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On the oscillation and asymptotic behavior of solutions of third order nonlinear differential equations with mixed nonlinear neutral terms

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Abstract: This paper is concerned with the oscillation and asymptotic behavior of solutions of third-order nonlinear neutral differential equations with a middle term and mixed nonlinear neutral terms in the case of the canonical operator. We establish several oscillation criteria that guarantee that all solutions are oscillatory or converge to zero. The given results are obtained by applying the comparison method, the Riccati transformation and the integral averaging technique. The results improve significantly and extend existing ones in the literature. Finally, illustrative examples are given.

Key words: Oscillation, third order differential equation, nonlinear neutral term, mixed neutral term, nonoscillation, canonical operator

1. Introduction

In the recent years, there has been increasing attention by scholars in studying the oscillatory behavior of solutions of third order differential and dynamic equations (see for example [3–5, 8–11, 13, 14, 17–19, 21, 22, 25–27]) due to its importance in real life, steam turbine regulation, neutral networks, and in the governing equations which describe the variation of hormones with time. Also, there are many advanced equations which used as an application or a direct representation of some problems that depend on the rate of change not only at the present but also in the future. The presence of an advanced argument describes the influence of potential future actions and usually appear in some phenomena like population dynamics and economical problems; see [7, 16]. Therefore, we study a third order differential equation which ensure the presence of an advanced term.

In this paper, we discuss the oscillatory and the asymptotic behavior of third-order nonlinear neutral differential equations with a middle term and mixed nonlinear neutral terms of the type,

$$\left(a(t)\left(z''(t)\right)^{\alpha}\right)' + b(t)\left(z''(t)\right)^{\alpha} + q(t)x^{\gamma}(\sigma(t)) + r(t)x^{\beta}(\zeta(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $z(t) = x^{\eta}(t) + p_1(t)x^{\lambda}(\tau_1(t)) + \delta p_2(t)x^{\nu}(\tau_2(t))$ and $\delta = \pm 1$. Throughout this article, we assume that the following hypotheses hold

(H$_1$) $\alpha, \eta, \lambda, \nu, \gamma$ and β are ratios of odd positive integers;

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(H$_2$) a, p_1, p_2, q and $r \in C([t_0,\infty),(0,\infty))$, $b \in C([t_0,\infty),[0,\infty))$ and (1.1) is in canonical form, i.e.

$$\int_{t_0}^{\infty} \frac{\exp\left(\frac{-1}{\alpha}\int_{t_0}^{t}\frac{b(s)}{a(s)}\mathrm{d}s\right)}{a^{\frac{1}{\alpha}}(t)}\mathrm{d}t = \infty; \tag{1.2}$$

(H$_3$) $\tau_1, \tau_2, \sigma, \zeta \in C([t_0,\infty),\mathbb{R})$ such that τ_1, τ_2 are strictly increasing, $\sigma(t) \le t$ and $\zeta(t) \ge t$ with $\lim_{t\to\infty}\tau_1(t) = \lim_{t\to\infty}\tau_2(t) = \lim_{t\to\infty}\sigma(t) = \infty$.

By a solution of Eq. (1.1), we mean a nontrivial function $x(t) \in C([T_x,\infty))$, $T_x \ge t_0$, which has the properties $z(t) \in C^2([T_x,\infty))$, $a(t)(z''(t))^\alpha \in C^1([T_x,\infty))$ and satisfies (1.1) on $[T_x,\infty)$. Our attention is restricted to those solutions $x(t)$ of (1.1) satisfying $\sup\{|x(t)| : t \ge T\} > 0$, for all $T \ge T_x$. We assume that (1.1) possesses such a solution. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is termed nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In 2019, Tunç et al. [23] established new sufficient conditions that guarantee the oscillation of solutions of the second-order half-linear differential equation

$$(a(t)(z_1'(t))^\alpha)' + q(t)x^\alpha(h(t)) = 0,$$

where $z_1(t) = x(t) + p_1(t)x(g_1(t)) + p_2(t)x(g_2(t))$, $g_1(t) < t$, $g_2(t) > t$, $p_1(t) \ge 0$ and $p_2(t) \ge 1, p_2(t) \not\equiv 1$ for large t. In 2020, Tunç et al. [24] obtained new oscillation criteria by using Riccati transformations for the second-order damped neutral differential equation

$$y''(t) + d(t)y'(t) + q(t)x^\beta(\sigma(t)) = 0,$$

where $y(t) = x(t) + p(t)x^{\alpha_1}(\tau(t))$, $\alpha_1 \ge 1$, $p(t) \ge 1, p(t) \not\equiv 1$ for large t and $\sigma(t) \le \tau(t) \le t$. In 2021, Bohner et al. [6] used Riccati transformations and comparison method to study a generalized case of the previous equation of the form

$$(ay')'(t) + d(t)y'(t) + q(t)x^\beta(\sigma(t)) = 0$$

under the assumptions $p(t) < 1$, $\sigma(t) \le t$ and $\tau(t) \ge t$. In [9, 25], Chatzarakis et al. and Tunç et al., respectively investigated the special case of (1.1) with $a(t) = 1$, $p_2(t) = 0$, $\eta = 1$, $\lambda = 1$, $b(t) = 0$, $r(t) = 0$, $p_1(t) \ge 1$, and $p_1(t) \not\equiv 1$ for large t. In [14, 19], Grace et al. and Liu et al., respectively studied the oscillatory behavior of (1.1) when $\eta = 1$, $p_2(t) = 0$, $\lambda \ge 1$, $b(t) = 0$, $r(t) = 0$, $p_1(t) \ge 1, p_1(t) \not\equiv 1$ for large t. In [22], Thandapani et al. established some sufficient conditions which ensure that all solutions of (1.1) are almost oscillatory in case when $\tau_1(t) = t - \tau_1$, $\tau_2(t) = t + \tau_2$, $\eta = \lambda = \nu = \delta = 1$, $p_1(t) \le p_1$, $p_2(t) \le p_2$, $p_1 + p_2 < 1$, $\sigma(t) = t - \sigma_1$ and $\zeta(t) = t + \sigma_2$.

Recently, Wang et al. [26] investigated the third order differential equation (1.1) in the case $\eta = 1$, $p_1(t) = p_2(t) = 0$ and $r(t) = 0$ and obtained new oscillation criteria by using generalized Riccati technique. On the other hand, number of authors focused on studying the oscillatory behavior of second and third-order damped differential and dynamic equations such as in [5, 6, 13, 24].

Very recently, Alzabut et al. [3] obtained new oscillation comparison theorems and new integral conditions for Eq. (1.1) when $\eta = 1$, $\delta = -1$, $b(t) = 0$, $\alpha \ge 1$ and $a'(t) \ge 0$. To the best of our knowledge, there are no results regarding the oscillation of solutions of third-order differential equations with a middle term and mixed neutral terms of the form (1.1) under the assumption $\alpha \neq \beta \neq \gamma$.

The purpose of this paper is to establish new oscillation criteria for the more general equation (1.1) in the case when either $\delta = -1$ or $\delta = +1$ holds, η differs from 1 and the condition $a'(t) \geq 0$ is neglected.

2. Preliminaries

In talking about oscillation results for Eq. (1.1), we assume that any functional inequality holds for all large t. We first outline some lemmas which will be needed for our main results. For simplicity, we shall use the following notations:

$$\bar{z} = -z(t), \quad \eta'_+(t) = \max\{0, \eta'(t)\}, \quad B_1(t, t_1) = \int_{t_1}^{t} \frac{\exp\left(\frac{-1}{\alpha}\int_{t_0}^{s} \frac{b(u)}{a(u)}\mathrm{d}u\right)}{a^{\frac{1}{\alpha}}(s)}\mathrm{d}s, \quad B_2(t, t_2, t_1) = \int_{t_2}^{t} B_1(s, t_1)\,\mathrm{d}s,$$

$$M_1(t) = \frac{(1-\lambda)p_1^{\frac{1}{1-\lambda}}(t)}{\lambda^{\frac{\lambda}{\lambda-1}}}, \quad V(t) = \exp\left(\int_{t_0}^{t} \frac{b(s)}{a(s)}\mathrm{d}s\right), \quad M_2(t) = \frac{(\nu-1)p_2^{\frac{1}{1-\nu}}(t)}{\nu^{\frac{\nu}{\nu-1}}}, \quad P(t) = \frac{\nu-\lambda}{\lambda}p_1^{\frac{\nu}{\nu-\lambda}}(t)p_2^{\frac{\lambda}{\lambda-\nu}}(t),$$

$$\varrho_1(t) = \tau_1^{-1}\left(\tau_2\left(\tau_1^{-1}(t)\right)\right), \quad \varrho_2(t) = \tau_1^{-1}\left(\tau_1^{-1}(t)\right), \quad \varrho_3(t) = \tau_2^{-1}\left(\tau_1\left(\tau_2^{-1}(t)\right)\right),$$

$$\varrho_4(t) = \tau_2^{-1}\left(\tau_2^{-1}(t)\right), \quad \varrho_5(t) = \tau_2^{-1}\left(\tau_1^{-1}(t)\right),$$

$$G_1(t) = \frac{\left(\frac{\eta}{\lambda}\right)B_2\left(\varrho_2(t), t_2, t_1\right)}{B_2\left(\tau_1^{-1}(t), t_2, t_1\right)p_1^{\frac{\eta}{\lambda}}(\varrho_2(t))} + \frac{\left(\frac{\nu}{\lambda}\right)p_2(\tau_1^{-1}(t))B_2\left(\varrho_1(t)), t_2, t_1\right)}{B_2\left(\tau_1^{-1}(t), t_2, t_1\right)p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))} + \frac{1-\frac{\eta}{\lambda}}{c_* p_1^{\frac{\eta}{\lambda}}(\varrho_2(t))} + \frac{\left(1-\frac{\nu}{\lambda}\right)p_2(\tau_1^{-1}(t))}{c_* p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))},$$

$$G_2(t) = \frac{\left(\frac{\eta}{\nu}\right)}{p_2^{\frac{\eta}{\nu}}(\varrho_4(t))} + \frac{\left(\frac{\lambda}{\nu}\right)p_1(\tau_2^{-1}(t))}{p_2^{\frac{\lambda}{\nu}}(\varrho_3(t))} + \frac{1-\frac{\eta}{\nu}}{c_* p_2^{\frac{\eta}{\nu}}(\varrho_4(t))} + \frac{\left(1-\frac{\lambda}{\nu}\right)p_1(\tau_2^{-1}(t))}{c_* p_2^{\frac{\lambda}{\nu}}(\varrho_3(t))}$$

and

$$G_3(t) = \frac{\left(\frac{\eta}{\nu}\right)}{p_2^{\frac{\eta}{\lambda}}(\varrho_5(t))} + \frac{\left(\frac{\nu}{\lambda}\right)p_2(\tau_1^{-1}(t))B_2\left(\varrho_1(t), t_2, t_1\right)}{B_2\left(\tau_1^{-1}(t), t_2, t_1\right)p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))} + \frac{1-\frac{\eta}{\nu}}{c_* p_2^{\frac{\eta}{\nu}}(\varrho_5(t))} + \frac{\left(1-\frac{\nu}{\lambda}\right)p_2(\tau_1^{-1}(t))}{c_* p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))},$$

where $t_2 > t_1 \geq t_0$ is sufficiently large, c_* is any positive constant and $\tau_1^{-1}(t), \tau_2^{-1}(t)$ are the inverse functions of $\tau_1(t), \tau_2(t)$, respectively.

Lemma 2.1 *Let $q_1 : [t_0, \infty) \to (0, \infty)$, $g_1 : [t_0, \infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions, f is nondecreasing with $xf(x) > 0$ for $x \neq 0$ and $g_1(t) \to \infty$ as $t \to \infty$. If*

(i) *the first-order delay differential inequality (i.e. $g_1(t) \leq t$)*

$$y_1'(t) + q_1(t)f(y_1(g_1(t))) \leq 0$$

has an eventually positive solution, then so does the corresponding delay differential equation.

(ii) *the first-order advanced differential inequality (i.e. $g_1(t) \geq t$)*

$$y_1'(t) - q_1(t)f(y_1(g_1(t))) \geq 0$$

has an eventually positive solution, then so does the corresponding advanced differential equation.

Proof The proof extends those of Lemma 2.3 of [4] and Corollary 1 of [20] and so it is omitted. □

Lemma 2.2 *[15]If $X, Y \geq 0$, then*

$$X^{\xi} + (\xi - 1)Y^{\xi} - \xi XY^{\xi-1} \geq 0, \quad \text{for } \xi > 1 \tag{2.1}$$

and

$$X^{\xi} - (1 - \xi)Y^{\xi} - \xi XY^{\xi-1} \leq 0, \quad \text{for } 0 < \xi < 1, \tag{2.2}$$

where equalities hold if and only if $X = Y$.

Lemma 2.3 *[15]Let B, L be nonnegative numbers and $m, n > 1$ are real numbers such that $\frac{1}{n} + \frac{1}{m} = 1$, then*

$$BL \leq \frac{1}{n}B^n + \frac{1}{m}L^m. \tag{2.3}$$

The equality holds if and only if $B^n = L^m$.

Lemma 2.4 *[28]Let $G(U) = AU - B(U - R)^{\frac{\alpha_1+1}{\alpha_1}}$, where $B > 0$, A and B are constants, α_1 is a ratio of odd positive integers. Then G attains its maximum value at $U_* = R + (\alpha_1 A/((\alpha_1 + 1)B)^{\alpha_1})$ and*

$$\max_{U \in \mathbb{R}} G(U) = G(U_*) = AR + \frac{\alpha_1^{\alpha_1}}{(\alpha_1 + 1)^{\alpha_1+1}} \frac{A^{\alpha_1+1}}{B^{\alpha_1}}. \tag{2.4}$$

3. The case $\delta = -1$

In this section, we study the oscillatory behavior of solutions of Eq. (1.1) in case when $\delta = -1$ and $\tau_1(t) = \tau_2(t) = \tau(t)$ and in which either of the two conditions $\lambda < 1$ and $\nu > 1$ or $\lambda < \nu \leq 1$ holds.

Theorem 3.1 *Assume that $\tau_1(t) = \tau_2(t) = \tau(t)$, $\lambda < 1$, $\nu > 1$, and (H_1)-(H_3) hold. Furthermore, assume that*

$$\lim_{t\to\infty} [M_1(t) + M_2(t)] = 0 \tag{3.1}$$

and there exists a nondecreasing function $\mu \in C([t_0, \infty), \mathbb{R})$ such that

$$\sigma(t) \leq \mu(t) \leq t,\ \tau^{-1}(\sigma(t)) \leq t,\ \tau^{-1}(\zeta(t)) \geq t \text{ and } \tau^{-1}(\zeta(\mu(\mu(t)))) \geq t \tag{3.2}$$

for $t \geq t_0$. If there exist numbers $k_1, k_2 \in (0, 1)$ such that the first-order differential equations

$$W'(t) + k_1^{\gamma} q(t)V(t)B_2^{\frac{\gamma}{\eta}}(\sigma(t), t_2, t_1)\, W^{\frac{\gamma}{\alpha\eta}}(\sigma(t)) = 0, \tag{3.3}$$

$$\Phi'(t) + k_2 q(t)V(t)\Big[\frac{\sigma(\tau^{-1}(\sigma(t)))B_1\left(\mu(\tau^{-1}(\sigma(t))), \sigma(\tau^{-1}(\sigma(t)))\right)}{p_2(\tau^{-1}(\sigma(t)))}\Big]^{\frac{\gamma}{\nu}} \Phi^{\frac{\gamma}{\alpha\nu}}(\mu(\tau^{-1}(\sigma(t)))) = 0 \tag{3.4}$$

and

$$\dot{z}'(t) - \Big[\int_{\mu(t)}^{t} V^{\frac{-1}{\alpha}}(s)\Big[\frac{1}{a(s)}\int_{\mu(s)}^{s} \frac{r(u)V(u)}{p_2^{\frac{\beta}{\nu}}(\tau^{-1}(\zeta(u)))}du\Big]^{\frac{1}{\alpha}} ds\Big] \left[\dot{z}(\tau^{-1}(\zeta(\mu(\mu(t)))))\right]^{\frac{\beta}{\alpha\nu}} = 0 \tag{3.5}$$

are oscillatory for sufficiently large $t_2 > t_1 \geq t_0$, then every solution of Eq. (1.1) is either oscillatory or converges to zero.

Proof Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Then, without loss of generality, assume that $x(t)$ is eventually positive with $\lim_{t\to\infty} x(t) \neq 0$ for $t \geq t_0$. Therefore, $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$ and $x(\zeta(t)) > 0$ for $t \geq t_1 \geq t_0$. The proof when $x(t)$ is eventually negative is similar, so it is omitted. Now from (1.1), it follows that

$$\left(a(t)\left(z''(t)\right)^{\alpha}\right)' + b(t)\left(z''(t)\right)^{\alpha} = -q(t)x^{\gamma}(\sigma(t)) - r(t)x^{\beta}(\zeta(t)) < 0.$$

Consequently, we have

$$\left[V(t)a(t)\left(z''(t)\right)^{\alpha}\right]' < 0 \quad \text{for } t \geq t_1,$$

which means that $V(t)a(t)\left(z''(t)\right)^{\alpha}$ is strictly decreasing and is of one sign. Then, there exists $t_2 \geq t_1$ such that $z''(t)$ is either positive or negative for $t \geq t_2$. We shall study in details the four possible cases:

case (1) : $z(t) > 0,\ z''(t) < 0$,

case (2) : $z(t) > 0,\ z''(t) > 0$,

case (3) : $z(t) < 0,\ z''(t) > 0$, and

case (4) : $z(t) < 0,\ z''(t) < 0$.

Firstly, suppose that **case (1)** holds. Depending on the fact that $z''(t) < 0$ and $\left[V(t)a(t)\left(z''(t)\right)^{\alpha}\right]' < 0$, we observe that this case is impossible. *Secondly*, suppose that **case (2)** holds. In this case, we have the two subcases when $z'(t) < 0$ and $z'(t) > 0$. In the first, we can easily make sure that the subcase $z'(t) < 0$ is excluded due to the fact that $\lim_{t\to\infty} x(t) \neq 0$. Now, assume that the second subcase $z'(t) > 0$ holds for $t \geq t_2$. From the definition of $z(t)$, we have

$$z(t) = \left(x^{\eta}(t) - p_2(t)x^{\nu}(\tau(t))\right) + p_1(t)x^{\lambda}(\tau(t)). \tag{3.6}$$

Applying Lemma 2.4 to $\left(x^{\eta}(t) - p_2(t)x^{\nu}(\tau(t))\right)$ with $U = x^{\eta}(t)$, $A = 1$, $B = p_2(t)$, $R = x^{\eta}(t) - x(\tau(t))$ and $\alpha_1 = \frac{1}{\nu-1}$, it follows that

$$x^{\eta}(t) - p_2(t)x^{\nu}(\tau(t)) \leq x^{\eta}(t) - x(\tau(t)) + \frac{\left(\frac{1}{\nu-1}\right)^{\frac{1}{\nu-1}}}{p_2^{\frac{1}{\nu-1}}(t)\left(\frac{\nu}{\nu-1}\right)^{\frac{\nu}{\nu-1}}}.$$

Substituting into (3.6), we get

$$x^{\eta}(t) \geq z(t) + x(\tau(t)) - p_1(t)x^{\lambda}(\tau(t)) - \frac{\left(\frac{1}{\nu-1}\right)^{\frac{1}{\nu-1}}}{p_2^{\frac{1}{\nu-1}}(t)\left(\frac{\nu}{\nu-1}\right)^{\frac{\nu}{\nu-1}}}. \tag{3.7}$$

Now, using the inequality (2.2) to $\left[p_1(t)x^{\lambda}(\tau(t)) - x(\tau(t))\right]$ with $\xi = \lambda$, $X = x(\tau(t))$ and $Y = \frac{1}{(\lambda p_1(t))^{\frac{1}{\lambda-1}}}$, we have

$$p_1(t)x^{\lambda}(\tau(t)) - x(\tau(t)) = p_1(t)\left[x^{\lambda}(\tau(t)) - \frac{\lambda}{\lambda p_1(t)}x(\tau(t))\right] \leq \frac{(1-\lambda)p_1^{\frac{1}{1-\lambda}}(t)}{\lambda^{\frac{\lambda}{\lambda-1}}}.$$

It follows from (3.7) that

$$x^{\eta}(t) \geq z(t) - \frac{\left(\frac{1}{\nu-1}\right)^{\frac{1}{\nu-1}}}{p_2^{\frac{1}{\nu-1}}(t)\left(\frac{\nu}{\nu-1}\right)^{\frac{\nu}{\nu-1}}} - \frac{(1-\lambda)p_1^{\frac{1}{1-\lambda}}(t)}{\lambda^{\frac{\lambda}{\lambda-1}}} = \left[1 - \frac{(M_1(t)+M_2(t))}{z(t)}\right]z(t).$$

Since $z(t)$ is positive and increasing, then there exists a constant $c>0$ such that $z(t) \geq c$ for $t \geq t_2$ and consequently we have

$$x^{\eta}(t) \geq \left[1 - \frac{(M_1(t)+M_2(t))}{c}\right]z(t). \tag{3.8}$$

Hence, from (3.1) and (3.8), there exists a constant $c_1 \in (0,1)$ such that

$$x(t) \geq c_1^{\frac{1}{\eta}} z^{\frac{1}{\eta}}(t) = c_2 z^{\frac{1}{\eta}}(t); \quad c_2 = c_1^{\frac{1}{\eta}} \quad \text{for } t \geq t_3 \geq t_2. \tag{3.9}$$

Now, it follows from (1.1) that

$$\left[V(t)a(t)\left(z''(t)\right)^{\alpha}\right]' = -q(t)V(t)x^{\gamma}(\sigma(t)) - r(t)V(t)x^{\beta}(\zeta(t)). \tag{3.10}$$

Thus, from (3.9), we have

$$\left[V(t)a(t)\left(z''(t)\right)^{\alpha}\right]' \leq -c_2^{\gamma}q(t)V(t)z^{\frac{\gamma}{\eta}}(\sigma(t)) - c_2^{\beta}r(t)V(t)z^{\frac{\beta}{\eta}}(\zeta(t)). \tag{3.11}$$

Since $\left[V(t)a(t)\left(z''(t)\right)^{\alpha}\right]$ is positive and decreasing, we have

$$z'(t) \geq z'(t) - z'(t_3) = \int_{t_3}^{t} \frac{(V(s)a(s)z''(s))^{\frac{1}{\alpha}}}{V^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)} \mathrm{d}s \geq V^{\frac{1}{\alpha}}(t)B_1(t,t_3)\,a^{\frac{1}{\alpha}}(t)z''(t), \tag{3.12}$$

which means that $\left(\frac{z'(t)}{B_1(t,t_3)}\right)' \leq 0$. Therefore, there exists $t_4 > t_3$ such that

$$z(t) \geq \int_{t_4}^{t} \frac{z'(s)B_1(s,t_3)}{B_1(s,t_3)} \mathrm{d}s \geq \frac{z'(t)}{B_1(t,t_3)} B_2(t,t_4,t_3).$$

This with (3.12) yields

$$z(t) \geq \left[V(t)\,a(t)\left(z''(t)\right)^{\alpha}\right]^{\frac{1}{\alpha}} B_2(t,t_4,t_3). \tag{3.13}$$

Substituting into (3.11), we get

$$\begin{aligned} [V(t)\,a(t)\left(z''(t)\right)^{\alpha}]' &\leq -c_2^{\gamma}q(t)V(t)z^{\frac{\gamma}{\eta}}(\sigma(t)) \\ &\leq -c_2^{\gamma}q(t)V(t)B_2^{\frac{\gamma}{\eta}}(\sigma(t),t_4,t_3)\left[V(\sigma(t))\,a(\sigma(t))\left(z''(\sigma(t))\right)^{\alpha}\right]^{\frac{\gamma}{\alpha\eta}}. \end{aligned} \tag{3.14}$$

Putting $W(t) = V(t)a(t)\left(z''(t)\right)^{\alpha}$, we have

$$W'(t) + c_2^{\gamma}q(t)V(t)B_2^{\frac{\gamma}{\eta}}(\sigma(t),t_4,t_3)\,W^{\frac{\gamma}{\alpha\eta}}(\sigma(t)) \leq 0. \tag{3.15}$$

From Lemma 2.1 (i), the differential equation (3.3) corresponds the inequality (3.15) also has a positive solution, which is a contradiction. *Thirdly*, suppose that **case (3)** holds. Clearly, we see that $z'(t) < 0$ for $t \geq t_2$. Setting

$$\bar{z}(t) = -z(t) = -x^{\eta}(t) - p_1(t)x^{\lambda}(\tau(t)) + p_2(t)x^{\nu}(\tau(t)) \leq p_2(t)x^{\nu}(\tau(t)),$$

which means that

$$x(t) \geq \Big(\frac{\bar{z}(\tau^{-1}(t))}{p_2(\tau^{-1}(t))}\Big)^{\frac{1}{\nu}}. \tag{3.16}$$

Substituting into (3.10), we get

$$\begin{aligned}\left[V(t)a(t)\left(\bar{z}''(t)\right)^{\alpha}\right]' &= q(t)V(t)x^{\gamma}(\sigma(t)) + r(t)V(t)x^{\beta}(\zeta(t)) \geq q(t)V(t)x^{\gamma}(\sigma(t)) \\ &\geq q(t)V(t)\Big(\frac{\bar{z}(\tau^{-1}(\sigma(t)))}{p_2(\tau^{-1}(\sigma(t)))}\Big)^{\frac{\gamma}{\nu}}. \end{aligned} \tag{3.17}$$

Since $\bar{z}''(t) < 0$ and $\bar{z}'(t) > 0$ for $t_2 \leq u_1 \leq v_1$, then we have

$$\bar{z}'(u_1) \geq -\bar{z}'(v_1) + \bar{z}'(u_1) = -\int_{u_1}^{v_1} \frac{V^{\frac{-1}{\alpha}}(s)}{a^{\frac{1}{\alpha}}(s)} V^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)\bar{z}''(s)\mathrm{d}s \geq B_1\left(v_1, u_1\right)\Big[-\left[V(v_1)a(v_1)\left(\bar{z}''(v_1)\right)^{\alpha}\right]^{\frac{1}{\alpha}}\Big].$$

By letting $u_1 = \sigma(t)$ and $v_1 = \mu(t)$, then

$$\bar{z}'(\sigma(t)) \geq B_1\left(\mu(t), \sigma(t)\right)\Big[-\left[V(\mu(t))a(\mu(t))\left(\bar{z}''(\mu(t))\right)^{\alpha}\right]^{\frac{1}{\alpha}}\Big]. \tag{3.18}$$

According to [1, Lemma 2.2.3], there exists a constant $c_3 \in (0,1)$ such that

$$\bar{z}(\sigma(t)) \geq c_3\sigma(t)\bar{z}'(\sigma(t)). \tag{3.19}$$

Substituting into (3.18), we get

$$\bar{z}(t) \geq \bar{z}(\sigma(t)) \geq -c_3\sigma(t)B_1\left(\mu(t), \sigma(t)\right)\left[V(\mu(t))a(\mu(t))\left(\bar{z}''(\mu(t))\right)^{\alpha}\right]^{\frac{1}{\alpha}}. \tag{3.20}$$

It follows from (3.17) and (3.20) that

$$\begin{aligned}\left[V(t)a(t)\left(\bar{z}''(t)\right)^{\alpha}\right]' \geq -c_3^{\frac{\gamma}{\nu}}q(t)V(t)\Big[\frac{\sigma(\tau^{-1}(\sigma(t)))B_1\left(\mu(\tau^{-1}(\sigma(t))), \sigma(\tau^{-1}(\sigma(t)))\right)}{p_2(\tau^{-1}(\sigma(t)))}\Big]^{\frac{\gamma}{\nu}} \times \\ \Big[V(\mu(\tau^{-1}(\sigma(t))))a(\mu(\tau^{-1}(\sigma(t))))\left(\bar{z}''(\mu(\tau^{-1}(\sigma(t))))\right)^{\alpha}\Big]^{\frac{\gamma}{\alpha\nu}},\end{aligned}$$

i.e.

$$\Phi'(t) + c_3^{\frac{\gamma}{\nu}}q(t)V(t)\Big[\frac{\sigma(\tau^{-1}(\sigma(t)))B_1\left(\mu(\tau^{-1}(\sigma(t))), \sigma(\tau^{-1}(\sigma(t)))\right)}{p_2(\tau^{-1}(\sigma(t)))}\Big]^{\frac{\gamma}{\nu}}\Phi^{\frac{\gamma}{\alpha\nu}}(\mu(\tau^{-1}(\sigma(t)))) \leq 0, \tag{3.21}$$

where $\Phi(t) = -V(t)a(t)\left(\bar{z}''(t)\right)^{\alpha} > 0$. Applying Lemma 2.1 (i), we deduce that equation (3.4) corresponds the inequality (3.21) also has a positive solution, which is a contradiction. *Finally*, suppose that **case (4)** holds. Clearly, $z'(t) < 0$ for $t \geq t_2$. As in the proof of case (3), we have (3.16). It follows from (3.10) that

$$\left[V(t)a(t)\left(\bar{z}''(t)\right)^{\alpha}\right]' \geq V(t)r(t)\left(\frac{\bar{z}(\tau^{-1}(\zeta(t)))}{p_2(\tau^{-1}(\zeta(t)))}\right)^{\frac{\beta}{\nu}}.$$

Integrating from $\mu(t)$ to t and using the fact that $\bar{z}'(t) > 0$, we get

$$V(t)a(t)\left(\bar{z}''(t)\right)^{\alpha} \geq \left[\bar{z}(\tau^{-1}(\zeta(\mu(t))))\right]^{\frac{\beta}{\nu}} \int_{\mu(t)}^{t} \frac{r(s)V(s)}{p_2^{\frac{\beta}{\nu}}(\tau^{-1}(\zeta(s)))} \mathrm{d}s,$$

i.e.

$$\bar{z}''(t) \geq V^{\frac{-1}{\alpha}}(t)\Big[\frac{1}{a(t)} \int_{\mu(t)}^{t} \frac{r(s)V(s)}{p_2^{\frac{\beta}{\nu}}(\tau^{-1}(\zeta(s)))} \mathrm{d}s\Big]^{\frac{1}{\alpha}} \left[\bar{z}(\tau^{-1}(\zeta(\mu(t))))\right]^{\frac{\beta}{\alpha\nu}}.$$

Integrating again from $\mu(t)$ to t, we obtain

$$\bar{z}'(t) \geq \left[\bar{z}(\tau^{-1}(\zeta(\mu(\mu(t)))))\right]^{\frac{\beta}{\alpha\nu}} \int_{\mu(t)}^{t} V^{\frac{-1}{\alpha}}(s) \left[\frac{1}{a(s)} \int_{\mu(s)}^{s} \frac{r(u)V(u)}{p_2^{\frac{\beta}{\nu}}(\tau^{-1}(\zeta(u)))} \mathrm{d}u\right]^{\frac{1}{\alpha}} \mathrm{d}s.$$

i.e.

$$\bar{z}'(t) - \left[\int_{\mu(t)}^{t} V^{\frac{-1}{\alpha}}(s) \left[\frac{1}{a(s)} \int_{\mu(s)}^{s} \frac{r(u)V(u)}{p_2^{\frac{\beta}{\nu}}(\tau^{-1}(\zeta(u)))} \mathrm{d}u\right]^{\frac{1}{\alpha}} \mathrm{d}s\right] \left[\bar{z}(\tau^{-1}(\zeta(\mu(\mu(t)))))\right]^{\frac{\beta}{\alpha\nu}} \geq 0, \qquad (3.22)$$

which has a positive solution $\bar{z}(t)$. It follows from Lemma 2.1 (ii) that equation (3.5) corresponds to the inequality (3.22) also has a positive solution. This completes the proof. □

Corollary 3.2 *Assume that $\tau_1(t) = \tau_2(t) = \tau(t)$, $\lambda < 1$, $\nu > 1$, and (H_1)-(H_3) hold. Furthermore, assume that (3.1) holds and there exists a nondecreasing function $\mu \in C([t_0, \infty), \mathbb{R})$ satisfies (3.2). If*

$$\lim_{t\to\infty} \int_{t_0}^{t} q(s)V(s)B_2^{\frac{\gamma}{\eta}}(\sigma(s), t_2, t_1)\,\mathrm{d}s = \infty, \quad \text{for } \gamma < \alpha\eta, \qquad (3.23)$$

$$\lim_{t\to\infty} \int_{t_0}^{t} q(s)V(s)\Big[\frac{\sigma(\tau^{-1}(\sigma(s)))B_1\left(\mu(\tau^{-1}(\sigma(s))), \sigma(\tau^{-1}(\sigma(s)))\right)}{p_2(\tau^{-1}(\sigma(s)))}\Big]^{\frac{\gamma}{\nu}} \mathrm{d}s = \infty, \quad \text{for } \gamma < \alpha\nu, \qquad (3.24)$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} \int_{\mu(s)}^{s} V^{\frac{-1}{\alpha}}(u)\Big[\frac{1}{a(u)} \int_{\mu(u)}^{u} \frac{r(v)V(v)}{p_2^{\frac{\beta}{\nu}}(\tau^{-1}(\zeta(v)))} \mathrm{d}v\Big]^{\frac{1}{\alpha}} \mathrm{d}u\ \mathrm{d}s = \infty, \quad \text{for } \beta > \alpha\nu \qquad (3.25)$$

hold for sufficiently large $t_2 > t_1 \geq t_0$, then every solution of Eq. (1.1) is either oscillatory or converges to zero.

Proof Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Then, without loss of generality, assume that $x(t)$ is eventually positive with $\lim_{t\to\infty} x(t) \neq 0$ for $t \geq t_0$. Therefore, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$ and $x(\zeta(t)) > 0$ for $t \geq t_1$. Going through as in the proof of Theorem 3.1, we arrive at (3.15) for $t \geq t_4 \geq t_3$, (3.21) for $t \geq t_2$ and (3.22) for $t \geq t_2$. Since $W(t)$ is positive decreasing function and $\sigma(t) \leq t$, then (3.15) becomes

$$W'(t) \leq -c_2^{\gamma} q(t)V(t)B_2^{\frac{\gamma}{\eta}}(\sigma(t), t_4, t_3)\, W^{\frac{\gamma}{\alpha\eta}}(\sigma(t)) \leq -c_2^{\gamma} q(t)V(t)B_2^{\frac{\gamma}{\eta}}(\sigma(t), t_4, t_3)\, W^{\frac{\gamma}{\alpha\eta}}(t),$$

i.e.

$$\frac{W'(t)}{W^{\frac{\gamma}{\alpha\eta}}(t)} + c_2^{\gamma} q(t) V(t) B_2^{\frac{\gamma}{\eta}} (\sigma(t), t_4, t_3) \leq 0.$$

Integrating from $t_5 \geq t_4$ to t, we get

$$\int_{t_5}^{t} c_2^{\gamma} q(s) V(s) B_2^{\frac{\gamma}{\eta}} (\sigma(s), t_4, t_3) \, \mathrm{d}s \leq \frac{W(t_5)}{\left(1 - \frac{\gamma}{\alpha\eta}\right)}.$$

Letting t tends to ∞, we get a contradiction with (3.23). Since $\Phi(t)$ is positive decreasing function and $\bar{z}(t)$ is positive increasing with $\mu(\tau^{-1}(\sigma(t)) \leq t$ and $\tau^{-1}(\zeta(\mu(\mu(t)))) \geq t$ in (3.21) and (3.22) respectively, then going through as in (3.15) for $t \geq t_3 \geq t_2$, we get the conclusion. This completes the proof. □

Theorem 3.3 *Assume that $\tau_1(t) = \tau_2(t) = \tau(t)$, $\lambda < 1$, $\nu > 1$, $\beta > \alpha\eta$, and (H_1)-(H_3) hold. Furthermore, assume that (3.1) holds and there exists a nondecreasing function $\mu \in C([t_0, \infty), \mathbb{R})$ satisfies (3.2). If there exist a function $\rho \in C^1([t_0, \infty), (0, \infty))$ and numbers $k_1, k_2 \in (0, 1)$, $k_* > 0$ such that*

$$\limsup_{t\to\infty} \int_{t_*}^{t} \left[k_1^{\beta} r(s) \rho(s) V(s) - \left(\frac{\alpha\eta}{k_*^{\frac{\beta}{\alpha\eta}-1} \beta B_1(s, t_1) \rho(s)} \right)^{\alpha} \left(\frac{\rho'_+(s)}{\alpha + 1} \right)^{\alpha+1} \right] \mathrm{d}s = \infty \tag{3.26}$$

and the differential equations (3.4) and (3.5) are oscillatory for sufficiently large $t_1 \geq t_0$, then every solution of Eq. (1.1) is either oscillatory or converges to zero.

Proof Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Following the same proof of Theorem 3.1, we arrive at (3.13). Define the Riccati transformation

$$\omega(t) = \rho(t) \frac{V(t) a(t) (z''(t))^{\alpha}}{z^{\frac{\beta}{\eta}}(t)}. \tag{3.27}$$

Then, $\omega(t) > 0$ and

$$\begin{aligned} \omega'(t) &= \frac{\rho'(t)}{\rho(t)} \omega(t) + \rho(t) \frac{\left(V(t) a(t) (z''(t))^{\alpha}\right)'}{z^{\frac{\beta}{\eta}}(t)} - \rho(t) \frac{V(t) a(t) (z''(t))^{\alpha} \frac{\beta}{\eta} z^{\frac{\beta}{\eta}-1}(t) z'(t)}{z^{\frac{2\beta}{\eta}}(t)} \\ &= \frac{\rho'(t)}{\rho(t)} \omega(t) + \rho(t) \frac{\left(V(t) a(t) (z''(t))^{\alpha}\right)'}{z^{\frac{\beta}{\eta}}(t)} - \frac{\beta}{\eta} \frac{z'(t)}{z(t)} \omega(t). \end{aligned}$$

It follows from (3.11), (3.12) and the monotonicity of z that

$$\begin{aligned} \omega'(t) &\leq \frac{\rho'(t)}{\rho(t)} \omega(t) - c_2^{\beta} r(t) \rho(t) V(t) \frac{z^{\frac{\beta}{\eta}}(\zeta(t))}{z^{\frac{\beta}{\eta}}(t)} - \frac{\beta}{\eta} \frac{\left[V(t) a(t) (z''(t))^{\alpha}\right]^{\frac{1}{\alpha}} B_1(t, t_3)}{z(t)} \omega(t) \\ &\leq \frac{\rho'(t)}{\rho(t)} \omega(t) - c_2^{\beta} r(t) \rho(t) V(t) - \frac{\beta}{\eta} \frac{\left[V(t) a(t) (z''(t))^{\alpha}\right]^{\frac{1}{\alpha}} B_1(t, t_3)}{z(t)} \omega(t). \end{aligned} \tag{3.28}$$

From (3.27), it is clear that

$$\left[V(t)a(t)\left(z''(t)\right)^{\alpha}\right]^{\frac{1}{\alpha}} = \left(\frac{\omega(t)z^{\frac{\beta}{\eta}}(t)}{\rho(t)}\right)^{\frac{1}{\alpha}}.$$

Thus, from (3.28), we have

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - c_2^{\beta}r(t)\rho(t)V(t) - \frac{\beta}{\eta}\frac{B_1\left(t,t_3\right)}{\rho^{\frac{1}{\alpha}}(t)}z^{\frac{\beta}{\alpha\eta}-1}(t)\omega^{\frac{1}{\alpha}+1}(t). \tag{3.29}$$

Since $z(t)>0$ and $z'(t)>0$, then there exists $c_*>0$ such that $z(t)\geq c_*$ for $t\geq t_4$ and consequently, we have

$$\omega'(t) \leq \frac{\rho'_+(t)}{\rho(t)}\omega(t) - \frac{c_*^{\frac{\beta}{\alpha\eta}-1}\beta}{\eta}\frac{B_1\left(t,t_3\right)}{\rho^{\frac{1}{\alpha}}(t)}\omega^{\frac{1}{\alpha}+1}(t) - c_2^{\beta}r(t)\rho(t)V(t).$$

Applying Lemma 2.4 to above inequality, we get

$$\omega'(t) \leq \left(\frac{\alpha\eta}{c_*^{\frac{\beta}{\alpha\eta}-1}\beta B_1\left(t,t_3\right)\rho(t)}\right)^{\alpha}\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\alpha+1} - c_2^{\beta}r(t)\rho(t)V(t).$$

Integrating from t_5 $(>t_4)$ to t, we obtain

$$\int_{t_5}^{t}\left[c_2^{\beta}r(s)\rho(s)V(s) - \left(\frac{\alpha\eta}{c_*^{\frac{\beta}{\alpha\eta}-1}\beta B_1\left(s,t_3\right)\rho(s)}\right)^{\alpha}\left(\frac{\rho'_+(s)}{\alpha+1}\right)^{\alpha+1}\right]\mathrm{d}s \leq \omega(t_5),$$

which contradicts (3.26). Completing the proof of the two cases (3) and (4) as in the proof of Theorem 3.1, we get the conclusion of the theorem. □

Based on Theorem 3.3 using a similar steps as in the proof of Corollary 3.2 with condition (3.26) instead of (3.23), we have the following result.

Corollary 3.4 *Assume that $\tau_1(t)=\tau_2(t)=\tau(t)$, $\lambda<1$, $\nu>1$, $\beta>\alpha\eta$, and (H_1)-(H_3) hold. Furthermore, assume that (3.1) holds and there exists a nondecreasing function $\mu\in C\left([t_0,\infty),\mathbb{R}\right)$ satisfies (3.2). If there exist a function $\rho\in C^1\left([t_0,\infty),(0,\infty)\right)$ and numbers $k_1\in(0,1)$, $k_*>0$ such that (3.24), (3.25) and (3.26) hold for sufficiently large $t_1\geq t_0$, then every solution of Eq. (1.1) is either oscillatory or converges to zero.*

Theorem 3.5 *Assume that $\tau_1(t)=\tau_2(t)=\tau(t)$, $\lambda<\ \nu\leq1$, and (H_1)-(H_3) hold. Furthermore, assume that the rest of all hypotheses of Theorem 3.1 hold by replacing*

$$\lim_{t\to\infty}P(t)=0 \tag{3.30}$$

instead of (3.1), then the conclusion of Theorem 3.1 holds.

Proof Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Then, without loss of generality, assume that $x(t)$ is eventually positive with $\lim_{t\to\infty}x(t)\neq0$ for $t\geq t_0$. Therefore, there exists $t_1\geq t_0$ such that $x(t)>0$, $x(\tau(t))>0$, $x(\sigma(t))>0$ and $x(\zeta(t))>0$ for $t\geq t_1$. Following the same manner of the proof that used for

Theorem 3.1, we shall consider the same four cases (1)–(4) for $z(t)$. *Firstly*, suppose that **case (1)** holds. As in the proof of case (1) of Theorem 3.1, we can easily observe that this case is impossible. *Secondly*, suppose that **case (2)** holds. Putting $\left[p_1(t)x^{\lambda}(\tau(t)) - p_2(t)x^{\nu}(\tau(t))\right]$ in the form

$$p_1(t)x^{\lambda}(\tau(t)) - p_2(t)x^{\nu}(\tau(t)) = \frac{\nu}{\lambda}p_2(t)\Big[x^{\lambda}(\tau(t))\frac{\lambda}{\nu}\frac{p_1(t)}{p_2(t)} - \frac{\lambda}{\nu}\left(x^{\lambda}(\tau(t))\right)^{\frac{\nu}{\lambda}}\Big]$$

and applying the inequality (2.3) with $n = \frac{\nu}{\lambda}$, $B = x^{\lambda}(\tau(t))$, $L = \frac{\lambda}{\nu}\frac{p_1(t)}{p_2(t)}$ and $m = \frac{\nu}{\nu-\lambda}$, we get

$$p_1(t)x^{\lambda}(\tau(t)) - p_2(t)x^{\nu}(\tau(t)) \leq \frac{\nu}{\lambda}p_2(t)\Big[\frac{\nu-\lambda}{\nu}\Big]\Big(\frac{\lambda}{\nu}\frac{p_1(t)}{p_2(t)}\Big)^{\frac{\nu}{\nu-\lambda}} = \frac{\nu-\lambda}{\lambda}p_1^{\frac{\nu}{\nu-\lambda}}(t)p_2^{\frac{\lambda}{\lambda-\nu}}(t) = P(t).$$

Substituting into (3.6), we obtain

$$x(t) \geq \Big[1 - \frac{P(t)}{z(t)}\Big]^{\frac{1}{\eta}} z^{\frac{1}{\eta}}(t).$$

Thus, in view of (3.30), and from the fact that $z(t)$ is positive and increasing there exists $c_3 \in (0,1)$ such that

$$x(t) \geq c_3^{\frac{1}{\eta}} z^{\frac{1}{\eta}}(t). \tag{3.31}$$

Completing the proof as in the proof of Theorem 3.1 by using (3.31) instead of (3.9), we get the conclusion of the theorem. □

Remark 3.6 *The result of Corollary 3.2, Theorem 3.3 and Corollary 3.4 can be extracted directly to Theorem 3.5 by using* (3.30) *instead of* (3.1) *and* $\lambda < \nu \leq 1$ *instead of* $\lambda < 1, \nu > 1$ *and the details are left to the reader.*

Example 3.7 *Consider the mixed neutral third-order differential equation*

$$\left(\left[\left(x^{\frac{5}{3}}(t) + \frac{1}{t^{\frac{5}{3}}}x^{\frac{1}{3}}(\frac{t}{2}) - t^{2/3}x^{\frac{7}{5}}(\frac{t}{2})\right)''\right]^3\right)'$$
$$+\frac{1}{t}\left[\left(x^{\frac{5}{3}}(t) + \frac{1}{t^{\frac{5}{3}}}x^{\frac{1}{3}}(\frac{t}{2}) - t^{2/3}x^{\frac{7}{5}}(\frac{t}{2})\right)''\right]^3 + \frac{1}{t^{\frac{8}{3}}}x(\frac{t}{4}) + t^{\frac{57}{7}}x^5(12t) = 0. \tag{3.32}$$

Here, $a(t) = 1$, $\eta = \frac{5}{3}$, $p_1(t) = \frac{1}{t^{\frac{5}{3}}}$, $\lambda = \frac{1}{3}$, $\tau_1(t) = \tau_2(t) = \tau(t) = \frac{t}{2}$, $\delta = -1$, $p_2(t) = t^{2/3}$, $\nu = \frac{7}{5}$, $b(t) = \frac{1}{t}$, $\alpha = 3$, $q(t) = \frac{1}{t^{\frac{8}{3}}}$, $\gamma = 1$, $\sigma(t) = \frac{t}{4}$, $r(t) = t^{\frac{57}{7}}$, $\beta = 5$ *and* $\zeta(t) = 12t$. *Thus,*

$$\int_{t_0}^{\infty}\frac{V^{\frac{-1}{\alpha}}(t)}{a^{\frac{1}{\alpha}}(t)}\mathrm{d}t = \int_{t_0}^{\infty}\exp\left(\frac{-1}{3}\int_{t_0}^{t}\frac{1}{s}\mathrm{d}s\right)\mathrm{d}t = t_0^{\frac{1}{3}}\int_{t_0}^{\infty}t^{\frac{-1}{3}}\mathrm{d}t = \infty$$

and

$$\lim_{t\to\infty}[M_1(t) + M_2(t)] = \lim_{t\to\infty}\left[\frac{(1-1/3)\left(\frac{1}{t^{\frac{5}{3}}}\right)^{\frac{3}{2}}}{(1/3)^{\frac{-1}{2}}} + \frac{(7/5-1)(t^{2/3})^{\frac{-5}{2}}}{(7/5)^{\frac{7}{2}}}\right] = \lim_{t\to\infty}\left[\frac{2}{3\sqrt{3}t^{\frac{5}{2}}} + \frac{2/5}{(7/5)^{\frac{7}{2}}t^{5/3}}\right] = 0.$$

Choosing $\mu(t) = \frac{t}{2}$, *then clearly* $\sigma(t) < \mu(t) < t$, $\tau^{-1}(\sigma(t)) = \tau^{-1}(t/4) = \frac{t}{2} < t$, $\tau^{-1}(\zeta(t)) = \tau^{-1}(12t) = 24t > t$ *and* $\tau^{-1}(\zeta(\mu(\mu(t)))) = \tau^{-1}(\zeta(\mu(t/2))) = \tau^{-1}(\zeta(t/4)) = \tau^{-1}(3t) = 6t > t$. *We can easily observe that*

$$\lim_{t\to\infty}\int_{t_0}^{t} q(s)V(s)B_2^{\frac{\gamma}{\eta}}\left(\sigma(s),t_2,t_1\right)\mathrm{d}s = \lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{8}{3}}}\exp\left(\int_{t_0}^{s}\frac{1}{u}\mathrm{d}u\right)\left[\int_{t_2}^{s/4}B_1\left(u,t_1\right)\mathrm{d}u\right]^{3/5}\mathrm{d}s$$

$$= \lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{8}{3}}}\frac{s}{t_0}\left[\int_{t_2}^{s/4}\int_{t_1}^{u}\frac{t_0^{1/3}}{v^{1/3}}\mathrm{d}v\mathrm{d}u\right]^{3/5}\mathrm{d}s = (3/2)^{3/5}\frac{1}{t_0^{4/5}}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{5}{3}}}\left[\int_{t_2}^{s/4}(u^{2/3}-t_1^{2/3})\mathrm{d}u\right]^{3/5}\mathrm{d}s$$

$$> \frac{(3/2)^{3/5}}{t_0^{4/5}}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{5}{3}}}\left[\int_{t_2}^{s/4}(u^{2/3}-t_2^{2/3})\mathrm{d}u\right]^{3/5}\mathrm{d}s$$

$$> \frac{(3/2)^{3/5}}{t_0^{4/5}}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{5}{3}}}\left[\frac{3}{5}\left(\frac{s}{4}\right)^{\frac{5}{3}} - t_2^{\frac{2}{3}}\left(\frac{s}{4}\right)\right]^{3/5}\mathrm{d}s$$

$$= \left(\frac{3}{5}\right)^{\frac{3}{5}}\frac{(3/2)^{3/5}}{t_0^{4/5}}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{5}{3}}}\frac{s}{4}\left[1-\frac{5}{3}\left(\frac{4t_2}{s}\right)^{\frac{2}{3}}\right]^{3/5}\mathrm{d}s$$

$$= \left(\frac{3}{5}\right)^{\frac{3}{5}}\frac{(3/2)^{3/5}}{4t_0^{4/5}}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{2/3}}\left(1-\left(\frac{4t_2}{s}\right)^{\frac{2}{3}}-\frac{1}{3}\left(\frac{4t_2}{s}\right)^{\frac{4}{3}}-\frac{7}{27}\left(\frac{4t_2}{s}\right)^{2}-\dots\right)\mathrm{d}s = \infty,$$

$$\lim_{t\to\infty}\int_{t_0}^{t} q(s)V(s)\left[\frac{\sigma(\tau^{-1}(\sigma(s)))B_1\left(\mu(\tau^{-1}(\sigma(s))),\sigma(\tau^{-1}(\sigma(s)))\right)}{p_2(\tau^{-1}(\sigma(s)))}\right]^{\frac{\gamma}{\nu}}\mathrm{d}s$$

$$= \lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{8}{3}}}\frac{s}{t_0}\left[\frac{\frac{s}{8}\int_{s/8}^{s/4}\exp\left(\frac{-1}{3}\int_{t_0}^{u}\frac{1}{v}\mathrm{d}v\right)\mathrm{d}u}{(s/2)^{2/3}}\right]^{\frac{5}{7}}\mathrm{d}s = \left[\frac{1/8}{(1/2)^{2/3}}\right]^{5/7}\frac{1}{t_0}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{5}{3}}}\left[s^{1/3}\int_{s/8}^{s/4}\frac{t_0^{1/3}}{u^{1/3}}\mathrm{d}u\right]^{\frac{5}{7}}\mathrm{d}s$$

$$= \left[\frac{1/8}{(1/2)^{2/3}}\right]^{5/7}(3/2)^{5/7}\frac{1}{t_0^{16/21}}\lim_{t\to\infty}\int_{t_0}^{t}s^{-10/7}\left[\left(\frac{s}{4}\right)^{2/3}-\left(\frac{s}{8}\right)^{2/3}\right]^{\frac{5}{7}}\mathrm{d}s$$

$$= \left[\frac{1/8}{(1/2)^{2/3}}\right]^{5/7}\left(\frac{3}{2}\right)^{\frac{5}{7}}\frac{1}{t_0^{\frac{16}{21}}}\left((1/4)^{2/3}-(1/8)^{2/3}\right)^{\frac{5}{7}}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{20}{21}}}\mathrm{d}s = \infty$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\int_{\mu(s)}^{s}V^{\frac{-1}{\alpha}}(u)\left[\frac{1}{a(u)}\int_{\mu(u)}^{u}V(v)\frac{r(v)}{p_2^{\frac{\beta}{\nu}}(\tau^{-1}(\zeta(v)))}\mathrm{d}v\right]^{\frac{1}{\alpha}}\mathrm{d}u\ \mathrm{d}s$$

$$= \lim_{t\to\infty}\int_{t_0}^{t}\int_{s/2}^{s}\exp\left(\frac{-1}{3}\int_{t_0}^{u}\frac{1}{v}dv\right)\left[\int_{u/2}^{u}\exp\left(\int_{t_0}^{v}\frac{1}{\psi}\mathrm{d}\psi\right)\frac{v^{\frac{57}{7}}}{(24v)^{\frac{50}{21}}}\mathrm{d}v\right]^{\frac{1}{3}}\mathrm{d}u\ \mathrm{d}s$$

$$= \lim_{t\to\infty}\int_{t_0}^{t}\int_{s/2}^{s}t_0^{\frac{1}{3}}u^{\frac{-1}{3}}\left[\int_{u/2}^{u}\frac{v^{\frac{64}{7}}}{t_0(24v)^{\frac{50}{21}}}\mathrm{d}v\right]^{\frac{1}{3}}\mathrm{d}u\ \mathrm{d}s = \lim_{t\to\infty}\frac{1}{(24)^{\frac{50}{63}}}\int_{t_0}^{t}\int_{s/2}^{s}u^{\frac{-1}{3}}\left[\int_{u/2}^{u}v^{142/21}\mathrm{d}v\right]^{\frac{1}{3}}\mathrm{d}u\ \mathrm{d}s = \infty.$$

Thus all the hypotheses of Corollary 3.2 hold, and so every solution $x(t)$ *of Eq.* (3.32) *is either oscillatory or converges to zero.*

Example 3.8 *Consider the third-order differential equation*

$$\left(\frac{1}{t}\left(x^{\frac{7}{3}}(t)+\frac{1}{t}x^{\frac{3}{5}}(2t)-tx^{3}(2t)\right)''\right)'+\frac{1}{t^{2}}\left(x^{\frac{7}{3}}(t)+\frac{1}{t}x^{\frac{3}{5}}(2t)-tx^{3}(2t)\right)''+tx^{\frac{5}{7}}(\frac{t}{4})+t^{\frac{2}{9}}x^{\frac{11}{3}}(36t)=0. \quad (3.33)$$

Here, $a(t)=\frac{1}{t}$, $\eta=\frac{7}{3}$, $p_1(t)=\frac{1}{t}$, $\lambda=\frac{3}{5}$, $\tau_1(t)=\tau_2(t)=\tau(t)=2t$, $\delta=-1$, $p_2(t)=t$, $\nu=3$, $b(t)=\frac{1}{t^2}$, $\alpha=1$, $q(t)=t$, $\gamma=\frac{5}{7}$, $\sigma(t)=\frac{t}{4}$, $r(t)=t^{\frac{2}{9}}$, $\beta=\frac{11}{3}$ *and* $\zeta(t)=36t$. *Thus,*

$$\int_{t_0}^{\infty}\frac{V^{\frac{-1}{\alpha}}(t)}{a^{\frac{1}{\alpha}}(t)}\mathrm{d}t=\int_{t_0}^{\infty}\frac{\exp\left(-\int_{t_0}^{t}\frac{1}{s}\mathrm{d}s\right)}{1/t}\mathrm{d}t=t_0\int_{t_0}^{\infty}\mathrm{d}t=\infty$$

and

$$\lim_{t\to\infty}[M_1(t)+M_2(t)]=\lim_{t\to\infty}\left[\frac{(1-3/5)\left(\frac{1}{t}\right)^{\frac{5}{2}}}{(3/5)^{\frac{-3}{2}}}+\frac{2(t^{-1/2})}{(3)^{\frac{3}{2}}}\right]=\lim_{t\to\infty}\left[\frac{2\sqrt{27}}{\sqrt{3125}t^{\frac{5}{2}}}+\frac{2}{\sqrt{27t}}\right]=0.$$

Choosing $\mu(t)=\frac{t}{3}$ *and* $\rho(t)=1$, *we observe that* $\sigma(t)=\frac{t}{4}<\mu(t)=\frac{t}{3}<t$, $\tau^{-1}(\sigma(t))=\tau^{-1}(t/4)=\frac{t}{8}<t$, $\tau^{-1}(\zeta(t))=\tau^{-1}(36t)=18t>t$ *and* $\tau^{-1}(\zeta(\mu(\mu(t))))=\tau^{-1}(\zeta(\mu(t/3)))=\tau^{-1}(\zeta(t/9))=\tau^{-1}(4t)=2t>t$ *and consequently we have*

$$\limsup_{t\to\infty}\int_{t_*}^{t}\Big[k_1^{\beta}r(s)\rho(s)V(s)-\Big(\frac{\alpha\eta}{k_*^{\frac{\beta}{\alpha\eta}-1}\beta B_1(s,t_1)\rho(s)}\Big)^{\alpha}\Big(\frac{\rho'_+(s)}{\alpha+1}\Big)^{\alpha+1}\Big]\mathrm{d}s$$

$$=\limsup_{t\to\infty}\int_{t_*}^{t}\Big[k_1^{\frac{11}{3}}s^{2/9}\exp\left(\int_{t_0}^{s}\frac{1}{u}\mathrm{d}u\right)-\Big(\frac{7/3}{k_*^{\frac{4}{7}}\frac{11}{3}\int_{t_1}^{s}t_0\mathrm{d}u}\Big)(0)\Big]\mathrm{d}s=\limsup_{t\to\infty}\int_{t_*}^{t}\Big[k_1^{\frac{11}{3}}s^{2/9}\frac{s}{t_0}\Big]\mathrm{d}s=\infty,$$

$$\lim_{t\to\infty}\int_{t_0}^{t}q(s)V(s)\left[\frac{\sigma(\tau^{-1}(\sigma(s)))B_1\left(\mu(\tau^{-1}(\sigma(s))),\sigma(\tau^{-1}(\sigma(s)))\right)}{p_2(\tau^{-1}(\sigma(s)))}\right]^{\frac{\gamma}{\nu}}\mathrm{d}s=\lim_{t\to\infty}\int_{t_0}^{t}s\frac{s^2}{t_0}\left[\frac{t_0}{4}\frac{s}{96}\right]^{\frac{5}{21}}\mathrm{d}s=\infty$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\int_{\mu(s)}^{s}V^{\frac{-1}{\alpha}}(u)\left[\frac{1}{a(u)}\int_{\mu(u)}^{u}V(v)\frac{r(v)}{p_2^{\frac{\beta}{\nu}}(\tau^{-1}(\zeta(v)))}\mathrm{d}v\right]^{\frac{1}{\alpha}}\mathrm{d}u\ \mathrm{d}s$$

$$=\lim_{t\to\infty}\int_{t_0}^{t}\int_{s/3}^{s}\exp\left(-\int_{t_0}^{u}\frac{1}{v}dv\right)\left[u\int_{u/3}^{u}\exp\left(\int_{t_0}^{v}\frac{1}{\psi}\mathrm{d}\psi\right)\frac{v^{\frac{2}{9}}}{(18v)^{\frac{11}{9}}}\mathrm{d}v\right]\mathrm{d}u\ \mathrm{d}s$$

$$=\lim_{t\to\infty}\int_{t_0}^{t}\int_{s/3}^{s}t_0\int_{u/3}^{u}\frac{v^{\frac{2}{9}}}{(18v)^{\frac{11}{9}}}\frac{v}{t_0}\mathrm{d}v\ \mathrm{d}u\ \mathrm{d}s=\infty.$$

Thus all the hypotheses of Corollary 3.4 hold, and so every solution $x(t)$ *of Eq.* (3.33) *is either oscillatory or converges to zero.*

Example 3.9 *Consider the third order differential equation*

$$\left(t^2\left[\left(x^3(t)+t^{\frac{2}{15}}x^{\frac{1}{5}}(\frac{t}{3})-t^{1/3}x^{\frac{1}{3}}(\frac{t}{3})\right)''\right]^{11/3}\right)'$$
$$+\,t\left[\left(x^3(t)+t^{\frac{2}{15}}x^{\frac{1}{5}}(\frac{t}{3})-t^{1/3}x^{\frac{1}{3}}(\frac{t}{3})\right)''\right]^{11/3}+\frac{1}{t^{\frac{76}{33}}}x(\frac{t}{5})+t^5x^5(4t)=0. \tag{3.34}$$

Here, $a(t)=t^2$, $\eta=3$, $p_1(t)=t^{\frac{2}{15}}$, $\lambda=\frac{1}{5}$, $\tau_1(t)=\tau_2(t)=\tau(t)=\frac{t}{3}$, $\delta=-1$, $p_2(t)=t^{1/3}$, $\nu=\frac{1}{3}$, $b(t)=t$, $\alpha=11/3$, $q(t)=\frac{1}{t^{\frac{76}{33}}}$, $\gamma=1$, $\sigma(t)=\frac{t}{5}$, $r(t)=t^5$, $\beta=5$ *and* $\zeta(t)=4t$. *Thus,*

$$\int_{t_0}^{\infty}\frac{V^{\frac{-1}{\alpha}}(t)}{a^{\frac{1}{\alpha}}(t)}\mathrm{d}t=\int_{t_0}^{\infty}\frac{\exp\left(\frac{-3}{11}\int_{t_0}^{t}\frac{1}{s}\mathrm{d}s\right)}{t^{6/11}}\mathrm{d}t=t_0^{\frac{3}{11}}\int_{t_0}^{\infty}t^{\frac{-9}{11}}\mathrm{d}t=\infty$$

and

$$\lim_{t\to\infty}P(t)=\lim_{t\to\infty}\left[\frac{(1/3-1/5)}{1/5}\left(t^{2/15}\right)^{5/2}\left(t^{1/3}\right)^{-3/2}\right]=\lim_{t\to\infty}\left[\frac{2}{3}t^{-1/6}\right]=0.$$

Choosing $\mu(t)=\frac{t}{2}$, *then clearly* $\sigma(t)<\mu(t)<t$, $\tau^{-1}(\sigma(t))=\tau^{-1}(t/5)=\frac{3t}{5}<t$, $\tau^{-1}(\zeta(t))=\tau^{-1}(4t)=12t>t$ *and* $\tau^{-1}(\zeta(\mu(\mu(t))))=\tau^{-1}(\zeta(\mu(t/2)))=\tau^{-1}(\zeta(t/4))=\tau^{-1}(t)=3t>t$. *We can easily observe that*

$$\lim_{t\to\infty}\int_{t_0}^{t}q(s)V(s)B_2^{\frac{\gamma}{\eta}}\left(\sigma(s),t_2,t_1\right)\mathrm{d}s=\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{76}{33}}}\exp\left(\int_{t_0}^{s}\frac{1}{u}\mathrm{d}u\right)\left[\int_{t_2}^{s/5}B_1\left(u,t_1\right)\mathrm{d}u\right]^{1/3}\mathrm{d}s$$

$$=\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{76}{33}}}\frac{s}{t_0}\left[\int_{t_2}^{s/5}\int_{t_1}^{u}\frac{\exp\left(\frac{-3}{11}\int_{t_0}^{v}\frac{1}{\psi}\mathrm{d}\psi\right)}{v^{6/11}}\mathrm{d}v\mathrm{d}u\right]^{1/3}\mathrm{d}s=\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{t_0s^{\frac{43}{33}}}\left[\int_{t_2}^{s/5}\int_{t_1}^{u}\frac{t_0^{3/11}}{v^{9/11}}\mathrm{d}v\mathrm{d}u\right]^{1/3}\mathrm{d}s$$

$$=(11/2)^{1/3}\frac{1}{t_0^{10/11}}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{43}{33}}}\left[\int_{t_2}^{s/5}(u^{2/11}-t_1^{2/11})\mathrm{d}u\right]^{1/3}\mathrm{d}s$$

$$>\frac{(11/2)^{1/3}}{t_0^{10/11}}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{43}{33}}}\left[\int_{t_2}^{s/5}(u^{2/11}-t_2^{2/11})\mathrm{d}u\right]^{1/3}\mathrm{d}s$$

$$=\frac{(11/2)^{1/3}}{t_0^{10/11}}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{43}{33}}}\left[\frac{11}{13}\left(\frac{s}{5}\right)^{\frac{13}{11}}-t_2^{\frac{2}{11}}\left(\frac{s}{5}\right)-\frac{11}{13}t_2^{\frac{13}{11}}+t_2^{\frac{13}{11}}\right]^{1/3}\mathrm{d}s$$

$$>\frac{(11/2)^{1/3}}{t_0^{10/11}}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{43}{33}}}\left[\frac{11}{13}\left(\frac{s}{5}\right)^{\frac{13}{11}}-t_2^{\frac{2}{11}}\left(\frac{s}{5}\right)\right]^{1/3}\mathrm{d}s$$

$$=\left(\frac{11}{13}\right)^{\frac{1}{3}}\frac{(11/2)^{1/3}}{t_0^{10/11}}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{43}{33}}}(\frac{s}{5})^{13/33}\left[1-\frac{13}{11}\left(\frac{5t_2}{s}\right)^{\frac{2}{11}}\right]^{1/3}\mathrm{d}s$$

$$=\left(\frac{1}{5}\right)^{\frac{13}{33}}\frac{(\frac{11}{13})^{\frac{1}{3}}(\frac{11}{2})^{\frac{1}{3}}}{t_0^{\frac{10}{11}}}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{10}{11}}}\left(1-\frac{13}{33}\left(\frac{5t_2}{s}\right)^{\frac{2}{11}}-\frac{169}{1089}\left(\frac{5t_2}{s}\right)^{\frac{4}{11}}-\frac{5}{81}(\frac{13}{11})^3\left(\frac{5t_2}{s}\right)^{\frac{6}{11}}-\ldots\right)\mathrm{d}s=\infty,$$

$$\lim_{t\to\infty}\int_{t_0}^{t} q(s)V(s)\left[\frac{\sigma(\tau^{-1}(\sigma(s)))B_1\left(\mu(\tau^{-1}(\sigma(s))),\sigma(\tau^{-1}(\sigma(s)))\right)}{p_2(\tau^{-1}(\sigma(s)))}\right]^{\frac{\gamma}{\nu}} \mathrm{d}s$$

$$=\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{76}{33}}}\frac{s}{t_0}\left[\frac{\frac{3s}{25}\int_{\frac{3s}{25}}^{\frac{3s}{10}}\frac{\exp\left(\frac{-3}{11}\int_{t_0}^{u}\frac{1}{v}\mathrm{d}v\right)\mathrm{d}u}{u^{\frac{6}{11}}}}{\frac{3s}{5}}\right]^3 \mathrm{d}s=\frac{1/125}{t_0}\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\frac{43}{33}}}\left[\int_{3s/25}^{3s/10}\frac{t_0^{3/11}}{u^{9/11}}\mathrm{d}u\right]^3 \mathrm{d}s=\infty$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\int_{\mu(s)}^{s} V^{\frac{-1}{\alpha}}(u)\left[\frac{1}{a(u)}\int_{\mu(u)}^{u} V(v)\frac{r(v)}{p_2^{\frac{\beta}{\nu}}(\tau^{-1}(\zeta(v)))}\mathrm{d}v\right]^{\frac{1}{\alpha}} \mathrm{d}u\ \mathrm{d}s$$

$$=\lim_{t\to\infty}\int_{t_0}^{t}\int_{s/2}^{s}\exp\left(\frac{-3}{11}\int_{t_0}^{u}\frac{1}{v}dv\right)\left[\frac{1}{u^2}\int_{u/2}^{u}\exp\left(\int_{t_0}^{v}\frac{1}{\psi}\mathrm{d}\psi\right)\frac{v^5}{(12v)^5}\mathrm{d}v\right]^{\frac{3}{11}} \mathrm{d}u\ \mathrm{d}s=\infty.$$

It is clear that all the conditions of Corollary 3.2 which extracted to Theorem 3.5 by replacing (3.30) instead of (3.1) and $\lambda<\nu\leq 1$ instead of $\lambda<1,\nu>1$ are satisfied due to Remark 3.6. Hence, every solution of Eq. (3.34) is either oscillatory or converges to zero.

4. The case $\delta=+1$

In this section, we study the oscillatory behavior of solutions of Eq. (1.1) in the case when $\delta=+1$, $\tau_1(t)\leq t$ and $\tau_2(t)\geq t$ in which either of the three conditions $\nu<\lambda$ with $\eta<\lambda$ or $\lambda<\nu$ with $\eta<\nu$ or $\eta<\nu<\lambda$ holds.

Theorem 4.1 *Assume that $\tau_1(t)\leq t$, $\tau_2(t)\geq t$, $\nu<\lambda$, $\eta<\lambda$, and (H_1)-(H_3) hold. Furthermore, assume that*

$$\lim_{t\to\infty} G_1(t)=0, \quad \text{for any positive number } c_*. \tag{4.1}$$

If there exists a number $k_3\in(0,1)$ such that the first-order delay differential equation

$$W'(t)+k_3\frac{q(t)V(t)B_2^{\frac{\gamma}{\lambda}}(\sigma(t),t_2,t_1)}{p_1^{\frac{\gamma}{\lambda}}(\tau_1^{-1}(\sigma(t)))}W^{\frac{\gamma}{\alpha\lambda}}(\sigma(t))=0 \tag{4.2}$$

is oscillatory for sufficiently large $t_2>t_1\geq t_0$, then every solution of Eq. (1.1) is either oscillatory or converges to zero.

Proof Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Then, without loss of generality, assume that $x(t)$ is eventually positive with $\lim_{t\to\infty} x(t)\neq 0$ for $t\geq t_0$. Therefore, there exists $t_1\geq t_0$ such that $x(t)>0$, $x(\tau_1(t))>0$, $x(\tau_2(t))>0$, $x(\sigma(t))>0$ and $x(\zeta(t))>0$ for $t\geq t_1$. It follows that $z(t)>0$ for $t\geq t_1$, which means that case (3) and case (4) mentioned before in the proof of Theorem 3.1 are impossible here. Consequently, we shall study the two cases: case (1) and case (2) in detail. From (1.1), we have (3.10). In the first, suppose that **case (1)** holds. Since $z''(t)<0$ and $\left(V(t)a(t)\left(z''(t)\right)^{\alpha}\right)'<0$, then $z(t)$ must be negative, which contradicts the positivity of $z(t)$, so it is impossible. Now, suppose that **case (2)** holds. In this case, we have two possibilities $z'(t) < 0$ or $z'(t) > 0$. We can easily observe that $z'(t) < 0$ is excluded due to the fact that $\lim_{t\to\infty} x(t) \neq 0$. Now, consider the case $z'(t) > 0$ for $t \geq t_1$. From the definition of $z(t)$, we have

$$x^{\lambda}(\tau_1(t)) = \frac{1}{p_1(t)}\left[z(t) - x^{\eta}(t) - p_2(t)x^{\nu}(\tau_2(t))\right].$$

i.e.

$$x^{\lambda}(t) = \frac{1}{p_1(\tau_1^{-1}(t))}\left[z(\tau_1^{-1}(t)) - x^{\eta}(\tau_1^{-1}(t)) - p_2(\tau_1^{-1}(t))x^{\nu}(\tau_2(\tau_1^{-1}(t)))\right], \tag{4.3}$$

which means that

$$x(t) = \frac{1}{p_1^{\frac{1}{\lambda}}(\tau_1^{-1}(t))}\left[z(\tau_1^{-1}(t)) - x^{\eta}(\tau_1^{-1}(t)) - p_2(\tau_1^{-1}(t))x^{\nu}(\tau_2(\tau_1^{-1}(t)))\right]^{\frac{1}{\lambda}}.$$

i.e.

$$x^{\nu}(\tau_2(\tau_1^{-1}(t))) = \frac{1}{p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))}\left[z(\varrho_1(t)) - x^{\eta}(\varrho_1(t)) - p_2(\varrho_1(t))x^{\nu}(\tau_2(\varrho_1(t)))\right]^{\frac{\nu}{\lambda}} \tag{4.4}$$

and

$$x^{\eta}(\tau_1^{-1}(t)) = \frac{1}{p_1^{\frac{\eta}{\lambda}}(\varrho_2(t))}\left[z(\varrho_2(t)) - x^{\eta}(\varrho_2(t)) - p_2(\varrho_2(t))x^{\nu}(\tau_2(\varrho_2(t)))\right]^{\frac{\eta}{\lambda}}. \tag{4.5}$$

Substituting from (4.4) and (4.5) into (4.3), we get

$$\begin{aligned} x^{\lambda}(t) =& \frac{1}{p_1(\tau_1^{-1}(t))}[z(\tau_1^{-1}(t)) - \frac{1}{p_1^{\frac{\eta}{\lambda}}(\varrho_2(t))}\left[z(\varrho_2(t)) - x^{\eta}(\varrho_2(t)) - p_2(\varrho_2(t))x^{\nu}(\tau_2(\varrho_2(t)))\right]^{\frac{\eta}{\lambda}} \\ &- \frac{p_2(\tau_1^{-1}(t))}{p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))}\left[z(\varrho_1(t)) - x^{\eta}(\varrho_1(t)) - p_2(\varrho_1(t))x^{\nu}(\tau_2(\varrho_1(t)))\right]^{\frac{\nu}{\lambda}}]. \end{aligned} \tag{4.6}$$

Applying inequality (2.2) with $Y = 1$, we conclude from (4.6) that

$$\begin{aligned} x^{\lambda}(t) \geq& \frac{1}{p_1(\tau_1^{-1}(t))}[z(\tau_1^{-1}(t)) - \frac{1-\frac{\eta}{\lambda}}{p_1^{\frac{\eta}{\lambda}}(\varrho_2(t))} - \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(t))}{p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))} \\ &- \frac{\frac{\eta}{\lambda}}{p_1^{\frac{\eta}{\lambda}}(\varrho_2(t))}\left[z(\varrho_2(t)) - x^{\eta}(\varrho_2(t)) - p_2(\varrho_2(t))x^{\nu}(\tau_2(\varrho_2(t)))\right] \\ &- \frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(t))}{p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))}\left[z(\varrho_1(t)) - x^{\eta}(\varrho_1(t)) - p_2(\varrho_1(t))x^{\nu}(\tau_2(\varrho_1(t)))\right]] \\ \geq& \frac{1}{p_1(\tau_1^{-1}(t))}[z(\tau_1^{-1}(t)) - \frac{\frac{\eta}{\lambda}z(\varrho_2(t))}{p_1^{\frac{\eta}{\lambda}}(\varrho_2(t))} - \frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(t))z(\varrho_1(t))}{p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))} - \Big(\frac{1-\frac{\eta}{\lambda}}{p_1^{\frac{\eta}{\lambda}}(\varrho_2(t))} + \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(t))}{p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))}\Big)]. \end{aligned} \tag{4.7}$$

Proceeding as in the proof of Theorem 3.1, we have (3.13) for $t \geq t_3 > t_2 \geq t_1$, which means that

$$\left(\frac{z(t)}{B_2(t, t_3, t_2)}\right)' \leq 0, \tag{4.8}$$

i.e. $\frac{z(t)}{B_2(t,t_3,t_2)}$ is a positive decreasing function. Since $\varrho_2(t) \geq \tau_1^{-1}(t)$ and $\varrho_1(t) \geq \tau_2(\tau_1^{-1}(t)) \geq \tau_1^{-1}(t)$, then we have

$$z(\varrho_2(t)) \leq \frac{B_2\left(\varrho_2(t), t_3, t_2\right)}{B_2\left(\tau_1^{-1}(t), t_3, t_2\right)} z(\tau_1^{-1}(t))$$

and

$$z(\varrho_1(t)) \leq \frac{B_2\left(\varrho_1(t)), t_3, t_2\right)}{B_2\left(\tau_1^{-1}(t), t_3, t_2\right)} z(\tau_1^{-1}(t)).$$

Substituting into (4.7), we get

$$\begin{aligned} x^{\lambda}(t) \geq & \frac{z(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(t))}[1 - \frac{\frac{\eta}{\lambda} B_2\left(\varrho_2(t)), t_3, t_2\right)}{B_2\left(\tau_1^{-1}(t), t_3, t_2\right) p_1^{\frac{\eta}{\lambda}}(\varrho_2(t))} - \frac{\frac{\nu}{\lambda} p_2(\tau_1^{-1}(t)) B_2\left(\varrho_1(t), t_3, t_2\right)}{B_2\left(\tau_1^{-1}(t), t_3, t_2\right) p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))} \\ & - \Big(\frac{1-\frac{\eta}{\lambda}}{p_1^{\frac{\eta}{\lambda}}(\varrho_2(t))} + \frac{(1-\frac{\nu}{\lambda}) p_2(\tau_1^{-1}(t))}{p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))}\Big) \frac{1}{z(\tau_1^{-1}(t))}]. \end{aligned}$$

Since $z(t)$ is positive and increasing, then there exists a constant $c_* > 0$ such that $z(t) \geq c_*$ for $t \geq t_4 \geq t_3$ and consequently, we have

$$\begin{aligned} x^{\lambda}(t) \geq & \frac{z(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(t))}[1 - (\frac{\frac{\eta}{\lambda} B_2\left(\varrho_2(t), t_3, t_2\right)}{B_2\left(\tau_1^{-1}(t), t_3, t_2\right) p_1^{\frac{\eta}{\lambda}}(\varrho_2(t))} + \frac{\frac{\nu}{\lambda} p_2(\tau_1^{-1}(t)) B_2\left(\varrho_1(t), t_3, t_2\right)}{B_2\left(\tau_1^{-1}(t), t_3, t_2\right) p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))} \\ & + \frac{1-\frac{\eta}{\lambda}}{c_* p_1^{\frac{\eta}{\lambda}}(\varrho_2(t))} + \frac{(1-\frac{\nu}{\lambda}) p_2(\tau_1^{-1}(t))}{c_* p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))})]. \end{aligned}$$

It follows from (4.1) that there exists a constant $c_4 \in (0,1)$ such that

$$x^{\lambda}(t) \geq \frac{c_4 z(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(t))} \text{ for } t \geq t_4,$$

i.e.

$$x(t) \geq \frac{c_4^{\frac{1}{\lambda}} z^{\frac{1}{\lambda}}(\tau_1^{-1}(t))}{p_1^{\frac{1}{\lambda}}(\tau_1^{-1}(t))}. \tag{4.9}$$

Substituting into (3.10), we get

$$\begin{aligned} \left(V(t)a(t)\left(z''(t)\right)^{\alpha}\right)' \leq & - c_4^{\frac{\gamma}{\lambda}} q(t) V(t) \frac{z^{\frac{\gamma}{\lambda}}(\tau_1^{-1}(\sigma(t)))}{p_1^{\frac{\gamma}{\lambda}}(\tau_1^{-1}(\sigma(t)))} - c_4^{\frac{\beta}{\lambda}} r(t) V(t) \frac{z^{\frac{\beta}{\lambda}}(\tau_1^{-1}(\zeta(t)))}{p_1^{\frac{\beta}{\lambda}}(\tau_1^{-1}(\zeta(t)))} \\ \leq & - c_4^{\frac{\gamma}{\lambda}} q(t) V(t) \frac{z^{\frac{\gamma}{\lambda}}(\tau_1^{-1}(\sigma(t)))}{p_1^{\frac{\gamma}{\lambda}}(\tau_1^{-1}(\sigma(t)))}. \end{aligned} \tag{4.10}$$

Since $z(t)$ is increasing and $\tau_1^{-1}(\sigma(t)) \geq \sigma(t)$, then we have $z(\tau_1^{-1}(\sigma(t))) \geq z(\sigma(t))$ and it follows from (4.10) that

$$\left(V(t)a(t)\left(z''(t)\right)^{\alpha}\right)' \leq -c_4^{\frac{\gamma}{\lambda}} q(t) V(t) \frac{z^{\frac{\gamma}{\lambda}}(\sigma(t))}{p_1^{\frac{\gamma}{\lambda}}(\tau_1^{-1}(\sigma(t)))}. \tag{4.11}$$

Since $\left(V(t)a(t)\left(z''(t)\right)^{\alpha}\right)$ is positive and decreasing, then as in the proof of Theorem 3.1, we arrive at (3.13). Substituting from (3.13) into (4.11), we get

$$\left(V(t)a(t)\left(z''(t)\right)^{\alpha}\right)' \le -c_4^{\frac{\gamma}{\lambda}} \frac{q(t)V(t)B_2^{\frac{\gamma}{\lambda}}(\sigma(t),t_3,t_2)}{p_1^{\frac{\gamma}{\lambda}}(\tau_1^{-1}(\sigma(t)))}\left[V(\sigma(t))a(\sigma(t))\left(z''(\sigma(t))\right)^{\alpha}\right]^{\frac{\gamma}{\alpha\lambda}},$$

i.e.

$$W'(t) + c_4^{\frac{\gamma}{\lambda}} \frac{q(t)V(t)B_2^{\frac{\gamma}{\lambda}}(\sigma(t),t_3,t_2)}{p_1^{\frac{\gamma}{\lambda}}(\tau_1^{-1}(\sigma(t)))} W^{\frac{\gamma}{\alpha\lambda}}(\sigma(t)) \le 0 \tag{4.12}$$

has a positive solution $W(t)$. From Lemma 2.1 (i), the differential equation (4.2) corresponds to the inequality (4.12) also has a positive solution, which is a contradiction. This completes the proof. □

Theorem 4.2 *Assume that $\tau_1(t) \le t$, $\tau_2(t) \ge t$, $\lambda < \nu$, $\eta < \nu$, and (H_1)-(H_3) hold. Furthermore, assume that*

$$\lim_{t\to\infty} G_2(t) = 0. \tag{4.13}$$

If there exists a number $k_4 \in (0,1)$ such that the first-order delay differential equation

$$W'(t) + k_4 \frac{q(t)V(t)B_2^{\frac{\gamma}{\nu}}\left(\tau_2^{-1}(\sigma(t)),t_2,t_1\right)}{p_2^{\frac{\gamma}{\nu}}(\tau_2^{-1}(\sigma(t)))} W^{\frac{\gamma}{\alpha\nu}}(\tau_2^{-1}(\sigma(t))) = 0 \tag{4.14}$$

is oscillatory for sufficiently large $t_2 > t_1 \ge t_0$, then every solution of Eq. (1.1) is either oscillatory or converges to zero.

Proof Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Proceeding as in the proof of Theorem 4.1, we conclude that the possible case of $z(t)$ is $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$ and $\left(V(t)a(t)\left(z''(t)\right)^{\alpha}\right)' < 0$ holds for $t \ge t_1 \ge t_0$. From the definition of $z(t)$, we have

$$x^{\nu}(\tau_2(t)) = \frac{1}{p_2(t)}\left[z(t) - x^{\eta}(t) - p_1(t)x^{\lambda}(\tau_1(t))\right],$$

i.e.

$$x^{\nu}(t) = \frac{1}{p_2(\tau_2^{-1}(t))}\left[z(\tau_2^{-1}(t)) - x^{\eta}(\tau_2^{-1}(t)) - p_1(\tau_2^{-1}(t))x^{\lambda}(\tau_1(\tau_2^{-1}(t)))\right]. \tag{4.15}$$

Thus,

$$x(t) = \frac{1}{p_2^{\frac{1}{\nu}}(\tau_2^{-1}(t))}\left[z(\tau_2^{-1}(t)) - x^{\eta}(\tau_2^{-1}(t)) - p_1(\tau_2^{-1}(t))x^{\lambda}(\tau_1(\tau_2^{-1}(t)))\right]^{\frac{1}{\nu}}.$$

It follows that

$$x^{\lambda}(\tau_1(\tau_2^{-1}(t))) = \frac{1}{p_2^{\frac{\lambda}{\nu}}(\varrho_3(t))}\left[z(\varrho_3(t)) - x^{\eta}(\varrho_3(t)) - p_1(\varrho_3(t))x^{\lambda}(\tau_1(\varrho_3(t)))\right]^{\frac{\lambda}{\nu}} \tag{4.16}$$

and

$$x(\tau_2^{-1}(t)) = \frac{1}{p_2^{\frac{\eta}{\nu}}(\varrho_4(t))}\left[z(\varrho_4(t)) - x^{\eta}(\varrho_4(t)) - p_1(\varrho_4(t))x^{\lambda}(\tau_1(\varrho_4(t)))\right]^{\frac{\eta}{\nu}}. \tag{4.17}$$

Substituting from (4.16) and (4.17) into (4.15), we get

$$x^{\nu}(t) = \frac{1}{p_2(\tau_2^{-1}(t))}\Big[z(\tau_2^{-1}(t)) - \frac{1}{p_2^{\frac{\eta}{\nu}}(\varrho_4(t))}\left[z(\varrho_4(t)) - x^{\eta}(\varrho_4(t)) - p_1(\varrho_4(t))x^{\lambda}(\tau_1(\varrho_4(t)))\right]^{\frac{\eta}{\nu}}$$
$$- \frac{p_1(\tau_2^{-1}(t))}{p_2^{\frac{\lambda}{\nu}}(\varrho_3(t))}\left[z(\varrho_3(t)) - x^{\eta}(\varrho_3(t)) - p_1(\varrho_3(t))x^{\lambda}(\tau_1(\varrho_3(t)))\right]^{\frac{\lambda}{\nu}}\Big]. \tag{4.18}$$

Applying inequality (2.2) with $Y = 1$, we conclude from (4.18) that

$$x^{\nu}(t) \geq \frac{1}{p_2(\tau_2^{-1}(t))}\Big[z(\tau_2^{-1}(t)) - \frac{\frac{\eta}{\nu}}{p_2^{\frac{\eta}{\nu}}(\varrho_4(t))}\left[z(\varrho_4(t)) - x^{\eta}(\varrho_4(t)) - p_1(\varrho_4(t))x^{\lambda}(\tau_1(\varrho_4(t)))\right]$$
$$- \frac{\frac{\lambda}{\nu}p_1(\tau_2^{-1}(t))}{p_2^{\frac{\lambda}{\nu}}(\varrho_3(t))}\left[z(\varrho_3(t)) - x^{\eta}(\varrho_3(t)) - p_1(\varrho_3(t))x^{\lambda}(\tau_1(\varrho_3(t)))\right] - \frac{(1-\frac{\eta}{\nu})}{p_2^{\frac{\eta}{\nu}}(\varrho_4(t))} - \frac{(1-\frac{\lambda}{\nu})p_1(\tau_2^{-1}(t))}{p_2^{\frac{\lambda}{\nu}}(\varrho_3(t))}\Big]$$
$$\geq \frac{1}{p_2(\tau_2^{-1}(t))}\Big[z(\tau_2^{-1}(t)) - \frac{\frac{\eta}{\nu}z(\varrho_4(t))}{p_2^{\frac{\eta}{\nu}}(\varrho_4(t))} - \frac{\frac{\lambda}{\nu}p_1(\tau_2^{-1}(t))z(\varrho_3(t))}{p_2^{\frac{\lambda}{\nu}}(\varrho_3(t))} - \frac{(1-\frac{\eta}{\nu})}{p_2^{\frac{\eta}{\nu}}(\varrho_4(t))} - \frac{(1-\frac{\lambda}{\nu})p_1(\tau_2^{-1}(t))}{p_2^{\frac{\lambda}{\nu}}(\varrho_3(t))}\Big]. \tag{4.19}$$

As in the proof of Theorem 4.1, we have (3.13) and (4.8). Thus, $\frac{z(t)}{B_2(t,t_3,t_2)}$ is a positive decreasing function for $t \geq t_3 > t_2 \geq t_1$. Since $z(t)$ is increasing, $\varrho_4(t) \leq \tau_2^{-1}(t)$ and $\varrho_3(t) \leq \tau_1(\tau_2^{-1}(t)) \leq \tau_2^{-1}(t)$, then we have $z(\varrho_4(t)) \leq z(\tau_2^{-1}(t))$ and $z(\varrho_3(t)) \leq z(\tau_2^{-1}(t))$. Substituting into (4.19), we get

$$x^{\nu}(t) \geq \frac{z(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(t))}\Big[1 - \Big(\frac{\frac{\eta}{\nu}}{p_2^{\frac{\eta}{\nu}}(\varrho_4(t))} + \frac{\frac{\lambda}{\nu}p_1(\tau_2^{-1}(t))}{p_2^{\frac{\lambda}{\nu}}(\varrho_3(t))} + \Big(\frac{(1-\frac{\eta}{\nu})}{p_2^{\frac{\eta}{\nu}}(\varrho_4(t))} - \frac{(1-\frac{\lambda}{\nu})p_1(\tau_2^{-1}(t))}{p_2^{\frac{\lambda}{\nu}}(\varrho_3(t))}\Big)\frac{1}{z(\tau_2^{-1}(t))}\Big)\Big].$$

Then as in the proof of Theorem 4.1, there exists a positive constant c_* such that $z(t) \geq c_*$ for $t \geq t_4 \geq t_3$ and consequently, we have

$$x^{\nu}(t) \geq \frac{z(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(t))}\Big[1 - \Big(\frac{\frac{\eta}{\nu}}{p_2^{\frac{\eta}{\nu}}(\varrho_4(t))} + \frac{\frac{\lambda}{\nu}p_1(\tau_2^{-1}(t))}{p_2^{\frac{\lambda}{\nu}}(\varrho_3(t))} + \frac{(1-\frac{\eta}{\nu})}{c_* p_2^{\frac{\eta}{\nu}}(\varrho_4(t))} - \frac{(1-\frac{\lambda}{\nu})p_1(\tau_2^{-1}(t))}{c_* p_2^{\frac{\lambda}{\nu}}(\varrho_3(t))}\Big)\Big].$$

It follows from (4.13) that there exists a constant $c_5 \in (0,1)$ such that

$$x^{\nu}(t) \geq \frac{c_5 z(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(t))} \quad \text{for } t \geq t_4,$$

i.e.

$$x(t) \geq \frac{c_5^{\frac{1}{\nu}} z^{\frac{1}{\nu}}(\tau_2^{-1}(t))}{p_2^{\frac{1}{\nu}}(\tau_2^{-1}(t))}. \tag{4.20}$$

Substituting into (3.10), we get

$$\left(V(t)a(t)\left(z''(t)\right)^{\alpha}\right)' \leq -c_5^{\frac{\gamma}{\nu}} q(t)V(t)\frac{z^{\frac{\gamma}{\nu}}(\tau_2^{-1}(\sigma(t)))}{p_2^{\frac{\gamma}{\nu}}(\tau_2^{-1}(\sigma(t)))} - c_5^{\frac{\beta}{\nu}} r(t)V(t)\frac{z^{\frac{\beta}{\nu}}(\tau_2^{-1}(\zeta(t)))}{p_2^{\frac{\beta}{\nu}}(\tau_2^{-1}(\zeta(t)))}$$
$$\leq -c_5^{\frac{\gamma}{\nu}} q(t)V(t)\frac{z^{\frac{\gamma}{\nu}}(\tau_2^{-1}(\sigma(t)))}{p_2^{\frac{\gamma}{\nu}}(\tau_2^{-1}(\sigma(t)))}. \tag{4.21}$$

It follows from (3.13) that

$$\left(V(t)a(t)\left(z''(t)\right)^{\alpha}\right)' \leq -c_5^{\frac{\gamma}{\nu}}\frac{q(t)V(t)B_2^{\frac{\gamma}{\nu}}\left(\tau_2^{-1}(\sigma(t)),t_3,t_2\right)}{p_2^{\frac{\gamma}{\nu}}(\tau_2^{-1}(\sigma(t)))} \times \left[V(\tau_2^{-1}(\sigma(t)))a(\tau_2^{-1}(\sigma(t)))\left(z''(\tau_2^{-1}(\sigma(t)))\right)^{\alpha}\right]^{\frac{\gamma}{\alpha\nu}}.$$

i.e.

$$W'(t)+c_5^{\frac{\gamma}{\nu}}\frac{q(t)V(t)B_2^{\frac{\gamma}{\nu}}\left(\tau_2^{-1}(\sigma(t)),t_3,t_2\right)}{p_2^{\frac{\gamma}{\nu}}(\tau_2^{-1}(\sigma(t)))}W^{\frac{\gamma}{\alpha\nu}}(\tau_2^{-1}(\sigma(t))) \leq 0 \tag{4.22}$$

has a positive solution $W(t)$. From Lemma 2.1 (i), the delay differential equation (4.14) corresponds to the inequality (4.22) also has a positive solution. This completes the proof. □

Theorem 4.3 *Assume that $\tau_1(t) \leq t$, $\tau_2(t) \geq t$, $\eta < \nu < \lambda$, and (H_1)-(H_3) hold. Furthermore, assume that*

$$\lim_{t\to\infty} G_3(t) = 0. \tag{4.23}$$

If there exists a number $k_3 \in (0,1)$ such that the first-order delay differential equation (4.2) is oscillatory for sufficiently large $t_2 > t_1 \geq t_0$, then every solution of Eq. (1.1) is either oscillatory or converges to zero.

Proof Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Proceeding as in the proof of Theorem 4.1, we conclude that the possible case is $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$ and $\left(V(t)a(t)\left(z''(t)\right)^{\alpha}\right)' < 0$ for $t \geq t_1 \geq t_0$. From the definition of $z(t)$, we have (4.3). Now, we have

$$x^{\eta}(\tau_1^{-1}(t)) = \frac{1}{p_2^{\frac{\eta}{\nu}}(\varrho_5(t))}\left[z(\varrho_5(t)) - x^{\eta}(\varrho_5(t)) - p_1(\varrho_5(t))x^{\lambda}(\tau_1(\varrho_5(t)))\right]^{\frac{\eta}{\nu}} \tag{4.24}$$

and

$$x^{\nu}(\tau_2(\tau_1^{-1}(t))) = \frac{1}{p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))}\left[z(\varrho_1(t)) - x^{\eta}(\varrho_1(t)) - p_2(\varrho_1(t))x^{\nu}(\tau_2(\varrho_1(t)))\right]^{\frac{\nu}{\lambda}}. \tag{4.25}$$

Substituting from (4.24) and (4.25) into (4.3), we get

$$\begin{aligned} x^{\lambda}(t) =& \frac{z(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(t))} - \frac{1}{p_1(\tau_1^{-1}(t))}\left[\frac{\left[z(\varrho_5(t)) - x^{\eta}(\varrho_5(t)) - p_1(\varrho_5(t))x^{\lambda}(\tau_1(\varrho_5(t)))\right]^{\frac{\eta}{\nu}}}{p_2^{\frac{\eta}{\nu}}(\varrho_5(t))}\right] \\ &- \frac{p_2(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(t))}\left[\frac{\left[z(\varrho_1(t)) - x^{\eta}(\varrho_1(t)) - p_2(\varrho_1(t))x^{\nu}(\tau_2(\varrho_1(t)))\right]^{\frac{\nu}{\lambda}}}{p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))}\right]. \end{aligned}$$

Applying the inequality (2.2) with $Y = 1$, we obtain

$$x^{\lambda}(t) = \frac{z(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(t))} - \frac{\frac{\eta}{\nu}z(\varrho_5(t))}{p_1(\tau_1^{-1}(t))p_2^{\frac{\eta}{\nu}}(\varrho_5(t))} - \frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(t))z(\varrho_1(t))}{p_1(\tau_1^{-1}(t))p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))} - \frac{(1-\frac{\eta}{\nu})}{p_1(\tau_1^{-1}(t))p_2^{\frac{\eta}{\nu}}(\varrho_5(t))} - \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(t))p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))}. \tag{4.26}$$

As in the proof of Theorem 4.1, we have (4.8) for $t \geq t_3 > t_2 \geq t_1$. Since $z(t)$ is increasing, $\frac{z(t)}{B_2(t,t_3,t_2)}$ is decreasing, $\varrho_5(t) \leq \tau_1^{-1}(t)$ and $\varrho_1(t) \geq \tau_1^{-1}(t)$, we have $z(\varrho_5(t)) \leq z(\tau_1^{-1}(t))$ and $z(\varrho_1(t)) \leq \frac{B_2(\varrho_1(t),t_3,t_2)}{B_2(\tau_1^{-1}(t),t_3,t_2)} z(\tau_1^{-1}(t))$. Therefore, from (4.26), we have

$$x^\lambda(t) = \frac{z(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(t))}[1 - \frac{\frac{\eta}{\nu}}{p_2^{\frac{\eta}{\nu}}(\varrho_5(t))} - \frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(t))B_2(\varrho_1(t),t_3,t_2)}{p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))B_2(\tau_1^{-1}(t),t_3,t_2)} - (\frac{(1-\frac{\eta}{\nu})}{p_2^{\frac{\eta}{\nu}}(\varrho_5(t))} + \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(t))}{p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))})\frac{1}{z(\tau_1^{-1}(t))}]$$

$$\geq \frac{z(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(t))}[1 - \frac{\frac{\eta}{\nu}}{p_2^{\frac{\eta}{\nu}}(\varrho_5(t))} - \frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(t))B_2(\varrho_1(t),t_3,t_2)}{p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))B_2(\tau_1^{-1}(t),t_3,t_2)} - \frac{(1-\frac{\eta}{\nu})}{c_* p_2^{\frac{\eta}{\nu}}(\varrho_5(t))} - \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(t))}{c_* p_1^{\frac{\nu}{\lambda}}(\varrho_1(t))}]$$

for $t \geq t_4 \geq t_3$. Now from (4.23), it follows that there exists a constant $c_6 \in (0,1)$ such that

$$x^\lambda(t) \geq \frac{c_6 z(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(t))} \quad \text{for } t \geq t_4,$$

i.e.

$$x(t) \geq \frac{c_6^{\frac{1}{\lambda}} z^{\frac{1}{\lambda}}(\tau_1^{-1}(t))}{p_1^{\frac{1}{\lambda}}(\tau_1^{-1}(t))}. \tag{4.27}$$

Completing the proof as in the proof of Theorem 4.1 by replacing (4.9) by(4.27), we get the same conclusion of the theorem. This completes the proof. □

Following the same proof of Theorem 4.1, by replacing (4.10) by the inequality

$$\left(V(t)a(t)\left(z''(t)\right)^\alpha\right)' \leq -c_4^{\frac{\beta}{\lambda}} r(t)V(t)\frac{z^{\frac{\beta}{\lambda}}(\tau_1^{-1}(\zeta(t)))}{p_1^{\frac{\beta}{\lambda}}(\tau_1^{-1}(\zeta(t)))}$$

and using the fact that $z(t)$ is increasing, we can easily get the following result.

Corollary 4.4 *Assume that all conditions of Theorem 4.1 hold by replacing (4.2) by the following equation*

$$W'(t) + k_3 \frac{r(t)V(t)B_2^{\frac{\beta}{\lambda}}(t,t_2,t_1)}{p_1^{\frac{\beta}{\lambda}}(\tau_1^{-1}(\zeta(t)))} W^{\frac{\beta}{\alpha\lambda}}(t) = 0, \tag{4.28}$$

we get the same conclusion of Theorem 4.1.

Following the same proof of Theorem 4.2, by replacing (4.21) by the inequality

$$\left(V(t)a(t)\left(z''(t)\right)^\alpha\right)' \leq -c_5^{\frac{\beta}{\nu}} r(t)V(t)\frac{z^{\frac{\beta}{\nu}}(\tau_2^{-1}(\zeta(t)))}{p_2^{\frac{\beta}{\nu}}(\tau_2^{-1}(\zeta(t)))}$$

and using the fact that $z(t)$ is increasing, we can easily get the following result.

Corollary 4.5 *Assume that all conditions of Theorem 4.2 hold by replacing the following equation*

$$W'(t) + k_4 r(t)V(t)\left[\frac{B_2\left(\tau_2^{-1}(\zeta(t)),t_2,t_1\right)B_2(t,t_2,t_1)}{p_2(\tau_2^{-1}(\zeta(t)))B_2(\zeta(t),t_2,t_1)}\right]^{\frac{\beta}{\nu}} W^{\frac{\beta}{\alpha\nu}}(t) = 0 \tag{4.29}$$

instead of (4.14), we get the same conclusion.

Following the same proof of Theorem 4.3, by replacing the inequality

$$\left(V(t)a(t)\left(z''(t)\right)^{\alpha}\right)' \leq -c_6^{\frac{\gamma}{\lambda}} q(t)V(t)\frac{z^{\frac{\gamma}{\lambda}}(\tau_1^{-1}(\sigma(t)))}{p_1^{\frac{\gamma}{\lambda}}(\tau_1^{-1}(\sigma(t)))}$$

by the inequality

$$\left(V(t)a(t)\left(z''(t)\right)^{\alpha}\right)' \leq -c_6^{\frac{\beta}{\lambda}} r(t)V(t)\frac{z^{\frac{\beta}{\lambda}}(\tau_1^{-1}(\zeta(t)))}{p_1^{\frac{\beta}{\lambda}}(\tau_1^{-1}(\zeta(t)))}$$

and using the fact that $z(t)$ is increasing, we can easily get the following result.

Corollary 4.6 *Assume that all conditions of Theorem 4.3 hold by replacing equation (4.28) instead of (4.2), we get the same conclusion.*

Based on Theorem 4.1, Theorem 4.2, Theorem 4.3, Corollary 4.4, Corollary 4.5 and Corollary 4.6, respectively with the same manner of the proof of Corollary 3.2, we have directly the following results.

Corollary 4.7 *Assume that all the hypotheses of Theorem 4.1 hold by replacing the condition*

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{q(s)V(s)B_2^{\frac{\gamma}{\lambda}}(\sigma(s),t_2,t_1)}{p_1^{\frac{\gamma}{\lambda}}(\tau_1^{-1}(\sigma(s)))}\mathrm{d}s=\infty \text{ for } \gamma<\alpha\lambda \tag{4.30}$$

instead of the condition on equation (4.2), then the conclusion of Theorem 4.1 holds.

Corollary 4.8 *Assume that all the hypotheses of Theorem 4.2 hold by replacing the condition*

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{q(s)V(s)B_2^{\frac{\gamma}{\nu}}\left(\tau_2^{-1}(\sigma(s)),t_2,t_1\right)}{p_2^{\frac{\gamma}{\nu}}(\tau_2^{-1}(\sigma(s)))}\mathrm{d}s=\infty \text{ for } \gamma<\alpha\nu$$

instead of the condition on equation (4.14), then the conclusion of Theorem 4.2 holds.

Corollary 4.9 *Assume that all the hypotheses of Theorem 4.3 hold by replacing condition (4.30) instead of the condition on equation (4.2), then the conclusion of Theorem 4.3 holds.*

Corollary 4.10 *Assume that all the hypotheses of Corollary 4.4 hold by replacing the condition*

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{r(s)V(s)B_2^{\frac{\beta}{\lambda}}(s,t_2,t_1)}{p_1^{\frac{\beta}{\lambda}}(\tau_1^{-1}(\zeta(s)))}\mathrm{d}s=\infty \text{ for } \beta<\alpha\lambda \tag{4.31}$$

instead of the condition on equation (4.28), then the conclusion of Corollary 4.4 holds.

Corollary 4.11 *Assume that all the hypotheses of Corollary 4.5 hold by replacing*

$$\lim_{t\to\infty}\int_{t_0}^{t} r(s)V(s)\left[\frac{B_2\left(\tau_2^{-1}(\zeta(s)),t_2,t_1\right)B_2\left(s,t_2,t_1\right)}{p_2(\tau_2^{-1}(\zeta(s)))B_2\left(\zeta(s),t_2,t_1\right)}\right]^{\frac{\beta}{\nu}}\mathrm{d}s=\infty \text{ for } \beta<\alpha\nu$$

instead of the condition on equation (4.29), then the conclusion of Corollary 4.5 holds.

Corollary 4.12 *Assume that all the hypotheses of Corollary 4.6 hold by replacing condition (4.31) instead of the condition on equation (4.2), then the conclusion of Corollary 4.6 holds.*

Example 4.13 *Consider the third-order differential equation*

$$\left(t^3\left[\left(x^{\frac{1}{5}}(t)+t^5x^5(\frac{t}{2})+\frac{1}{t}x^{\frac{1}{3}}(3t)\right)''\right]^5\right)' + t^2\left[\left(x^{\frac{1}{5}}(t)+t^5x^5(\frac{t}{2})+\frac{1}{t}x^{\frac{1}{3}}(3t)\right)''\right]^5 + \frac{1}{t^{\frac{157}{75}}}x^{\frac{7}{3}}(\frac{t}{3})+\frac{1}{t^2}x^{\frac{3}{11}}(2t)=0, \quad t\geq 2. \tag{4.32}$$

Here, $a(t)=t^3,\ \eta=\frac{1}{5},\ p_1(t)=t^5,\ \lambda=5,\ \tau_1(t)=\frac{t}{2},\ \tau_2(t)=3t,\ \delta=+1,\ p_2(t)=\frac{1}{t},\ \nu=\frac{1}{3},\ b(t)=t^2,$ $\alpha=5,\ q(t)=\frac{1}{t^{\frac{157}{75}}},\ \gamma=\frac{7}{3},\ \sigma(t)=\frac{t}{3},\ r(t)=\frac{1}{t^2},\ \beta=\frac{3}{11}$ *and* $\zeta(t)=2t$. *Thus,*

$$\int_{t_0}^{\infty}\frac{V^{\frac{-1}{\alpha}}(t)}{a^{\frac{1}{\alpha}}(t)}\mathrm{d}t=\int_{t_0}^{\infty}\frac{\exp\left(\frac{-1}{5}\int_{t_0}^{t}\frac{1}{s}\mathrm{d}s\right)}{t^{\frac{3}{5}}}\mathrm{d}t=t_0^{\frac{1}{5}}\int_{t_0}^{\infty}t^{\frac{-4}{5}}\mathrm{d}t=\infty,$$

$$\lim_{t\to\infty}G_1(t)=\lim_{t\to\infty}\Bigg[\frac{(1/25)(5t_0^{1/5})\left[\frac{5}{6}(4t)^{\frac{6}{5}}-t_1^{\frac{1}{5}}(4t)-\frac{5}{6}t_2^{\frac{6}{5}}+t_1^{\frac{1}{5}}t_2\right]}{(5t_0^{1/5})(4t)^{\frac{1}{5}}\left[\frac{5}{6}(2t)^{\frac{6}{5}}-t_1^{\frac{1}{5}}(2t)-\frac{5}{6}t_2^{\frac{6}{5}}+t_1^{\frac{1}{5}}t_2\right]}$$
$$+\frac{(1/15)\frac{1}{2t}(5t_0^{1/5})\left[\frac{5}{6}(12t)^{\frac{6}{5}}-t_1^{\frac{1}{5}}(12t)-\frac{5}{6}t_2^{\frac{6}{5}}+t_1^{\frac{1}{5}}t_2\right]}{(5t_0^{1/5})(12t)^{\frac{1}{3}}\left[\frac{5}{6}(2t)^{\frac{6}{5}}-t_1^{\frac{1}{5}}(2t)-\frac{5}{6}t_2^{\frac{6}{5}}+t_1^{\frac{1}{5}}t_2\right]}+\frac{24/25}{c_*(4t)^{\frac{1}{5}}}+\frac{(14/15)\frac{1}{2t}}{c_*(12t)^{\frac{1}{3}}}\Bigg]=0$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{q(s)V(s)B_2^{\frac{\gamma}{\lambda}}(\sigma(s),t_2,t_1)}{p_1^{\frac{\gamma}{\lambda}}(\tau_1^{-1}(\sigma(s)))}\mathrm{d}s=\lim_{t\to\infty}\int_{t_0}^{t}\frac{\frac{1}{s^{\frac{157}{75}}}\frac{s}{t_0}(5t_0^{\frac{1}{5}})^{\frac{7}{15}}\left[\frac{5}{6}(s/3)^{\frac{6}{5}}-t_1^{\frac{1}{5}}(s/3)-\frac{5}{6}t_2^{\frac{6}{5}}+t_1^{\frac{1}{5}}t_2\right]^{\frac{7}{15}}}{\left(\frac{2s}{3}\right)^{7/15}}\mathrm{d}s$$
$$>\lim_{t\to\infty}\int_{t_0}^{t}\frac{(5)^{7/15}\frac{1}{t_0^{\frac{68}{75}}}\frac{1}{s^{82/75}}\left[\frac{5}{6}(s/3)^{\frac{6}{5}}-\frac{5}{6}t_2^{\frac{6}{5}}-t_2^{\frac{1}{5}}(s/3-t_2)\right]^{\frac{7}{15}}}{\left(\frac{2s}{3}\right)^{7/15}}\mathrm{d}s$$
$$>\lim_{t\to\infty}\int_{t_0}^{t}\frac{(5)^{7/15}\frac{1}{t_0^{\frac{68}{75}}}\frac{1}{s^{82/75}}\left[\frac{5}{6}(s/3)^{\frac{6}{5}}-t_2^{\frac{1}{5}}(s/3)\right]^{\frac{7}{15}}}{\left(\frac{2s}{3}\right)^{7/15}}\mathrm{d}s=\infty.$$

It is clear that all the hypotheses of Corollary 4.7 are satisfied, and so every solution of Eq. (4.32) is either oscillatory or converges to zero.

Example 4.14 *Consider the third order differential equation*

$$\left(t\left[\left(x^3(t)+\frac{1}{t^{\frac{3}{2}}}x(\frac{t}{4})+5tx^7(6t)\right)''\right]^3\right)'$$
$$+\left[\left(x^3(t)+\frac{1}{t^{\frac{3}{2}}}x(\frac{t}{4})+5tx^7(6t)\right)''\right]^3+\frac{1}{t^{\frac{31}{15}}}x^{\frac{7}{5}}(t)+\frac{1}{2t}x^{11}(7t)=0,\quad t\geq 1. \tag{4.33}$$

Here, $a(t)=t\,,\ \eta=3\,,\ p_1(t)=\frac{1}{t^{\frac{3}{2}}}\,,\ \lambda=1\,,\ \tau_1(t)=\frac{t}{4}\,,\ \tau_2(t)=6t\,,\ \delta=+1\,,\ p_2(t)=5t\,,\ \nu=7\,,\ b(t)=1\,,\ \alpha=3\,,$ $q(t)=\frac{1}{t^{\frac{31}{15}}}\,,\ \gamma=\frac{7}{5}\,,\ \sigma(t)=t\,,\ r(t)=\frac{1}{2t}\,,\ \beta=11$ *and* $\zeta(t)=7t$. *Thus,*

$$\int_{t_0}^{\infty}\frac{V^{\frac{-1}{\alpha}}(t)}{a^{\frac{1}{\alpha}}(t)}\mathrm{d}t=\int_{t_0}^{\infty}\frac{\exp\left(\frac{-1}{3}\int_{t_0}^{t}\frac{1}{s}\mathrm{d}s\right)}{t^{\frac{1}{3}}}\mathrm{d}t=t_0^{\frac{1}{3}}\int_{t_0}^{\infty}t^{\frac{-2}{3}}\mathrm{d}t=\infty,$$

$$\lim_{t\to\infty}G_2(t)=\lim_{t\to\infty}\left[\frac{\frac{3}{7}}{\left(\frac{5t}{36}\right)^{\frac{3}{7}}}+\frac{(1/7)\frac{1}{(t/6)^{\frac{3}{2}}}}{\left(\frac{5t}{144}\right)^{\frac{1}{7}}}+\frac{1-\frac{3}{7}}{c_*\left(\frac{5t}{36}\right)^{\frac{3}{7}}}+\frac{(1-\frac{1}{7})\frac{1}{(t/6)^{\frac{3}{2}}}}{c_*\left(\frac{5t}{144}\right)^{\frac{1}{7}}}\right]=0$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{q(s)V(s)B_2^{\frac{\gamma}{\nu}}\left(\tau_2^{-1}(\sigma(s)),t_2,t_1\right)}{p_2^{\frac{\gamma}{\nu}}(\tau_2^{-1}(\sigma(s)))}\mathrm{d}s=\lim_{t\to\infty}\int_{t_0}^{t}\frac{\frac{1}{s^{\frac{31}{15}}}\frac{s}{t_0}B_2^{\frac{1}{5}}(s/6,t_2,t_1)}{\left(\frac{5s}{6}\right)^{1/5}}\mathrm{d}s$$
$$>\lim_{t\to\infty}\int_{t_0}^{t}\frac{\frac{1}{s^{\frac{31}{15}}}\frac{s}{t_0}(3t_0^{\frac{1}{3}})^{\frac{1}{5}}\left[\frac{3}{4}(s/6)^{\frac{4}{3}}-t_2^{\frac{1}{3}}(s/6)\right]^{\frac{1}{5}}}{\left(\frac{5s}{6}\right)^{1/5}}\mathrm{d}s=\infty.$$

It is clear that all the hypotheses of Corollary 4.8 are satisfied and so every solution of Eq. (4.33) is either oscillatory or converges to zero.

Example 4.15 *Consider the third-order differential equation*

$$\left(\frac{1}{t}\left[\left(x^{\frac{1}{7}}(t)+2tx^3(\frac{t}{2})+3t^{\frac{5}{11}}x^{\frac{5}{3}}(4t)\right)''\right]^{\frac{7}{3}}\right)'$$
$$+\frac{1}{t^2}\left[\left(x^{\frac{1}{7}}(t)+2tx^3(\frac{t}{2})+3t^{\frac{5}{11}}x^{\frac{5}{3}}(4t)\right)''\right]^{\frac{7}{3}}+\frac{1}{t^3}x^5(\frac{t}{3})+\frac{1}{\sqrt{t}}x^7(7t)=0,\quad t\geq 1. \tag{4.34}$$

Here, $a(t)=\frac{1}{t}\,,\ \eta=\frac{1}{7}\,,\ p_1(t)=2t,\ \lambda=3,\ \tau_1(t)=\frac{t}{2},\ \tau_2(t)=4t,\ \delta=+1,\ p_2(t)=3t^{\frac{5}{11}}\,,\ \nu=\frac{5}{3},\ b(t)=\frac{1}{t^2}\,,$ $\alpha=\frac{7}{3},\ q(t)=\frac{1}{t^3}\,,\ \gamma=5,\ \sigma(t)=\frac{t}{3},\ r(t)=\frac{1}{\sqrt{t}}\,,\ \beta=7$ *and* $\zeta(t)=7t$. *Thus,*

$$\int_{t_0}^{\infty}\frac{V^{\frac{-1}{\alpha}}(t)}{a^{\frac{1}{\alpha}}(t)}\mathrm{d}t=\int_{t_0}^{\infty}\frac{\exp\left(\frac{-3}{7}\int_{t_0}^{t}\frac{1}{s}\mathrm{d}s\right)}{(\frac{1}{t})^{\frac{3}{7}}}\mathrm{d}t=t_0^{\frac{3}{7}}\int_{t_0}^{\infty}\mathrm{d}t=\infty,$$

$$\lim_{t\to\infty} G_3(t) = \lim_{t\to\infty}\Big[\frac{(3/35)}{(3(\frac{t}{2})^{\frac{5}{11}})^{\frac{3}{35}}} + \frac{(3)(5/9)(2t)^{\frac{5}{11}}(t_0^{3/7})\left[\frac{1}{2}(16t)^2 - t_1(16t) - \frac{1}{2}t_2^2 + t_1t_2\right]}{(t_0^{3/7})(32t)^{\frac{5}{9}}\left[\frac{1}{2}(2t)^2 - t_1(2t) - \frac{1}{2}t_2^2 + t_1t_2\right]}$$

$$+ \frac{1-\frac{3}{35}}{c_*(3(\frac{t}{2})^{\frac{5}{11}})^{\frac{3}{35}}} + \frac{1-\frac{5}{9}}{c_*(32t)^{\frac{5}{9}}}\Big] = 0$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{q(s)V(s)B_2^{\frac{\gamma}{\lambda}}(\sigma(s),t_2,t_1)}{p_1^{\frac{\gamma}{\lambda}}(\tau_1^{-1}(\sigma(s)))}\mathrm{d}s = \lim_{t\to\infty}\int_{t_0}^{t}\frac{\frac{1}{s^3}\frac{s}{t_0}B_2^{\frac{5}{3}}(s/3,t_2,t_1)}{\left(\frac{4s}{3}\right)^{5/3}}\mathrm{d}s$$

$$> \lim_{t\to\infty}\int_{t_0}^{t}\frac{\frac{1}{s^3}\frac{s}{t_0}(t_0^{\frac{3}{7}})^{\frac{5}{3}}\left[\frac{1}{2}(s/3)^2 - t_2(s/3)\right]^{\frac{5}{3}}}{\left(\frac{4s}{3}\right)^{5/3}}\mathrm{d}s = \infty.$$

It is clear that all the hypotheses of Corollary 4.9 hold and so every solution of Eq. (4.34) *is either oscillatory or converges to zero.*

5. Conclusion and discussion

The results of this paper are presented in a fundamentally innovative and broadly applicable manner. These discoveries not only enhance but also supplement the existing literature see [3, 6, 19, 22–26]. In the light of these findings, the following conclusions can be drawn:

- Eq. (1.1) is a more general equation such that it contains a middle term unlike in [3] and contains neutral terms unlike in [26]. Additionally, Eq. (1.1) was investigated for $\delta = -1$ or $\delta = 1$ and $\alpha \neq \beta \neq \gamma$ and η differ from 1.
- The results were obtained with fewer conditions, where the condition $a'(t) \geq 0$ was neglected unlike in [3] and the condition $\sigma'(t) \geq 0$ was neglected unlike in [26].
- Novel criteria for oscillation are derived, presenting previously unpublished insights, particularly when $\eta \neq 1$. Six illustrative examples are provided to substantiate and validate these results. Furthermore, the findings can be applied to higher-order equations of the form

$$\left(a(t)\left(z^{n-1}(t)\right)^{\alpha}\right)' + b(t)\left(z^{n-1}(t)\right)^{\alpha} + q(t)x^{\gamma}(\sigma(t)) + r(t)x^{\beta}(\zeta(t)) = 0,$$

  where $n \geq 3$ is an odd natural integer, $z(t) = x^{\eta}(t) + p_1(t)x^{\lambda}(\tau_1(t)) + \delta p_2(t)x^{\nu}(\tau_2(t))$ and $\delta = \pm 1$. The details are left to the reader.
- Our results not only generalize the existing ones documented in the literature but also offer new insights for formulating more comprehensive research paper concepts than those specified in references [2, 12]. This can be achieved by employing diverse techniques in the context of higher-order scenarios, characterized by considering

$$z(t) = x^{\eta}(t) + p_1(t)x^{\lambda}(\tau_1(t)) + \delta p_2(t)x^{\nu}(\tau_2(t))$$

  with either $b(t) \neq 0$, $\delta = \pm 1$, $\tau_1(t) \leq t$ and $\tau_2(t) \geq t$ or with $b(t) \neq 0$, $\delta = \pm 1$, $\tau_1(t) \leq t$ and $\tau_2(t) \leq t$ or under the condition

$$\int_{t_0}^{\infty}\frac{\exp\left(\frac{-1}{\alpha}\int_{t_0}^{t}\frac{b(s)}{a(s)}\mathrm{d}s\right)}{a^{\frac{1}{\alpha}}(t)}\mathrm{d}t < \infty.$$

References

[1] Agarwal RP, Grace SR, ÓRegan D. Oscillation Theory for Difference and Functional Differential Equations. Dordrecht Kluwer: Academic Publisher, 2000.

[2] Alzabut J, Grace SR, Chhatria GN. New oscillation results for higher-order nonlinear differential equations with a nonlinear neutral terms. Journal of Mathematics and Computer Science 2023; 28 (3): 294-305. https://doi.org/10.22436/jmcs.028.03.07

[3] Alzabut J, Grace SR, Santra SS, Chhatria GN. Asymptotic and oscillatory behaviour of third order non-linear differential equations with canonical operator and mixed neutral terms. Qualitative Theory of Dynamical Systems 2023; 22 (15). https://doi.org/10.1007/s12346-022-00715-6

[4] Baculíková B. Properties of third-order nonlinear functional differential equations with mixed arguments. Abstract and Applied Analysis 2011, 2011 (857860): 1-15. https://doi.org/10.1155/2011/857860

[5] Bohner M, Grace SR, Sağer I, Tunç E. Oscillation of third-order nonlinear damped delay differential equations. Applied Mathematics and Computation 2016; 278 (3): 21-32. https://doi.org/10.1016/j.amc.2015.12.036

[6] Bohner M, Srinivasan R, Thandapani E. Oscillation of second-order damped noncanonical differential equations with superlinear neutral term. Journal of Inequalities and Special Functions 2021; 12 (3): 44-53.

[7] Chang GS. PWR advanced fuel high burnup-dependent xenon oscillation controllability assessment. Annals of Nuclear Energy 2019; 124 (4): 592-605. https://doi.org/10.1016/j.anucene.2018.10.035

[8] Chatzarakis GE, Džurina J, Jadlovská I. Oscillatory and asymptotic properties of third-order quasilinear delay differential equations. Journal of Inequalities and Applications 2019; 2019 (23): 1-17. https://doi.org/10.1186/s13660-019-1967-0

[9] Chatzarakis GE, Grace SR, Jadlovská I, Li T, Tunç E. Oscillation criteria for third-order Emden–Fowler differential equations with unbounded neutral coefficients. Complexity 2019; 2019 (5691758): 1–7. https://doi.org/10.1155/2019/5691758

[10] Feng L, Han Z. Oscillation of a class of third-order neutral differential equations with noncanonical operators. Bulletin of the Malaysian Mathematical Sciences Society 2021; 44: 2519–2530. https://doi.org/10.1007/s40840-021-01079-x

[11] Grace SR, Alzabut J, Özbekler A. New criteria on oscillatory and asymptotic behavior of third-order nonlinear dynamic equations with nonlinear neutral terms. Entropy 2021; 23 (227): 1-11. https://doi.org/10.3390/e23020227

[12] Grace SR, Chhatria GN. Oscillation of higher order nonlinear dynamic equations with a nonlinear neutral term. Mathematical Methods in the Applied Sciences 2023; 46 (2): 2373-2388. https://doi.org/10.1002/mma.8650

[13] Grace SR, Graef JR, Tunç E. On the oscillation of certain third order nonlinear dynamic equations with a nonlinear damping term. Mathematica Slovaca 2017; 67 (2): 501-508. https://doi.org/10.1515/ms-2016-0284

[14] Grace SR, Jadlovská I, Tunç E. Oscillatory and asymptotic behavior of third-order nonlinear differential equations with a superlinear neutral term. Turkish Journal of Mathematics 2020; 44 (4): 1317-1329. https://doi.org/10.3906/mat-2004-85

[15] Hardy GH, Littlewood JE, Pólya G. Inequalities. Cambridge, USA: Cambridge University Press, 1952.

[16] Houtte PV, Li S, Seefeldt M, Delannay L. Deformation texture prediction: from the Taylor model to the advanced Lamel model. International Journal of Plasticity 2005; 21 (3): 589-624. https://doi.org/10.1016/j.ijplas.2004.04.011

[17] Jiang C, Li T. Oscillation criteria for third-order nonlinear neutral differential equations with distributed deviating arguments. Journal of Nonlinear Science and Applications 2016; 9 (2016): 6170-6182.

[18] Li W, Yu Y. Oscillatory behavior of third-order nonlinear differential equations with a sublinear neutral term. Acta Mathematicae Applicatae Sinica 2022; 38 (2): 484-496. https://doi.org/10.1007/s10255-022-1089-1

[19] Liu Q, Grace SR, Jadlovská I, Tunç E, Li T. On the asymptotic behavior of noncanonical third-order Emden–Fowler delay differential equations with a superlinear neutral term. Mathematics 2022; 10 (2902): 1-12. https://doi.org/10.3390/math10162902

[20] Philos CG. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. Archiv der Mathematik 1981; 36: 168-178. https://doi.org/10.1007/BF01223686

[21] Thandapani E, Göktürk B, Özdemir O, Tunç E. Oscillatory behaviour of semi-canonical nonlinear neutral differential equations of third-order via comparison principles. Qualitative Theory of Dynamical Systems 2023; 22 (30): 1-13. https://doi.org/10.1007/s12346-022-00731-6

[22] Thandapani E, Rama R. Oscillatory behavior of solutions of certain third order mixed neutral differential equations. Tamkang Journal of Mathematics 2013; 44 (1): 99-112. https://doi.org/10.5556/j.tkjm.44.2013.1150

[23] Tunç E, Özdemir O. On the oscillation of second-order half-linear functional differential equations with mixed neutral term. Journal of Taibah University for Science 2019; 13 (1): 481-489. https://doi.org/10.1080/16583655.2019.1595948

[24] Tunç E, Özdemir O. Oscillatory behavior of second-order damped differential equations with a superlinear neutral term. Opuscula Mathematica 2020; 40 (5): 629–639. https://doi.org/10.7494/OpMath.2020.40.5.629

[25] Tunç E, Şahin S, Graef JR, Pinelas S. New oscillation criteria for third-order differential equations with bounded and unbounded neutral coefficients. Electronic Journal of Qualitative Theory of Differential Equations 2021; 1 (46): 1-13. https://doi.org/10.14232/ejqtde.2021.1.46

[26] Wang M, Dong L. Some oscillation criteria for a Class of third-order delay differential equations. Journal of Mathematics 2022; 2022 (9347942): 1-11. https://doi.org/10.1155/2022/9347942

[27] Wei M, Jiang C, Li T. Oscillation of third-order neutral differential equations with damping and distributed delay. Advances in Difference Equations 2019; 2019 (426): 1-11. https://doi.org/10.1186/s13662-019-2363-2

[28] Wu H, Erbe L, Peterson A. Oscillation of solution to second-order half-linear delay dynamic equations on time scales. Electronic Journal of Differential Equations 2016; 2016 (71): 1-15.