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On the reconstruction of an integro-differential Dirac operator with parameter-dependent nonlocal integral boundary conditions from the nodal data

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Abstract: We consider the integro-differential Dirac operator with parameter-dependent nonlocal integral boundary conditions. We derive the asymptotic expressions for the eigenvalues and the zeros of eigenfunctions (nodal points or nodes) and develop a constructive procedure for solving the inverse nodal problem for this operator.

Key words: Integro-differential Dirac operators, nonlocal integral boundary conditions, inverse nodal problem

1. Introduction

We consider the following integro-differential Dirac system

$$BY'(\varkappa)+\Omega(\varkappa)Y(\varkappa)+\int\limits_0^{\varkappa}K(\varkappa,\varsigma)Y(\varsigma)d\varsigma=\lambda Y(\varkappa),\ \ \varkappa\in[0,\pi]\,, \tag{1}$$

subject to the parameter-dependent nonlocal integral boundary conditions

$$W_1(Y)\quad:\quad=\sin\delta y_1(0)+\cos\delta y_2(0)+\int\limits_0^{\pi}Y(\varsigma)^t\sigma_1(\varsigma)d\varsigma=0, \tag{2}$$

$$W_2(Y)\quad:\quad=\lambda\sin\beta y_1(\pi)+\lambda\cos\beta_2 y_2(\pi)+\int\limits_0^{\pi}Y(\varsigma)^t\sigma_2(\varsigma)d\varsigma=0, \tag{3}$$

where

$$\Omega(\varkappa)=\begin{bmatrix}p(\varkappa)&0\\0&r(\varkappa)\end{bmatrix},\ B=\begin{bmatrix}0&1\\-1&0\end{bmatrix},\ Y(\varkappa)=\begin{bmatrix}y_1(\varkappa)\\y_2(\varkappa)\end{bmatrix},$$

$$K(\varkappa,\varsigma)=\left[K_{ij}(\varkappa,\varsigma)\right]_{2\times2},\ \ \sigma_i(\varkappa)=\begin{bmatrix}\sigma_{i,1}(\varkappa)\\\sigma_{i,2}(\varkappa)\end{bmatrix},$$

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the functions $K_{ij}(\varkappa,\varsigma)$, $\sigma_{i,j}(\varkappa)$, $(i,j=1,2)$, $p(\varkappa)$, and $r(\varkappa)$ are real-valued in $W_2^1[0,\pi]$, $0<\delta,\beta<\pi$, and λ is the spectral parameter.

Boundary value problems with different nonlocal boundary conditions have been one of the most researched areas in various applications in mathematical physics. The nonlocal boundary conditions arise when values of the functions at the boundary are connected with values inside the domain. The problem relates to various areas of mathematical physics, biology, biotechnology, mechanics, and geophysics [4, 14, 18, 50]. The nonlocal boundary condition was first introduced by Samarskii and Bitsadze in 1969 [5]. In that study, the researchers investigated and formulated the nonlocal problem for a general elliptic equation and proved the existence and uniqueness of the solution.

Inverse spectral problems consist of reconstructing the coefficients of the operator from their spectral characteristics. The results of different types of inverse problems for various nonlocal operators can be found in [3, 13, 31, 32, 37, 48].

Boundary value problems, with boundary conditions containing the spectral parameter in different ways, are frequently encountered in applications as well as in various problems of mathematical physics. In 1973, Walter discussed eigenvalue problems with the eigenvalue parameter in the boundary conditions and gave expansion theorems for this problem. [41]. Fulton's [16, 17] studies and the references in this study can be cited as examples of studies conducted on this subject until 1980. Fulton's research suggested that the results from Titchmarsh's book ([40]) extend to regular problems involving the spectral parameter in the boundary condition. Direct and inverse problems for various operators with spectral parameter-dependent boundary conditions have been well-studied (see [1, 15, 19–21, 23, 24], and the references therein).

The problem of determining the coefficients of the operator using the nodes is called the inverse nodal problem. Mclaughlin (1988) was the first researcher who dealt with the problem and solved it for a specific Sturm Liouville problem [36]. The researcher showed in her work that a dense subset of nodes alone could determine the potential function of the Sturm-Liouville problem up to a constant. After Mclaughlin's work, using similar methods, Hald and McLaughlin provided some numerical schemes for the reconstruction of the potential of the operator with some general boundary conditions [25]. Yang was one of the researchers who also proposed a constructive procedure to reconstruct the coefficients of the operator from the nodes [44]. Many researchers have extensively investigated the inverse nodal problems for some operators with different boundary conditions [2, 9, 12, 22, 35, 39, 42, 43, 45–47, 49, 51].

Integro-differential operators appear in many areas of the applied sciences [34]. Inverse problems for such operators are of great interest and are currently studied by many researchers because of their importance in applications [6–8, 10, 11, 26–30, 33]. In [38], the authors considered the Dirac operator $BY'(\varkappa)+\Omega(\varkappa)Y(\varkappa)+\int\limits_0^{\varkappa}K(\varkappa,\varsigma)Y(\varsigma)d\varsigma=\lambda Y(\varkappa)$ with nonlocal integral boundary conditions. In this study, the potential function was regarded as $\Omega(\varkappa)=\begin{bmatrix} V(\varkappa)+m & 0 \\ 0 & V(\varkappa)-m \end{bmatrix}$ and the special case $V(\varkappa)$ was reconstructed.

The present study aims to discuss the inverse nodal problem for an integro-differential Dirac system with more general potential function and parameter-dependent nonlocal integral boundary conditions.

Consider the solutions

$$S(\varkappa,\lambda)=\begin{bmatrix} S_1(\varkappa,\lambda) \\ S_2(\varkappa,\lambda) \end{bmatrix},\ S(0,\lambda)=\begin{bmatrix} 0 \\ -1 \end{bmatrix},\ \text{and}$$

$$C(\varkappa,\lambda)=\begin{bmatrix} C_1(\varkappa,\lambda) \\ C_2(\varkappa,\lambda) \end{bmatrix},\ C(0,\lambda)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

of Eq. (1.1). It is easy to show that the following asymptotic expressions exist

$$S_1(\varkappa,\lambda)=\left(1+\frac{\Gamma(\varkappa)}{2\lambda}\right)\sin(\vartheta(\varkappa,\lambda)-\delta) \tag{4}$$
$$+\frac{1}{2\lambda}\left(H_-(\varkappa)-\int_0^{\varkappa}\gamma^2(\varsigma)d\varsigma\right)\cos(\vartheta(\varkappa,\lambda)-\delta)+o\left(\frac{\exp(|\tau|\varkappa)}{\lambda}\right),$$

$$S_2(\varkappa,\lambda)=-\left(1+\frac{\Gamma(\varkappa)}{2\lambda}\right)\cos(\vartheta(\varkappa,\lambda)-\delta) \tag{5}$$
$$+\frac{1}{2\lambda}\left(H_-(\varkappa)-\int_0^{\varkappa}\gamma^2(\varsigma)d\varsigma\right)\sin(\vartheta(\varkappa,\lambda)-\delta)+o\left(\frac{\exp(|\tau|\varkappa)}{\lambda}\right),$$

$$C_1(\varkappa,\lambda)=\left(1+\frac{\Gamma(\varkappa)}{2\lambda}\right)\cos(\vartheta(\varkappa,\lambda)-\delta) \tag{6}$$
$$-\frac{1}{2\lambda}\left(H_-(\varkappa)-\int_0^{\varkappa}\gamma^2(\varsigma)d\varsigma\right)\sin(\vartheta(\varkappa,\lambda)-\delta)+o\left(\frac{\exp(|\tau|\varkappa)}{\lambda}\right),$$

$$C_2(\varkappa,\lambda)=\left(1+\frac{\Gamma(\varkappa)}{2\lambda}\right)\sin(\vartheta(\varkappa,\lambda)-\delta) \tag{7}$$
$$+\frac{1}{2\lambda}\left(H_-(\varkappa)-\int_0^{\varkappa}\gamma^2(\varsigma)d\varsigma\right)\cos(\vartheta(\varkappa,\lambda)-\delta)+o\left(\frac{\exp(|\tau|\varkappa)}{\lambda}\right),$$

for sufficiently large $|\lambda|$, uniformly in $\varkappa$. Here, $H_+(\varkappa)=\int_0^{\varkappa}(K_{11}(\varsigma,\varsigma)+K_{22}(\varsigma,\varsigma))d\varsigma$, $H_-(\varkappa)=\int_0^{\varkappa}(K_{12}(\varsigma,\varsigma)-K_{21}(\varsigma,\varsigma))d\varsigma$, $2\chi(\varkappa)=\int_0^{\varkappa}(p(\varsigma)+r(\varsigma))d\varsigma$, $\gamma(\varkappa)=p(\varkappa)-r(\varkappa)$, $\psi(\varkappa)=\int_0^{\varkappa}\gamma^2(\varsigma)\,d\varsigma$, $\vartheta(\varkappa,\lambda)=\lambda\varkappa-\chi(\varkappa)+\delta$, $\Gamma(\varkappa)=\gamma(\varkappa)-\gamma(0)-H_+(\varkappa)$, and $\tau=Im\lambda$.

2. Main results

In this section, we will first obtain asymptotics for the eigenvalue, eigenfunction, and nodal points. Then, we will give an algorithm on how to find the coefficients of the operator with the help of nodal points.

2.1. Eigenvalues of the problem L

Let us denote the problem (1.1-(1.3) by L, and let $y(\varkappa,\lambda)=\rho_1(\lambda)C(\varkappa,\lambda)+\rho_2(\lambda)S(\varkappa,\lambda)$ be a nontrivial solution of the Eq (1.1). Together with (1.2) and (1.3), we have

$$\begin{aligned}\rho_1(\lambda)W_1(C(\varkappa,\lambda))+\rho_2(\lambda)W_1(S(\varkappa,\lambda)) &= 0,\\ \rho_1(\lambda)W_2(C(\varkappa,\lambda))+\rho_2(\lambda)W_2(S(\varkappa,\lambda)) &= 0.\end{aligned}$$

The zeros denoted by λ_ν of the characteristic function $\Lambda(\lambda):=\det\begin{pmatrix} W_1(C) & W_1(S) \\ W_2(C) & W_2(S)\end{pmatrix}$ are the eigenvalues of L.

Introduce the function

$$\varphi(\varkappa,\lambda)=(\varphi_1(\varkappa,\lambda),\varphi_2(\varkappa,\lambda))^t=W_1(S(\varkappa,\lambda))C(\varkappa,\lambda)-W_1(C(\varkappa,\lambda))S(\varkappa,\lambda),$$

note that this is the solution of the L. Without loss of generality, this is equivalent to choosing $\rho_1(\lambda)=W_1(S(\varkappa,\lambda))$ and $\rho_2(\lambda)=-W_1(C(\varkappa,\lambda))$.

Hence,

$\varphi(\varkappa,\lambda_\nu)=W_1(S(\varkappa,\lambda_\nu))C(\varkappa,\lambda_\nu)-W_1(C(\varkappa,\lambda_\nu))S(\varkappa,\lambda_\nu)$

are the eigenfunctions corresponding to the eigenvalues λ_ν.

Lemma 1 *Let $\{\lambda_\nu\}_{\nu\in\mathbb{Z}}$ be the eigenvalues and $\varphi(\varkappa,\lambda_\nu)$ be the eigenfunctions of L. For sufficiently large $|\nu|$, the following asymptotic relations hold:*

$$\begin{aligned}\lambda_\nu&=(\nu-1)+\frac{\chi(\pi)-\delta-\beta}{\pi}\\&+\frac{1}{(\nu-1)2\pi}\left(\gamma(0)-H_-(\pi)+\int_0^\pi\gamma^2(\varsigma)d\varsigma\right)+o\left(\frac{1}{\nu}\right),\quad \nu\geq 2,\end{aligned}\tag{8}$$

and similarly,

$$\begin{aligned}\lambda_\nu&=(\nu+1)+\frac{\chi(\pi)-\delta-\beta}{\pi}\\&+\frac{1}{(\nu+1)2\pi}\left(\gamma(0)-H_-(\pi)+\int_0^\pi\gamma^2(\varsigma)d\varsigma\right)+o\left(\frac{1}{\nu}\right),\ \nu\leq -2.\end{aligned}\tag{9}$$

Moreover, the first components $\varphi_1(\varkappa,\lambda_\nu)$ of eigenfunctions satisfy

$$\begin{aligned}\varphi_1(\varkappa,\lambda_\nu)&=\left(-1-\frac{\Gamma(\varkappa)}{2\lambda_\nu}\right)\cos\vartheta(\varkappa,\lambda_\upsilon)+\left(H_-(\varkappa)-\int_0^\varkappa\gamma^2(\varsigma)d\varsigma\right)\frac{\sin\vartheta(\varkappa,\lambda_\upsilon)}{2\lambda_\nu}\\&+\frac{I_\nu}{\lambda_\nu}\cos\left(\vartheta(\varkappa,\lambda_\upsilon)-\delta\right)-\frac{J_\nu}{\lambda_\nu}\sin\left(\vartheta(\varkappa,\lambda_\upsilon)-\delta\right)+o\left(\frac{\exp(|\tau|\,\varkappa)}{\lambda_\nu}\right),\end{aligned}$$

where $I_\nu=(-1)^\nu\left(\sin(\delta+\beta)\sigma_{1,2}(\pi)-\cos(\delta+\beta)\sigma_{1,1}(\pi)\right)+\sigma_{1,1}(0)$,

$J_\nu=(-1)^\nu\left(\cos(\delta+\beta)\sigma_{1,2}(\pi)+\sin(\delta+\beta)\sigma_{1,1}(\pi)\right)-\sigma_{1,2}(0)$.

Proof Since the following expressiontion holds

$$\begin{aligned}\Lambda(\lambda)&=-\lambda\Bigg\{\sin\left(\vartheta(\pi,\lambda)+\beta\right)+\frac{\gamma(\pi)-H_+(\pi)}{2\lambda}\sin\left(\vartheta(\pi,\lambda)+\beta\right)\\&-\frac{1}{2\lambda}\left(\gamma(0)-H_-(\pi)+\int_0^\pi\gamma^2(\varsigma)d\varsigma\right)\cos\left(\vartheta(\pi,\lambda)+\beta\right)\\&+o\left(\frac{e^{|\tau|\pi}}{\lambda}\right)\Bigg\},\end{aligned}$$

we can obtain equations (2.1) and (2.2) from the above asymptotic, i.e., the first part of Lemma 2.1 is proven.

Let $\varphi(\varkappa,\lambda_\nu)$ be the eigenfunctions of the problem L, it follows from (1.2), (1.4)-(1.7) that

$$\begin{aligned}
&\varphi_1(\varkappa,\lambda_\nu) = -\left(1+\frac{\Gamma(\varkappa)}{2\lambda_\nu}\right)\cos\vartheta(x,\lambda_\upsilon) + \frac{1}{2\lambda_\nu}\left(H_-(\varkappa) - \int_0^\varkappa \gamma^2(\varsigma)d\varsigma\right)\sin\vartheta(\varkappa,\lambda_\upsilon) \\
&+\frac{I_2-\sigma_{1,1}(0)}{\lambda_\nu}\cos\left(\vartheta(\varkappa,\lambda_\upsilon)+\vartheta(\pi,\lambda)-\delta+\beta\right) - \frac{J_2+\sigma_{1,2}(0)}{\lambda_\nu}\sin\left(\vartheta(\varkappa,\lambda_\upsilon)+\vartheta(x,\lambda_\upsilon)-\delta+\beta\right) \\
&+\frac{\sigma_{1,1}(0)}{\lambda_\nu}\cos(\vartheta(\varkappa,\lambda_\upsilon)-\delta) + \frac{\sigma_{1,2}(0)}{\lambda_\nu}\sin(\vartheta(\varkappa,\lambda_\upsilon)-\delta) + o\left(\frac{\exp(|\tau|\,\varkappa)}{\lambda}\right) \\
&= -\left(1+\frac{\Gamma(\varkappa)}{2\lambda_\nu}\right)\cos\vartheta(\varkappa,\lambda_\upsilon) + \frac{1}{2\lambda_\nu}\left(H_-(\varkappa) - \int_0^\varkappa \gamma^2(\varsigma)d\varsigma\right)\sin\vartheta(\varkappa,\lambda_\upsilon) \\
&+\frac{I_\nu}{\lambda_\nu}\cos(\vartheta(\varkappa,\lambda_\upsilon)-\delta) - \frac{J_\nu}{\lambda_\nu}\sin(\vartheta(\varkappa,\lambda_\upsilon)-\delta) + o\left(\frac{\exp(|\tau|\,\varkappa)}{\lambda_\upsilon}\right)
\end{aligned}$$

Hence, the second part of Lemma 2.1 is proved.

2.2. Nodal points of the problem L

Let $\varkappa_\nu^i$ be the position of the ith zero of the characteristic function $\varphi_1(\varkappa,\lambda_\nu)$, $\varkappa\in(0,\pi)$. The following Lemma describes the asymptotic behavior of $\varkappa_\nu^i$ when ν is large enough.

Lemma 2 *As ν is large enough, the function $\varphi_1(\varkappa,\lambda_\nu)$ has $\nu-2$ nodes $\left\{\varkappa_\nu^i : i=0,1,...,\nu-3\right\}$ in $(0,\pi)$: $0<\varkappa_\nu^0<\varkappa_\nu^1<...<\varkappa_\nu^{\nu-3}<\pi$. Moreover, The following asymptotic expression is provided*

$$\begin{aligned}
\varkappa_\nu^i \;=\;& \frac{(2i+1)\,\pi}{2\,(\nu-1)} + \frac{\chi(\varkappa_\nu^i)-\delta}{\nu-1} - \frac{(2i+1)\,\pi}{2\,(\nu-1)}\frac{\chi\,(\pi)-\delta-\beta}{(\nu-1)\,\pi} \\
&-\frac{(2i+1)\,\pi}{2\,(\nu-1)}\frac{\Phi}{(\nu-1)^2\,\pi} - \frac{\left(\chi(\varkappa_\nu^i)-\delta\right)(\chi\,(\pi)-\delta-\beta)}{(\nu-1)^2\,\pi} \\
&-\frac{1}{2\,(\nu-1)^2}\left(H_-\left(\varkappa_\nu^i\right) - \int_0^{\varkappa_\nu^i}\gamma^2(\varsigma)d\varsigma\right) - \frac{\sigma_{1,2}(0)\cos\delta-\sigma_{1,1}(0)\sin\delta}{(\nu-1)^2} \\
&+\frac{(-1)^\nu\,(\sigma_{1,2}(\pi)\cos\beta+\sigma_{1,1}(\pi)\sin\beta)}{(\nu-1)^2} - \frac{\left(\chi(\varkappa_\nu^i)-\delta\right)\Phi}{(\nu-1)^3\,\pi} \\
&-\frac{\chi\,(\pi)-\delta-\beta-H_-\left(\varkappa_\nu^i\right)}{(\nu-1)^3\,\pi}\int_0^{\varkappa_\nu^i}\gamma^2(\varsigma)d\varsigma \\
&-\frac{2\left((-1)^\nu\,(\sigma_{1,2}(\pi)\cos\beta+\sigma_{1,1}(\pi)\sin\beta)\right)(\chi\,(\pi)-\delta-\beta)}{(\nu-1)^3} \\
&+\frac{2\,(\sigma_{1,2}(0)\cos\delta-\sigma_{1,1}(0)\sin\delta)\,(\chi\,(\pi)-\delta-\beta)}{(\nu-1)^3} + o\left(\frac{1}{\nu^3}\right)
\end{aligned} \tag{10}$$

for sufficiently large $\nu>0$, uniformly with respect to i, where $\Phi = \pi\left(\gamma(0)-H_-(\pi)+\int_0^\pi\gamma^2(\varsigma)d\varsigma\right) + (\chi\,(\pi)-\delta-\beta)^2$.

Proof Consider the equation $\varphi_1(\varkappa_\nu^i, \lambda_\nu) = 0$ on $(0, \pi)$, then

$$
\begin{aligned}
&-\cos\vartheta(\varkappa_\nu^i, \lambda_\upsilon) - \frac{\Gamma(\varkappa_\nu^i)}{2\lambda_\nu}\cos\vartheta(\varkappa_\nu^i, \lambda_\upsilon) + \frac{1}{2\lambda_\nu}\left(H_-\left(\varkappa_\nu^i\right) - \int_0^{\varkappa_\nu^i}\gamma^2(\varsigma)d\varsigma\right)\sin\vartheta(\varkappa_\nu^i, \lambda_\upsilon) \\
&+\frac{(-1)^\nu\left(\sigma_{1,2}(\pi)\sin\beta - \sigma_{1,1}(\pi)\cos\beta\right)}{\lambda_\nu}\cos\vartheta(\varkappa_\nu^i, \lambda_\upsilon) \\
&-\frac{(-1)^\nu\left(\sigma_{1,2}(\pi)\cos\beta + \sigma_{1,1}(\pi)\sin\beta\right)}{\lambda_\nu}\sin\vartheta(\varkappa_\nu^i, \lambda_\upsilon) \\
&+\frac{\left(\sigma_{1,2}(0)\sin\delta + \sigma_{1,1}(0)\cos\delta\right)}{\lambda_\nu}\cos\vartheta(\varkappa_\nu^i, \lambda_\upsilon) \\
&+\frac{\left(\sigma_{1,2}(0)\cos\delta - \sigma_{1,1}(0)\sin\delta\right)}{\lambda_\nu}\sin\vartheta(\varkappa_\nu^i, \lambda_\upsilon) + o\left(\frac{1}{\lambda_\nu}\exp(|\tau|\,\varkappa_\nu^i)\right) = 0\,.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\cot\vartheta(\varkappa_\nu^i, \lambda_\upsilon)\left(1 + \frac{\Gamma(\varkappa_\nu^i)}{2\lambda_\nu} + \frac{(-1)^\nu\left(\sigma_{1,1}(\pi)\cos\beta - \sigma_{1,2}(\pi)\sin\beta\right)}{\lambda_\nu}\right. \\
&\left.-\frac{\sigma_{1,2}(0)\sin\delta + \sigma_{1,1}(0)\cos\delta}{\lambda_\nu}\right) \\
&= -\frac{1}{2\lambda_\nu}\left(-H_-\left(\varkappa_\nu^i\right) + \int_0^{\varkappa_\nu^i}\gamma^2(\varsigma)d\varsigma\right) - \frac{(-1)^\nu\left(\sigma_{1,2}(\pi)\cos\beta + \sigma_{1,1}(\pi)\sin\beta\right)}{\lambda_\nu} \\
&+\frac{\sigma_{1,2}(0)\cos\delta - \sigma_{1,1}(0)\sin\delta}{\lambda_\nu} + o\left(\frac{1}{\lambda_\nu}\exp(|\tau|\,\varkappa_\nu^i)\right)
\end{aligned}
$$

which also imply the estimates

$$
\begin{aligned}
&\tan\left(\vartheta(\varkappa_\nu^i, \lambda_\upsilon) - \frac{\pi}{2}\right)\left(1 + O\left(\frac{1}{\lambda_\nu}\right)\right) \\
&= \frac{-1}{2\lambda_\nu}\left(H_-\left(\varkappa_\nu^i\right) - \int_0^{\varkappa_\nu^i}\gamma^2(\varsigma)d\varsigma\right) + \frac{(-1)^\nu\left(\sigma_{1,2}(\pi)\cos\beta + \sigma_{1,1}(\pi)\sin\beta\right)}{\lambda_\nu} \\
&-\frac{\sigma_{1,2}(0)\cos\delta - \sigma_{1,1}(0)\sin\delta}{\lambda_\nu} + o\left(\frac{1}{\lambda_\nu}\exp(|\tau|\,\varkappa_\nu^i)\right)
\end{aligned}
$$

using Taylor's expansion formula, we get

$$
\begin{aligned}
\varkappa_\nu^i \quad = \quad & \frac{(2i+1)\pi}{2\lambda_\nu} + \frac{\chi(\varkappa_\nu^i) - \delta}{\lambda_\nu} + \frac{1}{2\lambda_\nu^2}\int_0^{\varkappa_\nu^i}\gamma^2(\varsigma)d\varsigma - \frac{1}{2\lambda_\nu^2}H_-\left(\varkappa_\nu^i\right) \\
&+\frac{(-1)^\nu\left(\sigma_{1,2}(\pi)\cos\beta + \sigma_{1,1}(\pi)\sin\beta\right)}{\lambda_\nu^2} - \frac{\sigma_{1,2}(0)\cos\delta - \sigma_{1,1}(0)\sin\delta}{\lambda_\nu^2} \\
&+o\left(\frac{1}{\lambda_\nu^2}\exp(|\tau|\,\varkappa_\nu^i)\right)
\end{aligned}
$$

If we substitute the following expression in the above equation

$$\lambda_\nu^{-1} = \frac{1}{\nu-1}\left\{1 - \frac{\chi(\pi)-\delta-\beta}{(\nu-1)\pi} - \frac{\Phi}{(\nu-1)^2\pi} + o\left(\frac{1}{\nu^2}\right)\right\},$$

we have completed the proof of Lemma 2.2

2.3. Reconstruction of problem of the problem L

In this section, we develop a constructive algorithm for determining the coefficients of problem L. Denote the set of nodal points by Ξ and the dense subset of the set Ξ by F.

Theorem 1 *For $\varkappa \in (0,\pi)$, let $\left(\varkappa_\nu^i\right)_{\nu\in\mathbb{N}} \subset \Xi$, and $\lim \varkappa_\nu^i = \varkappa$. Then, the following limits exist and are finite*

$$h(\varkappa) := \lim_{\nu\to\infty}\left(\varkappa_\nu^i - \frac{(2i+1)\pi}{2(\nu-1)}\right)(\nu-2) = \chi(\varkappa) - \delta - \varkappa\frac{\chi(\pi)-\delta-\beta}{\pi} \tag{11}$$

and

$$\begin{aligned} g_\nu(\varkappa) \quad : &= \lim_{\nu\to\infty}\left(\varkappa_\nu^i - \frac{(2i+1)\pi}{2(\nu-2)} - \frac{\chi(\varkappa_\nu^i)-\delta}{\nu-1} + \frac{(2i+1)\pi}{2(\nu-1)}\frac{\chi(\pi)-\delta-\beta}{(\nu-1)\pi}\right)\pi(\nu-2)^2 \\ &= -\varkappa\Phi - (\chi(\varkappa)-\delta)(\chi(\pi)-\delta-\beta) + \frac{\pi}{2}\int_0^\varkappa \gamma^2(\varsigma)d\varsigma - \frac{\pi}{2}H_-(\varkappa) \\ &\quad -\pi(\sigma_{1,2}(0)\cos\delta - \sigma_{1,1}(0)\sin\delta) + \pi\left((-1)^\nu(\sigma_{1,2}(\pi)\cos\beta + \sigma_{1,1}(\pi)\sin\beta)\right) \end{aligned} \tag{12}$$

$$\begin{aligned} f_\nu(\varkappa) \quad : &= \lim_{\nu\to\infty}\left\{\varkappa_\nu^i - \frac{(2i+1)\pi}{2(\nu-2)} - \frac{\chi(\varkappa_\nu^i)-\delta}{\nu-2} + \frac{(2i+1)\pi}{2(\nu-1)}\frac{\chi(\pi)-\delta-\beta}{(\nu-1)\pi}\right. \\ &\quad + \frac{(2i+1)\pi}{2(\nu-1)}\frac{\Phi}{(\nu-1)^2\pi} + \frac{\left(\chi(\varkappa_\nu^i)-\delta\right)(\chi(\pi)-\delta-\beta)}{(\nu-1)^2\pi} \\ &\quad - \frac{1}{2(\nu-1)^2}\int_0^{\varkappa_\nu^i}\gamma^2(\varsigma)d\varsigma + \frac{H_-\left(\varkappa_\nu^i\right)\pi(\nu-2)^2}{2(\nu-1)^2} \\ &\quad \left. + \frac{\sigma_{1,2}(0)\cos\delta - \sigma_{1,1}(0)\sin\delta}{(\nu-1)^2} - \frac{(-1)^\nu(\sigma_{1,2}(\pi)\cos\beta + \sigma_{1,1}(\pi)\sin\beta)}{(\nu-1)^2}\right\}\frac{(\nu-2)^3}{2} \\ &= -(\chi(\varkappa)-\delta)\Phi - (\chi(\pi)-\delta-\beta)\int_0^\varkappa\gamma^2(\varsigma)d\varsigma + (\chi(\pi)-\delta-\beta)H_-(\varkappa) + \Upsilon_\nu \end{aligned} \tag{13}$$

where $\Upsilon_\nu = (\chi(\pi)-\delta-\beta)\left(\sigma_{1,2}(0)\cos\delta - \sigma_{1,1}(0)\sin\delta - (-1)^\nu(\sigma_{1,2}(\pi)\cos\beta + \sigma_{1,1}(\pi)\sin\beta)\right)$

Proof By using (2.3) and $\lim \varkappa_\nu^i = \varkappa$. For any $\nu \in \mathbb{Z}$, (2.4), (2.5), and (2.6) can be easily obtained by direct calculation.

Theorem 2 *Let us assume that* $\chi(\pi)=0$. *The given set* F *uniquely determines* $p(\varkappa)$, $r(\varkappa)$ *almost everywhere on* $(0,\pi)$, μ_1, μ_2, *and the coefficients* δ *and* β. *Also, we have the algorithm below for reconstructing* $p(\varkappa)$, $r(\varkappa)$, μ_1, μ_2, δ *and* β *:*

(1) For each set F *and* $\varkappa\in(0,\pi)$, $\left\{\varkappa_\nu^{i(\nu)}\right\}\subset F$, *then* $\lim_{|\nu|\to\infty}\varkappa_\nu^{i(\nu)}=\varkappa$;

(2) Find $h(\varkappa)$ *from the Eq. (2.4), and calculate*

$$\begin{aligned}\delta &= -h(0)\\ \beta &= h(\pi)\\ \chi'(\varkappa) &= \frac{1}{2}(r(\varkappa)+p(\varkappa))=h'(\varkappa)-\frac{\delta+\beta}{\pi}\end{aligned}$$

(3) If $\chi(\pi)=0$, *and* $K_{12}(x,x)-K_{21}(x,x)$ *is known, find the functions* $g_\nu(\varkappa)$ *and* $f_\nu(\varkappa)$ *via (2.5) and (2.6) and calculate*

$$\Upsilon_\nu=-2(h(0)+h(\pi))g_\nu(0)-2h(0)(h(0)+h(\pi)^2$$

$$\Phi=-\frac{f_\nu(0)-\Upsilon_\nu}{h(0)}$$

$$p(\varkappa)=h'(\varkappa)-\frac{\delta+\beta}{\pi}+\frac{\xi(\varkappa)}{2}$$

$$r(\varkappa)=h'(\varkappa)-\frac{\delta+\beta}{\pi}-\frac{\xi(\varkappa)}{2}$$

$$\mu_1=\frac{g_{2k+1}(0)+g_{2k}(0)+2\delta(\delta+\beta)}{-2\pi}$$

$$\mu_2=\frac{g_{2k}(0)-g_{2k+1}(0)}{2\pi},$$

where

$$\begin{aligned}\xi^2(\varkappa) &= \frac{\pi f_\nu'(\varkappa)+\pi(\delta+\beta)H_-'(\varkappa)+(\pi h'(\varkappa)-(\delta+\beta))\Phi}{\pi}\\ \mu_1 &= \sigma_{1,2}(0)\cos\delta-\sigma_{1,1}(0)\sin\delta\\ \mu_2 &= \sigma_{1,2}(\pi)\cos\beta+\sigma_{1,1}(\pi)\sin\beta.\end{aligned}$$

3. Conclusion

In this work, we study inverse nodal problems for Dirac-type integral differential systems with parameter-dependent nonlocal integral boundary conditions. Asymptotic expressions were obtained for the eigenfunctions, eigenvalues, and nodes of the problem under consideration. With the help of these data, a constructive procedure for solving inverse nodal problems is proposed.This study differs significantly from studies in the literature in that it includes both a more general potential function and more general boundary conditions.

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