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MUSTAFA AYDIN
mustafaydin.bs@hotmail.com

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Langevin delayed equations with Prabhakar derivatives involving two generalized fractional distinct orders

Mustafa AYDIN*

Department of Medical Services and Techniques, Muradiye Vocational School, Van Yüzüncü Yıl University, Van, Turkiye

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Abstract: This paper is devoted to defining the delayed analogue of the Mittag-Leffler type function with three parameters and investigating a representation of a solution to Langevin delayed equations with Prabhakar derivatives involving two generalized fractional distinct orders, which are first introduced and investigated, by means of the Laplace integral transform. It is verified by showing the solution satisfies the introduced system. Special cases which are also novel are presented as examples. The findings are illustrated with the help of the RLC circuits.

Key words: Fractional Langevin type equation, Mittag-Leffler type function, Prabhakar fractional derivative, RLC circuit

1. Introduction

Fractional calculus is an extension of integer calculus. This difference (extension) creates great opportunities, that integer calculus does not have, for fractional calculus, such as modeling social and physical problems more adequately. This and reasons like this have encouraged many researchers to study in this area. In a very short time like about 30 years, it has become the center of attention. This has led to the use of fractional calculus in many fields such as signal, electrochemistry, engineering, control theory, biophysics, mathematical physics, etc; see[25][8][15][30][9][21][32]. A fractional differential equation, which is a differential equation with fractional orders, is the most important subject of fractional calculus, and it has two major aspects; theoretical analysis[9][32] and numerical simulations[4][6][17][18][29][31]. It is easily noticed that Riemann-Liouville and Caputo fractional derivatives are mostly used in the literature even though there are lots of definitions of fractional derivatives. This makes working with the Prabhakar fractional derivative more reasonable because the Prabhakar fractional derivative contains both Caputo and Riemann-Liouville derivatives. The fractional operator which was described in [26], and profoundly examined in [20] causes the Prabhakar fractional derivative[13] to emerge as of late. Properties of Prabhakar fractional derivative and integral such as the semi-group, the inverse, the commutativity, the linearity, and more information are discussed in the articles[12][13][20][27]. In a short period, it has started to be used in several applications[11][33] and applied and pure mathematical subjects[14][28]. In addition, the Prabhakar fractional derivative includes many available derivatives such as the Gorenflo-Minardi, the Miller-Ros, Riemann-Liouville, Caputo, the Lorenzo–Hartley fractional derivatives, etc; which makes it stand out.

*Correspondence: m.aydin@yyu.edu.tr

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Paul Langevin gave an elaborative description, under the name of the Langevin equation, of Brownian motion in the 1900s. Langevin equations[5],[7] can describe many of the stochastic problems in fluctuating environments. But, today the traditional Langevin-type equation could not adequately formulate some sophisticated physical phenomena[19]. This creates the need to generalize the classical Langevin equation in order to better describe the sophisticated physical problems. There is no doubt that one of them is the fractional Langevin type system acquired from the traditional Langevin equation by using a fractional-order derivative instead of an integer-order derivative. As seen in the works[37][22][23][2][1][35], many researchers have tried to both introduce and examine the Langevin-type systems consisting of two different fractional-order derivatives. In this context, although there are several studies about fractional Langevin-type systems, there is no study about fractional Langevin-type delayed systems excluding a small number of studies. For instance, Mahmudov[24] defines the delayed Mittag-Leffler type function generated by λ_1, λ_2 of two parameters by drawing inspiration from Mittag-Leffler function with two parameters in order to solve Langevin delayed equations with Riemann- Liouville fractional derivatives as shown in (1.1) replacing Prabhakar fractional derivatives by Riemman- Liouville fractional derivatives. Huseynov et al.[16] solve Langevin delayed equations with Caputo fractional derivatives with the help of the same delayed Mittag-Leffler type function generated by λ_1, λ_2 of two parameters. The above explanations and the pioneer works[16][24] inspire us to consider the following inhomogeneous linear fractional Langevin delayed equations with Prabhakar fractional derivatives of Caputo type involving two distinct general fractional orders

$$\begin{cases} {}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1} z\left(x\right) - \lambda_2 {}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_2} z\left(x\right) - \lambda_1 z\left(x-h\right) = \zeta(x), \quad x \in (0,T], \quad h>0, \\ z(x) = \psi(x), \quad x \in [-h,0] \end{cases} \tag{1.1}$$

where ${}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}$ and ${}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_2}$ stand for the Prabhakar derivative of Caputo type of fractional orders α_1 and α_2 in distinct intervals $m-1<\alpha_1\leq m$ and $m-2<\alpha_2\leq m-1$ with $m\geq 2$, $\psi:[-h,0]\to\mathbb{R}$ is $(m-1)$-times continuously differentiable, the disturb function $\zeta:[0,T]\to\mathbb{R}$, $\lambda_1,\lambda_2\in\mathbb{R}$, and $T=nh$ for a fixed natural number $n\in\mathbb{N}$.

2. Preliminaries

In the present section, we will remind basic notions to help the readers easily understand all of the details of this paper.

$\mathbb{R}^n$ is an Euclidean space whose dimension is $n\in\mathbb{N}$. $AC^n(a,T)$ with $T>a$ consists of such a real-valued function g that it owns derivatives up to order $n-1$ on (a,T), and $g^{(n-1)}$ is absolutely continuous.

For $\eta,\alpha_1,\delta,w\in\mathbb{C}$ with $Re(\alpha_1), Re(\eta)>0$, the Prabhakar fractional integral[26][11][20] is given as noted below

$$\left(\mathcal{I}^{w,\delta}_{\eta,\alpha_1}\zeta\right)(x) = \int_0^x (x-s)^{\alpha_1-1} E^{\delta}_{\eta,\alpha_1}\left(w\left(x-s\right)^{\eta}\right)\zeta\left(s\right)ds \tag{2.1}$$

where the famous Mittag-Leffler function with three parameters

$$E^{\delta}_{\eta,\alpha_1}\left(x\right) = \sum_{i=0}^{\infty} \frac{(\delta)_i}{\Gamma(i\eta+\alpha_1)}\frac{x^i}{i!}.$$

here $\Gamma(.)$ is the well-known Gamma function and $(\delta)_i$ is the Pochhammer symbol, that is, $(\delta)_i=\frac{\Gamma(\delta+i)}{\Gamma(\delta)}$ or

$$(\delta)_0=1, \quad (\delta)_i=\delta(\delta+1)...(\delta+i-1), \quad i=0,1,2,... .$$

Remark 2.1 *The Mittag-Leffler function with three parameters* $E^{\delta}_{\eta,\alpha_1}(t)$ *for* $\delta = 0$ *is equal to* $\frac{1}{\Gamma(\alpha_1)}$, *that is,* $E^{0}_{\eta,\alpha_1}(t) = \frac{1}{\Gamma(\alpha_1)}$.

Remark 2.2 *The Prabhakar fractional integral* $\mathcal{I}^{w,\delta}_{\eta,\alpha_1}$ *for* $\delta = 0$ *reduces to Riemann-Liouville fractional integral of order* α_1.

In the work[13], the Prabhakar derivatives of Caputo types is given as follows

$$\left({}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\zeta\right)(x) = \mathcal{I}^{w,-\delta}_{\eta,m-\alpha_1}\left(\frac{d^m}{dx^m}\zeta\right)(x) = \int_0^x (x-s)^{m-\alpha_1-1} E^{-\delta}_{\eta,m-\alpha_1}\left(w(x-s)^{\alpha_1}\right)\frac{d^m}{ds^m}\zeta(s)\,ds, \tag{2.2}$$

where $\eta, \alpha_1, \delta, w \in \mathbb{C}$ with $Re(\eta) > 0, Re(\alpha_1) \geq 0$, and $m = \lfloor Re(\alpha_2) \rfloor + 1$ (here $\lfloor . \rfloor$ is the floor function) and $\zeta \in AC^m(0,T)$.

Remark 2.3 *The Prabhakar fractional derivative of Caputo type* ${}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}$ *for* $\delta = 0$ *reduces to Caputo fractional derivative of order* α_1.

Definition 2.4 *[36] If* ζ *is a both exponentially bounded and measurable function from* $[0,\infty)$ *to* $\mathbb{R}$, *then the Laplace transform of the function* ζ; $\mathcal{L}\{\zeta(x)\}(s)$, *is defined by*

$$\mathcal{L}\{\zeta(x)\}(s) = \int_0^\infty e^{-sx}\zeta(x)dx, \qquad s \in \mathbb{C}.$$

Lemma 2.5 *[36] The shifting feature for the Laplace integral transform is given by*

$$\mathcal{L}\{\zeta(x-h)\mathcal{H}(x-h)\}(s) = e^{-hs}\mathcal{L}\{\zeta(x)\}(s),$$

where the heaviside $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ *is given by*

$$\mathcal{H}(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Lemma 2.6 *[36] The Laplace integral transform of the convolution of* ζ *and* ψ *which are two functions on* $[0,\infty)$ *is given by*

$$\mathcal{L}\{(\zeta * \psi)(x)\}(s) = \mathcal{L}\{\zeta(x)\}(s)\,\zeta L\{\psi(x)\}(s), \qquad s \in \mathbb{C}.$$

Lemma 2.7 *[36] Suppose that* B *is an operator on a Banach space that is bounded and linear with* $\|B\| < 1$. $(I-B)^{-1}$ *is also so bounded and linear that*

$$(I-B)^{-1} = \sum_{i=0}^{\infty} B^i,$$

where I *is the identity operator.*

Lemma 2.8 *[10] For any $\eta, \alpha_1, w > 0$, the Laplace integral transform of the Mittag-Leffler function with three parameters $E^{\delta}_{\eta,\alpha_1}(wx^{\eta})$ is*

$$\mathcal{L}\left\{x^{\alpha_1-1}E^{\delta}_{\eta,\alpha_1}(wx^{\eta})\right\}(s) = s^{-\alpha_1}\left(1-ws^{-\eta}\right)^{-\delta},$$

which holds for $Re(s) > \|w\|^{\frac{1}{\alpha_1}}$.

Lemma 2.9 *[10] The Laplace integral transform of Prabhakar fractional derivative of Caputo-type is represented by*

$$\mathcal{L}\left\{{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\zeta(x)\right\}(s) = s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}\mathcal{L}\left\{f(x)\right\}(s) - \sum_{i=0}^{m-1} s^{\alpha_1-i-1}\left(1-ws^{-\eta}\right)^{\delta}\zeta^{(i)}(0).$$

where $m-1 \leq Re(\alpha_1) < m$.

From now on, all of the below sharing contributions will be novel.

3. A representation of a solution to system (1.1)

In this section, we will investigate a representation of a solution to the system (1.1). For this, we need to define a new Mittag-Leffler type function and make new preparations.

Firstly, we will extend the Mittag-Leffler function with three parameters to shorten the coming notations.

Remark 3.1 *The extended three-parameter Mittag-Leffler function $\mathfrak{E}^{w,\delta}_{\eta,\alpha_1}(x)$ is given by*

$$\mathfrak{E}^{w,\delta}_{\eta,\alpha_1}(x) = (x)_+^{\alpha_1-1}E^{\delta}_{\eta,\alpha_1}(wx^{\eta}),$$

where $\eta, \alpha_1, \delta, w \in \mathbb{C}$, $h > 0$, $(x)_+ = \max\{x, 0\}$.

Definition 3.2 *Delayed analogue of Mittag-Leffler type function generated by λ_1, λ_2 of three parameters $\mathbb{E}^{w,\delta,\theta}_{\eta,\alpha_1,\alpha_2,\gamma}(\lambda_1,\lambda_2;.) : \mathbb{R} \to \mathbb{R}$ is defined by*

$$\mathbb{E}^{w,\delta,\theta}_{\eta,\alpha_1,\alpha_2,\gamma}(\lambda_1,\lambda_2;x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}{\lambda_1}^i{\lambda_2}^j\mathfrak{E}^{w,i\delta+\theta}_{\eta,i\alpha_1+j\alpha_2+\gamma}(x-ih)\mathcal{H}(x-ih)$$

for $\eta, \alpha_1, \theta, \gamma, \delta, w \in \mathbb{C}$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and $h > 0$.

Lemma 3.3 *The delayed analogue of the three-parameter Mittag-Leffler type function produced by λ_1, λ_2 under the choices $\delta = 0$ reduces to the delayed two-parameter Mittag-Leffler type function produced by λ_1, λ_2 of [16, Definition 3.1.] and [24, Definition 2].*

Remark 3.4 *Delayed Mittag-Leffler type function generated by λ_1, λ_2 of three parameters in Definition 3.2 is in the closed form. It is understandable that the keystone of the delayed Mittag-Leffler type function generated by λ_1, λ_2 of three parameters is the three-parameter Mittag-Leffler function. If it is necessary to write it clearly, one can write it down as follows, $\mathbb{E}^{w,\delta,\theta}_{\eta,\alpha_1,\alpha_2,\gamma}(\lambda_1,\lambda_2;x)$:*

$$\mathbb{E}^{w,\delta,\theta}_{\eta,\alpha_1,\alpha_2,\gamma}(\lambda_1,\lambda_2;x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}{\lambda_1}^i{\lambda_2}^j(x-ih)^{i\alpha_1+j\alpha_2+\gamma-1}E^{i\delta+\theta}_{\eta,i\alpha_1+j\alpha_2+\gamma}\left(w(x-ih)^{\eta}\right)\mathcal{H}(x-ih).$$

Lemma 3.5 *The following expression is true:*

$$\frac{1}{\left(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}\right)^{i+1}}=\sum_{j=0}^{\infty}\binom{i+j}{j}\frac{{\lambda_2}^j}{s^{(i+1)\alpha_1+j(\alpha_1-\alpha_2)}\left(1-ws^{-\eta}\right)^{(i+1)\delta}},$$

provided that $|\lambda_2|<|s^{\alpha_1-\alpha_2}|$.

Proof If $|\lambda_2|<|s^{\alpha_1-\alpha_2}|$, according to the Taylor series representation, we have

$$\begin{aligned}\frac{1}{\left(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}\right)^{i+1}}&=\frac{1}{\left(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}\right)^{i+1}}\frac{1}{\left(1-\frac{\lambda_2}{s^{\alpha_1-\alpha_2}}\right)^{i+1}}\\&=\frac{1}{\left(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}\right)^{i+1}}\sum_{j=0}^{\infty}\binom{i+j}{j}\left(\frac{\lambda_2}{s^{\alpha_1-\alpha_2}}\right)^j\\&=\sum_{j=0}^{\infty}\binom{i+j}{j}\frac{{\lambda_2}^j}{s^{(i+1)\alpha_1+j(\alpha_1-\alpha_2)}\left(1-ws^{-\eta}\right)^{(i+1)\delta}}.\end{aligned}$$

□

Lemma 3.6 *The following inverse Laplace transform expression is true:*

$$\mathcal{L}^{-1}\left\{\left(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}\right)^{-1}\right\}(x)=\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x\right),$$

provided that $|\lambda_1(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta})^{-1}e^{-sh}|<1$.

Proof If the stated condition holds, according to the Neumann Series we have

$$\begin{aligned}\left(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}\right)^{-1}&=\left(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}\right)^{-1}\\&\times\left(1-\lambda_1\left(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}\right)^{-1}e^{-sh}\right)^{-1}\\&=\left(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}\right)^{-1}\\&\times\sum_{i=0}^{\infty}{\lambda_1}^i\left(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}\right)^{-i}e^{-ish}\\&=\sum_{i=0}^{\infty}{\lambda_1}^i\left(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}\right)^{-(i+1)}e^{-ish}.\end{aligned}$$

If Lemma 3.5 is applied, one can get

$$\begin{aligned}\left(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}\right)^{-1}&=\sum_{i=0}^{\infty}{\lambda_1}^i\sum_{j=0}^{\infty}\binom{i+j}{j}\frac{{\lambda_2}^j}{s^{(i+1)\alpha_1+j(\alpha_1-\alpha_2)}\left(1-ws^{-\eta}\right)^{(i+1)\delta}}e^{-ish}\\&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}{\lambda_1}^i{\lambda_2}^j\frac{1}{s^{(i+1)\alpha_1+j(\alpha_1-\alpha_2)}\left(1-ws^{-\eta}\right)^{(i+1)\delta}}e^{-ish}.\end{aligned}$$

If Lemmas 2.5 and 2.8 are employed, one can get

$$\mathcal{L}^{-1}\left\{\left(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}\right)^{-1}\right\}(x)$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}{\lambda_1}^i{\lambda_2}^j\mathfrak{E}^{w,i\delta+\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+\alpha_1}(x-ih)\,\mathcal{H}(x-ih),$$

which is the desired result. □

Lemma 3.7 *The following inverse Laplace transform expression is true:*

$$\mathcal{L}^{-1}\left\{\frac{s^{k+1}e^{-sh}}{s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}}\right\}(x)=\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1+k+1}(\lambda_1,\lambda_2;x-h),$$

for k=0,1, ..., provided that $|\lambda_1(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta})^{-1}e^{-sh}|<1$.

Proof By receiving help from the proof of Lemma 3.6, we get

$$\frac{s^{k+1}e^{-sh}}{s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}}$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}{\lambda_1}^i{\lambda_2}^j\frac{1}{s^{(i+1)\alpha_1+j(\alpha_1-\alpha_2)+k+1}\left(1-ws^{-\eta}\right)^{(i+1)\delta}}e^{-(i+1)sh}.$$

If Lemmas 2.5 and 2.8 are employed, one can get

$$\mathcal{L}^{-1}\left\{\frac{s^{k+1}e^{-sh}}{s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}}\right\}(x)$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}{\lambda_1}^i{\lambda_2}^j\mathfrak{E}^{w,i\delta+\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+\alpha_1+k+1}(x-(i+1)h)\,\mathcal{H}(x-(i+1)h),$$

which is the desired result. □

Lemma 3.8 *The following inverse Laplace transform expression is true:*

$$\mathcal{L}^{-1}\left\{\frac{s^{\gamma}\left(1-ws^{-\eta}\right)^{\delta}}{s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}}\right\}(x)=\mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1-\gamma}(\lambda_1,\lambda_2;x),$$

provided that $|\lambda_1(s^{\alpha_1}\left(1-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(1-ws^{-\eta}\right)^{\delta})^{-1}e^{-sh}|<1$.

Proof Since the proof is similar to the proofs of Lemma 3.6 and 3.7, we omit it. □

Now, we especially calculate the Laplace integral transform of the delayed term $z(x-h)$. Based on the substitution $\varepsilon = x - h$, we have

$$\begin{aligned}
\mathcal{L}\left\{z\left(x-h\right)\right\}(s) &= \int_0^\infty e^{-sx} z\left(x-r\right) dx \\
&= e^{-sh} \int_{-h}^\infty e^{-s\varepsilon} z\left(\varepsilon\right) d\varepsilon \\
&= e^{-sh} \left(\int_{-h}^0 e^{-s\varepsilon} z\left(\varepsilon\right) d\varepsilon + \int_0^\infty e^{-s\varepsilon} z\left(\varepsilon\right) d\varepsilon \right) \\
&= e^{-sh} \mathcal{L}\left\{z\left(x\right)\right\}(s) + \int_{-h}^0 e^{-s(\varepsilon+h)} \psi\left(\varepsilon\right) d\varepsilon.
\end{aligned}$$

In the light of the substitution $\varepsilon + h = x$, one can obtain

$$\begin{aligned}
\mathcal{L}\left\{z\left(x-h\right)\right\}(s) &= e^{-sh} \mathcal{L}\left\{z\left(x\right)\right\}(s) + \int_0^h e^{-sx} \psi\left(x-h\right) dx \\
&= e^{-sh} \mathcal{L}\left\{z\left(x\right)\right\}(s) + \int_0^\infty e^{-sx} \tilde{\psi}\left(x-h\right) dx \\
&= e^{-sh} \mathcal{L}\left\{z\left(x\right)\right\}(s) + \mathcal{L}\left\{\bar{\psi}\left(x-h\right)\right\}(s), \qquad (3.1)
\end{aligned}$$

where the unit-step $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ is defined as follows

$$\tilde{\psi}(x) = \begin{cases} \psi(x), & -h \leq x \leq 0, \\ 0, & x > 0. \end{cases}$$

It is time to offer the first main theorem.

Theorem 3.9 *Under the condition that the Laplace transforms of all terms in (1.1) exist, an analytical solution to system (1.1) is given by*

$$\begin{aligned}
z(t) = {} & \sum_{k=0}^{m-2} \left(\frac{x^k}{\Gamma(k+1)} + \lambda_1 \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1+k+1}\left(\lambda_1, \lambda_2; x-h\right) \right) \psi^{(k)}(0) \\
& + \mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,m}\left(\lambda_1, \lambda_2; x\right) \psi^{(m-1)}(0) \\
& + \lambda_1 \int_{-h}^{\min\{x-h,0\}} \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1, \lambda_2; x-h-s\right) \psi(s) ds \\
& + \int_0^x \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1, \lambda_2; x-s\right) \zeta(s) ds.
\end{aligned}$$

Proof If the Laplace integral function is applied to both sides of the system (1.1)

$$\begin{aligned}
& s^{\alpha_1} \left(I - ws^{-\eta}\right)^\delta Z\left(s\right) - \sum_{i=0}^{m-1} s^{\alpha_1-i-1} \left(I - ws^{-\eta}\right)^\delta \psi^{(i)}(0) \\
& - \lambda_2 \left(s^{\alpha_2} \left(I - ws^{-\eta}\right)^\delta Z\left(s\right) - \sum_{i=0}^{m-2} s^{\alpha_2-i-1} \left(I - ws^{-\eta}\right)^\delta \psi^{(i)}(0) \right) \\
& - \lambda_1 \left(e^{-sh} Z\left(s\right) + \mathcal{L}\left\{\bar{\psi}\left(x-h\right)\right\}(s)\right) = \mathcal{L}\left\{\zeta\left(x\right)\right\}(s).
\end{aligned}$$

One can rearrange the just-above equation as follows

$$\left(s^{\alpha_1}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}\right)Z\left(s\right)$$
$$=\sum_{i=0}^{m-2}\left(s^{\alpha_1-i-1}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2-i-1}\left(I-ws^{-\eta}\right)^{\delta}\right)\psi^{(i)}(0)$$
$$s^{\alpha_1-m}\left(I-ws^{-\eta}\right)^{\delta}\psi^{(m-1)}(0)+\lambda_1\mathcal{L}\left\{\bar{\psi}\left(x-h\right)\right\}\left(s\right)+\mathcal{L}\left\{\zeta\left(x\right)\right\}\left(s\right).$$

Divide the whole equation by the coefficient of $Z\left(s\right)$,

$$Z\left(s\right)=\sum_{i=0}^{m-2}\frac{s^{\alpha_1-i-1}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2-i-1}\left(I-ws^{-\eta}\right)^{\delta}}{s^{\alpha_1}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}}\psi^{(i)}(0)$$
$$+\frac{s^{\alpha_1-m}\left(I-ws^{-\eta}\right)^{\delta}}{s^{\alpha_1}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}}\psi^{(m-1)}(0)$$
$$+\frac{\lambda_1}{s^{\alpha_1}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}}\mathcal{L}\left\{\bar{\psi}\left(x-h\right)\right\}\left(s\right)$$
$$+\frac{1}{s^{\alpha_1}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}}\mathcal{L}\left\{\zeta\left(x\right)\right\}\left(s\right).$$

By adding and subtracting $\lambda_1 e^{-sh}$ in the nominator of the first term, we get

$$Z\left(s\right)=\sum_{i=0}^{m-2}\left(s^{\alpha_1-i-1}+\frac{s^{\alpha_1-i-1}\lambda_1 e^{-sh}}{s^{\alpha_1}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}}\right)\psi^{(i)}(0)$$
$$+\frac{s^{\alpha_1-m}\left(I-ws^{-\eta}\right)^{\delta}}{s^{\alpha_1}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}}\psi^{(m-1)}(0)$$
$$+\frac{\lambda_1}{s^{\alpha_1}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}}\mathcal{L}\left\{\bar{\psi}\left(x-h\right)\right\}\left(s\right)$$
$$+\frac{1}{s^{\alpha_1}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_2 s^{\alpha_2}\left(I-ws^{-\eta}\right)^{\delta}-\lambda_1 e^{-sh}}\mathcal{L}\left\{\zeta\left(x\right)\right\}\left(s\right).$$

Now if one takes the inverse of the Laplace transform and uses Lemmas 2.6 3.5, 3.6, and 3.7, the following is acquired

$$z(t)=\sum_{k=0}^{m-2}\left(\frac{x^k}{\Gamma(k+1)}+\lambda_1\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1+k+1}\left(\lambda_1,\lambda_2;x-h\right)\right)\psi^{(k)}(0)$$
$$+\mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,m}\left(\lambda_1,\lambda_2;x\right)\psi^{(m-1)}(0)+\lambda_1\int_{-h}^{x-h}\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-h-s\right)\tilde{\psi}(s)ds$$
$$+\int_{0}^{x}\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-s\right)\zeta(s)ds.$$

We consider that if $x\geq h$, then

$$\int_{-h}^{x-h}\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-h-s\right)\psi(s)ds=\int_{-h}^{0}\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-h-s\right)\psi(s)ds,$$

and if $x < h$, then

$$\int_{-h}^{x-h} \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-h-s\right)\psi(s)ds = \int_{-h}^{x-h} \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-h-s\right)\psi(s)ds,$$

which gives

$$\begin{aligned}&\int_{-h}^{x-h} \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-h-s\right)\psi(s)ds\\ =&\int_{-h}^{\min\{x-h,0\}} \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-h-s\right)\psi(s)ds.\end{aligned}$$

This is the last point for this lemma, which means the proof is completed. □

Remark 3.10 *The condition that the Laplace transforms of all terms in (1.1) exist is a setback for this theorem. Theorem 3.16 shows that this condition could be removed.*

Theorem 3.11 *An explicit solution to the system (1.1) with $\zeta = 0$ is given by*

$$\begin{aligned}z(t) =& \sum_{k=0}^{m-2}\left(\frac{x^k}{\Gamma(k+1)} + \lambda_1 \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1+k+1}\left(\lambda_1,\lambda_2;x-h\right)\right)\psi^{(k)}(0)\\ &+ \mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,m}\left(\lambda_1,\lambda_2;x\right)\psi^{(m-1)}(0)\\ &+\lambda_1\int_{-h}^{\min\{x-h,0\}} \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-h-s\right)\psi(s)ds.\end{aligned}$$

In order to prove this theorem shortly and understandably, we calculate some expressions we will face in the proof.

Lemma 3.12 *The Prabhakar fractional derivative of Caputo type ${}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}$ of $\mathfrak{E}^{w,\theta}_{\eta,\alpha_2}(x)$ is given as follows:*

$${}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\mathfrak{E}^{w,\theta}_{\eta,\alpha_2}(x) = \mathfrak{E}^{w,\theta-\delta}_{\eta,\alpha_2-\alpha_1}(x).$$

Proof We will use the related definition and the properties of the gamma functions,

$$\begin{aligned}{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\mathfrak{E}^{w,\theta}_{\eta,\alpha_2}(x) &= {}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}x^{\alpha_2-1}E^{\theta}_{\eta,\alpha_2}(wx^{\eta})\\ &= \sum_{i=0}^{\infty}\frac{(\theta)_i\ w^i}{\Gamma(i\eta+\alpha_2)i!}\mathcal{I}^{w,-\delta}_{\eta,m-\alpha_1}\left(\frac{d^m}{dx^m}x^{i\eta+\alpha_2-1}\right)\\ &= \mathcal{I}^{w,-\delta}_{\eta,m-\alpha_1}x^{\alpha_2-m-1}E^{\theta}_{\eta,\alpha_2-m}(wx^{\eta}).\end{aligned}$$

From [20, Theorem 2], one can obtain

$$\begin{aligned}{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\mathfrak{E}^{w,\theta}_{\eta,\alpha_2}(x) &= x^{\alpha_2-\alpha_1-1}E^{\theta-\delta}_{\eta,\alpha_2-\alpha_1}(wx^{\eta})\\ &= \mathfrak{E}^{w,\theta-\delta}_{\eta,\alpha_2-\alpha_1}(x).\end{aligned}$$

□

Corollary 3.13 *We note that*

$$
\begin{aligned}
{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\mathfrak{E}^{w,\theta}_{\eta,\alpha_2}(x) &= \left(\sum_{k=0}^{m-2}\frac{x^k}{\Gamma(k+1)}\psi^{(k)}(0)\right)\\
&= \int_0^x (x-s)^{m-\alpha_1-1} E^{-\delta}_{\eta,m-\alpha_1}\left(w(x-s)^{\alpha_1}\right)\frac{d^m}{ds^m}\sum_{k=0}^{m-2}\frac{s^k}{\Gamma(k+1)}ds\psi^{(k)}(0)\\
&= 0.
\end{aligned}
$$

Lemma 3.14 *The following equation holds true:*

$$
\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\binom{l+k}{k}b_{lk} = b_{00} + \sum_{l=0}^{\infty}\sum_{k=1}^{\infty}\binom{l+k-1}{k-1}b_{lk} + \sum_{l=1}^{\infty}\sum_{k=0}^{\infty}\binom{l+k-1}{k}b_{lk}
$$

where $b_{lk}\in\mathbb{R}$ *for each* $l,k\in\mathbb{N}$*.*

Proof It can be easily proved based on these known information $\binom{0}{0}=1$, $\binom{l}{k}=0$, for $k>l$, and $\binom{l}{k}=\binom{l-1}{k}+\binom{l-1}{k-1}$, for $0<k<l$. □

Proof of Theorem 3.11: We will calculate the Prabhakar fractional derivatives of Caputo type ${}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}$ of all terms one by one in order to simplify the proof. Applying Lemma 3.12 and Corollary 3.13 to the just-above expression, we get

$$
\begin{aligned}
&{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\sum_{k=0}^{m-2}\left(\frac{x^k}{\Gamma(k+1)} + \lambda_1\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1+k+1}(\lambda_1,\lambda_2;x-h)\right)\psi^{(k)}(0)\\
=&\lambda_1\sum_{k=0}^{m-2}\mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,k+1}(\lambda_1,\lambda_2;x-h)\psi^{(k)}(0)\\
=&\lambda_1\sum_{k=0}^{m-2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}{\lambda_1}^i{\lambda_2}^j\mathfrak{E}^{w,i\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+k+1}(x-(i+1)h)\mathcal{H}(x-(i+1)h)\psi^{(k)}(0).
\end{aligned}
$$

From Lemma 3.14 one can easily acquire

$$
\begin{aligned}
&{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\sum_{k=0}^{m-2}\left(\frac{x^k}{\Gamma(k+1)} + \lambda_1\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1+k+1}(\lambda_1,\lambda_2;x-h)\right)\psi^{(k)}(0)\\
=&\lambda_1\sum_{k=0}^{m-2}\frac{(x-h)^k}{\Gamma(k+1)}\psi^{(k)}(0)\\
&+\lambda_1\sum_{k=0}^{m-2}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\binom{i+j-1}{j-1}{\lambda_1}^i{\lambda_2}^j\mathfrak{E}^{w,i\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+k+1}(x-(i+1)h)\mathcal{H}(x-(i+1)h)\psi^{(k)}(0)\\
&+\lambda_1\sum_{k=0}^{m-2}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\binom{i+j-1}{j}{\lambda_1}^i{\lambda_2}^j\mathfrak{E}^{w,i\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+k+1}(x-(i+1)h)\mathcal{H}(x-(i+1)h)\psi^{(k)}(0)
\end{aligned}
$$

$$
\begin{aligned}
=&\lambda_1 \sum_{k=0}^{m-2} \frac{(x-h)^k}{\Gamma(k+1)} \psi^{(k)}(0) \\
&+ \lambda_1\lambda_2 \sum_{k=0}^{m-2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{i+j}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w,i\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+\alpha_1-\alpha_2+k+1}\left(x-(i+1)h\right) \mathcal{H}\left(x-(i+1)h\right) \psi^{(k)}(0) \\
&+ {\lambda_1}^2 \sum_{k=0}^{m-2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{i+j}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w,i\delta+\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+\alpha_1+k+1}\left(x-(i+2)h\right) \mathcal{H}\left(x-(i+2)h\right) \psi^{(k)}(0) \\
=&\lambda_1 \sum_{k=0}^{m-2} \frac{(x-h)^k}{\Gamma(k+1)} \psi^{(k)}(0) + \lambda_1\lambda_2 \sum_{k=0}^{m-2} \mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1-\alpha_2+k+1}\left(\lambda_1,\lambda_2;x-h\right)\psi^{(k)}(0) \\
&+ {\lambda_1}^2 \sum_{k=0}^{m-2} \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1+k+1}\left(\lambda_1,\lambda_2;x-2h\right)\psi^{(k)}(0). \qquad (3.2)
\end{aligned}
$$

Secondly, we will calculate the following expression

$$
\begin{aligned}
&{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1} \mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,m}\left(\lambda_1,\lambda_2;x\right)\psi^{(m-1)}(0) \\
=&{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{i+j}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w,i\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+m}\left(x-ih\right)\mathcal{H}\left(x-ih\right)\psi^{(m-1)}(0) \\
=&{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1} \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \binom{i+j-1}{j-1} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w,i\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+m}\left(x-ih\right)\mathcal{H}\left(x-ih\right)\psi^{(m-1)}(0) \\
+&{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1} \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \binom{i+j-1}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w,i\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+m}\left(x-ih\right)\mathcal{H}\left(x-ih\right)\psi^{(m-1)}(0) \\
=&\lambda_2 {}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{i+j}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w,i\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+\alpha_1-\alpha_2+m}\left(x-ih\right)\mathcal{H}\left(x-ih\right)\psi^{(m-1)}(0) \\
+&\lambda_1 {}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{i+j}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w,i\delta+\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+\alpha_1+m}\left(x-(i+1)h\right) \\
&\times \mathcal{H}\left(x-(i+1)h\right)\psi^{(m-1)}(0) \\
=&\lambda_2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{i+j}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w,i\delta-\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+m-\alpha_2}\left(x-ih\right)\mathcal{H}\left(x-ih\right)\psi^{(m-1)}(0) \\
+&\lambda_1 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{i+j}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w,i\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+m}\left(x-(i+1)h\right)\mathcal{H}\left(x-(i+1)h\right)\psi^{(m-1)}(0) \\
=&\lambda_2 \mathbb{E}^{w,\delta,-\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,m-\alpha_2}\left(\lambda_1,\lambda_2;x\right)\psi^{(m-1)}(0) + \lambda_1 \mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,m}\left(\lambda_1,\lambda_2;x-h\right)\psi^{(m-1)}(0). \qquad (3.3)
\end{aligned}
$$

Thirdly, we will calculate the Prabhakar fractional derivative of the last term in the solution equation

$$
\begin{aligned}
&{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\left(\lambda_1\int_{-h}^{\min\{x-h,0\}}\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-h-s\right)\psi(s)ds\right)\\
=&\lambda_1{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\int_{-h}^{\min\{x-h,0\}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}{\lambda_1}^i{\lambda_2}^j\mathfrak{E}^{w,i\delta+\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+\alpha_1}\left(x-(i+1)h-s\right)\\
&\times\mathcal{H}\left(x-(i+1)h-s\right)\psi(s)ds\\
=&\lambda_1{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\int_{-h}^{\min\{x-h,0\}}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\binom{i+j-1}{j-1}{\lambda_1}^i{\lambda_2}^j\mathfrak{E}^{w,i\delta+\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+\alpha_1}\left(x-(i+1)h-s\right)\\
&\times\mathcal{H}\left(x-(i+1)h-s\right)\psi(s)ds\\
+&\lambda_1{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\int_{-h}^{\min\{x-h,0\}}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\binom{i+j-1}{j}{\lambda_1}^i{\lambda_2}^j\mathfrak{E}^{w,i\delta+\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+\alpha_1}\left(x-(i+1)h-s\right)\\
&\times\mathcal{H}\left(x-(i+1)h-s\right)\psi(s)ds\\
=&\lambda_1\lambda_2{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\int_{-h}^{\min\{x-h,0\}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}{\lambda_1}^i{\lambda_2}^j\mathfrak{E}^{w,i\delta+\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+2\alpha_1-\alpha_2}\left(x-(i+1)h-s\right)\\
&\times\mathcal{H}\left(x-(i+1)h-s\right)\psi(s)ds\\
+&{\lambda_1}^2{}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}\int_{-h}^{\min\{x-h,0\}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}{\lambda_1}^i{\lambda_2}^j\mathfrak{E}^{w,i\delta+2\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+2\alpha_1}\left(x-(i+2)h-s\right)\\
&\times\mathcal{H}\left(x-(i+2)h-s\right)\psi(s)ds\\
=&\lambda_1\lambda_2\int_{-h}^{\min\{x-h,0\}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}{\lambda_1}^i{\lambda_2}^j\mathfrak{E}^{w,i\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+\alpha_1-\alpha_2}\left(x-(i+1)h-s\right)\\
&\times\mathcal{H}\left(x-(i+1)h-s\right)\psi(s)ds\\
+&{\lambda_1}^2\int_{-h}^{\min\{x-2h,0\}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}{\lambda_1}^i{\lambda_2}^j\mathfrak{E}^{w,i\delta+\delta}_{\eta,i\alpha_1+j(\alpha_1-\alpha_2)+\alpha_1}\left(x-(i+2)h-s\right)\\
&\times\mathcal{H}\left(x-(i+2)h-s\right)\psi(s)ds\\
=&\lambda_1\lambda_2\int_{-h}^{\min\{x-h,0\}}\mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1-\alpha_2}\left(\lambda_1,\lambda_2;x-h-s\right)\psi(s)ds\\
+&{\lambda_1}^2\int_{-h}^{\min\{x-2h,0\}}\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-2h-s\right)\mathcal{H}\left(x-(i+2)h-s\right)\psi(s)ds.
\end{aligned}\tag{3.4}
$$

Lastly, one can easily calculate the following expressions

$$\begin{aligned}
& -\lambda_2 {}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_2} z(x) \\
&= -\lambda_2 {}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_2} \Bigg[\sum_{k=0}^{m-2} \left(\frac{(x-h)^k}{\Gamma(k+1)} + \lambda_1 \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1+k+1}(\lambda_1,\lambda_2;x-h) \right) \psi^{(k)}(0) \\
& + \mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,m}(\lambda_1,\lambda_2;x)\, \psi^{(m-1)}(0) \\
& + \lambda_1 \int_{-h}^{\min\{x-h,0\}} \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}(\lambda_1,\lambda_2;x-h-s)\, \psi(s)ds \Bigg] \\
&= -\lambda_2\lambda_1 \sum_{k=0}^{m-2} \mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,k+1}(\lambda_1,\lambda_2;x-h)\, \psi^{(k)}(0) - \lambda_2 \mathbb{E}^{w,\delta,-\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,m-\alpha_2}(\lambda_1,\lambda_2;x)\, \psi^{(m-1)}(0) \\
& - \lambda_2\lambda_1 \int_{-h}^{\min\{x-h,0\}} \mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1-\alpha_2}(\lambda_1,\lambda_2;x-h-s)\, \psi(s)ds, \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
-\lambda_1 z(x-h) &= -\lambda_1 \sum_{k=0}^{m-2} \left(\frac{(x-h)^k}{\Gamma(k+1)} + \lambda_1 \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1+k+1}(\lambda_1,\lambda_2;x-2h) \right) \psi^{(k)}(0) \\
& - \lambda_1 \mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,m}(\lambda_1,\lambda_2;x-h)\, \psi^{(m-1)}(0) \\
& - {\lambda_1}^2 \int_{-h}^{\min\{x-2h,0\}} \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}(\lambda_1,\lambda_2;x-2h-s)\, \psi(s)ds. \qquad (3.6)
\end{aligned}$$

A linear combination of equations (3.2), (3.3), (3.4), (3.5), and (3.6) gives the desired equation

$${}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1} z(x) - \lambda_2 {}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_2} z(x) - \lambda_1 z(x-h) = 0.$$

□

Theorem 3.15 *An explicit solution to system (1.1) with zero initial condition is given by*

$$z(x) = \int_0^x \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}(\lambda_1,\lambda_2;x-s)\, \zeta(s)ds.$$

Proof Taking the Prabhakar fractional derivative of Caputo type ${}^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1}$ of $\int_0^x \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}(\lambda_1,\lambda_2;x-s)\, \zeta(s)ds$ as done in Theorem 3.11, one can easily acquire the result. □

A combination of Theorems 3.11 and 3.15 provides the following result.

Theorem 3.16 *An analytical whole solution to inhomogeneous linear fractional Langevin delayed equations with Prabhakar fractional derivatives of Caputo type involving two distinct general fractional orders in (1.1) is* *given by*

$$z(x)=\sum_{k=0}^{m-2}\left(\frac{x^k}{\Gamma(k+1)}+\lambda_1\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1+k+1}\left(\lambda_1,\lambda_2;x-h\right)\right)\psi^{(k)}(0)$$
$$+\mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,m}\left(\lambda_1,\lambda_2;x\right)\psi^{(m-1)}(0)$$
$$+\lambda_1\int_{-h}^{\min\{x-h,0\}}\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-h-s\right)\psi(s)ds$$
$$+\int_0^x\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-s\right)\zeta(s)ds.$$

Remark 3.17 *This paper is a quite comprehensive study because it is novel and inholds quite different novel results(see; special cases' section) as well as containing results of the available works(see; Lemma 3.18) in the literature.*

Lemma 3.18 *Under the special selections of the existent parameters, our findings correspond to some works in the literature.*

a) *An analytical solution expressed in Theorem 3.16 for system (1.1) under the choices $\delta=0$ reduces to that of [16, Theorem 4.2.].*

b) *An analytical solution expressed in Theorem 3.16 for system (1.1) under the choices $\delta=0$, $h=0$, and $m=2$ corresponds to that of [3, Theorem 3.1.].*

4. Examples

In this section, as examples, we offer a couple of special cases of our findings, which are also new.

Example 4.1 *Let us consider inhomogeneous linear fractional Langevin delayed equations (1.1) with Prabhakar fractional derivatives of Caputo type involving two fractional distinct orders $0<\alpha_2\leq 1$, $1<\alpha_1\leq 2$. In this case, Theorem 3.16 can be reexpressed as follows.*

Proposition 4.2 *An explicit solution formula of the initial value system (1.1) with $m=2$ has the following form*

$$z(x)=1+\lambda_1\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1+1}\left(\lambda_1,\lambda_2;x-h\right)\psi(0)+\mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,2}\left(\lambda_1,\lambda_2;x\right)\psi^{'}(0)$$
$$+\lambda_1\int_{-h}^{\min\{x-h,0\}}\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2\alpha_1}\left(\lambda_1,\lambda_2;x-h-s\right)\psi(s)ds$$
$$+\int_0^x\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(\lambda_1,\lambda_2;x-s\right)\zeta(s)ds$$

Example 4.3 *If $h=0$ is taken, then the inhomogeneous linear fractional Langevin delayed equations (1.1) with Prabhakar fractional derivatives of Caputo type involving two distinct general fractional orders transforms to the inhomogeneous linear fractional Langevin equations with Prabhakar fractional derivatives of Caputo type involving two distinct general fractional orders which is also not studied before. In this case, Theorem 3.16 can be restated as noted below.*

Proposition 4.4 *An explicit solution formula of the initial value system (1.1) with $h = 0$ transforms to the following form*

$$z(x) = \sum_{k=0}^{m-2} \frac{x^k}{\Gamma(k+1)} \psi^{(k)}(0)$$

$$= \sum_{k=0}^{m-2} \lambda_1 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{i+j}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w, i\delta+\delta}_{\eta, i\alpha_1 + j(\alpha_1-\alpha_2)+\alpha_1+k+1}(x) \, \psi^{(k)}(0)$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{i+j}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w, i\delta}_{\eta, i\alpha_1 + j(\alpha_1-\alpha_2)+m}(x) \, \psi^{(m-1)}(0)$$

$$+ \int_0^x \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{i+j}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w, i\delta+\delta}_{\eta, i\alpha_1 + j(\alpha_1-\alpha_2)+\alpha_1}(x-s) \, \zeta(s) ds.$$

Example 4.5 *If $h = 0$ and $m = 2$ are taken, then the inhomogeneous linear fractional Langevin delayed equations (1.1) with Prabhakar fractional derivatives of Caputo type involving two generalized fractional distinct orders $m-2 < \alpha_2 \leq m-1$, $m-1 < \alpha_1 \leq m$ transforms to the inhomogeneous linear fractional Langevin equations with Prabhakar fractional derivatives of Caputo type involving two fractional distinct orders $0 < \alpha_2 \leq 1$, $1 < \alpha_1 \leq 2$ which is also not studied before. In this case, Theorem 3.16 can be restated as noted below.*

Proposition 4.6 *An explicit solution formula of the initial value system (1.1) with $h = 0$ transforms to the following form*

$$z(x) = 1 + \lambda_1 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{i+j}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w, i\delta+\delta}_{\eta, i\alpha_1 + j(\alpha_1-\alpha_2)+\alpha_1+1}(x) \, \psi(0)$$

$$+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{i+j}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w, i\delta}_{\eta, i\alpha_1 + j(\alpha_1-\alpha_2)+m}(x) \, \psi^{(m-1)}(0)$$

$$+ \int_0^x \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{i+j}{j} {\lambda_1}^i {\lambda_2}^j \mathfrak{E}^{w, i\delta+\delta}_{\eta, i\alpha_1 + j(\alpha_1-\alpha_2)+\alpha_1}(x-s) \, \zeta(s) ds.$$

5. An application to the RLC circuits

RLC circuits are exploited for radio receivers, signal processing, the tuning process of television, etc. Due to its widespread usage, we illustrate our findings with the help of RLC circuits.

RLC circuits as seen in Figure 1 have four main elements: the resistance(R), the inductance(L), the capacitance(C), and the voltage(E) in addition to the current(I). The voltage drops of resistor, inductor, and capacitor in series are equal to $V_R = IR$, $V_L = L\frac{dI}{dt}$, and $V_C = \frac{Q}{C}$ which are acquired from experimental data and physics, here Q stands for the charge of the capacitor so that $\frac{d}{dt}Q(t) = I(t)$.

Based on Kirchoff's law, one can get

$$V_L + V_R + V_C = E(t)$$

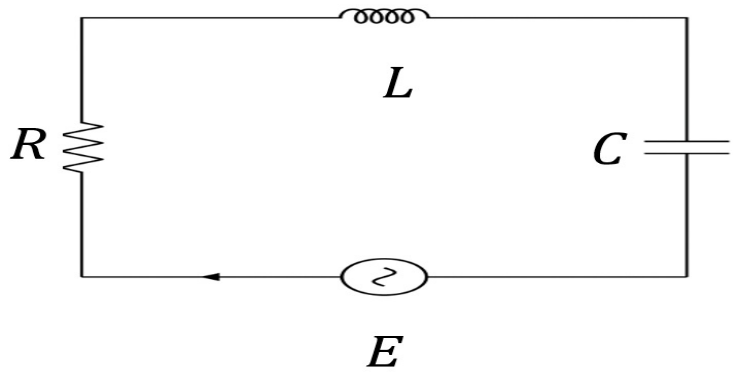

Figure 1. An RLC series circuit.

where $E(t)$ is the voltage. Thus, the differential equation is as follows

$$L\frac{d}{dt}I(t) + RI(t) + \frac{1}{C}Q(t) = E(t).$$

Then, the second-order linear ordinary differential equation is noted below

$$L\frac{d^2}{dt^2}I(t) + R\frac{d}{dt}I(t) + \frac{1}{C}I(t) = \frac{d}{dt}E(t).$$

We will examine an initial value problem for a fractional Langevin delayed equation with fractional orders $1 < \alpha_1 \geq 2$, $0 < \alpha_2 \geq 1$ as a special case. The main principle needs to formalize an initial value problem for a fractional Langevin delayed equation that models the RLC circuits in series in the following form of

$$\begin{cases} L^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_1} I\left(t\right) + R^{PC}\mathcal{D}^{w,\delta}_{\eta,\alpha_2} I\left(t\right) \frac{1}{C} I\left(t-h\right) = \zeta(t), \quad t \in (0,T], \quad h > 0, \\ \qquad I(t) = \psi(t), \quad t \in [-h, 0]. \end{cases} \tag{5.1}$$

In order to find the current $I(t)$, we use Theorem 3.16, it can be written as noted below

$$\begin{aligned} I(t) = 1 &- \frac{1}{RL}\mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1+1}\left(-\frac{1}{RL}, -\frac{R}{L}; t-h\right)\psi(0) \\ &+ \int_0^t \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(-\frac{1}{RL}, -\frac{R}{L}; t-s\right)E^{'}(s)ds \\ &+ \frac{1}{L}\mathbb{E}^{w,\delta,0}_{\eta,\alpha_1,\alpha_1-\alpha_2,2}\left(-\frac{1}{RL}, -\frac{R}{L}; t\right)\psi^{'}(0) \\ &- \frac{1}{RL}\int_{-h}^{\min\{t-h,0\}} \mathbb{E}^{w,\delta,\delta}_{\eta,\alpha_1,\alpha_1-\alpha_2,\alpha_1}\left(-\frac{1}{RL}, -\frac{R}{L}; t-h-s\right)\psi(s)ds. \end{aligned}$$

For common parameters $\alpha_1 = 1,4$, $\alpha_2 = 0.7$, $\eta = 1$, $\delta = 1$, $L = 2$, $R = 40$, $C = 16 \times 10^{-4}$, $\psi(t) = 2t^2$, $w = 1$, $h = 0.5$, $T = 2$, the graphs of the currents I for different frequencies $\theta = 5, 10, 25$ in $E(t) = 20\sin(\theta t)$ are plotted in Figure 2.

Remark 5.1 *In Figure 1, there are four components in an RLC series circuit: the resistance* (R)*, the inductance* (L)*, the capacitance* (C)*, and the voltage* (E)*. There are two types of voltages, alternating voltage and direct* *voltage. The alternating voltage changes the polarity of the connections at regular intervals, but the direct voltage does not. The alternating voltage E is chosen in this paper because it is more widely used in various applications of houses, office buildings, etc.*

Remark 5.2 *As seen in Figure 2, in the considered RLC circuit with the alternating voltage E, the current I is the alternating current which describes the flow of charge that changes direction periodically. The alternating current is of a sinusoidal structure. The corresponding peak currents for the alternating voltage $E(t) = 20\sin(\theta t)$ with $\theta = 5$, $\theta = 10$, and $\theta = 25$ in the discussed system are approximately equal to $I_{\theta=5} = 10$, $I_{\theta=10} = 7$, and $I_{\theta=25} = 4$, respectively. This shows that the increases in the frequencies of the system cause the decreases in the corresponding peak currents. It is also easily observed that the corresponding wavelengths of the currents decrease as the frequencies increase.*

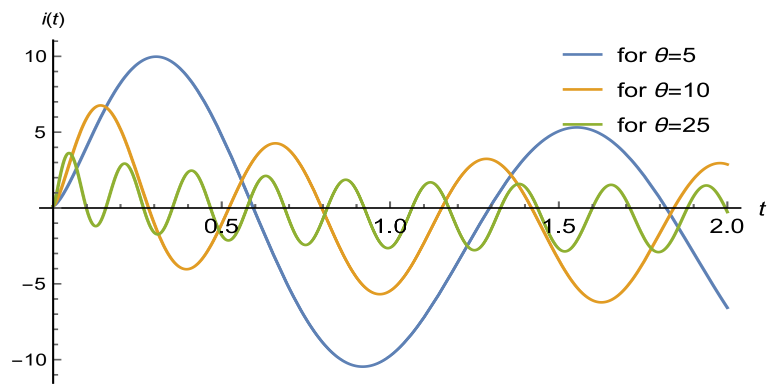

Figure 2. Graphs of the currents $I(t)$ for $\theta = 5, 10, 25$ in $E(t) = 20\sin(\theta t)$.

6. Conclusion

We introduce the Langevin delayed equations with Prabhakar derivatives involving two distinct general fractional orders and investigate its explicit solution using the Laplace transform. It is shown that the obtained solution satisfies the introduced system. A couple of special cases which are also new are offered. Lastly, we exemplify our theoretical results via RLC circuits.

As a next work, one can investigate not only the system's stabilities such as Lyapunov, finite-time, Ulam-Hyers stabilities, etc; but also the system's controllability such as approximate controllability, relative controllability, iterative learning controllability, etc.

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