12, Definition 3.1, Lemma 3.2]). Thus, for an arbitrary Hopf brace $\mathbb{H}$, a left $\mathbb{H}$-module in the sense of Zhu is a left $\mathbb{H}$-module in our sense. Moreover, if $\mathbb{H}$ is cocommutative, (19) hold for any left $\mathbb{H}$-module as in Definition 11. As a consequence, in the cocommutative setting, [12, Definition 3.1] and Definition 11 are equivalent. Moreover, if we use Definition 11, trivially, (H, μ_H^1, μ_H^2) is a left $\mathbb{H}$-module but, if we work with the definition introduced by Zhu, the condition of left $\mathbb{H}$-module for (H, μ_H^1, μ_H^2) implies that the following identity

$$(\Gamma_{H_1} \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) = (\Gamma_{H_1} \otimes H) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \circ \delta_H) \otimes H)$$

holds. Therefore, if C is symmetric, for example the category of vector spaces over a field $\mathbb{K}$, (H, Γ_{H_1}) is in the cocommutativity class of H (see [2] for the definition). In other words, under certain circumstances, for example, the lack of cocommutativity, the category of left modules over a Hopf brace introduced by Zhu may not contain the obvious object (H, μ_H^1, μ_H^2).

Remark 9 *It is easy to show that (18) is equivalent to*

$$\psi_M^2 \circ (H \otimes \psi_M^1) = \psi_M^1 \circ (\Gamma'_{H_1} \otimes \psi_M^2) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes H \otimes M).$$

and the following equality holds:

$$\Gamma_M \circ (H \otimes \psi_M^1) = \psi_M^1 \circ (\Gamma_{H_1} \otimes \Gamma_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes H \otimes M). \tag{20}$$

Moreover, (M, Γ_M) *is a left* H_2*-module because trivially* $\Gamma_M \circ (\eta_H \otimes M) = id_M$ *and, on the other hand,*

$$\Gamma_M \circ (H \otimes \Gamma_M)$$

$$= \Gamma_M \circ (H \otimes (\psi_M^1 \circ (\lambda_H^1 \otimes \psi_M^2) \circ (\delta_H \otimes M))) \text{ (by definition of } \Gamma_M\text{)}$$

$$= \psi_M^1 \circ (\Gamma_{H_1} \otimes \Gamma_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes ((\lambda_H^1 \otimes \psi_M^2) \circ (\delta_H \otimes M))) \text{ (by (20))}$$

$$= \psi_M^1 \circ ((\Gamma_{H_1} \circ (H \otimes \lambda_H^1)) \otimes \Gamma_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes ((H \otimes \psi_M^2) \circ (\delta_H \otimes M))) \text{ (by naturality of } c\text{)}$$

$$= \psi_M^1 \circ ((\mu_H^1 \circ ((\lambda_H^1 \circ \mu_H^2) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)) \otimes (\psi_M^1 \circ (\lambda_H^1 \otimes \psi_M^2) \circ (\delta_H \otimes M)))$$

$$\circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes ((H \otimes \psi_M^2) \circ (\delta_H \otimes M))) \text{ (by (7) and the definition of } \Gamma_M\text{)}$$

$$= \psi_M^1 \circ ((\mu_H^1 \circ ((\lambda_H^1 \circ \mu_H^2) \otimes (\mu_H^1 \circ (H \otimes \lambda_H^1))) \otimes (\psi_M^2 \circ (\mu_H^2 \otimes M))) \circ (H \otimes c_{H,H} \otimes \delta_H \otimes H \otimes M)$$

$$\circ (\delta_H \otimes c_{H,H} \otimes H \otimes M) \circ (\delta_H \otimes \delta_H \otimes M) \text{ (by the condition of left } H_1 \text{ and } H_2\text{-module for } M \text{ and the associativity of } \mu_H^1\text{)}$$

$$= \psi_M^1 \circ ((\mu_H^1 \circ ((\lambda_H^1 \circ \mu_H^2) \otimes (\mu_H^1 \circ (H \otimes \lambda_H^1))) \otimes (\psi_M^2 \circ (\mu_H^2 \otimes M))) \circ (((H \otimes c_{H,H} \otimes H \otimes H \otimes H)$$

$$\circ (\delta_H \otimes c_{H,H} \otimes H \otimes H) \circ (H \otimes \delta_{H\otimes H}) \circ (\delta_H \otimes H)) \otimes M) \text{ (by naturality of } c\text{)}$$

$$= \psi_M^1 \circ ((\mu_H^1 \circ ((\lambda_H^1 \circ \mu_H^2) \otimes (id_H * \lambda_H^1)) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)) \otimes (\psi_M^2 \circ (\mu_H^2 \otimes M))) \circ (\delta_{H\otimes H} \otimes M)$$

$$\text{(by naturality of } c \text{ and coassociativity of } \delta_H\text{)}$$

$$= \psi_M^1 \circ (\lambda_H^1 \otimes \psi_M^2) \circ (((\mu_H^2 \otimes \mu_H^2) \circ \delta_{H\otimes H}) \otimes M) \text{ (by (1) and unit and counit properties)}$$

$$= \Gamma_M \circ (\mu_H^2 \otimes M) \text{ (by the condition of coalgebra morphism for } \mu_H^2\text{)}$$

Theorem 6 *Let* $\mathbb{H}$ *be a Hopf brace and let* $\mathsf{E}(\mathbb{H})$ *be the invertible 1-cocycle induced by the functor* E *introduced in the proof of Theorem 2. There exists a functor*

$$\mathsf{G}_{\mathbb{H}} : {}_{\mathbb{H}}\mathsf{Mod} \to {}_{(id_H, \Gamma_{H_1})}\mathsf{Mod}$$

defined on objects by

$$\mathsf{G}_{\mathbb{H}}((M, \psi_M^1, \psi_M^2)) = (M, M, \widehat{\Phi}_M = \Gamma_M, \widehat{\varphi}_M = \psi_M^1, \overline{\Phi}_M = \psi_M^2, id_M)$$

and on morphisms by $\mathsf{G}_{\mathbb{H}}(f) = (f, f)$.

Proof By Remark 9, we know that $(M, \widehat{\Phi}_M = \Gamma_M)$ is a left H_2-module and, by assumption, $(M, \widehat{\varphi}_M = \psi^1_M)$ is a left H_1-module and $(M, \overline{\Phi}_M = \psi^2_M)$ is a left H_2-module. On the other hand, by (20) we have that

$$\widehat{\Phi}_M \circ (H \otimes \widehat{\varphi}_M) = \widehat{\varphi}_M \circ (\Gamma_{H_1} \otimes \widehat{\Phi}_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes H \otimes M)$$

and then, (13) holds. Also,

$$\widehat{\varphi}_M \circ (H \otimes \widehat{\Phi}_M) \circ (\delta_H \otimes M)$$

$= \psi^1_M \circ (H \otimes \Gamma_M) \circ (\delta_H \otimes M)$ (by definition of $\widehat{\varphi}_M$ and $\widehat{\Phi}_M$)

$= \psi^1_M \circ (H \otimes (\psi^1_M \circ (\lambda^1_H \otimes \psi^2_M) \circ (\delta_H \otimes M))) \circ (\delta_H \otimes M)$ (by definition of Γ_M)

$= \psi^1_M \circ ((id_H * \lambda^1_H) \otimes \psi^2_M) \circ (\delta_H \otimes M)$ (by the condition of left H_1-module of (M, ψ^1_M) and the coassociativity of δ_H)

$= \overline{\Phi}_M$ (by (1), the counit properties, the condition of left H_1-module of (M, ψ^1_M) and the definition of $\overline{\Phi}_M$).

Finally, it is easy to show that if f is a morphism in $_{\mathbb{H}}\mathsf{Mod}$ between the objects (M, ψ^1_M, ψ^2_M) and $(M', \psi^1_{M'}, \psi^2_{M'})$, the pair (f, f) is a morphism in $_{(id_H, \Gamma_{H_1})}\mathsf{Mod}$ between the objects $\mathsf{G}_{\mathbb{H}}((M, \psi^1_M, \psi^2_M))$ and $\mathsf{G}_{\mathbb{H}}((M', \psi^1_{M'}, \psi^2_{M'}))$. □

Theorem 7 *Let $\pi : A \to H$ be an invertible 1-cocycle. Then the categories $_{(\pi,\Phi_H)}\mathsf{Mod}$ and $_{\mathbb{H}_\pi}\mathsf{Mod}$ are equivalent.*

Proof First of all, we will prove that there exists a functor

$$\mathsf{H}^{\pi}_{\mathsf{br}} : \ _{(\pi,\Phi_H)}\mathsf{Mod} \ \to \ _{\mathbb{H}_\pi}\mathsf{Mod}$$

defined on objects by

$$\mathsf{H}^{\pi}_{\mathsf{br}}((M, N, \Phi_M, \varphi_M, \Phi_N, \gamma)) = (M, \overline{\psi}^1_M = \varphi_M, \overline{\psi}^2_M = \gamma \circ \Phi_N \circ (\pi^{-1} \otimes \gamma^{-1}))$$

and on morphisms by $\mathsf{H}^{\pi}_{\mathsf{br}}((h, l)) = h$. Indeed: By assumption, $(M, \overline{\psi}^1_M = \varphi_M)$ is a left H-module and, using the condition of left A-module of N, we obtain that $(M, \overline{\psi}^2_M = \gamma \circ \Phi_N \circ (\pi^{-1} \otimes \gamma^{-1}))$ is a left H_π-module. Also, by (17), we have that the identity

$$\Phi_M \circ (\pi^{-1} \otimes M) = \overline{\Gamma}_M \tag{21}$$

holds, where $\overline{\Gamma}_M = \overline{\psi}^1_M \circ (\lambda_H \otimes \overline{\psi}^2_M) \circ (\delta_H \otimes M)$. Then, (18) holds because:

$$\overline{\psi}^2_M \circ (H \otimes \overline{\psi}^1_M)$$

$= \varphi_M \circ (\pi \otimes \Phi_M) \circ ((\delta_A \circ \pi^{-1}) \otimes \varphi_M)$ (by (16))

$= \varphi_M \circ (\pi \otimes (\varphi_M \circ (\Phi_H \otimes \Phi_M) \circ (A \otimes c_{A,H} \otimes M) \circ (\delta_A \otimes H \otimes M))) \circ ((\delta_A \circ \pi^{-1}) \otimes H \otimes M)$ (by (13))

$= \varphi_M \circ ((\mu_H \circ (\pi \otimes \Phi_H) \circ (\delta_A \otimes \pi)) \otimes \Phi_M) \circ (A \otimes c_{A,A} \otimes M) \circ ((\delta_A \circ \pi^{-1}) \otimes \pi^{-1} \otimes M)$ (by the condition of left H-module for M, the coassociativity of δ_A, the naturality of c and the condition of isomorphism for π)

$= \varphi_M \circ ((\pi \circ \mu_A) \otimes \Phi_M) \circ (A \otimes c_{A,A} \otimes M) \circ ((\delta_A \circ \pi^{-1}) \otimes \pi^{-1} \otimes M)$ (by (3))

$= \varphi_M \circ (\mu_{H_\pi} \otimes (\Phi_M \circ (\pi^{-1} \otimes M))) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes H \otimes M)$ (by the condition of coalgebra isomorphism for π and the naturality of c)

$= \overline{\psi}^1_M \circ (\mu_{H_\pi} \otimes \overline{\Gamma}_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes H \otimes M)$ (by (21))

On the other hand, if (h,l) is a morphisms in ${}_{(\pi,\Phi_H)}\mathsf{Mod}$ between the objects $(M, N, \Phi_M, \varphi_M, \Phi_N, \gamma)$ and $(M', N', \Phi_{M'}, \varphi_{M'}, \Phi_{N'}, \gamma')$, we have that h is a morphism in ${}_{\mathbb{H}_\pi}\mathsf{Mod}$ between $(M, \overline{\psi}^1_M, \overline{\psi}^2_M)$ and $(M', \overline{\psi}^1_{M'}, \overline{\psi}^2_{M'})$ because, using that h is a morphism of left H-modules, we have $h \circ \overline{\psi}^1_M = \overline{\psi}^1_{M'} \circ (H \otimes h)$ and, by (15) and the condition of morphism of left A-modules for h, we have that $h \circ \overline{\psi}^2_M = \overline{\psi}^2_{M'} \circ (H \otimes h)$.

Taking into account the functors $\mathsf{H}^\pi_{\mathsf{br}}$, $\mathsf{G}_{\mathbb{H}_\pi}$ and $\mathsf{M}_{(\pi, id_H)}$, it is easy to show that

$$\mathsf{H}^\pi_{\mathsf{br}} \circ (\mathsf{M}_{(\pi,id_H)} \circ \mathsf{G}_{\mathbb{H}_\pi}) = \mathsf{id}_{\,{}_{\mathbb{H}_\pi}\mathsf{Mod}}$$

and

$$((\mathsf{M}_{(\pi,id_H)} \circ \mathsf{G}_{\mathbb{H}_\pi}) \circ \mathsf{H}^\pi_{\mathsf{br}})((M, N, \Phi_M, \varphi_M, \Phi_N, \gamma)) = (M, M, \Phi_M, \varphi_M, \overline{\Phi}^\pi_M = \gamma \circ \Phi_N \circ (A \otimes \gamma^{-1}), id_M)$$

hold. Then,

$$(\mathsf{M}_{(\pi,id_H)} \circ \mathsf{G}_{\mathbb{H}_\pi}) \circ \mathsf{H}^\pi_{\mathsf{br}} \backsimeq \mathsf{id}_{\,{}_{(\pi,\Phi_H)}\mathsf{Mod}}$$

because (id_M, γ) is an isomorphism in the category ${}_{(\pi,\Phi_H)}\mathsf{Mod}$ between the objects $(M, N, \Phi_M, \varphi_M, \Phi_N, \gamma)$ and $(M, M, \Phi_M, \varphi_M, \overline{\Phi}^\pi_M, id_M)$.

□

As a consequence of this result, we have the following corollary whose proof is an immediate consequence of the preceding theorems.

Corollary 1 *Let $\mathbb{H}$ be a Hopf brace. Then, the categories ${}_{(id_H,\Gamma_{H_1})}\mathsf{Mod}$ and ${}_{\mathbb{H}}\mathsf{Mod}$ are equivalent.*

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