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Isometries of length 1 in purely loxodromic free kleinian groups and trace inequalities

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Abstract: In this paper, we prove a generalization of a discreteness criteria for a large class of subgroups of $\mathrm{PSL}_2(\mathbb{C})$. In particular, given a finitely generated purely loxodromic free Kleinian group $\Gamma = \langle \xi_1, \xi_2, \dots, \xi_n \rangle$ for $n \geq 2$, we show that

$$|\mathrm{trace}^2(\xi_i) - 4| + |\mathrm{trace}(\xi_i \xi_j \xi_i^{-1} \xi_j^{-1}) - 2| \geq 2 \sinh^2\left(\frac{1}{4} \log \alpha_n\right)$$

for some ξ_i and ξ_j for $i \neq j$ in Γ provided that certain conditions on the hyperbolic displacements given by ξ_i , ξ_j and their length 3 conjugates formed by the generators are satisfied. Above, the constant α_n turns out to be the real root strictly larger than $(2n-1)^2$ of a fourth degree integer coefficient polynomial obtained by solving a family of optimization problems via the Karush-Kuhn-Tucker theory. The use of this theory in the context of hyperbolic geometry is another novelty of this work.

Key words: Free Kleinian groups, Jorgensen's inequality, trace inequalities, hyperbolic displacements, log3 theorem, Karush-Kuhn-Tucker theory

1. Introduction

By the work of Thurston [16, 17] and Jorgensen and Gromov [10], it is known that the volume is a geometric invariant of finite-volume hyperbolic 3-manifolds. Furthermore, by the Mostow-Prasad rigidity theorem (see [3, 14]), it is also a topological invariant. Therefore, investigation of the connections between the volume and the usual invariants of topology is one of the main topics of research in the study of finite-volume hyperbolic 3-manifolds.

A particular approach practiced in this area of research is to explore the implications of discreteness criteria such as the Jorgensen's inequality [11] which states that $|\mathrm{trace}^2(\xi_1) - 4| + |\mathrm{trace}(\xi_1 \xi_2 \xi_1^{-1} \xi_2^{-1}) - 2| \geq 1$ if $\Gamma = \langle \xi_1, \xi_2 \rangle$ is a Kleinian group. Trace inequalities of this sort can be used to calculate not only estimates for the volumes but also estimates for geometric quantities like the injectivity radius and diameter. A seminal result along this line due to Meyerhoff [13] which establishes that 0.104 is a Margulis constant for $n = 3$. In other words, for isometries ξ and η of \mathbb{H}^3 generating a nonelementary discrete group the inequality $\max\{d_\xi z, d_\eta z\} \geq 0.104$ holds for any $z \in \mathbb{H}^3$, where $d_\gamma z$ denotes the hyperbolic distance between $z \in \mathbb{H}^3$

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and $\gamma(z)$ for an isometry γ of \mathbb{H}^3 . The existence of such constants is implied by the Margulis Lemma [3]. They have implications on the volumes of closed hyperbolic 3-manifolds [13].

An analogous lower bound independent from the Jorgensen's inequality for the maximum of the hyperbolic displacements by ξ and η known as the log 3 Theorem is proved by Culler and Shalen in [8]. This theorem states that $\max\{d_\xi z, d_\eta z\} \geq \log 3$ holds for any $z \in \mathbb{H}^3$ provided that ξ and η generate a purely loxodromic free Kleinian group [8, 15]. A brief survey by Shalen on the implications of the log 3 theorem and its generalization called the $\log(2k - 1)$ Theorem [1] on the volume of the closed hyperbolic 3-manifolds can be found in [15].

The lower bound given in the Jorgensen's inequality [11] is the best possible. It is conceivable that if one considers more restrictive Kleinian groups such as the classes of purely loxodromic and free Kleinian groups as a particular example the class of all finitely generated Schottky groups [12], a larger universal lower bound in the Jorgensen's inequality for these classes may be achieved. Consequently, the implications of this lower bound on the geometric and topological invariants of finite-volume hyperbolic 3-manifolds can be investigated for finite-volume hyperbolic 3-manifolds whose fundamental groups are k -free. A group Γ is called k -free if every finitely generated subgroup of Γ of rank at most k is free. With this motivation, the following refinement of the Jorgensen's Inequality is proved in [20]:

Theorem 1 *Let $\Gamma = \langle \xi_1, \xi_2 \rangle$ be a purely loxodromic free Kleinian group. If the following inequalities hold*

- (i) $d_\gamma z_2 < 1.6068\dots$ for every $\gamma \in \{\xi_2, \xi_1^{-1}\xi_2\xi_1, \xi_1\xi_2\xi_1^{-1}\}$ for the midpoint z_2 of the shortest geodesic segment connecting the axis of ξ_1 to the axis of $\xi_2^{-1}\xi_1\xi_2$ and
- (ii) $d_{\xi_2\xi_1\xi_2^{-1}}z_2 \leq d_{\xi_2\xi_1\xi_2^{-1}}z_1$ for the midpoint z_1 of the shortest geodesic segment connecting the axis of ξ_1 to the axis of $\xi_2\xi_1\xi_2^{-1}$

then, we have $|\text{trace}^2(\xi_1) - 4| + |\text{trace}(\xi_1\xi_2\xi_1^{-1}\xi_2^{-1}) - 2| \geq 1.5937\dots$

In this paper, we shall prove the statement below, also proposed in [20], as a generalization of Theorem 1:

Theorem 2 *Let $\Gamma = \langle \xi_1, \xi_2, \dots, \xi_n \rangle$ be a purely loxodromic free Kleinian group. If there exist generators ξ_i and ξ_j for $i \neq j$ such that*

- (i) $d_\gamma z_2 < \frac{1}{2} \log \alpha_n$ for every γ a generator or a length 3 conjugate formed by the generators other than ξ_i , $\xi_j^{-1}\xi_i\xi_j$ and $\xi_j\xi_i\xi_j^{-1}$ for the midpoint z_2 of the shortest geodesic segment connecting the axis of ξ_i to the axis of $\xi_j^{-1}\xi_i\xi_j$ and
- (ii) $d_{\xi_j\xi_i\xi_j^{-1}}z_2 \leq d_{\xi_j\xi_i\xi_j^{-1}}z_1$ for the midpoint z_1 of the shortest geodesic segment connecting the axis of ξ_i to the axis of $\xi_j\xi_i\xi_j^{-1}$, then the inequality

$$|\text{trace}^2(\xi_i) - 4| + |\text{trace}(\xi_i\xi_j\xi_i^{-1}\xi_j^{-1}) - 2| \geq 2 \sinh^2 \left(\frac{1}{4} \log \alpha_n \right) \tag{1}$$

holds. Above, α_n is the only root greater than $(2n - 1)^2$ of the polynomial

$$\begin{aligned} \mathcal{P}(\lambda) = & (8n^3 - 12n^2 + 2n + 1) \lambda^4 + \\ & (-64n^6 + 192n^5 - 192n^4 + 64n^3 + 4n^2 + 2n - 4) \lambda^3 + \\ & (-96n^5 + 224n^4 - 168n^3 + 52n^2 - 18n + 6) \lambda^2 + \\ & (32n^5 - 112n^4 + 128n^3 - 68n^2 + 22n - 4) \lambda + \\ & 16n^4 - 32n^3 + 24n^2 - 8n + 1. \end{aligned} \tag{2}$$

Both Theorems 1 and 2 follow from some lower bounds for the maximum of hyperbolic displacements given by the conjugates formed by isometries of length 1 in purely loxodromic free Kleinian groups. These lower bounds are calculated in Theorem 4.2 in [20] and Theorem 7 in Section 6. In fact, both of these theorems may be used in their own rights to investigate if the log 3 Theorem and its generalization can be improved using the techniques in [7] which will be left to future studies.

The proof of Theorem 1 also uses a result of Beardon [2, Theorem 5.4.5] which connects upper bounds for trace inequalities to a lower bound for the maximum of hyperbolic displacements by loxodromic isometries of \mathbb{H}^3 . The proof of Theorem 2 requires the use of the same tools with one major challenge due to the generalization of Theorem 1 to an arbitrary number of generators. This makes the necessary calculations for a lower bound for the maximum of hyperbolic displacements by loxodromics much more complicated. A direct analogue of the paper [20] for even $n = 3$ would have been technically very cumbersome. To overcome this difficulty, a new notational system is introduced in Section 2. With this new notational system, a simplified version of Theorem 2 is rephrased as Theorem 8 in Section 6.

A brief summary of the proof of Theorem 2 and the organization of this paper can be given as follows: The main step in the proof of Theorem 2 is the calculation of a lower bound for the maximum of the hyperbolic displacements given by the generators of $\Gamma = \langle \xi_1, \xi_2, \dots, \xi_n \rangle$ and their length 3 conjugates. This is achieved in Theorem 7 in Section 6 by considering two cases: Γ is geometrically infinite or Γ is geometrically finite. The machinery introduced by Culler and Shalen to prove the log 3 theorem [8] is used in both cases. The methods developed in [18, 19] are needed in the first case.

Assuming Γ is geometrically infinite, a carefully chosen decomposition denoted by $\Gamma_{\mathcal{D}}$ of $\Gamma = \langle \xi_1, \xi_2, \dots, \xi_n \rangle$ concentrating on the length 3 conjugates formed by the generators is defined in Section 2. This decomposition leads to a decomposition of the Patterson-Sullivan measure which is the area measure on the limit set of Γ homeomorphic to the sphere at infinity. The group-theoretical relations of $\Gamma_{\mathcal{D}}$ provide some measure-theoretical relations given in Theorem 3. With a key lemma [8, Lemma 5.5], these measure-theoretical relations provide lower bounds for the hyperbolic displacements by the isometries determined by the decomposition $\Gamma_{\mathcal{D}}$ of Γ . The lower bounds are considered as the values of a certain set of functions $\{f_r\}$, referred to as the displacement functions, evaluated at a point m in a certain simplex Δ . This is phrased in Proposition 1. The infimums α_n of the maximum of the displacement functions $\{f_r\}$ over Δ provide the lower bounds $(1/2) \log \alpha_n$ for the maximum of the hyperbolic displacements by the generators of Γ and their length 3 conjugates. In Section 3, we summarize the relevant parts of the Culler-Shalen machinery for this case of the proof, particularly, by underlining the importance and applications of the group theoretical relations.

All of Section 5 is devoted to the calculation of the infimums α_n , obtained by solving a family of nonlinear optimization problems via the Karush-Kuhn-Tucker theory [4], completing the proof of Theorem 7 when Γ is

geometrically infinite. The increase in the number of displacement functions $\{f_r\}$ due to the increase in the number of generators is tackled by the use of inherent symmetries of the decomposition $\Gamma_{\mathcal{D}}$ of Γ from the start rather than towards the end as it was done in [20]. This simplifies the entire process as the symmetries are not encrypted into the displacement functions. The use of these symmetries is achieved via classifications of the decomposition elements and group theoretical relations as they are discussed and presented in Section 4. On a side note, Section 5 is largely a simplified and generalized version of the calculations in [20] as we reap the fruits of our efforts in these classifications. Almost all the ideas that are used in this section are from [20] as they were presented in this paper but they are translated to the language of the classifications in Section 4.

The second case, when $\Gamma = \langle \xi_1, \xi_2, \dots, \xi_n \rangle$ is geometrically finite, in the proof of Theorem 7 is considered in Section 6. This case can be reduced to geometrically infinite case by the use of a number of deep results [8, Propositions 8.2 and 9.3], [6, Main Theorem] and [5] from the deformation theory of Kleinian groups completing the proof of this theorem. Once Theorem 7 is at hand, the main result Theorem 2 (Theorem 8 in Section 6) follows from a proof by contradiction.

Under the hypothesis of Theorem 2, assuming the strictly less than inequality in (1) implies the existence of a point in \mathbb{H}^3 so that the infimum of the maximum of the hyperbolic displacements given by the generators and their length 3 conjugates in $\Gamma = \langle \xi_1, \xi_2, \dots, \xi_n \rangle$ at that point becomes strictly less than $(1/2) \log \alpha_n$ contradicting with Theorem 7. This is done by reversing the inequalities in the proof of a theorem by Beardon [2, Theorem 5.4.5 (iii)] which states that if ξ_1 is elliptic or strictly loxodromic and $|\text{trace}(\xi_1) - 4| < 1/4$, then $\max \left\{ \sinh \frac{1}{2} d_{\xi_1} z, \sinh \frac{1}{2} d_{\xi_2 \xi_1 \xi_2^{-1}} z \right\} \geq 1/4$ for all $z \in \mathbb{H}^3$ provided that $\langle \xi_1, \xi_2 \rangle$ is discrete and non-elementary. A number of auxiliary lemmas needed in the proof of Theorem 8 are also proved in the final section.

2. Basic definitions and notations

Throughout this text, we will work with a fixed subset $\Xi = \{\xi_1, \xi_2, \dots, \xi_n\}$ of $PSL(2, \mathbb{C})$ such that $\Gamma = \langle \Xi \rangle$ is a purely loxodromic group freely generated by Ξ . Let $\Xi^{-1} = \{\xi_1^{-1}, \xi_2^{-1}, \dots, \xi_n^{-1}\}$. The sets of particular interest in this paper will be the subsets Γ_1, Ψ and Γ_* of Γ defined as $\Gamma_1 = \{1\} \cup \Xi \cup \Xi^{-1}$,

$$\begin{aligned} \Psi &= \{ \xi_i^{2t}, \xi_i^t \xi_j^{2s}, \xi_i^t \xi_j^s \xi_k^p \mid i, j, k \in \{1, 2, \dots, n\}, i \neq j, j \neq k, t, s, p \in \{-1, +1\} \}, \\ \Gamma_* &= \{ 1, \xi_i^t, \xi_i^t \xi_j^s \xi_i^{-t}, \mid i, j \in \{1, 2, \dots, n\}, i \neq j, t, s \in \{-1, +1\} \}. \end{aligned}$$

In general, we will consider i, j, k, t, s, p varying in their respective ranges arbitrarily and use subindexes such as i_0, j_0, i_1 to represent their values once they are fixed. We will also refrain from stating obvious equalities or inequalities such as $i \neq j$ and $t = \pm 1$. This notational compromise leads to the following relaxations of the definitions above

$$\Psi = \{ \xi_i^{2t}, \xi_i^t \xi_j^{2s}, \xi_i^t \xi_j^s \xi_k^p \} \quad \text{and} \quad \Gamma_* = \{ 1, \xi_i^t, \xi_i^t \xi_j^s \xi_i^{-t} \}.$$

For $\psi \in \Psi$, let J_ψ represents the elements of Γ that start with ψ in their simplified form. Notice that $(J_\psi)_{\psi \in \Psi}$ gives a collection of disjoint subsets of Γ . Moreover, we have the following partition of Γ :

$$\Gamma = \{1\} \sqcup \Xi \sqcup \Xi^{-1} \sqcup \bigsqcup_{\psi \in \Psi} J_\psi. \tag{3}$$

The setting $\mathcal{D} = (\Psi, \Gamma_*)$ together with the partition above is said to be a decomposition $\Gamma_{\mathcal{D}}$ of Γ . A group

theoretical relation r of the decomposition $\Gamma_{\mathcal{D}}$ is defined to be a triple $(\gamma_r, \psi_r, \Psi_r)$ satisfying the following:

$$r : \gamma_r \in \Gamma_*, \psi_r \in \Psi, \Psi_r \subseteq \Psi \text{ and } \gamma_r J_{\psi_r} = \Gamma - (\{\cdot\} \cup J_{\Psi_r}). \tag{4}$$

To better understand this definition, consider r with $\gamma_r = \xi_1^{-1}$ and $\psi_r = \xi_1 \xi_2 \xi_3$. It is easy to see that $\gamma_r J_{\psi_r}$ will consist of all the elements of Γ starting with $\xi_2 \xi_3$. The term $(\{\cdot\} \cup J_{\Psi_r})$ in the relation equation should be all elements not starting with $\xi_2 \xi_3$. If we take $\Psi_r = \Psi - \{\xi_2 \xi_3, \xi_2 \xi_3 \xi_i^t\}$, then $J_{\Psi_r} = \bigcup_{\psi \in \Psi_r} J_{\psi}$ will exactly be what needs to be removed from Γ . Hence, the followings $r : \gamma_r = \xi_1^{-1}, \psi_r = \xi_1 \xi_2 \xi_3, \Psi_r = \Psi - \{\xi_2 \xi_3^2, \xi_2 \xi_3 \xi_i^t\}, \{\cdot\} = \Gamma_1$ define a group theoretical relation. Including the relation above, all relations are quite symmetrical. We can generalize them into families of relations as follows:

$$r : \gamma_r = \xi_{i_0}^{-t_0}, \psi_r = \xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{k_0}^{p_0}, \Psi_r = \Psi - \left\{ \xi_{j_0}^{s_0} \xi_{k_0}^{2p_0}, \xi_{j_0}^{s_0} \xi_{k_0}^{p_0} \xi_i^t \right\}, \{\cdot\} = \Gamma_1.$$

Next, we shall simplify the notation for displacement functions. Let $x \in \mathbb{R}^{\Psi}$ and r be a relation. We define

$$x_r = x(\psi_r), X_r = \sum_{\psi \in \Psi_r} x(\psi) \text{ and } X = \sum_{\psi \in \Psi} x(\psi).$$

Although this notation has been described above only for x , we shall use capital letters almost exclusively for this purpose. Accordingly, the formula of the displacement function f_r of the relation r is defined as:

$$f_r : \Delta \longrightarrow \mathbb{R}_{++} \text{ such that } f_r(x) = \frac{1 - x_r}{x_r} \cdot \frac{1 - X_r}{X_r},$$

where Δ will denote the simplex $\{x \in \mathbb{R}_{++}^{\Psi} \mid X = 1\}$. To define the α used in the rest of the paper, we use the polynomial $\mathcal{P}(\lambda)$ in (2) and introduce the lemma below:

Lemma 1 *The polynomial $\mathcal{P}(\lambda)$ has a unique root larger than $(2n - 1)^2$ and all other roots are less than 1.*

Proof All four roots of $\mathcal{P}(\lambda)$ are real and distinct for $n \geq 2$. Therefore, it suffices to find certain intervals such that the evaluation on either side have opposite signs.

Table 1. Roots of \mathcal{P} .

λ	$\mathcal{P}(\lambda)$	sign of \mathcal{P}
-2	$(2n - 3)(256n^5 - 608n^4 + 424n^3 - 36n^2 + 18n - 17)$	pos.
-1/n	$-(16n^8 - 48n^7 + 72n^6 - 84n^5 + 33n^4 + 10n^3 + 8n^2 - 6n - 1)/n^4$	neg.
-1/(2n - 1)	$32n^4(n - 1)/(2n - 1)^2$	pos.
1	$-64n^4(n - 1)^2$	neg.
$(2n - 1)^2$	$-128n^4(2n^2 - 3n + 1)^3(4n^2 - 8n + 5)$	neg.
$(2n - 1)^3$	$16(2n - 1)^4(n - 1)^2(32n^7 - 80n^6 + 56n^5 + 4n^4 - 22n^3 + 16n^2 - 5n + 1)$	pos.

The first 4 rows of Table 1 prove that $\mathcal{P}(\lambda)$ has three roots between -2 and 1. The last two rows prove that $\mathcal{P}(\lambda)$ has a root larger than $(2n - 1)^2$. Hence, we located all four roots.

By Lemma 1, we can formally give the definition of $\alpha = \alpha_n$: the unique root of $\mathcal{P}(\lambda)$ larger than $(2n - 1)^2$.

3. The Culler-Shalen machinery

In the rest of the paper, we will use S_∞ to denote the boundary of the canonical compactification of \mathbb{H}^3 . The decompositions $\Gamma_{\mathcal{D}}$ of Γ in (3) and their group theoretical relations defined in (4) in the previous section allow one to decompose the area measure of S_∞ into certain Borel measures and obtain some measure theoretical relations among these Borel measures, respectively. Consequently, measure theoretical relations lead to the lower bounds for the hyperbolic displacements by the isometries used in the definition of $\Gamma_{\mathcal{D}}$. This will be done in an almost identical way in [8, Lemma 5.3], [18, Lemma 3.3, Theorem 3.4], and [19, Theorem 2.1]. Therefore, we phrase the basic theorems below without any proofs:

Theorem 3 *Let $z \in \mathbb{H}^3$ and A_z be the area measure of S_∞ at z . If Γ is geometrically infinite, then there exists a family of Borel measures $\{\nu_\psi\}_{\psi \in \Psi}$ on S_∞ such that:*

$$(i) A_z = \sum_{\psi \in \Psi} \nu_\psi, \quad (ii) A_z(S_\infty) = 1, \quad (iii) \int_{S_\infty} \lambda_{\gamma_r, z}^2 d\nu_{\psi_r} = 1 - \sum_{\psi \in \Psi_r} \nu_\psi(S_\infty) \text{ for any relation } r.$$

With Theorem 3, we can define a special element $m(\psi) = \nu_\psi(S_\infty) \in \mathbb{R}^\Psi$ for all $\psi \in \Psi$ for any given $z \in \mathbb{H}^3$. We take $\{\nu_\psi\}_{\psi \in \Psi}$ as in Theorem 3. We also use the notations of [2] for the hyperbolic displacement of z . If γ is an isometry of \mathbb{H}^3 with respect to the hyperbolic metric ρ , then $d_\gamma z = \rho(z, \gamma z)$.

The crucial part of the Culler-Shalen machinery for this paper is that it gives lower bounds for the hyperbolic displacements given by the isometries. This is due to [8, Lemma 5.5] and its improved version [9, Lemma 2.1]. We present the latter:

Lemma 2 *Let a and b be numbers in $[0, 1]$ which are not both equal to 0 and are not both equal to 1. Let γ be a loxodromic isometry of \mathbb{H}^3 and let $z \in \mathbb{H}^3$. Suppose that ν is a measure on S_∞ such that*

$$\nu \leq A_z, \quad \nu(S_\infty) \leq a \text{ and } \int_{S_\infty} \lambda_{\gamma, z}^2 d\nu \geq b.$$

Then, we have

$$a > 0, \quad b < 1 \text{ and } d_\gamma z \geq \frac{1}{2} \log \frac{b(1-a)}{a(1-b)}.$$

Finally, we merge the results of Theorem 3 and Lemma 2 to obtain the lower bounds for the hyperbolic displacements that we will need in Section 6.

Proposition 1 *Let $z \in \mathbb{H}^3$ and m be the relevant vector in \mathbb{R}^Ψ . If Γ is geometrically infinite, then for any $\gamma \in \Gamma_*$ and for any relation r defined by Ψ we have:*

$$(i) m \in \Delta, \quad (ii) e^{2d_\gamma z} \geq f_r(m) \text{ for any relation } r \text{ with } \gamma_r = \gamma.$$

Proof First, we shall prove that $m(\psi) \neq 0$ for all $\psi \in \Psi$. We do so by starting with the conjugate elements of Ψ and then generalizing to the rest of the elements.

Assume $m(\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0}) = 0$. Let r_0 be the relation defined by $r_0 : \gamma_0 = \xi_{i_0}^{t_0} \xi_{j_0}^{-s_0} \xi_{i_0}^{-t_0}$, $\psi_0 = \xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0}$, $\Psi_0 = \{\xi_{i_0}^{2t_0}, \xi_{i_0}^{t_0} \xi_j^{2s}, \xi_{i_0}^{t_0} \xi_j^s \xi_k^p\}$. Since $m(\psi_0) = 0$, we have $M_0 = \sum_{\psi \in \Psi_0} m(\psi) = 1$ by Theorem 3(iii). This

implies that there exists a $\psi_1 \in \Psi_0$ such that $m(\psi_1) \neq 0$. Take $\psi_2 = \xi_{i_2}^{t_2} \xi_{j_2}^{s_2} \xi_{i_2}^{-t_2}$, where $i_2 \neq i_0$ so that $\psi_2 \notin \Psi_0$. Notice that $M = \sum_{\psi \in \Psi} m(\psi) = 1$ by Theorem 3 (i) and (ii). Now, as both M and M_0 are equal to 1, we have $\sum_{\psi \in \Psi - \Psi_0} m(\psi) = 0$. This implies that $m(\psi_2) = 0$.

Let r_2 be the relation defined by: $r_2 : \gamma_2 = \psi_2^{-1}$, $\psi_2 = \psi_2$, $\Psi_2 = \{\xi_{i_2}^{2t_2}, \xi_{i_2}^{t_2} \xi_j^{2s}, \xi_{i_2}^{t_2} \xi_j^s \xi_k^p\}$ so that $M_2 = 1$ and $M - M_2 = 0$. This gives that $m(\psi) = 0$ for all $\psi \in \Psi - \Psi_2$, which contradicts with the existence of ψ_1 as $\Psi_0 \subset \Psi - \Psi_2$. Hence, we have the following fact

$$m(\xi_i^t \xi_j^s \xi_i^{-t}) \neq 0. \tag{5}$$

Next, assume that $m(\xi_{i_0}^{2t_0}) = 0$. Take r_0 to be the group theoretical relation $r_0 : \gamma_0 = \xi_{i_0}^{-t_0}$, $\psi_0 = \xi_{i_0}^{2t_0}$, $\Psi_0 = \Psi - \{\xi_{i_0}^{2t_0}, \xi_{i_0}^{t_0} \xi_j^{2s}, \xi_{i_0}^{t_0} \xi_j^s \xi_k^p\}$. By Theorem 3, we obtain $m(\psi) = 0$ for all $\psi \in \Psi - \Psi_0$. Thus, $m(\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0}) = 0$ contradicting (5). This proves that $m(\xi_i^{2t}) \neq 0$.

Using similar arguments given above, we proceed as follows: assume that $m(\xi_{i_0}^{t_0} \xi_{j_0}^{2s_0}) = 0$. We consider the relation $r_0 : \gamma_0 = \xi_{i_0}^{t_0} \xi_{j_0}^{-s_0} \xi_{i_0}^{-t_0}$, $\psi_0 = \xi_{i_0}^{t_0} \xi_{j_0}^{2s_0}$, $\Psi_0 = \Psi - \{\xi_{i_0}^{t_0} \xi_{j_0}^{2s_0}, \xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_k^p\}$. Then, we derive that $m(\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0}) = 0$ which contradicts with (5). Assume that $m(\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{t_0}) = 0$. Consider the relation $r_0 : \gamma_0 = \xi_{i_0}^{-t_0}$, $\psi_0 = \xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^p$, $\Psi_0 = \Psi - \{\xi_{j_0}^{s_0} \xi_{i_0}^{2t_0}, \xi_{j_0}^{s_0} \xi_{i_0}^p \xi_k^p\}$. Then, we get $m(\xi_{j_0}^{s_0} \xi_{i_0}^{t_0} \xi_{j_0}^{-s_0}) = 0$, a contradiction by (5). Finally, assume that $m(\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{k_0}^p) = 0$. We consider the relation $r_0 : \gamma_0 = \xi_{i_0}^{-t_0}$, $\psi_0 = \xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{k_0}^p$, $\Psi_0 = \Psi - \{\xi_{j_0}^{s_0} \xi_{k_0}^{2p_0}, \xi_{j_0}^{s_0} \xi_{k_0}^p \xi_k^p\}$. Then, we find that $m(\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0}) = 0$, again a contradiction by (5). Therefore, we derive that $m(\xi_i^t \xi_j^{2s}) \neq 0$, $m(\xi_i^t \xi_j^s \xi_i^t) \neq 0$, and $m(\xi_i^t \xi_j^s \xi_k^p) \neq 0$. We cycled over all element types in Ψ showing $m \in \mathbb{R}_{++}^\Psi$. Since we already know that $M = 1$, this proves (i) of the current proposition.

To prove (ii), fix any relation r with $\gamma_r = \gamma$. Take $a = \nu_{\psi_r}(S_\infty) = m(\psi_r)$ and $b = \int_{S_\infty} \lambda_{\gamma,z}^2 d\nu_{\psi_r}$. Notice that a and b satisfy the conditions of Lemma 2. Thus, we obtain

$$d_\gamma z \geq \frac{1}{2} \log \left(\frac{1-a}{a} \cdot \frac{b}{1-b} \right).$$

Moreover, if $a = m_r$ and $b = 1 - \sum_{\psi \in \Psi_r} m(\psi) = 1 - M_r$ (by Theorem 3(iii)), then the inequality above turns into

$$d_\gamma z \geq \frac{1}{2} \log \left(\frac{1-m_r}{m_r} \cdot \frac{1-M_r}{M_r} \right) = \frac{1}{2} \log f_r(m).$$

Taking the exponential of the last expression yields (ii).

The proposition above shows that we can use the displacement functions of the relations r and m to find lower bounds for hyperbolic displacements. We could sharpen this bound by searching for the maximal value of $f_r(m)$ over a family of relations. Obviously, the larger the family, the better the lower bound for $\max\{f_r(m) \mid r\}$. Unfortunately, calculating values for m for every $z \in \mathbb{H}^3$ is not feasible. Therefore, we look at all possible values of m within the bounds set by Proposition 1(i). As we do not know which element of Δ is actually m , we need to calculate the infimum over all elements. Ultimately, we shall prove in Section 6 that what matters is the infimum of the max function of a collection of displacement functions over Δ . This optimization is performed in Section 5.

4. The max functions and classification of relations

In this section, we mainly present some tables illuminating the mechanics of the displacement functions for the relevant decompositions $\Gamma_{\mathcal{D}}$. We also discuss how these tables are used in later sections and the objectives behind their discovery. We count and classify elements of Ψ into 5 types in the table below:

While reading Table 2, remember the notational relaxations that are discussed in Section 2 and also assume $|\{i, j, k\}| = 3$ at any instance. The above classification (with the counts) will be useful in simplifying the calculations for the optimization question which is discussed in Section 1. However, what we really need is a set of coherent relations that will yield many symmetries of the simplex Δ .

Notice that for the purposes of finding a lower bound for $f(m)$ as discussed in Section 3, one can work with as small or as large a set of relations as one desires. Notice that if the set of relations is small, the resulting lower bound runs the risk of being too small as well and not be useful in any shape for future applications. On the opposite side, if the set of relations being considered is too large (or badly curated) the resulting optimization question will become too complicated to solve. The approach preferred in this paper is to work with and optimize a midsize collection of displacement functions but to choose that collection in such a way that the resulting lower bound is actually identical to that of a much larger collection. In the following table, we list, classify, and count all the relations in the large collection \mathcal{G} .

Table 2. Types of elements in Ψ .

	ψ	Count
type 1	ξ_i^{2t}	$2n$
type 2	$\xi_i^t \xi_j^{2s}$	$4n(n-1)$
type 3	$\xi_i^t \xi_j^s \xi_i^t$	$4n(n-1)$
type 4	$\xi_i^t \xi_j^s \xi_i^{-t}$	$4n(n-1)$
type 5	$\xi_i^t \xi_j^s \xi_k^p$	$8n(n-1)(n-2)$
$ \Psi $	$ \{i, j, k\} = 3$	$2n + 4n(n-1) + 8n(n-1)^2$

In fact, \mathcal{G} is the largest possible collection that can be worked with. Table 3 has been prepared in a ψ_r centric way, in the sense that to generate a relation, first ψ_r is fixed and then a suitable γ_r is chosen. The calculation of Ψ_r and $\{\cdot\}$ at every type is very similar to the example done in Section 2. With this collection at hand, we can define the function $G : \Delta \rightarrow \mathbb{R}_{++}$ as follows

$$G(x) = \max_{r \in \mathcal{G}} f_r(x).$$

It is the infimum of this function that we will ultimately obtain in Section 5. The midsize collection \mathcal{F} is the subset of \mathcal{G} consisting of Type 1a, 2b, 3a, 4b, and 5a relations. The relevant function $F : \Delta \rightarrow \mathbb{R}_{++}$ of this collection is defined as follows

$$F(x) = \max_{r \in \mathcal{F}} f_r(x).$$

In Section 5, we will mostly be optimizing F .

Table 3. Relations in the collection \mathcal{G} .

	γ_r	ψ_r	Ψ_r	$\{\cdot\}$	Count
type 1a	$\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{2t_0}$	$\Psi - \{\xi_{i_0}^{2t_0}, \xi_{i_0}^{t_0} \xi_j^{2s}, \xi_{i_0}^{t_0} \xi_j^s \xi_k^p\}$	$\Gamma_1 - \{\xi_{i_0}^{t_0}\}$	$2n$
type 1b	$\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0}$	$\xi_{i_0}^{2t_0}$	$\Psi - \{\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{t_0}\}$	Γ_1	$4n(n-1)$
type 2a	$\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0} \xi_{j_0}^{2s_0}$	$\Psi - \{\xi_{j_0}^{2s_0}\}$	Γ_1	$4n(n-1)$
type 2b	$\xi_{i_0}^{t_0} \xi_{j_0}^{-s_0} \xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0} \xi_{j_0}^{2s_0}$	$\Psi - \{\xi_{i_0}^{t_0} \xi_{j_0}^{2s_0}, \xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_k^p\}$	Γ_1	$4n(n-1)$
type 3a	$\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{t_0}$	$\Psi - \{\xi_{j_0}^{s_0} \xi_{i_0}^{2t_0}, \xi_{j_0}^{s_0} \xi_{i_0}^{t_0} \xi_k^p\}$	Γ_1	$4n(n-1)$
type 3b	$\xi_{i_0}^{t_0} \xi_{j_0}^{-s_0} \xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{t_0}$	$\Psi - \{\xi_{i_0}^{2t_0}\}$	Γ_1	$4n(n-1)$
type 4a	$\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0}$	$\Psi - \{\xi_{j_0}^{s_0} \xi_{i_0}^{-2t_0}, \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0} \xi_k^p\}$	Γ_1	$4n(n-1)$
type 4b	$\xi_{i_0}^{t_0} \xi_{j_0}^{-s_0} \xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0}$	$\{\xi_{i_0}^{2t_0}, \xi_{i_0}^{t_0} \xi_j^{2s}, \xi_{i_0}^{t_0} \xi_j^s \xi_k^p\}$	$\{\xi_{i_0}^{t_0}\}$	$4n(n-1)$
type 5a	$\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{k_0}^{p_0}$	$\Psi - \{\xi_{j_0}^{s_0} \xi_{k_0}^{2p_0}, \xi_{j_0}^{s_0} \xi_{k_0}^{p_0} \xi_k^p\}$	Γ_1	$8n(n-1)(n-2)$
type 5b	$\xi_{i_0}^{t_0} \xi_{j_0}^{-s_0} \xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{k_0}^{p_0}$	$\Psi - \{\xi_{i_0}^{t_0} \xi_{k_0}^{2p_0}, \xi_{i_0}^{t_0} \xi_{k_0}^{p_0} \xi_k^p\}$	Γ_1	$8n(n-1)(n-2)$
$ \mathcal{G} $					$2(2n + 4n(n-1) + 8n(n-1)^2) + 2n(2n-3)$

The choice of \mathcal{F} can be explained as follows: For example, for any type 1b relation there exists a type 1a relation such that the type 1a relation's displacement function is larger at every point of the simplex. This implies that the displacement functions for all the type 1b relations are unnecessary inside the max function. However, most of the other "cancellations" are not this straightforward and ultimately unnecessary as once the unique optimal solution for the infimum of F is calculated, it is straight forward to show that it must also be the unique optimal point for the infimum of G . As a result, the collection \mathcal{F} has been chosen largely due to the patterns observed in the $n = 2$ in [20] and the computer models observed by the authors. We list the elements of \mathcal{F} for ease of access in the table below:

In Table 4, we omitted the set $\{\cdot\}$ and γ_r columns (as they are irrelevant to the calculation of displacements) and expanded the description of Ψ_r as to distinguish (and count) type 3, 4, and 5 elements of Ψ_r better. Also notice that $|\mathcal{F}| = |\Psi|$.

In the rest of this section, we shall also calculate the gradients of the displacement functions defined in Section 2. They will be needed in Section 5. Recall that for a relation r , the definition of the displacement function f_r of r is

$$f_r(x) = \frac{(1-x_r)(1-X_r)}{x_r X_r},$$

where $x_r = x(\psi_r)$ and $X_r = \sum_{\psi \in \Psi_r} x(\psi)$. Thus, the partial derivative of f_r with respect to a coordinate $\psi \in \Psi$ depends on whether the coordinate is in Ψ_r and if it is equal to ψ_r or not. To further complicate the issue, we have $\psi_r \in \Psi_r$ for some displacements and $\psi_r \notin \Psi_r$ for some displacements. Therefore, we present the gradient of f_r for a given relation r in two cases. For relations of type 1a and 2b, we have $\psi_r \notin \Psi_r$ which implies that

$$\frac{df_r}{d\psi} = \begin{cases} \frac{1-x_r}{x_r} \cdot \frac{-1}{X_r^2} & \text{if } \psi \in \Psi_r \\ \frac{-1}{x_r^2} \cdot \frac{1-X_r}{X_r} & \text{if } \psi = \psi_r \\ 0 & \text{if } \psi \notin \Psi_r \cup \{\psi_r\}. \end{cases}$$

Table 4. Relations in the collection \mathcal{F} .

	ψ_r	Ψ	Ψ_r				Count
1a	$\xi_{i_0}^{2t_0}$	$\Psi - \{\xi_{i_0}^{2t_0},$ 1	$\xi_{i_0}^{t_0} \xi_j^{2s},$ $2(n-1)$	$\xi_{i_0}^{t_0} \xi_j^s \xi_{i_0}^{t_0},$ $2(n-1)$	$\xi_{i_0}^{t_0} \xi_j^s \xi_{i_0}^{-t_0},$ $2(n-1)$	$\xi_{i_0}^{t_0} \xi_j^s \xi_k^p \}$ $4(n-1)(n-2)$	$2n$
2b	$\xi_{i_0}^{t_0} \xi_{j_0}^{2s_0}$	$\Psi - \{$ 1	$\xi_{i_0}^{t_0} \xi_{j_0}^{2s_0},$ 1	$\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{t_0},$ 1	$\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0},$ 1	$\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_k^p \}$ $2(n-2)$	$4n(n-1)$
3a	$\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{t_0}$	$\Psi - \{$ 1	$\xi_{j_0}^{s_0} \xi_{i_0}^{2t_0},$ 1	$\xi_{j_0}^{s_0} \xi_{i_0}^{t_0} \xi_{j_0}^{s_0},$ 1	$\xi_{j_0}^{s_0} \xi_{i_0}^{t_0} \xi_{j_0}^{-s_0},$ 1	$\xi_{j_0}^{s_0} \xi_{i_0}^{t_0} \xi_k^p \}$ $2(n-2)$	$4n(n-1)$
4b	$\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0}$	$\{\xi_{i_0}^{2t_0},$ 1	$\xi_{i_0}^{t_0} \xi_j^{2s},$ $2(n-1)$	$\xi_{i_0}^{t_0} \xi_j^s \xi_{i_0}^{t_0},$ $2(n-1)$	$\xi_{i_0}^{t_0} \xi_j^s \xi_{i_0}^{-t_0},$ $2(n-1)$	$\xi_{i_0}^{t_0} \xi_j^s \xi_k^p \}$ $4(n-1)(n-2)$	$4n(n-1)$
5a	$\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{k_0}^{p_0}$	$\Psi - \{$ 1	$\xi_{j_0}^{s_0} \xi_{k_0}^{2p_0}$ 1	$\xi_{j_0}^{s_0} \xi_{i_0}^{t_0} \xi_{j_0}^{s_0},$ 1	$\xi_{j_0}^{s_0} \xi_{i_0}^{t_0} \xi_{j_0}^{-s_0},$ 1	$\xi_{j_0}^{s_0} \xi_{i_0}^{t_0} \xi_k^p \}$ $2(n-2)$	$\frac{8n!}{(n-3)!}$
$ \mathcal{F} $						$2n + 4n(n-1) + 8n(n-1)^2$	

Using the partial derivatives above, we see that

$$\nabla f_r(x) = -\frac{1-x_r}{x_r X_r^2} \cdot \sum_{\psi \in \Psi_r} e_\psi - \frac{1-X_r}{x_r^2 X_r} \cdot e_{\psi_r}. \tag{6}$$

For relations of type 3a, 4b, and 5a, we have $\psi_r \in \Psi_r$ so that

$$\frac{df_r}{d\psi} = \begin{cases} \frac{1-x_r}{x_r} \cdot \frac{-1}{X_r^2} & \text{if } \psi \in \Psi_r - \{\psi_r\} \\ -\frac{1}{x_r^2} \cdot \frac{1-X_r}{X_r} + \frac{1-x_r}{x_r} \cdot \frac{-1}{X_r^2} & \text{if } \psi = \psi_r \\ 0 & \text{if } \psi \notin \Psi_r \cup \{\psi_r\}. \end{cases}$$

As a result, we calculate the gradients as

$$\nabla f_r(x) = -\frac{1-x_r}{x_r X_r^2} \cdot \sum_{\psi \in \Psi_r} e_\psi - \left(\frac{1-X_r}{x_r^2 X_r} + \frac{1-x_r}{x_r X_r^2} \right) \cdot e_{\psi_r}. \tag{7}$$

In both (6) and (7), the expression $(e_\psi)_\Psi$ is the canonical basis for \mathbb{R}^Ψ .

5. Optimizing the infimum of the max function

The purpose of this section is to compute $\inf_{x \in \Delta}(G(x))$. However, for this purpose, we will calculate $\inf_{x \in \Delta}(F(x))$. This calculation can be summarized in three phases. The first phase is to show the attainment of optimality (for both G and F). The second phase is to prove the uniqueness of optimality. The final

phase is to calculate the point of optimality using the Karush-Kuhn-Tucker theorem. We would like to point out that the problem with the infimum is not that of existence, as the image is a bounded below subset of \mathbb{R} that part is obvious, but attainment in Δ . In fact, from here on we shall denote the infimums with α^* and β^* ; in other words,

$$\alpha^* = \inf_{x \in \Delta} (G(x)) \quad \text{and} \quad \beta^* = \inf_{x \in \Delta} (F(x)).$$

Lemma 3 *There exists $x^* \in \Delta$ such that $G(x^*) = \alpha^*$.*

Proof As α^* is the infimum of the set $G(\Delta)$, there must be a sequence $(x^n)_n$ in Δ such that $G(x^n) \rightarrow \alpha^*$. Moreover, $(x^n)_n$ must have a convergent subsequence since the closure of Δ is a bounded subset of \mathbb{R}^Ψ . Without loss of generality, assume $x^n \rightarrow \bar{x}$ for some $\bar{x} \in cl(\Delta)$. Notice that $cl(\Delta) = \{x \in \mathbb{R}_+^\Psi \mid X = 1\}$. The fact that the max function of continuous functions is also continuous is a simple exercise ([4] Exercise 2.3.1). Our aim is to prove $\bar{x} \notin cl(\Delta) - \Delta$. We shall do so by assuming the opposite and obtaining a contradiction.

Let $K = \{\psi \in \Psi \mid \bar{x}_\psi = 0\}$. The assumption that $\bar{x} \in cl(\Delta) - \Delta$ implies $K \neq \emptyset$. Now, if there exists a relation r such that $\psi_r \in K$ and $\bar{X}_r \in [0, 1)$, then $x_r^n \rightarrow \bar{x}_r = 0$ and $(1 - X_r^n)/X_r^n$ either converges or diverges to ∞ , which shows that $f_r(x^n) = (1 - x_r^n)/x_r^n \cdot (1 - X_r^n)/X_r^n \rightarrow \infty$. However, $f_r(x) \leq G(x)$ for any x . Thus, we also have $\lim f_r(x^n) \leq \lim G(x^n) = \alpha^*$, contradicting the existence of α^* .

The above paragraph shows that for all $r \in \mathcal{G}$ with $\psi_r \in K$, we have $\bar{X}_r = 1$. However, we know that $\bar{X}_r + \sum_{\psi \notin \Psi_r} \bar{x}(\psi) = \bar{X} = 1$ proving $\bar{x}(\psi) = 0$ for all $\psi \in \Psi - \Psi_r$ in any such relation. In summary, what we showed is that

$$\psi_r \in K \text{ implies } \Psi - \Psi_r \subseteq K \tag{8}$$

for a relation $r \in \mathcal{G}$. The contradiction that we seek will soon come in the form of $K = \Psi$ as that would imply $\bar{X} = 0$, contradicting the fact that $\bar{x} \in cl(\Delta)$. We already know $K \neq \emptyset$ by our assumption. We shall start with a random element of K . We will cycle over all possible types in Table 2 and use (8) together with Table 3 to show that $K = \Psi$.

If $\xi_{i_0}^{2t_0} \in K$, the relevant type 1a relation r_0 will satisfy $\psi_{r_0} \in K$. By (8), we get $\Psi - \Psi_{r_0} \subseteq K$, implying $\{\xi_{i_0}^{t_0} \xi_j^s \xi_{i_0}^{-t_0}\} \subseteq K$. Repeating the same process for the type 4b relation r_j with $\psi_{r_j} = \xi_{i_0}^{t_0} \xi_j^s \xi_{i_0}^{-t_0}$, we get $\{\xi_i^{2t}\} - \{\xi_{i_0}^{2t_0}\} \subseteq K$. Since we already know that $\xi_{i_0}^{2t_0} \in K$, we actually have $\{\xi_i^{2t}\} \subseteq K$. The arguments so far in this paragraph show that for any type 1a relation r , we obtain $\psi_r \in K$. Merging all this information together, we get $\Psi = \bigcup_{i=1}^n (\Psi - \Psi_{r_i}) \subseteq K$, proving what we wanted. We summarize this result below:

$$\text{if there exists a type 1 element in } K, \text{ then } \Psi = K. \tag{9}$$

If $\xi_{i_0}^{t_0} \xi_{j_0}^{2s_0} \in K$, the relevant type 2b relation r_0 will satisfy $\psi_{r_0} \in K$ and so by (8) we get $\Psi - \Psi_{r_0} \subseteq K$, implying $\{\xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0}\} \subseteq K$. Repeating the same process for the type 4b relation r_1 with $\psi_{r_1} = \xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0}$, we get $\{\xi_i^{2t}\} - \{\xi_{i_0}^{2t_0}\} \subseteq K$. This shows that K has a type 1 element and $K = \Psi$ by (9). We summarize this result below:

$$\text{if there exists a type 2 element in } K, \text{ then } \Psi = K. \tag{10}$$

If K has an element of the other types, we have the same conclusion $\Psi = K$. More specifically, we see that if K has a type 3 element and the relevant type 3a relation together with (10), then $K = \Psi$. The

assumption that K has a type 4 element and the relevant type 4b relation together with (9) implies $K = \Psi$. If K has a type 5 element and the relevant type 5a relation together with (10), then $K = \Psi$. This gives that as long as K has an element no matter what type, $K = \Psi$.

Notice that in the proof above, we only used the relations in \mathcal{F} . Therefore, the following lemma has an identical proof.

Lemma 4 *There exists $x^* \in \Delta$ such that $F(x^*) = \beta^*$.*

The next lemma gives us a rough location for β^* and α^* which proves very useful for the proofs in the second phase.

Lemma 5 *The infimums of $F(x)$ and $G(x)$ on Δ , β^* and α^* are greater than 1 and less than or equal to α . Moreover, $\beta^* \leq \alpha^*$.*

Proof First of all, we know that $1 < \alpha$ by Lemma 1. Therefore, at the very least the lemma makes sense. Also, since $\mathcal{F} \subset \mathcal{G}$, we have $F(x) \leq G(x)$ for any $x \in \Delta$. This proves $\beta^* \leq \alpha^*$.

To show that $1 < \beta^*$, let r be any type 1a relation. As $\psi_r \notin \Psi_r$, we have $1 = X > x_r + X_r$ for all $x \in \Delta$. Dividing both sides of this inequality with $x_r X_r$, we obtain:

$$\frac{1}{x_r X_r} > \frac{1}{x_r} + \frac{1}{X_r} \quad \text{or} \quad \frac{1}{x_r X_r} - \frac{1}{x_r} - \frac{1}{X_r} > 0.$$

Moreover, we have

$$f_r(x) = \frac{1 - x_r}{x_r} \cdot \frac{1 - X_r}{X_r} = \frac{1}{x_r X_r} - \frac{1}{x_r} - \frac{1}{X_r} + 1,$$

which must be greater than 1 by the first inequality. Since $F(x) \geq f_r(x)$ for any relation (and specifically any type 1a relation) in \mathcal{F} , we see that $\beta^* = F(x^*) \geq f_r(x^*) > 1$ for x^* in Lemma 4. To show that $\alpha^* \leq \alpha$, we shall need a special element $y \in \Delta$. Define

$$y(\psi) = \begin{cases} \frac{1}{(2n-1)\alpha + 1} & \text{if } \psi \text{ is type 1,} \\ \frac{(2n-1)(\alpha-1)}{(4n^2-4n-1)(2n-1)\alpha^2 + (4n^2-2)\alpha - (2n-1)} & \text{if } \psi \text{ is type 2,3,5,} \\ \frac{(2n-1)}{(2n-1) + \alpha} & \text{if } \psi \text{ is type 4.} \end{cases}$$

All the coordinates of y are obviously greater than 0. It is straightforward to show that $Y = 1$ by using Table 2 and the counts of element types therein. The resulting expression can easily be simplified using $\mathcal{P}(\alpha) = 0$. Similarly, one can also show that $f_r(y) = \alpha$ for all $r \in \mathcal{F}$ and $f_r(y) \leq \alpha$ for all $r \in \mathcal{G}$ by using Table 4 and the counts of element types within every Ψ_r . Therefore, $\alpha^* = \inf(G(x)) \leq G(y) = \alpha$. In summary, we have $1 < \beta^* \leq \alpha^* \leq \alpha$, completing the proof.

The previous lemma marks the end of phase one. The vector y given in the proof is in fact the unique optimization point for our problem. Proving this statement takes most of the technical work in this paper. What we shall do next is to prove the uniqueness of the optimal point using convexity arguments via a bank of lemmas.

Lemma 6 For all $x^* \in \Delta$ such that $F(x^*) = \beta^*$, there exists a type 4b relation r_0 such that $f_{r_0}(x^*) = \beta^*$.

Proof If we consider Δ as a submanifold of \mathbb{R}^Ψ , the tangent space $T_x\Delta$ at any point $x \in \Delta$ is the set of vectors whose coordinates sum to 0. Moreover, every displacement function is smooth on an open set containing Δ which implies that the directional derivative for $\vec{u} \in T_x\Delta$ will be $\nabla f(x) \cdot \vec{u}$.

Our plan of attack is to find a common direction of decrease for type 1a, 2b, 3a, and 5a relations, which will give us a very nice contradiction if the infimum was not attained by any type 4b relation at an optimal point. As the common direction of decrease, consider the vector \vec{u} described below.

$$u(\psi) = \begin{cases} 1 & \text{if } \psi \text{ is not type 4,} \\ -\frac{1 + 4(n - 1)^2}{2(n - 1)} & \text{if } \psi \text{ is type 4.} \end{cases}$$

It is easy to check that the sum of coordinates of \vec{u} is 0 with the help of Table 2; in turn, this shows that $\vec{u} \in T_x\Delta$. Moreover, we present the calculations below using (6), (7), and Table 4 proving that directional derivatives of type 1a, 2b, 3a, and 5a relation displacements are all negative in the direction \vec{u} . For r of type 1a, we see that

$$\begin{aligned} \nabla f_r(x) \cdot \vec{u} &= -\frac{1 - X_r}{x_r^2 X_r} \cdot u_{\psi_r} - \frac{1 - x_r}{x_r X_r^2} \sum_{\psi \in \Psi_r} u_\psi \\ &= -\frac{1 - X_r}{x_r^2 X_r} \cdot 1 - \frac{1 - x_r}{x_r X_r^2} \left(\sum_{\psi \in \Psi_r - \text{type 4}} u_\psi + \sum_{\psi \in \Psi_r \cap \text{type 4}} u_\psi \right) \\ &= -\frac{1 - X_r}{x_r^2 X_r} - \frac{1 - x_r}{x_r X_r^2} \left(\sum_{\psi \in \Psi_r - \text{type 4}} 1 + \sum_{\psi \in \Psi_r \cap \text{type 4}} -\frac{1 + 4(n - 1)^2}{2(n - 1)} \right) \\ &= -\frac{1 - X_r}{x_r^2 X_r} - \frac{1 - x_r}{x_r X_r^2} \left((2n - 1)(1 + 4(n - 1)^2) - \frac{1 + 4(n - 1)^2}{2(n - 1)} \cdot 2(n - 1)(2n - 1) \right) \\ &= -\frac{1 - X_r}{x_r^2 X_r} \\ &< 0. \end{aligned}$$

For the last inequality, recall that $x_r, X_r \in (0, 1)$. For type 2b, 3a, and 5a, we find that

$$\nabla f_r(x) \cdot \vec{u} = -\frac{1 - X_r}{x_r^2 X_r} - \frac{1 - x_r}{x_r X_r^2} \cdot \left(1 - \frac{1}{2(n - 1)} \right) < 0.$$

Assume that there exists an optimal point x^* such that $f_r(x^*) < \beta^*$ for all r of type 4b. Let $\varepsilon > 0$ be small enough so that $x^* + \varepsilon\vec{u} \in \Delta$ and $f_r(x^* + \varepsilon\vec{u}) < \beta^*$ for all r of type 4b. Such a choice is possible by the continuity of displacement functions. Under these conditions, we have $F(x^* + \varepsilon\vec{u}) < \beta^*$ as $f_r(x^* + \varepsilon\vec{u}) < f_r(x^*) \leq F(x^*) = \beta$ for all r of type 1a, 2b, 3a, 5a and $f_r(x^* + \varepsilon\vec{u}) < \beta$ for all r of type 4b, contradicting the minimality of β^* .

Unfortunately, the displacement functions are not convex on the entire simplex. Therefore, neither are the max functions. Any argument of uniqueness of the optimal solution must first be defended on a local level,

displacement by displacement. Fortunately, each displacement function is strictly convex on a relatively large set.

Lemma 7 For $r \in \mathcal{F}$ define $C_r = \{x \in \Delta \mid x_r + X_r - x_r X_r < 3/4\}$. The displacement function f_r is strictly convex on C_r .

Proof Consider the function $g : (0, 1)^2 \rightarrow \mathbb{R}$ defined by $g(z_1, z_2) = (1 - z_1)/z_1 \cdot (1 - z_2)/z_2$. As g is twice differentiable, g is strictly convex, where the Hessian of g is positive definite by [4, Theorem 3.1.11 and Exercise 3.1.12(c)]:

$$\text{Hess}(g) = \begin{pmatrix} 2(1 - z_2)/z_1^3 z_2 & 1/z_1^2 z_2^2 \\ 1/z_1^2 z_2^2 & 2(1 - z_1)/z_1 z_2^3 \end{pmatrix}.$$

Moreover, a 2×2 matrix is positive definite if and only if $\det(H) > 0$ and $\text{trace}(H) > 0$, where

$$\det(\text{Hess}(g)) = \frac{4(1 - z_1)(1 - z_2)}{z_1^4 z_2^4} - \frac{1}{z_1^4 z_2^4} \quad \text{and} \quad \text{trace}(\text{Hess}(g)) = \frac{2(1 - z_1)}{z_1 z_2^3} + \frac{2(1 - z_2)}{z_1^3 z_2}.$$

Since the trace of the Hessian is always positive in our domain, g is strictly convex, where the determinant is positive. Hence, g is strictly convex on $C_g = \{(z_1, z_2) \in (0, 1)^2 \mid z_1 + z_2 - z_1 z_2 < 3/4\}$. Now, notice that $f_r = g(x_r, X_r)$. Since the map $x \rightarrow (x_r, X_r) \in (0, 1)^2$ satisfies the conditions listed in [4, Exercise 1.1.9], we get f_r is strictly convex on $C_r = \{x \in \Delta \mid x_r + X_r - x_r X_r < 3/4\}$.

We already know that some type 4b displacement attains β^* at a point of optimality by Lemma 6. It is also true that any optimal point of a strictly convex function is unique [4, Exercise 2.1.8(a)]. Therefore, proving any optimal point must be in the domain of convexity of all type 4b relation displacements will take us most of the way to the uniqueness.

Theorem 4 If r is a type 4b relation, then $x^* \in C_r$ for all $x^* \in \Delta$ such that $F(x^*) = \beta^*$.

Proof Assume there exists a point of optimality x^* such that $x^* \notin C_{r_0}$ for a type 4b group theoretical relation $r_0 : \psi_0 = \xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0}$, $\Psi_0 = \{\xi_{i_0}^{2t_0}, \xi_{i_0}^{t_0} \xi_j^{2s}, \xi_{i_0}^{t_0} \xi_j^s \xi_k^p\}$, where

$$x_0^* + X_0^* - x_0^* X_0^* \geq 3/4. \tag{11}$$

Our aim is to obtain a contradiction under these conditions. Let T be the collection of type 4b relations such that $\psi_r = \xi_i^t \xi_{j_0}^{s_0} \xi_i^{-t}$ and $\psi_r = \xi_{j_0}^s \xi_{i_0}^{t_0} \xi_{j_0}^{-s}$. Notice that $(\Psi_r)_{r \in T}$ gives a disjoint partition of Ψ . This means that $\Psi = \bigsqcup_{r \in T} \Psi_r$, which implies that for any $x \in \Delta$, we get

$$1 = \sum_{r \in T} X_r. \tag{12}$$

Based on the possible location of x^* , we will separate the proof into the following three cases

$$(A) \frac{1}{2} \geq X_0^*, \quad (B) \frac{1}{2n+2} \geq X_{r_1}^* \text{ for some } r_1 \in T, \quad (C) \frac{1}{2} < X_0^* \text{ and } \frac{1}{2n+2} < X_r^* \text{ for all } r \in T.$$

Each one of these cases will end up with a contradiction, proving the theorem. Assume (A) is the case. Then, (11) gives us the following chain of inequalities:

$$X_0^*(1 - x_0^*) + x_0^* \geq 3/4 \quad \text{or} \quad \frac{1}{2}(1 - x_0^*) + x_0^* \geq 3/4 \quad \text{or} \quad x_0^* \geq 1/2.$$

However, as $\psi_0 \in \Psi_0$, we get the contradiction $1/2 \geq X_0^* > x_0^* \geq 1/2$.

Assume (B) is the case. Let $\psi_{r_1} = \xi_{i_1}^{t_1} \xi_{j_1}^{s_1} \xi_{i_1}^{-t_1}$. Consider any type 4b relation r with $\psi_r = \xi_{i_1}^{t_1} \xi_j^s \xi_{i_1}^{-t_1}$. By Lemma 5, we have

$$\frac{1 - x_r^*}{x_r^*} \cdot \frac{1 - X_r^*}{X_r^*} = f_r(x^*) \leq \beta^* \leq \alpha.$$

This, after a number of rearrangements, turns into the following inequality

$$\frac{1 - X_r^*}{X_r^*(\alpha - 1) + 1} \leq x_r^*.$$

Notice that $\Psi_{r_1} = \Psi_r$ and so $1/(2n + 1) \geq X_{r_1}^* = X_r^*$. In the above inequality, the left side is a decreasing function of X_r^* . Thus, by the case assumption, we preserve the inequality by replacing X_r^* with $1/(2n + 2)$. We obtain

$$\frac{2n + 1}{\alpha + 2n + 1} = \frac{1 - \frac{1}{2n + 2}}{\frac{1}{2n + 2}(\alpha - 1) + 1} \leq x_r^* = x^*(\xi_{i_1}^{t_1} \xi_j^s \xi_{i_1}^{-t_1}).$$

The following inequalities

$$\frac{1}{2n + 2} \geq X_{r_1}^* > \sum_{j \neq i_1, s = \pm 1} x^*(\xi_{i_1}^{t_1} \xi_j^s \xi_{i_1}^{-t_1}) \geq 2(n - 1) \cdot \frac{2n + 1}{\alpha + 2n + 1}$$

are obtained by using the previous inequality and the fact that $\xi_{i_1}^{t_1} \xi_j^s \xi_{i_1}^{-t_1} \in \Psi_{r_1}$ for all $j \neq i_1, s = \pm 1$. Again, after a number of rearrangements, we get $\alpha > (2n - 1)^3 + 8n^2 - 12n - 4$. However, we have $8n^2 - 12n - 3 > 0$ for $n \geq 2$ implying $\alpha > (2n - 1)^3$, contradicting the proof of Lemma 1.

Assume (C) is the case. By using (12) and the fact that $n \geq 2$, we derive that

$$1 = X_0^* + \sum_{r \in T - \{r_0\}} X_r^* > \frac{1}{2} + (2n - 1) \cdot \frac{1}{2n + 2} = 1 + \frac{1}{2} - \frac{3}{2n + 2} \geq 1,$$

which is the final contradiction that we needed.

It is a simple exercise to show that the max function of strictly convex functions is strictly convex. Hence, we have the fact that F is strictly convex on $\cap_{r \in \text{Type } 4b} C_r$ which is a nonempty set as it contains all the optimal points at the very least. As optimal points of strictly convex functions are unique, we prove the following corollary.

Corollary 1 *The optimal point of $\inf_{x \in \Delta} F(x)$ is unique.*

Henceforth, we shall use x^* to denote the unique optimal point. In other words, we let

$$x^* : \text{ the unique point in } \Delta \text{ with } F(x^*) = \beta^*,$$

which marks the end of phase two.

We start phase three by simplifying x^* . The uniqueness of x^* will be heavily used in this simplification. Then, we apply the Karush-Kuhn-Tucker Theorem.

Lemma 8 *There exists $a^*, b^*, c^* \in (0, 1)$ such that*

$$x^*(\psi) = \begin{cases} a^* & \text{for } \psi \in \text{type 1} \\ b^* & \text{for } \psi \in \text{type 2 or type 3 or type 5} \\ c^* & \text{for } \psi \in \text{type 4.} \end{cases}$$

Proof Let $i_1, i_2 \in \{1, 2, \dots, n\}$ and $t_1, t_2 \in \{-1, +1\}$ such that $i_1 \neq i_2$. Consider the symmetry $\tau \in \text{Sym}(\Psi)$, which switches $\xi_{i_1}^{t_1}$ with $\xi_{i_2}^{t_2}$ and vice versa in every element of Ψ . For clarification, we give the details of the action of τ in the following table:

Table 5. Actions of the symmetry $\tau \in \text{Sym}(\Psi)$.

type 1	$\xi_{i_1}^{\pm 2t_1} \leftrightarrow \xi_{i_2}^{\pm 2t_2}$	
type 2	$\xi_{i_1}^{t_1} \xi_{i_2}^{\pm 2t_2} \leftrightarrow \xi_{i_2}^{t_2} \xi_{i_1}^{\pm 2t_1}$	$\xi_{i_1}^{-t_1} \xi_{i_2}^{\pm 2t_2} \leftrightarrow \xi_{i_2}^{-t_2} \xi_{i_1}^{\pm 2t_1}$
	$\xi_i^t \xi_{i_1}^{\pm 2t_1} \leftrightarrow \xi_i^t \xi_{i_2}^{\pm 2t_2}$	$\xi_{i_1}^{\pm t_1} \xi_j^{2s} \leftrightarrow \xi_{i_2}^{\pm t_2} \xi_j^{2s}$
type 3	$\xi_{i_1}^{t_1} \xi_{i_2}^{\pm t_2} \xi_{i_1}^{t_1} \leftrightarrow \xi_{i_2}^{t_2} \xi_{i_1}^{\pm t_1} \xi_{i_2}^{t_2}$	$\xi_{i_1}^{-t_1} \xi_{i_2}^{\pm t_2} \xi_{i_1}^{-t_1} \leftrightarrow \xi_{i_2}^{-t_2} \xi_{i_1}^{\pm t_1} \xi_{i_2}^{-t_2}$
	$\xi_i^t \xi_{i_1}^{\pm t_1} \xi_i^t \leftrightarrow \xi_i^t \xi_{i_2}^{\pm t_2} \xi_i^t$	$\xi_{i_1}^{\pm t_1} \xi_j^s \xi_{i_1}^{\pm t_1} \leftrightarrow \xi_{i_2}^{\pm t_2} \xi_j^s \xi_{i_2}^{\pm t_2}$
type 4	$\xi_{i_1}^{t_1} \xi_{i_2}^{\pm t_2} \xi_{i_1}^{-t_1} \leftrightarrow \xi_{i_2}^{t_2} \xi_{i_1}^{\pm t_1} \xi_{i_2}^{-t_2}$	$\xi_{i_1}^{-t_1} \xi_{i_2}^{\pm t_2} \xi_{i_1}^{t_1} \leftrightarrow \xi_{i_2}^{-t_2} \xi_{i_1}^{\pm t_1} \xi_{i_2}^{t_2}$
	$\xi_i^t \xi_{i_1}^{\pm t_1} \xi_i^{-t} \leftrightarrow \xi_i^t \xi_{i_2}^{\pm t_2} \xi_i^{-t}$	$\xi_{i_1}^{\pm t_1} \xi_j^s \xi_{i_1}^{\mp t_1} \leftrightarrow \xi_{i_2}^{\pm t_2} \xi_j^s \xi_{i_2}^{\mp t_2}$
type 5	$\xi_{i_1}^{t_1} \xi_{i_2}^{\pm t_2} \xi_l^p \leftrightarrow \xi_{i_2}^{t_2} \xi_{i_1}^{\pm t_1} \xi_l^p$	$\xi_{i_1}^{-t_1} \xi_{i_2}^{\pm t_2} \xi_l^p \leftrightarrow \xi_{i_2}^{-t_2} \xi_{i_1}^{\pm t_1} \xi_l^p$
	$\xi_{i_1}^{t_1} \xi_j^s \xi_{i_2}^{\pm t_2} \leftrightarrow \xi_{i_2}^{t_2} \xi_j^s \xi_{i_1}^{\pm t_1}$	$\xi_{i_1}^{-t_1} \xi_j^s \xi_{i_2}^{\pm t_2} \leftrightarrow \xi_{i_2}^{-t_2} \xi_j^s \xi_{i_1}^{\pm t_1}$
	$\xi_i^t \xi_{i_1}^{t_1} \xi_{i_2}^{\pm t_2} \leftrightarrow \xi_i^t \xi_{i_2}^{t_2} \xi_{i_1}^{\pm t_1}$	$\xi_i^t \xi_{i_1}^{-t_1} \xi_{i_2}^{\pm t_2} \leftrightarrow \xi_i^t \xi_{i_2}^{-t_2} \xi_{i_1}^{\pm t_1}$
	$\xi_i^t \xi_j^s \xi_{i_1}^{\pm t_1} \leftrightarrow \xi_i^t \xi_j^s \xi_{i_2}^{\pm t_2}$	$\xi_i^t \xi_{i_1}^{\pm t_1} \xi_l^p \leftrightarrow \xi_i^t \xi_{i_2}^{\pm t_2} \xi_l^p$ $\xi_{i_1}^{\pm t_1} \xi_j^s \xi_l^p \leftrightarrow \xi_{i_2}^{\pm t_2} \xi_j^s \xi_l^p$

Using Table 4, one can prove that τ can also be considered as an element of $\text{Sym}(\mathcal{F})$ in the sense that $\psi_{\tau(r)} = \tau(\psi_r)$ and $\Psi_{\tau(r)} = \tau(\Psi_r)$ by observing the effects of τ on the elements of \mathcal{F} listed in Table 5. As a result, we have the following

$$F(\tau(x)) = \max_{r \in \mathcal{F}} f_r(\tau(x)) = \max_{r \in \mathcal{F}} f_{\tau(r)}(x) = \max_{r \in \mathcal{F}} f_r(x) = F(x).$$

In particular, we have $F(\tau(x^*)) = F(x^*)$ and hence $\tau(x^*) = x^*$ by Corollary 1. By repeated application of τ (with different values for (i_1, t_1) and (i_2, t_2)), we can show that any coordinate of x^* is equal to any other coordinate of the same type. We give the following example of type 5 coordinates for clarification

$$x^*(\xi_{i_1}^{t_1} \xi_{j_1}^{s_1} \xi_{l_1}^{p_1}) \underset{(i_1, t_1) \rightarrow (i_2, t_2)}{\overset{\tau}{=}} x^*(\xi_{i_2}^{t_2} \xi_{j_1}^{s_1} \xi_{l_1}^{p_1}) \underset{(j_1, s_1) \rightarrow (j_2, s_2)}{\overset{\tau}{=}} x^*(\xi_{i_2}^{t_2} \xi_{j_2}^{s_2} \xi_{l_1}^{p_1}) \underset{(l_1, p_1) \rightarrow (l_2, p_2)}{\overset{\tau}{=}} x^*(\xi_{i_2}^{t_2} \xi_{j_2}^{s_2} \xi_{l_2}^{p_2}).$$

This proves that there are $a^*, b_2^*, b_3^*, b_5^*, c^* \in (0, 1)$ such that:

$$x^*(\psi) = \begin{cases} a^* & \text{for } \psi \in \text{type 1} \\ b_2^* & \text{for } \psi \in \text{type 2} \\ b_3^* & \text{for } \psi \in \text{type 3} \\ b_5^* & \text{for } \psi \in \text{type 5} \\ c^* & \text{for } \psi \in \text{type 4.} \end{cases}$$

It remains to prove $b_2^* = b_3^* = b_5^*$. Fortunately, the above simplification makes it much easier.

Consider $\sigma_1 = (\xi_i^t \xi_j^{2s} \leftrightarrow \xi_j^s \xi_i^t \xi_j^s \mid \forall i, j, t, s) \in \text{Sym}(\Psi)$. It is not true that in general $F(\sigma_1(x)) = F(x)$. However, by using the previous simplification for x^* together with Table 4, it is easy to see that the equality $F(\sigma_1(x^*)) = F(x^*)$ holds. Thus, again by Corollary 1, we get

$$b_2^* = x^*(\xi_1 \xi_2^2) = x^*(\xi_2 \xi_1 \xi_2) = b_3^*.$$

Similarly, $\sigma_2 = (\xi_1 \xi_2^2 \leftrightarrow \xi_1 \xi_2 \xi_3) \in \text{Sym}(\Psi)$ gives us $b_2^* = b_5^*$ which completes the proof.

Now that we have simplified x^* as much as possible, we shall use the optimization question with the Karush-Kuhn-Tucker Theorem to get further equalities on x^* with a target of calculating exact values for its coordinates.

Theorem 5 For all $r_1, r_2 \in \mathcal{F}$, we have $f_{r_1}(x^*) = f_{r_2}(x^*)$.

Proof We will obtain the proof by applying the Karush-Kuhn-Tucker Theorem to the optimization question

$$\min F(x) \text{ for } x \in \Delta.$$

By Lemma 8, we can restrict our search for an optimal solution to those elements of Δ showing the same coordinate distribution as x^* . In other words, we solve the problem below:

$$\min F(x) \text{ for } x \in \tilde{\Delta}, \text{ where}$$

$$\tilde{\Delta} := \{x \in \Delta \mid \exists a, b, c \in (0, 1), x(\psi) = \begin{cases} a & \text{for } \psi \in \text{type 1} \\ b & \text{for } \psi \in \text{type 2 or type 3 or type 5} \\ c & \text{for } \psi \in \text{type 4.} \end{cases}$$

A more illuminating observation comes from the fact that $X = 1$ for all $x \in \tilde{\Delta}$ and the counts in Table 2. For any $x \equiv (a, b, c)$, we have

$$1 = 2na + 8n(n - 1)^2b + 4n(n - 1)c. \tag{13}$$

The displacement functions are simplified as it no longer matters what specific coordinate Ψ_r contains, but the number of types of coordinates. As an example, for any type 1a relation r and $x \equiv (a, b, c) \in \tilde{\Delta}$, we have (with the help of Table 4) $x_r = a$ and $X_r = 1 - (a + 4(n - 1)^2b + 2(n - 1)c)$. Moreover, using (13) to replace a , we obtain $X_r = (2n - 1)/(2n)$. This simplifies $f_r(x) = (1 - x_r)/x_r \cdot (1 - X_r)/X_r$ into $f_r(x) = (1 - a)/a \cdot 1/2n - 1$. We summarize these simplifications for all types of relations in \mathcal{F} listed in the following table:

By Lemma 6, we know that there exists a Type 4b relation r_0 such that $\beta^* = f_{r_0}(x^*) \geq f_r(x^*)$ for all $r \in \mathcal{F}$. Applying this knowledge in light of Table 6, we get two new inequalities further narrowing the search for the optimal solution. We obtain

$$\frac{1 - a}{a} \cdot \frac{1}{2n - 1} \leq \frac{1 - c}{c} \cdot (2n - 1) \quad \text{and} \quad \frac{1 - b}{b} \cdot \frac{1 - 2na}{4n(n - 1) - 1 + 2na} \leq \frac{1 - c}{c} \cdot (2n - 1).$$

This simplifies the optimization question to its ultimate form as described in the table below:

Table 6. $f_r(x)$ for $x \equiv (a, b, c) \in \tilde{\Delta}$.

r	x_r	X_r	$f_r(x)$
Type 1a	a	$\frac{2n - 1}{2n}$	$\frac{1 - a}{a} \cdot \frac{1}{2n - 1}$
Type 2b, 3a, and 5a	b	$1 - (2(n - 1)b + c)$	$\frac{1 - b}{b} \cdot \frac{1 - 2na}{4n(n - 1) - 1 + 2na}$
Type 4b	c	$\frac{1}{2n}$	$\frac{1 - c}{c} \cdot (2n - 1)$

From hereon in, we shall apply the optimization techniques detailed in [4, Chapter 7.2] to solve the optimization question in Table 7. Our goal is to prove that the Lagrange multipliers for g_1 and g_2 cannot be 0, which (by Karush-Kuhn-Tucker Theorem as in [4, Theorem 7.2.9]) will imply that the relevant inequalities in the constraints has to be active (equal) at the optimal point, proving our theorem.

Table 7. Simplified optimization question.

Target:	$f(a, b, c) = \frac{(2n - 1)(1 - c)}{c}$ (minimize)
Constraints:	$g_1(a, b, c) = \frac{1 - a}{a} \cdot \frac{c}{1 - c} - (2n - 1)^2 \leq 0$ $g_2(a, b, c) = \frac{1 - 2na}{4n(n - 1) - 1 + 2na} \cdot \frac{1 - b}{b} \cdot \frac{c}{1 - c} - (2n - 1) \leq 0$ $h(a, b, c) = 2na + 8n(n - 1)^2b + 4n(n - 1)c = 1$ $(a, b, c) \in U = (0, 1)^3.$

We shall first discuss the three prerequisites of Karush-Kuhn-Tucker Theorem. The existence of the

optimal solution is proven in Lemma 4. The target and constraint functions are obviously differentiable on U . We also need to show that the Mangasarian-Fromovitz constraint qualification [4, Assumption 7.2.3] holds. The qualification for our question can be checked by verifying the statements (1) if the function $p \mapsto \nabla h(x^*) \cdot p$ from \mathbb{R}^3 to \mathbb{R} is onto and, (2) $\{p \in \mathbb{R}^3 \mid \nabla h(x^*) \cdot p = 0 \text{ and } \nabla g_i(x^*) \cdot p < 0 \text{ for } i = 1, 2\} \neq \emptyset$. The first one is trivially true as ∇h is constant and nonzero. The second one also holds since $p = (2(n - 1), 0, -1)$ is in the set. Hence, by the Karush-Kuhn-Tucker Theorem, there exists $\lambda_1, \lambda_2 \in \mathbb{R}_+$ and $\lambda_3 \in \mathbb{R}$ such that:

$$(\nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \lambda_3 \nabla h)(x^* \equiv (a^*, b^*, c^*)) = \vec{0}$$

This gives us the following linear system of λ_1, λ_2 and λ_3

$$-\frac{c^*}{(a^*)^2(1 - c^*)} \lambda_1 + \frac{8n^2(n - 1)}{A} \lambda_2 + 2n\lambda_3 = 0, \tag{14}$$

$$-\frac{(1 - 2na^*)c^*}{A(b^*)^2(1 - c^*)} \lambda_2 + 8n(n - 1)^2 \lambda_3 = 0, \tag{15}$$

$$\frac{1 - a^*}{(a^*)(1 - c^*)^2} \lambda_1 + \frac{(1 - 2na^*)(1 - b^*)}{Ab^*(1 - c^*)^2} \lambda_2 + 4n(n - 1)\lambda_3 = \frac{2n - 1}{(c^*)^2}, \tag{16}$$

where $A = 4n(n - 1) - (1 - 2na)$. It is easy to see that $1 - 2na \in (0, 1)$ by using the constraint $h = 1$. Thus, every parenthesis in this system is positive. This implies that the signs in front of the coefficients are the signs of the coefficients.

If $\lambda_3 = 0$, then by (15) $\lambda_2 = 0$ as well, which implies $\lambda_1 = 0$ by (14). However, this is a contradiction to (16). Thus, $\lambda_3 \neq 0$. The fact that λ_3 is nonzero implies $\lambda_2 > 0$ by (15), which in turn, again via (15), shows that $\lambda_3 > 0$ as well. Finally, the positiveness of λ_2 and λ_3 on (14) shows that $\lambda_1 > 0$. Therefore, the Lagrange multipliers of g_1 and g_2 are greater than 0, proving what we wanted.

Theorem 6 *If $x^* \equiv (a^*, b^*, c^*)$ is as in Lemma 8 and α is as defined after Lemma 1, then we have*

$$(i) \ a^* = \frac{1}{(2n - 1) + \alpha}, \quad b^* = \frac{(2n - 1)(\alpha - 1)}{(4n^2 - 4n - 1)(2n - 1)\alpha^2 + 2(2n^2 - 1)\alpha - (2n - 1)}, \quad c^* = \frac{2n - 1}{2n - 1 + \alpha},$$

$$(ii) \ \beta^* = \alpha^* = \alpha.$$

Proof By Theorem 5, we know that $f_r(x^*) = \beta^*$ for all $r \in \mathcal{F}$. By the proof of the same theorem, we also know that there are really three functions at the point of optimality. Therefore, we have the following equations

$$\begin{aligned} \frac{1 - a^*}{a^*} \cdot \frac{1}{2n - 1} &= \beta^*, \\ \frac{1 - b^*}{b^*} \cdot \frac{1 - 2na^*}{4n(n - 1) - 1 + 2na^*} &= \beta^*, \\ \frac{1 - c^*}{c^*} \cdot (2n - 1) &= \beta^*. \end{aligned}$$

Simply solving this system for a^*, b^* , and c^* yields that

$$\begin{aligned}
 a^* &= \frac{1}{(2n-1) + \beta^*}, \\
 b^* &= \frac{(2n-1)(\beta^* - 1)}{(4n^2 - 4n - 1)(2n-1)(\beta^*)^2 + 2(2n^2 - 1)\beta^* - (2n-1)}, \\
 c^* &= \frac{2n-1}{2n-1 + \beta^*}.
 \end{aligned}$$

Putting these formulas into the equation (13) gives $\mathcal{P}(\beta^*) = 0$, where \mathcal{P} is the polynomial defined in Lemma 1. However, by Lemma 5, we know that $1 < \beta^*$. Since α is the only root of \mathcal{P} greater than 1, we find $\beta^* = \alpha$. Applying that to $\beta^* \leq \alpha^* \leq \alpha$ from Lemma 5 gives us $\alpha = \beta^* \leq \alpha^* \leq \alpha$, also proving that $\alpha^* = \alpha$.

6. Proof of the main theorem

We will first prove two lemmas that will simplify the proof of the main theorem.

Lemma 9 *Let g, h in $PSL(2, \mathbb{C})$ and $G = \langle g, h \rangle$. If G is freely generated by $\{g, h\}$, purely loxodromic and Kleinian, then G is nonelementary.*

Proof Assume G is elementary. By the classification of discrete elementary groups done in [2, Chapter 5.1], there exists a two element subset of $\hat{\mathbb{C}}$ that is invariant under the action of G . Without loss of generality, take the invariant subset to be $\{0, \infty\}$.

If the fixed points of g were not 0 and ∞ , we would have $g(0) = \infty$ and $g(\infty) = 0$. This implies that $g(z) = a/z$, contradicting with the freeness of G as $g^2 = id$. Similarly, the fixed points of h must be 0 and ∞ as well. Hence, the fixed point of g and h are the same. By [2, Theorem 4.3.5], we conclude that either $[g, h] = id$ or $[g, h]$ is parabolic. The first case contradicts with the freeness of G , and the second one contradicts the lemma hypothesis that G is purely loxodromic.

Lemma 10 *Let $g, h \in PSL(2, \mathbb{C})$ be noncommuting and loxodromic. If z_0 is the midpoint of the shortest geodesic segment connecting the axes of g and $h^{-1}gh$, then $d_g z_0 < d_{hgh^{-1}} z_0$.*

Proof For $\gamma \in PSL(2, \mathbb{C})$, let $Z_\lambda(\gamma) = \{z \in \mathbb{H}^3 \mid d_\gamma z \leq \lambda\}$, the displacement cylinder of γ with radius λ . By [2, 5.4.11], $d_\gamma z$ is an increasing function of $\rho(z, A_\gamma)$. Thus, $Z_\lambda(\gamma)$ really is a cylinder with axis A_γ . Fix $\lambda = d_g z_0$ so that $z_0 \in Z_\lambda(g)$. By lemma hypothesis, we have $\rho(z_0, A_g) = \rho(z_0, A_{h^{-1}gh})$. Moreover, the terms T_g and θ_g in [2, 5.4.11] are invariant under conjugation. Hence, $\rho(z_0, gz_0) = \rho(z_0, h^{-1}ghz_0)$ which proves that $z_0 \in Z_\lambda(h^{-1}gh)$.

We claim that z_0 is the only element in $Z_\lambda(g) \cap Z_\lambda(h^{-1}gh)$. If it was not, then there would be an element $z \neq z_0$ such that $\rho(z, gz) = \rho(z, h^{-1}gh)$ which (again by [2, 5.4.11]) would imply that $\rho(z, A_g) = \rho(z, A_{h^{-1}gh})$. By lemma hypothesis, we would have $\rho(z_0, A_g) < \rho(z, A_g)$ so that $\lambda = \rho(z_0, gz_0) < \rho(z, gz)$, a contradiction. Now that we know $Z_\lambda(g) \cap Z_\lambda(h^{-1}gh) = \{z_0\}$, one can easily show that $Z_\lambda(g) \cap Z_\lambda(hgh^{-1}) = \{hz_0\}$. Because $z_0 \in Z_\lambda(g)$, we can derive that $z_0 \notin Z_\lambda(hgh^{-1})$. Therefore, $\lambda = d_g z_0 < d_{hgh^{-1}} z_0$.

The following theorem will be the principal effect of the optimization that was done in Section 5 to our main result. The geometrically infinite case is quite straightforward due to the Culler-Shalen machinery.

However, the generalization to the geometrically finite case requires some high level theorems which we refer to inside the proof.

Theorem 7 *Let Γ be a Kleinian, purely loxodromic subgroup of $PSL(2, C)$ that is freely generated by $\Xi = \{\xi_1, \xi_2, \dots, \xi_n\}$. If $\Gamma_* = \{1, \xi_i^t, \xi_i^t \xi_j^s \xi_i^{-t}\}$ and α is the unique root of the polynomial \mathcal{P} greater than $(2n - 1)^2$, then for any $z \in \mathbb{H}^3$ we have*

$$\max\{d_\gamma z \mid \gamma \in \Gamma_*\} \geq \frac{1}{2} \log \alpha.$$

Proof Fix $z \in \mathbb{H}^3$ and let $\lambda = \max\{d_\gamma z \mid \gamma \in \Gamma_*\}$. Assume Γ is geometrically infinite. By Proposition 1(ii), we have

$$\lambda \geq d_{\gamma_r} z \geq \frac{1}{2} \log f_r(m)$$

for any relation r so that $\lambda \geq \frac{1}{2} \log G(m)$, where G is defined in Section 4. Moreover, by Proposition 1(i) and Theorem 6, we derive

$$\lambda \geq \frac{1}{2} \log G(m) \geq \frac{1}{2} \log (\inf\{G(x) \mid x \in \Delta\}) = \frac{1}{2} \log \alpha.$$

Assume that Γ is geometrically finite. Our plan of attack is to prove that the infimum of all such lower boundaries over all possible free groups is attained at a special place close to the geometrically infinite case. We will have to use yet another optimization and some prior knowledge. Consider the function f_z defined below

$$\begin{aligned} f_z : PSL(2, \mathbb{C})^n &\longrightarrow \mathbb{R}_+ \\ \xi &\longmapsto \max\{d_\gamma z \mid \gamma \in \Gamma_*(\xi)\}. \end{aligned}$$

Let \mathfrak{GF} denote the subset of elements of $PSL(2, \mathbb{C})^n$ whose components form a set of free generators of a purely loxodromic free subgroup of $PSL(2, \mathbb{C})$ that is geometrically finite. As the image $f_z(\mathfrak{GF})$ is bounded below and f is continuous, there exists a $\bar{\xi} \in \overline{\mathfrak{GF}}$ such that

$$f_z(\bar{\xi}) = \inf\{f_z(\xi) \mid \xi \in \mathfrak{GF}\}.$$

However, it has been shown in the proofs of [18, Theorem 5.1], [19, Theorem 4.1] and finally in [20, Theorem 4.2] that any such optimal point must satisfy $\bar{\xi} \in \overline{\mathfrak{GF}} - \mathfrak{GF}$. Moreover, the subset of elements of $PSL(2, \mathbb{C})^n$ whose components are generating purely loxodromic and geometrically infinite groups is dense in $\overline{\mathfrak{GF}} - \mathfrak{GF}$ as was shown in [8, Propositions 8.2 and 9.3], [6, Main Theorem], and [5]. Hence, the geometrically finite case folds into the geometrically infinite case, completing the proof.

The final theorem of this paper is more general and in appearance more complicated than its counterpart in [20]. However, most of the complications in the proof have been handled by the previous lemmas and theorems in this paper and the arguments inside the proof is largely identical to that of the counterpart.

Theorem 8 *Let Γ and α be as in Theorem 7. Assume that there exist i_0, j_0 such that the inequalities $d_{\xi_{i_0} \xi_{j_0} \xi_{i_0}^{-1}} z_2 \leq d_{\xi_{i_0} \xi_{j_0} \xi_{i_0}^{-1}} z_1$ and $d_\gamma z_2 < \frac{1}{2} \log \alpha$ for every $\gamma \in \Phi = \Gamma_* - \{\xi_{i_0}, \xi_{j_0}^t \xi_{i_0}^s \xi_{j_0}^{-t}\}$ hold, where z_1 and*

z_2 are the midpoints of the shortest geodesic segments connecting the axis of ξ_{j_0} to the axes of $\xi_{i_0}\xi_{j_0}\xi_{i_0}^{-1}$ and $\xi_{i_0}^{-1}\xi_{j_0}\xi_{i_0}$, respectively. Then, we have

$$|\text{trace}^2(\xi_{j_0}) - 4| + |\text{trace}(\xi_{j_0}\xi_{i_0}\xi_{j_0}^{-1}\xi_{i_0}^{-1}) - 2| \geq 2 \sinh^2\left(\frac{1}{4} \log \alpha\right).$$

Proof We shall make use of the following facts. The first one is a straightforward exercise to the teachings of [2] and the second one is due to Lemma 9. The theorem remains invariant under conjugation. The subgroup $\langle \xi_{i_0}, \xi_{j_0} \rangle$ must be nonelementary as it is free, Kleinian, and purely loxodromic.

Since the theorem remains invariant under conjugation, we may assume without loss of generality that ξ_{i_0} fixes 0 and ∞ . This gives us some useful matrix presentations

$$\xi_{i_0} = \begin{bmatrix} u & 0 \\ 0 & 1/u \end{bmatrix} \text{ for some } u = |u|e^{i\theta} \quad \text{and} \quad \xi_{j_0} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ with } ad - bc = 1$$

so that we can write

$$|\text{trace}^2(\xi_{i_0}) - 4| + |\text{trace}(\xi_{j_0}\xi_{i_0}\xi_{j_0}^{-1}\xi_{i_0}^{-1}) - 2| = \left(u - \frac{1}{u}\right)^2 (1 + |bc|) = 4 \sinh^2 \frac{1}{2} T_{\xi_{i_0}} + 4 \sin(\theta),$$

where the last equation is due to [2, equations 5.4.8 and 5.4.10].

By [2, proof of Theorem 5.1.3 case(i)], ξ_{i_0} and $\xi_{j_0}\xi_{i_0}\xi_{j_0}^{-1}$ have no common fixed point. This implies $a, b, c, d \neq 0$. Consider the cross-ratio equation $[1, -1, w, -w] = [0, \infty, b/d, a/c]$ which is true for any w that satisfies the following equation

$$bc = \frac{(1-w)^2}{4w}. \tag{17}$$

This equation has a unique solution for w with norm greater than 1. Let $w = e^{2z_0}$ (for $z_0 = x_0 + iy_0$) be that solution. Replacing w in (17) yields $bc = \sinh^2 z_0$ which we can use to obtain the following chain of (in)equalities

$$4|bc|^2 = |\cosh^2 z_0 - 1|^2 = (\cosh 2x_0 - \cos 2y_0)^2 \geq (\cosh 2x_0 - 1)^2 \geq (\cosh^2 x_0 - 1)^2. \tag{18}$$

Using the above inequality and number of simplifications give

$$1 + |bc| \geq \frac{\cosh^2 x_0}{2}. \tag{19}$$

Let \mathcal{A} be the translation axis of ξ_{i_0} and \mathcal{B} be the translation axis of $\xi_{j_0}\xi_{i_0}\xi_{j_0}^{-1}$. Also notice that $\mathcal{B} = \xi_{j_0}\mathcal{A}$. Since trace squared, translation length and the square of the sine square of translation angle are all invariant under conjugation, we have the following inequalities for all $z \in \mathbb{H}^3$

$$\sin^2 \frac{1}{2} d_{\xi_{i_0}} z = \sinh \frac{1}{2} T_{\xi_{i_0}} \cosh d(z, \mathcal{A}) + \sin^2 \theta_{\xi_{i_0}} \sinh^2 d(z, \mathcal{A}) \leq \left(\sinh^2 \frac{1}{2} T_{\xi_{i_0}} + \sin^2 \theta_{\xi_{i_0}} \right) \cosh^2 d(z, \mathcal{A}) \tag{20}$$

and

$$\begin{aligned} \sinh^2 \frac{1}{2} d_{\xi_{j_0} \xi_{i_0} \xi_{j_0}^{-1}} z &= \sinh \frac{1}{2} T_{\xi_{i_0}} \cosh d(z, \mathcal{B}) + \sin^2 \theta_{\xi_{i_0}} \sinh^2 d(z, \mathcal{B}) \\ &\leq \left(\sinh^2 \frac{1}{2} T_{\xi_{i_0}} + \sin^2 \theta_{\xi_{i_0}} \right) \cosh^2 d(z, \mathcal{B}). \end{aligned} \tag{21}$$

By using the above inequalities and the fact that the hyperbolic sine and cosine are increasing for $x \geq 0$, we conclude that

$$\sinh^2 \frac{1}{2} \max \left\{ d_{\xi_{i_0}} z, d_{\xi_{j_0} \xi_{i_0} \xi_{j_0}^{-1}} z \right\} \leq \frac{1}{4} \left| u - \frac{1}{u} \right|^2 \cosh^2 \max \{ d_z \mathcal{A}, d_z \mathcal{B} \}. \tag{22}$$

Let $\Psi \in \mathcal{M}$ be defined by taking $[0, \infty, h0, h\infty]$ to $[1, -1, w, -w]$, where w is defined in (17). Under this setting $\Psi \mathcal{A}$ will be the geodesic from -1 to 1 and $\Psi \mathcal{B}$ will be the geodesic from $-w$ to w so that we have

$$d(\mathcal{A}, \mathcal{B}) = d(j, |w|j) = \log |w| = 2x_0.$$

By the above equality and the definition of z_1 given in theorem hypothesis, we can establish the fact that $d(z_1, \mathcal{A}) = x_0 = d(z_1, \mathcal{B})$. Applying this fact to the equations in (20) and (21) gives us another fact $d_{\xi_{i_0}} z_1 = d_{\xi_{j_0} \xi_{i_0} \xi_{j_0}^{-1}} z_1$. These two facts can be used to simplify (22) into the first of the following inequalities. Using (19) gives us the second one, which is

$$\sinh^2 \frac{1}{2} d_{\xi_{i_0}} z_1 \leq \frac{1}{4} \left| u - \frac{1}{u} \right|^2 \cosh^2 x_0 \leq \frac{1}{2} \left| u - \frac{1}{u} \right|^2 (1 + |bc|). \tag{23}$$

Finally, assume that the opposite of the theorem is true. In other words, assume that we have the inequality

$$|\text{trace}^2(\xi_{j_0}) - 4| + |\text{trace}(\xi_{j_0} \xi_{i_0} \xi_{j_0}^{-1} \xi_{i_0}^{-1}) - 2| < 2 \sinh^2 \left(\frac{1}{4} \log \alpha \right).$$

Because we have $d_{\xi_{i_0} \xi_{j_0} \xi_{i_0}^{-1}} z_2 \leq d_{\xi_{i_0} \xi_{j_0} \xi_{i_0}^{-1}} z_1$ and $d_\gamma z_2 < \frac{1}{2} \log \alpha$ for every $\gamma \in \Phi$ by the hypothesis, we get $d_\gamma z_2 < \frac{1}{2} \log \alpha$ for all $\gamma \in \Gamma_*$ by (23) and Lemma 10. This is a contradiction to Theorem 7.

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