Turkish Journal of Mathematics

Volume 48 | Number 2 | Article 4

3-8-2024

Twisted Sasaki metric on the unit tangent bundle and harmonicity

LIANA LOTARETS
lyanalotarets@gmail.com

Recommended Citation

LOTARETS, LIANA (2024) "Twisted Sasaki metric on the unit tangent bundle and harmonicity," *Turkish Journal of Mathematics*: Vol. 48: No. 2, Article 4. https://doi.org/10.55730/1300-0098.3498
Available at: https://journals.tubitak.gov.tr/math/vol48/iss2/4

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2024) 48: 123 – 143
© TÜBİTAK
doi:10.55730/1300-0098.3498

Twisted Sasaki metric on the unit tangent bundle and harmonicity

Liana LOTARETS*
Department of Pure Mathematics, School of Mathematics and Computer Sciences,
V. N. Karazin Kharkiv National University, Kharkiv, Ukraine

Received: 13.07.2023 • **Accepted/Published Online:** 01.12.2023 • **Final Version:** 08.03.2024

Abstract: The paper deals with the twisted Sasaki metric on the unit tangent bundle of n–dimensional Riemannian manifold M^n. The main purpose of the research is to find deformations that preserve the existence harmonic left-invariant unit vector fields on 3-dimensional unimodular Lie groups G with the left invariant metric and harmonic maps $G \to T_1G$ in case of twisted Sasaki metric on the unit tangent bundle. The necessary and sufficient conditions for harmonicity of left-invariant unit vector field and map $M^n \to T_1M^n$ are obtained. The necessary and sufficient conditions for harmonicity of left-invariant unit vector field and map $M^2 \to T_1M^2$ with respect to some orthonormal frame are obtained. Left-invariant harmonic unit vector fields and harmonic maps $G \to T_1G$, where G is a three-dimensional unimodular Lie group with left-invariant metric, using some orthonormal frame are described. Left-invariant harmonic unit vector fields which determine harmonic maps $G \to T_1G$, where G is a three-dimensional unimodular Lie group with left-invariant metric in the particular case of twisted Sasaki metric, namely the vertical rescaled metric are classified.

Key words: Twisted Sasaki metric, vertical rescaled metric, unit tangent bundle, Lie group, harmonic vector field, harmonic map

1. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold, TM^n be its tangent bundle, $\mathfrak{X}(M^n)$ be the Lie algebra of smooth vector fields of a Riemannian manifold (M^n, g), ∇ be the Levi-Civita connection on M^n. The standard metric on the tangent bundle of Riemannian manifold (M^n, g) is the Sasaki metric [17, 18]. It can be completely defined by scalar products of various combinations of vertical and horizontal lifts of vector fields. The Sasaki metric weakly inherits the base manifold properties [16]. That is why the rigidity of the Sasaki metric motivates many authors to consider various deformations of the Sasaki metric (see [1, 9, 10, 14, 20, 22] and others).

Belarbi L. and El Hendi H. introduce in [2] the twisted Sasaki metric on the tangent bundle TM as a new natural metric nonrigid on TM. The authors were motivated by the studies of Cheeger J. and Gromoll D. (see [7]), Dida H.M., Hathout F., Azzouz A. (see [8]), and others. The twisted Sasaki metric is defined as follows.

Definition 1.1 *[2] Let (M^n, g) be a Riemannian manifold and $\delta, \varepsilon : M^n \longrightarrow \mathbb{R}$ be strictly positive smooth*

*Correspondence: lyanalotarets@gmail.com
2010 *AMS Mathematics Subject Classification:* 53B20, 53B21, 53B99

functions. On the tangent bundle TM^n, we define a twisted Sasaki metric noted $G^{\delta,\varepsilon}$ by

$$G^{\delta,\varepsilon}_{(x,\xi)}(X^h,Y^h)=e^{\delta(x)}g_x(X,Y),$$
$$G^{\delta,\varepsilon}_{(x,\xi)}(X^h,Y^t)=0,$$
$$G^{\delta,\varepsilon}_{(x,\xi)}(X^t,Y^t)=e^{\varepsilon(x)}g_x(X,Y),$$

for all vector fields $X,Y\in\mathfrak{X}(M^n)$ and $(x,\xi)\in TM^n$.

Note that, if $\delta=\varepsilon=0$, then $G^{0,0}$ is the *Sasaki metric* [18]. If $\delta=0$, then $G^{0,\varepsilon}$ is the *vertical rescaled metric* (see [3, 7, 8]).

For a unit vector field ξ on a compact Riemannian manifold (M^n,g), Gerrit Weigmink [19] considered a very natural geometric functional

$$\int_M ||A_\xi||^2 dVol(M^n),$$

where $||A_\xi||$ is a norm of the Nomizu operator $A_\xi X=-\nabla_X\xi$, i.e. $||A_\xi||^2=\sum_{i=1}^n g(A_\xi e_i,A_\xi e_i)$ relative to some orthonormal frame $(e_1,\dots,e_n)$. It was proved, that this functional is unbounded from above. The critical points of this functional were called *harmonic unit vector fields* (see [5, 19] and others for more details). Gerrit Wigmink proved, that a unit vector field ξ on a compact Riemannian manifold is harmonic if and only if

$$\bar{\Delta}\xi=||A_\xi||^2\xi,$$

where $\bar{\Delta}\xi$ is rough Laplacian (or Bochner Laplacian) of the field ξ defined as $\bar{\Delta}\xi=-trace\nabla^2\xi$, where $\nabla^2_{X,Y}=\nabla_X\nabla_Y-\nabla_{\nabla_XY}$.

On the other hand (see [11]), the energy of a *map* $\varphi:(M^n,g)\to(N^k,h)$ between Riemannian manifolds is defined as

$$E(\varphi):=\frac{1}{2}\int_M|d\varphi|^2dVol_{M^n}.$$

The mapping φ *is called harmonic* if it the critical point of the energy functional. It was proved that the mapping φ is harmonic if and only if the divergence of its differential vanishes, or equivalently its tension field $\tau(\varphi)=div(d\varphi)$ vanishes identically, where $|d\varphi|$ is a norm of 1-form $d\varphi$ in the cotangent bundle T^*M^n. Supposing on T_1M^n the Sasaki metric g_S, a unit vector field ξ as a mapping $\xi:(M^n,g)\to(T_1M^n,g_S)$ *defines a harmonic map* if and only if it is *harmonic* and, in addition, $\sum_{i=1}^n R(\xi,A_\xi e_i)e_i=0$ relative to some othonormal frame $\{e_i\}_{i=1}^n$.

González-Dávila J.C. and Vanhecke L. completely described left-invariant harmonic unit vector fields ξ and ones that define a harmonic map on 3-dimensional Lie groups equipped with Sasaki metric on the unit tangent bundle T_1M^3 (see [12]). The authors describe and classify ones, using orthonormal frame and classification of Milnor J. in [15].

In the present research, we consider the twisted Sasaki metric on the unit tangent bundle T_1M^n of n-dimensional Riemannian manifold. A particular case of a twisted Sasaki metric, namely the vertical rescaled metric (if $\delta=0$) on the unit tangent bundle, requires special attention because such deformation is a more natural generalization of the Sasaki metric than the twisted Sasaki metric. Namely, geometrically, the vertical rescaled metric performs point-wise homothetic deformation in the fibers. The main purpose of the research is to find deformations that preserve the existence harmonic left-invariant unit vector fields ξ on 3-dimensional unimodular Lie groups G with the left invariant metric and harmonic maps $\xi : G \to T_1 G$ in case of twisted Sasaki metric on the unit tangent bundle $T_1 G$, using orthonormal frame and classification of Milnor J. in [15]. Moreover, we want to describe and classify ones according to the classification of González-Dávila J.C. and Vanhecke L. in [12] for vertical rescaled metric. As the main results,

- we obtain the necessary and sufficient conditions for harmonicity of unit vector field ξ and map $\xi\colon (M^n, g) \to (T_1 M^n, G^{\delta,\varepsilon})$ (Theorem 3.4);
- we obtain the necessary and sufficient conditions for harmonicity of unit vector field ξ and map ξ: $(M^2, g) \to (T_1 M^2, G^{\delta,\varepsilon})$ with respect to orthonormal frame $\{e_1, e_2\}$ on M^2 such as $e_1 = \xi$, $e_2 = \eta$, where $\eta \in \mathfrak{X}(M^2)$, $g(\eta,\eta) = 1$, $g(\xi,\eta) = 0$ (Theorem 4.1);
- we describe left-invariant harmonic unit vector fields ξ and harmonic maps $\xi\colon (G, g) \to (T_1 G, G^{\delta,\varepsilon})$, where G is a three-dimensional unimodular Lie group with left-invariant metric g, using orthonormal frame of Milnor J. [15] (Theorem 5.1);
- we classify left-invariant harmonic unit vector fields ξ which determine harmonic maps $\xi\colon (G, g) \to (T_1 G, G^{0,\varepsilon})$, where G is a three-dimensional unimodular Lie group with left-invariant metric g according to the classification of González-Dávila J.C. and Vanhecke L. [12] (Theorem 5.3–5.8).

2. Preliminaries

Let (M^n, g) be n-dimensional Riemannian manifold with metric g. Denote by $g(\cdot,\cdot)$ a scalar product with respect to g. Denote by TM^n tangent bundle of (M^n, g). It is well known that at each point $(x,\xi) \in TM^n$ the tangent space $T_{(x,\xi)}TM^n$ splits into *vertical* and *horizontal* parts:

$$T_{(x,\xi)}TM^n = \mathcal{H}_{(x,\xi)}TM^n \oplus \mathcal{V}_{(x,\xi)}TM^n.$$

The vertical part $\mathcal{V}_{(x,\xi)}$ is tangent to the fiber, while the horizontal part $\mathcal{H}_{(x,\xi)}$ is transversal to it. Denote by $(x^1, \dots, x^n; \xi^1, \dots, \xi^n)$ the natural induced local coordinate system on TM. Denote $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{n+i} = \frac{\partial}{\partial \xi^i}$. Then for $\tilde X \in T_{(x,\xi)}TM^n$ we have $\tilde X = \tilde X^i \partial_i + \tilde X^{n+i}\partial_{n+i}$.

Denote by $\pi : TM \to M^n$ the tangent bundle projection. The *mapping* $\pi_* : T_{(x,\xi)}TM^n \to TM^n$ acts on $\tilde X$ by $\pi_* \tilde X = \tilde X^i \partial_i$. The mapping π_* defines a point-wise linear isomorphism between $\mathcal{H}_{(x,\xi)}(TM^n)$ and $T_x M^n$. Remark that $\ker \pi_*|_{(x,\xi)} = \mathcal{V}_{(x,\xi)}$. The *connection mapping* $\mathcal{K} : T_{(x,\xi)}TM^n \to T_x M^n$ acts on $\tilde X$ by $\mathcal{K}\tilde X = (\tilde X^{n+i} + \Gamma^i_{jk}\xi^j \tilde X^k)\partial_i$, where Γ^i_{jk} are the *Christoffel symbols* of g. The connection mapping $\mathcal{K}$ defines a point-wise linear isomorphism between $\mathcal{V}_{(x,\xi)}(TM^n)$ and $T_x M^n$. Remark that $\ker \mathcal{K}|_{(x,\xi)} = \mathcal{H}_{(x,\xi)}$. The images $\pi_* \tilde X$ and $\mathcal{K}\tilde X$ are called *horizontal* and *vertical projections* of $\tilde X$, respectively. The operations inverse to projections are called *lifts* (see [4]). Namely, if $X \in T_x M$, then

$$X^h = X^i \partial_i - \Gamma^i_{jk}\xi^j X^k \partial_{n+i}$$

is in $\mathcal{H}_{(x,\xi)}TM^n$ and is called the *horizontal lift* of X, and

$$X^v = X^i \partial_{n+i}$$

is in $\mathcal{V}_{(x,\xi)}TM^n$ and is called the *vertical lift* of X.

Let $\tilde{X}, \tilde{Y} \in T_{(x,\xi)}TM^n$. The standard *Sasaki metric* on TM^n is defined at each point $(x,\xi) \in TM^n$ by the following scalar product

$$G(\tilde{X}, \tilde{Y})\big|_{(x,\xi)} = g(\pi_* \tilde{X}, \pi_* \tilde{Y})\big|_x + g(K\tilde{X}, K\tilde{Y})\big|_x.$$

Horizontal and vertical subspaces are mutually orthogonal with respect to Sasaki metric. The Sasaki metric can be completely defined by the scalar product of various combinations of lifts by

$$G_{(x,\xi)}(X^h, Y^h) = g_x(X, Y), \quad G_{(x,\xi)}(X^h, Y^v) = 0, \quad G_{(x,\xi)}(X^v, Y^v) = g_x(X, Y).$$

Let T_1M^n be unit tangent bundle of (M^n, g), $\mathfrak{X}(M^n)$ be the Lie algebra of smooth vector fields of a Riemannian manifold (M^n, g), ∇ be the *Levi-Civita connection* on M^n. Note that the vertical lift of a vector field is not tangent to T_1M^n in general. The lifted frame on T_1M^n at $(x,\xi) \in T_1M^n$ is formed by horizontal lift and the *tangential lift* (see [6] for more details) defined by

$$X^t = X^v - g(X,\xi)\xi^v.$$

Evidently, if X is orthogonal to ξ, then $X^t = X^v$. *Nomizu operator* $A_\xi : \mathfrak{X}(M^n) \to \xi^\perp \subset \mathfrak{X}(M^n)$ for unit smooth vector field ξ is defined by

$$A_\xi X = -\nabla_X \xi.$$

Note that *norma of* A_ξ is given by

$$||A_\xi||^2 = \sum_{k=1}^{n} g(A_\xi e_k, A_\xi e_k), \tag{2.1}$$

with respect to some orthonormal frame $\{e_k\}_{k=1}^n$. The *tangent mapping* $\xi_* : \mathfrak{X}(M^n) \to T\xi(M^n)$ is defined by

$$\xi_* X = X^h - (A_\xi X)^t. \tag{2.2}$$

The *rough Hessian* and the ξ*-harmonicity* tensor (see [21]) are given by

$$Hess_\xi(X,Y) = \frac{1}{2}((\nabla_X A_\xi)Y + (\nabla_Y A_\xi)X), \tag{2.3}$$

$$Hm_\xi(X,Y) = \frac{1}{2}(R(\xi, A_\xi X)Y + R(\xi, A_\xi Y)X), \tag{2.4}$$

where $(\nabla_X A_\xi)Y = \nabla_X(A_\xi Y) - A_\xi(\nabla_X Y)$ and R is the *curvature tensor* of the base manifold (M^n, g). The *rough Laplacian* of the field ξ is given by

$$\bar{\Delta}\xi = trace(Hess_\xi) = \sum_{i=1}^{n} (\nabla_{e_i} A_\xi) e_i, \tag{2.5}$$

with respect to some orthonormal frame $\{e_i\}_{i=1}^n$. The equivalent definition of the rough Laplacian of the field ξ is given by

$$\bar{\Delta}\xi = -trace\nabla^2\xi \tag{2.6}$$

where $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla_X Y}$. *Second fundamental form of the mapping* $\varphi : (M^n, g) \to (N^k, h)$ between Riemannian manifolds is defined as

$$B_\varphi(X, Y) = \nabla^\varphi_{\varphi_* X}(\varphi_* Y) - \varphi_*(\nabla^g_X Y), \tag{2.7}$$

where ∇^φ is induced Levi-Civita connection on $\varphi(M^n)$ and ∇^g is Levi-Civita connection on M^n. *Tension field* of the mapping $\xi\colon\ M^n \to T_1M^n$ is given by

$$\tau(\xi) = trace(B_\xi). \tag{2.8}$$

Denote $hB_\xi(X, Y) = \pi_*(B_\xi(X, Y))$ and $vB_\xi(X, Y) = \mathcal{K}(B_\xi(X, Y))$. Then

$$\tau(\xi) = (trace(hB_\xi))^h + (trace(vB_\xi))^t.$$

Unit vector field ξ is *harmonic* if and only if

$$\exists\lambda \in C^\infty(M^n) : trace(vB_\xi) = \lambda\xi. \tag{2.9}$$

Unit vector field ξ defines a *harmonic map* $\xi\colon\ M^n \to T_1M^n$ if and only if ξ is harmonic and

$$trace(hB_\xi) = 0. \tag{2.10}$$

3. Twisted Sasaki metric on unit tangent bundle

Definition 3.1 *Let (M^n, g) be a Riemannian manifold and $\delta, \varepsilon : M^n \longrightarrow \mathbb{R}$ be smooth functions. On the unit tangent bundle T_1M^n, we define a twisted Sasaki metric noted $G^{\delta,\varepsilon}$ by*

$$G^{\delta,\varepsilon}_{(x,\xi)}(X^h, Y^h) = e^{\delta(x)}g_x(X, Y),$$
$$G^{\delta,\varepsilon}_{(x,\xi)}(X^h, Y^t) = 0,$$
$$G^{\delta,\varepsilon}_{(x,\xi)}(X^t, Y^t) = e^{\varepsilon(x)}g_x(X, Y).$$

for all vector fields $X, Y \in \mathfrak{X}(M^n)$ and $(x, \xi) \in T_1M^n$, where $g(\xi, \xi) = 1$.

Note that, if $\delta = \varepsilon = 0$, then $G^{0,0}$ is the Sasaki metric on T_1M^n. If $\varepsilon = 0$, then $G^{\delta,0}$ is the rescaled Sasaki metric on T_1M^n. If $\delta = 0$, then $G^{0,\varepsilon}$ is the vertical rescaled metric on T_1M^n.

The following lemma contains Kowalski-type formulas [13].

Lemma 3.2 *Let (M^n, g) be the Riemannian manifold. The Levi-Civita connection $\tilde{\nabla}$ of the unit tangent* *bundle* T_1M^n *equipped with twisted Sasaki metric* $G^{\delta,\varepsilon}$ *is completely defined by*

$$\tilde{\nabla}_{X^h}Y^h = (\nabla_X Y + F_\delta(X,Y))^h - \frac{1}{2}\left(R(X,Y)\xi\right)^t, \tag{3.1}$$

$$\tilde{\nabla}_{X^h}Y^t = \frac{e^{\varepsilon-\delta}}{2}\left(R(\xi,Y)X\right)^h + \left(\nabla_X Y + \frac{X(\varepsilon)}{2}Y\right)^t, \tag{3.2}$$

$$\tilde{\nabla}_{X^t}Y^h = \frac{e^{\varepsilon-\delta}}{2}\left(R(\xi,X)Y\right)^h + \frac{Y(\varepsilon)}{2}X^t, \tag{3.3}$$

$$\tilde{\nabla}_{X^t}Y^t = -\frac{e^{\varepsilon-\delta}}{2}g(X,Y)(\nabla\varepsilon)^h - g(Y,\xi)X^t, \tag{3.4}$$

where ∇ *is the Levi-Civita connection on* (M^n, g)*,* R *is the curvature tensor of* ∇*, and*

$$F_\delta(X,Y) = \frac{1}{2}(X(\delta)Y + Y(\delta)X - g(X,Y)\nabla\delta).$$

Proof Remark, first, that the following formulas are independent on the choice of tangent bundle metric and are analogous to Dombrowski formulas [4] in terms of horizontal and tangential lifts: at each point $(x,\xi) \in T_1M^n$ the brackets of lifts are

$$[X^h, Y^h] = [X,Y]^h - (R(X,Y)\xi)^t, \qquad [X^h, Y^t] = (\nabla_X Y)^t,$$

$$[X^t, Y^t] = g(X,\xi)Y^t - g(Y,\xi)X^t.$$

Using Definition 3.1, note that derivative of twisted Sasaki metric $G^{\delta,\varepsilon}$ along the lifts of vector fields are

$$X^h G^{\delta,\varepsilon}(Y^h, Z^h) = e^\delta X(\delta)g(Y,Z) + e^\delta g(\nabla_X Y, Z) + e^\delta g(Y, \nabla_X Z),$$

$$X^h G^{\delta,\varepsilon}(Y^t, Z^t) = e^\varepsilon X(\varepsilon)g(Y,Z) + e^\varepsilon g(\nabla_X Y, Z) + e^\varepsilon g(Y, \nabla_X Z),$$

$$X^t G^{\delta,\varepsilon}(Y^h, Z^h) = X^t G^{\delta,\varepsilon}(Y^t, Z^t) = 0.$$

Now we can use Koszul formula. Prove formula (3.4).

$$\begin{aligned}
2G^{\delta,\varepsilon}(\tilde{\nabla}_{X^t}Y^t, Z^h) &= X^t G^{\delta,\varepsilon}(Y^t, Z^h) + Y^t G^{\delta,\varepsilon}(X^t, Z^h) - Z^h G^{\delta,\varepsilon}(X^t, Y^t) \\
&- G^{\delta,\varepsilon}(X^t, [Y^t, Z^h]) - G^{\delta,\varepsilon}(Y^t, [X^t, Z^h]) + G^{\delta,\varepsilon}(Z^h, [X^t, Y^t]) \\
&= -e^\varepsilon Z(\varepsilon)g(X,Y) = -e^\varepsilon g(\nabla\varepsilon, Z)g(X,Y) = g(-e^\varepsilon g(X,Y)\nabla\varepsilon, Z) \\
&= 2e^\delta g\left(-\frac{e^\varepsilon}{2e^\delta}g(X,Y)\nabla\varepsilon, Z\right) = 2G^{\delta,\varepsilon}\left(-\frac{e^{\varepsilon-\delta}}{2}g(X,Y)(\nabla\varepsilon)^h, Z^h\right); \\
2G^{\delta,\varepsilon}(\tilde{\nabla}_{X^t}Y^t, Z^t) &= X^t G^{\delta,\varepsilon}(Y^t, Z^t) + Y^t G^{\delta,\varepsilon}(X^t, Z^t) - Z^t G^{\delta,\varepsilon}(X^t, Y^t) \\
&- G^{\delta,\varepsilon}(X^t, [Y^t, Z^t]) - G^{\delta,\varepsilon}(Y^t, [X^t, Z^t]) + G^{\delta,\varepsilon}(Z^t, [X^t, Y^t]) \\
&= -e^\varepsilon g(Y,\xi)g(X,Z) + e^\varepsilon g(Z,\xi)g(X,Y) - e^\varepsilon g(X,\xi)g(Y,Z) \\
&+ e^\varepsilon g(Z,\xi)g(Y,X) + e^\varepsilon g(X,\xi)g(Z,Y) - e^\varepsilon g(Y,\xi)g(Z,X) \\
&= 2e^\varepsilon g(g(X,Y)\xi - g(Y,\xi)X, Z) = 2G^{\delta,\varepsilon}(-g(Y,\xi)X^t, Z^t).
\end{aligned}$$

Thus, we obtain (3.4). In a similar way we can obtain formulas (3.1)–(3.3). □

Theorem 3.3 *Let (M^n, g) be n-dimensional Riemannian manifold equipped with twisted Sasaki metric $G^{\delta,\varepsilon}$ on the unit tangent bundle T_1M^n. Tension field of the map $\xi\colon (M^n, g) \to (T_1M^n, G^{\delta,\varepsilon})$ is given by*

$$\tau(\xi) = -\frac{e^{\varepsilon-\delta}}{2}\Big(2\cdot trace(Hm_\xi) + (n-2)e^{\delta-\varepsilon}\nabla\delta + ||A_\xi||^2\nabla\varepsilon\Big)^h - \Big(\bar{\Delta}\xi + A_\xi(\nabla\varepsilon)\Big)^t.$$

Proof Consider some orthonormal frame $\{e_k\}_{k=1}^n$. According to (2.8), we have $\tau(\xi) = trace(B_\xi) = \sum_{k=1}^n B_\xi(e_k, e_k)$. At first, we find the second fundamental form of the mapping $\xi\colon M^n \to T_1M^n$. Substituting $\xi_*X = X^h - (A_\xi X)^t$ in $B_\xi(X,Y) = \tilde{\nabla}_{\xi_*X}(\xi_*Y) - \xi_*(\nabla_X Y)$, we get

$$\xi_*(\nabla_X Y) = (\nabla_X Y)^h - (A_\xi(\nabla_X Y))^t,$$

using (2.4), (2.3) and Lemma 3.2, we have

$$\begin{aligned}
\tilde{\nabla}_{\xi_*X}(\xi_*Y) &= \tilde{\nabla}_{X^h}Y^h - \tilde{\nabla}_{X^h}(A_\xi Y)^t - \tilde{\nabla}_{(A_\xi X)^t}Y^h + \tilde{\nabla}_{(A_\xi X)^t}(A_\xi Y)^t \\
&= \tilde{\nabla}_{X^h}Y^h + \tilde{\nabla}_{X^h}(\nabla_Y\xi)^t + \tilde{\nabla}_{(\nabla_X\xi)^t}Y^h + \tilde{\nabla}_{(\nabla_X\xi)^t}(\nabla_Y\xi)^t \\
&= \Big(\nabla_X Y + \frac{1}{2}(X(\delta)Y + Y(\delta)X - g(X,Y)\nabla\delta)\Big)^h - \frac{1}{2}(R(X,Y)\xi)^t + \frac{e^{\varepsilon-\delta}}{2}(R(\xi,\nabla_Y\xi)X)^h \\
&+ \Big(\nabla_X\nabla_Y\xi + \frac{X(\varepsilon)}{2}\nabla_Y\xi\Big)^t + \frac{e^{\varepsilon-\delta}}{2}(R(\xi,\nabla_X\xi)Y)^h + \frac{Y(\varepsilon)}{2}(\nabla_X\xi)^t - \frac{e^{\varepsilon-\delta}}{2}g(\nabla_X\xi,\nabla_Y\xi)(\nabla\varepsilon)^h \\
&= \Big(\nabla_X Y - e^{\varepsilon-\delta}Hm_\xi(X,Y) + \frac{1}{2}(X(\delta)Y + Y(\delta)X - g(X,Y)\nabla\delta - e^{\varepsilon-\delta}g(A_\xi X, A_\xi Y)\nabla\varepsilon)\Big)^h \\
&- \Big(A_\xi(\nabla_X Y) + Hess_\xi(X,Y) + \frac{1}{2}(Y(\varepsilon)A_\xi X + X(\varepsilon)A_\xi Y\Big)^t.
\end{aligned}$$

Thus, the second fundamental form of the mapping $\xi\colon M^n \to T_1M^n$ is defined as

$$\begin{aligned}
B_\xi(X,Y) = \Big(-e^{\varepsilon-\delta}Hm_\xi(X,Y) + \frac{1}{2}(X(\delta)Y + Y(\delta)X - g(X,Y)\nabla\delta - e^{\varepsilon-\delta}g(A_\xi X, A_\xi Y)\nabla\varepsilon)\Big)^h \\
- \Big(Hess_\xi(X,Y) + \frac{1}{2}(Y(\varepsilon)A_\xi X + X(\varepsilon)A_\xi Y)\Big)^t. \quad (3.5)
\end{aligned}$$

Substituting (3.5) in (2.8), using (2.5) and (2.1), we obtain

$$\begin{aligned}
\tau(\xi) = \sum_{k=1}^n B_\xi(e_k,e_k) &= \sum_{k=1}^n \Big(-e^{\varepsilon-\delta}Hm_\xi(e_k,e_k) + \frac{1}{2}(2e_k(\delta)e_k - \nabla\delta - e^{\varepsilon-\delta}g(A_\xi e_k, A_\xi e_k)\nabla\varepsilon)\Big)^h \\
&- \sum_{k=1}^n \Big(Hess_\xi(e_k,e_k) + A_\xi(e_k(\varepsilon)e_k)\Big)^t = \Big(-e^{\varepsilon-\delta}trace(Hm_\xi) + \frac{1}{2}(2\nabla\delta - n\nabla\delta - e^{\varepsilon-\delta}||A_\xi||^2\nabla\varepsilon)\Big)^h \\
&- \Big(\bar{\Delta}\xi + A_\xi(\nabla\varepsilon)\Big)^t = -\frac{e^{\varepsilon-\delta}}{2}\Big(2\cdot trace(Hm_\xi) + (n-2)e^{\delta-\varepsilon}\nabla\delta + ||A_\xi||^2\nabla\varepsilon\Big)^h - \Big(\bar{\Delta}\xi + A_\xi(\nabla\varepsilon)\Big)^t.
\end{aligned}$$

This completes the proof of Theorem 3.3. □

As a consequence of the Theorem 3.3, we have the following theorem.

Theorem 3.4 *Let* (M^n, g) *be n-dimensional Riemannian manifold equipped with twisted Sasaki metric* $G^{\delta,\varepsilon}$ *on the unit tangent bundle* T_1M^n. *Unit vector field* ξ *is harmonic on* (M^n, g) *if and only if*

$$\bar{\Delta}\xi + A_\xi(\nabla\varepsilon) = ||A_\xi||^2\xi. \tag{3.6}$$

Harmonic unit vector field ξ *defines a harmonic map* $\xi\colon (M^n, g) \to (T_1M^n, G^{\delta,\varepsilon})$ *if and only if*

$$2 \cdot trace(Hm_\xi) + (n-2)e^{\delta-\varepsilon}\nabla\delta + ||A_\xi||^2\nabla\varepsilon = 0. \tag{3.7}$$

Proof Using (2.9), we have that unit vector field ξ is harmonic if and only if

$$\exists\lambda \in C^\infty(M^n) : -\bar{\Delta}\xi - A_\xi(\nabla\varepsilon) = \lambda\xi.$$

Note that $\lambda = -g(\bar{\Delta}\xi, \xi)$, and, using (2.6), we get $\lambda = -||A_\xi||^2$. Thus, we obtain (3.6). Using (2.10), we obtain (3.7). □

Note, that if $\nabla_{e_i}\xi = 0$ for $i = \overline{1,n}$, then the Riemannian manifold (M^n, g) equipped with twisted Sasaki metric on the unit tangent bundle T_1M^n admits harmonic unit vector fields which determine the harmonic maps regardless of the deformation function $\varepsilon(x)$, if $\nabla\delta = 0$ or $n = 2$. On the contrary, if $\nabla\delta = 0$ or $n = 2$, and the Riemannian manifold (M^n, g) equipped with twisted Sasaki metric on the unit tangent bundle T_1M^n admits harmonic unit vector fields which determine the harmonic maps regardless of the deformation function $\varepsilon(x)$, then $\nabla_{e_i}\xi = 0$ for $i = \overline{1,n}$. Therefore, we have the following corollary.

Corollary 3.5 *The Riemannian manifold* (M^n, g) *equipped with twisted Sasaki metric* $G^{\delta,\varepsilon}$ *on the unit tangent bundle* T_1M^n *admits harmonic unit vector field* ξ *which determines the harmonic map regardless of the deformation function* $\varepsilon(x)$ *if and only if* $\nabla\delta = 0$ *or* $n = 2$, *and vector field* ξ *is parallel, that is* $M^n = M^{n-1}\times\mathbb{E}^1$.

4. Two-dimensional Riemannian manifolds

Consider orthonormal frame $\{e_1, e_2\}$ on two-dimensional Riemannian manifold M^2 is given by

$$\nabla_{e_1}e_1 = ke_2, \qquad \nabla_{e_1}e_2 = -ke_1, \qquad \nabla_{e_2}e_1 = -\varkappa e_2, \qquad \nabla_{e_2}e_2 = \varkappa e_1, \tag{4.1}$$

where k and $\varkappa$ are oriented geodesic curvatures of the integral curves of the fields e_1 and e_2, respectively. Then nonzero components of the curvature tensor are given by

$$R(e_1, e_2)e_2 = Ke_1, \qquad R(e_2, e_1)e_1 = Ke_2,$$

where K is Gaussian curvature of M^2,

$$K = e_1(\varkappa) + e_2(k) - k^2 - \varkappa^2.$$

Consider $e_1 = \xi$, $e_2 = \eta$, where $\eta \in \mathfrak{X}(M^2)$, $g(\eta, \eta) = 1$, $g(\xi, \eta) = 0$. Then

$$A_\xi e_1 = -ke_2, \qquad A_\xi e_2 = \varkappa e_2,$$

and, using (2.1), we have

$$||A_\xi||^2 = k^2 + \varkappa^2.$$

Theorem 4.1 *Let (M^2, g) be a two-dimensional Riemannian manifold equipped with twisted Sasaki metric $G^{\delta,\varepsilon}$ on the unit tangent bundle T_1M^2. Let ξ and η, where $\eta \in \mathfrak{X}(M^2)$, $g(\eta,\eta) = 1$, $g(\xi,\eta) = 0$ be orthonormal frame on M^2. Let k and $\varkappa$ be oriented geodesic curvatures of the integral curves of the fields ξ and η, respectively. Then unit vector field ξ is harmonic on (M^2, g) if and only if*

$$k\xi(\varepsilon) - \varkappa\eta(\varepsilon) = \eta(\varkappa) - \xi(k). \tag{4.2}$$

Unit vector field ξ defines a harmonic map $\xi\colon (M^2, g) \to (T_1M^2, G^{\delta,\varepsilon})$ if and only if

$$\begin{cases} k\xi(\varepsilon) - \varkappa\eta(\varepsilon) = 0, \\ \xi(k) - \eta(\varkappa) = 0. \end{cases} \tag{4.3}$$

Proof Using (2.3), (2.5) and orthonormal frame (4.1), we have

$$(\nabla_{e_1}A_\xi)e_1 = k^2e_1 - (e_1(k) + k\varkappa)e_2, \quad (\nabla_{e_2}A_\xi)e_2 = \varkappa^2e_1 + (e_2(\varkappa) + k\varkappa)e_2,$$

$$\bar{\Delta}\xi = (\nabla_{e_1}A_\xi)e_1 + (\nabla_{e_2}A_\xi)e_2 = (k^2 + \varkappa^2)e_1 + (e_2(\varkappa) - e_1(k))e_2,$$

$$A_\xi(\nabla\varepsilon) = (\varkappa e_2(\varepsilon) - ke_1(\varepsilon))e_2.$$

Using Theorem 3.4, we have

$$(k^2 + \varkappa^2)e_1 + (e_2(\varkappa) - e_1(k))e_2 + (\varkappa e_2(\varepsilon) - ke_1(\varepsilon))e_2 = (k^2 + \varkappa^2)e_1,$$

$$ke_1(\varepsilon) - \varkappa e_2(\varepsilon) = e_2(\varkappa) - e_1(k). \tag{4.4}$$

Using (2.4) and orthonormal frame (4.1), we have

$$trace(Hm_\xi) = K(\varkappa e_1 + ke_2).$$

Using Theorem 3.4, we have

$$2K(\varkappa e_1 + ke_2) + (k^2 + \varkappa^2)(e_1(\varepsilon)e_1 + e_2(\varepsilon)e_2) = 0,$$

$$(2K\varkappa + (k^2 + \varkappa^2)e_1(\varepsilon))e_1 + (2Kk + (k^2 + \varkappa^2)e_2(\varepsilon))e_2 = 0,$$

$$\begin{cases} 2K\varkappa + (k^2 + \varkappa^2)e_1(\varepsilon) = 0 \\ 2Kk + (k^2 + \varkappa^2)e_2(\varepsilon) = 0 \end{cases}.$$

If $k^2 + \varkappa^2 \neq 0$, then $e_1(\varepsilon) = -\frac{2K\varkappa}{k^2+\varkappa^2}$ and $e_2(\varepsilon) = -\frac{2Kk}{k^2+\varkappa^2}$. Substituting these ones in (4.4), we get

$$e_1(k) - e_2(\varkappa) = 0 \quad \text{for any } k \text{ and } \varkappa.$$

This completes the proof of Theorem. □

Example 4.2 *If integral curves of the field ξ are geodesic, i.e. $k = 0$, then unit vector field ξ is harmonic and defines a harmonic map $\xi\colon (M^2, g) \to (T_1M^2, G^{\delta,\varepsilon})$ if and only if integral curves of the field η are Darboux* *circles and deformation function* ε *does not depend on the points on integral curves of the field* η, *that is* $\eta(\varkappa) = 0$ *and* $\eta(\varepsilon) = 0$.

For example, let M^2 *be a surface of revolution with the first fundamental form* $ds^2 = du^2 + f(u)^2 dv^2$, *where* u *is the arclength parameter of a meridional section* $f(u)$. *Take* $\xi = \frac{\partial}{\partial u}$, *then* $\eta = \frac{1}{f}\frac{\partial}{\partial v}$ *and* $k = 0$, $\varkappa = -\frac{f'_u}{f}$, $\eta(\varkappa) = 0$. *Thus,* ξ *is harmonic and defines a harmonic map* $\xi\colon (M^2, g) \to (T_1M^2, G^{\delta,\varepsilon})$ *if and only if* $\varepsilon = \varepsilon(u)$.

Remark 4.3 *Using Corollary 3.5, two-dimensional Riemannian manifold* M^2 *admits harmonic unit vector fields* ξ *which determine harmonic maps* $\xi\colon (M^2, g) \to (T_1M^2, G^{\delta,\varepsilon})$ *regardless of the deformation function* $\varepsilon(x)$ *if* $k = \varkappa = 0$, *that is if* M^2 *is a flat manifold.*

5. Three-dimensional unimodular Lie groups

Let G be a three-dimensional unimodular Lie group, g be a left-invariant metric on G. Let $\{e_i\}_{i=1}^3$ be an orthonormal frame of the Lie algebra of a Lie group G satisfying [15]

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3, \tag{5.1}$$

where $\lambda_1, \lambda_2, \lambda_3$ are structure constants and $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Following [15] and according to the signs of $\lambda_1, \lambda_2, \lambda_3$, we have six kinds of Lie algebras as described in Table 1.

Denote connection numbers by $\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$. Consider any left-invariant unit vector field $\xi = \xi^1 e_1 + \xi^2 e_2 + \xi^3 e_3$. The matrix of the operator A_ξ has the form

$$A_\xi = \begin{pmatrix} 0 & -\mu_2\xi^3 & \mu_3\xi^2 \\ \mu_1\xi^3 & 0 & -\mu_3\xi^1 \\ -\mu_1\xi^2 & \mu_2\xi^1 & 0 \end{pmatrix}.$$

Therefore,

$$||A_\xi||^2 = (\mu_2^2 + \mu_3^2)(\xi^1)^2 + (\mu_1^2 + \mu_3^2)(\xi^2)^2 + (\mu_1^2 + \mu_2^2)(\xi^3)^2. \tag{5.2}$$

Using Theorem 3.4, we have the following theorem.

Theorem 5.1 *Let* G *be a three-dimensional unimodular Lie group with left-invariant metric* g *equipped with twisted Sasaki metric* $G^{\delta,\varepsilon}$ *on the unit tangent bundle* T_1G. *Left-invariant unit vector fields* ξ *is harmonic on* (G, g) *with respect to orthonormal frame of the Lie algebra of a Lie group* G *satisfying* (5.1) *if and only if*

$$\begin{cases} \xi^2\mu_3 e_3(\varepsilon) - \xi^3\mu_2 e_2(\varepsilon) = ((\mu_1^2 - \mu_2^2)(\xi^2)^2 + (\mu_1^2 - \mu_3^2)(\xi^3)^2)\xi^1 \\ \xi^3\mu_1 e_1(\varepsilon) - \xi^1\mu_3 e_3(\varepsilon) = ((\mu_2^2 - \mu_3^2)(\xi^3)^2 + (\mu_2^2 - \mu_1^2)(\xi^1)^2)\xi^2 \\ \xi^1\mu_2 e_2(\varepsilon) - \xi^2\mu_1 e_1(\varepsilon) = ((\mu_3^2 - \mu_1^2)(\xi^1)^2 + (\mu_3^2 - \mu_2^2)(\xi^2)^2)\xi^3 \end{cases}. \tag{5.3}$$

Harmonic left-invariant unit vector field ξ defines a harmonic map $\xi\colon (G,g) \to (T_1G, G^{\delta,\varepsilon})$ with respect to orthonormal frame of the Lie algebra of a Lie group G satisfying (5.1) if and only if

$$\begin{cases} e^{\delta-\varepsilon}e_1(\delta) + ||A_\xi||^2 e_1(\varepsilon) = 2\xi^2\xi^3(\mu_2-\mu_3)\sigma_{23} \\ e^{\delta-\varepsilon}e_2(\delta) + ||A_\xi||^2 e_2(\varepsilon) = 2\xi^3\xi^1(\mu_3-\mu_1)\sigma_{31} \\ e^{\delta-\varepsilon}e_3(\delta) + ||A_\xi||^2 e_3(\varepsilon) = 2\xi^1\xi^2(\mu_1-\mu_2)\sigma_{12} \end{cases}, \tag{5.4}$$

where $\sigma_{ij} = \mu_i\mu_k + \mu_j\mu_k - \mu_i\mu_j$ and (5.2).

Proof Using (2.3), (2.5) and orthonormal frame of the Lie algebra of a Lie group G satisfying (5.1), we have

$$\begin{aligned} \bar{\Delta}\xi &= \sum_{k=1}^{3} Hess_\xi(e_k,e_k) = \sum_{k=1}^{3}(\nabla_{e_k}A_\xi)e_k = \mu_1^2(\xi^2e_2+\xi^3e_3) + \mu_2^2(\xi^1e_1+\xi^3e_3) + \mu_3^2(\xi^1e_1+\xi^3e_3) \\ &= \xi^1(\mu_2^2+\mu_3^2)e_1 + \xi^2(\mu_1^2+\mu_3^2)e_2 + \xi^3(\mu_1^2+\mu_2^2)e_3, \\ A_\xi(\nabla\varepsilon) &= (\xi^2\mu_3e_3(\varepsilon) - \xi^3\mu_2e_2(\varepsilon))e_1 + (\xi^3\mu_1e_1(\varepsilon) - \xi^1\mu_3e_3(\varepsilon))e_2 + (\xi^1\mu_2e_2(\varepsilon) - \xi^2\mu_1e_1(\varepsilon))e_3. \end{aligned}$$

Using Theorem 3.4, we have

$$\begin{aligned} &\xi^1(\mu_2^2+\mu_3^2)e_1 + \xi^2(\mu_1^2+\mu_3^2)e_2 + \xi^3(\mu_1^2+\mu_2^2)e_3 + (\xi^2\mu_3e_3(\varepsilon) - \xi^3\mu_2e_2(\varepsilon))e_1 \\ &+ (\xi^3\mu_1e_1(\varepsilon) - \xi^1\mu_3e_3(\varepsilon))e_2 + (\xi^1\mu_2e_2(\varepsilon) - \xi^2\mu_1e_1(\varepsilon))e_3 = ||A_\xi||^2(\xi^1e_1+\xi^2e_2+\xi^3e_3), \end{aligned}$$

$$\begin{cases} \xi^2\mu_3e_3(\varepsilon) - \xi^3\mu_2e_2(\varepsilon) = (||A_\xi||^2 - (\mu_2^2+\mu_3^2))\xi^1 \\ \xi^3\mu_1e_1(\varepsilon) - \xi^1\mu_3e_3(\varepsilon) = (||A_\xi||^2 - (\mu_1^2+\mu_3^2))\xi^2 \\ \xi^1\mu_2e_2(\varepsilon) - \xi^2\mu_1e_1(\varepsilon) = (||A_\xi||^2 - (\mu_1^2+\mu_2^2))\xi^3 \end{cases},$$

where $||A_\xi||^2 - (\mu_j^2+\mu_k^2) = (\mu_i^2-\mu_j^2)(\xi^j)^2 + (\mu_i^2-\mu_k^2)(\xi^k)^2$, because of (5.2), and we obtain the system (5.3). Using (2.4) and orthonormal frame of the Lie algebra of a Lie group G satisfying (5.1), we have

$$trace(Hm_\xi) = \xi^2\xi^3(\mu_3-\mu_2)\sigma_{23}e_1 + \xi^3\xi^1(\mu_1-\mu_3)\sigma_{31}e_2 + \xi^1\xi^2(\mu_2-\mu_1)\sigma_{12}e_3.$$

Using Theorem 3.4, we have

$$\begin{aligned} &2(\xi^2\xi^3(\mu_3-\mu_2)\sigma_{23}e_1 + \xi^3\xi^1(\mu_1-\mu_3)\sigma_{31}e_2 + \xi^1\xi^2(\mu_2-\mu_1)\sigma_{12}e_3) \\ &+ e^{\delta-\varepsilon}(e_1(\delta)e_1 + e_2(\delta)e_2 + e_3(\delta)e_3) + ||A_\xi||^2(e_1(\varepsilon)e_1 + e_2(\varepsilon)e_2 + e_3(\varepsilon)e_3) = 0, \end{aligned}$$

$$\begin{cases} 2\xi^2\xi^3(\mu_3-\mu_2)\sigma_{23} + e^{\delta-\varepsilon}e_1(\delta) + ||A_\xi||^2e_1(\varepsilon) = 0 \\ 2\xi^3\xi^1(\mu_1-\mu_3)\sigma_{31} + e^{\delta-\varepsilon}e_2(\delta) + ||A_\xi||^2e_2(\varepsilon) = 0 \\ 2\xi^1\xi^2(\mu_2-\mu_1)\sigma_{12} + e^{\delta-\varepsilon}e_3(\delta) + ||A_\xi||^2e_3(\varepsilon) = 0 \end{cases},$$

and we obtain the system (5.4). This completes the proof of the theorem. □

Note that for Sasaki metric $G^{0,0}$, we have $e_1(\delta) = e_2(\delta) = e_3(\delta) = 0$ and $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$. But if on the contrary, $e_1(\delta) = e_2(\delta) = e_3(\delta) = 0$ and $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$, then Theorem 5.1 is also held for all left-invariant harmonic unit vector fields ξ and harmonic map $\xi\colon (G,g) \to (T_1G, G^{0,0})$ on three-dimensional unimodular Lie groups G equipped with Sasaki metric $G^{0,0}$ on the unit tangent bundle (see Table 2). Moreover, we obtain the following corollary.

Corollary 5.2 *Let G be three-dimensional unimodular Lie group with left-invariant metric g equipped with twisted Sasaki metric $G^{\delta,\varepsilon}$ (Sasaki metric $G^{0,0}$ for $\delta=\varepsilon=0$) on the unit tangent bundle T_1G. Let ξ be a left-invariant harmonic unit vector field on (G,g) equipped $G^{0,0}$ on T_1G with respect to orthonormal frame $\{e_i\}_{i=1}^3$ of the Lie algebra of a Lie group G satisfying (5.1). Then twisted Sasaki metric $G^{\delta,\varepsilon}$ preserves the property of harmonicity of the vector field ξ if $e_1(\varepsilon)=e_2(\varepsilon)=e_3(\varepsilon)=0$. Moreover, let ξ define also a harmonic map $\xi\colon (G,g)\to(T_1G,G^{0,0})$. Then the map $\xi\colon (G,g)\to(T_1G,G^{\delta,\varepsilon})$ is also harmonic if $e_1(\delta)=e_2(\delta)=e_3(\delta)=0$ and $e_1(\varepsilon)=e_2(\varepsilon)=e_3(\varepsilon)=0$.*

Consider each three-dimensional unimodular Lie group separately in more detail for vertical rescaled metric $G^{0,\varepsilon}$, namely, find out when the group G admits left-invariant harmonic unit vector fields ξ which determine harmonic maps $\xi\colon (G,g)\to(T_1G,G^{0,\varepsilon})$ with respect to orthonormal frame of the Lie algebra of a Lie group G satisfying (5.1).

Theorem 5.3 *Let the groups $SU(2)$ and $SO(3)$ equip with vertical rescaled metric $G^{0,\varepsilon}$ on the unit tangent bundle T_1G, where $G=SU(2)$ or $SO(3)$ with left-invariant metric g. Then the groups $SU(2)$ and $SO(3)$ admit left-invariant harmonic unit vector fields ξ which determine harmonic maps $\xi\colon (G,g)\to(T_1G,G^{0,\varepsilon})$ with respect to orthonormal frame of the Lie algebra of a Lie group G satisfying (5.1) if and only if one of the following cases is held.*

- $\lambda_1=\lambda_2=\lambda_3$, $e_1(\varepsilon)=e_2(\varepsilon)=e_3(\varepsilon)=0$, $\xi\in\mathbb{S}$;
- $\lambda_1>\lambda_2=\lambda_3$, $e_1(\varepsilon)=e_2(\varepsilon)=e_3(\varepsilon)=0$, $\xi\in\{\pm e_1\}\cup(\mathbb{S}\cap\{e_2,e_3\}_{\mathbb{R}})$;
- $\lambda_1>\lambda_2=\lambda_3$, $\mu_1>0$, $\mu^4-2\mu_1\mu^3+\mu_1^4>0$, $e_1(\varepsilon)=0$, $e_2(\varepsilon)=const$, $e_3(\varepsilon)=const$, $e_2(\varepsilon)^2+e_3(\varepsilon)^2=\frac{2\mu_1(\mu-\mu_1)(\mu^4-2\mu_1\mu^3+\mu_1^4)}{(\mu^2-\mu_1^2)^2}$, $\xi=\pm\frac{\sqrt{\mu^4-2\mu_1\mu^3+\mu_1^4}}{\mu^2-\mu_1^2}e_1\mp\frac{e_3(\varepsilon)\mu\sqrt{\mu^4-2\mu_1\mu^3+\mu_1^4}}{\mu^4-2\mu_1\mu^3+\mu_1^4}e_2\pm\frac{e_2(\varepsilon)\mu\sqrt{\mu^4-2\mu_1\mu^3+\mu_1^4}}{\mu^4-2\mu_1\mu^3+\mu_1^4}e_3$, where $\mu=\mu_2=\mu_3$;
- $\lambda_1=\lambda_2>\lambda_3$, $e_1(\varepsilon)=e_2(\varepsilon)=e_3(\varepsilon)=0$, $\xi\in\{\pm e_3\}\cup(\mathbb{S}\cap\{e_1,e_2\}_{\mathbb{R}})$;
- $\lambda_1>\lambda_2>\lambda_3$, $e_1(\varepsilon)=e_2(\varepsilon)=e_3(\varepsilon)=0$, $\xi\in\{\pm e_1,\pm e_2,\pm e_3\}$;
- $\lambda_1>\lambda_2>\lambda_3$, $\mu_1\sigma_{23}>0$, $p_{1,23},q_{1,23}>0$, $e_2(\varepsilon)=e_3(\varepsilon)=0$, $e_1(\varepsilon)=\frac{\sqrt{p_{1,23}q_{1,23}}}{\mu_1(\mu_3^2-\mu_2^2)}$, $\xi=\pm\frac{\sqrt{p_{1,23}}e_2+\sqrt{q_{1,23}}e_3}{\mu_3^2-\mu_2^2}$;
- $\lambda_1>\lambda_2>\lambda_3$, $\mu_1\sigma_{23}>0$, $p_{1,23},q_{1,23}>0$, $e_2(\varepsilon)=e_3(\varepsilon)=0$, $e_1(\varepsilon)=-\frac{\sqrt{p_{1,23}q_{1,23}}}{\mu_1(\mu_3^2-\mu_2^2)}$, $\xi=\pm\frac{\sqrt{p_{1,23}}e_2-\sqrt{q_{1,23}}e_3}{\mu_3^2-\mu_2^2}$;
- $\lambda_1>\lambda_2>\lambda_3$, $\sigma_{31}>0$, $p_{2,31},q_{2,31}>0$, $e_1(\varepsilon)=e_3(\varepsilon)=0$, $e_2(\varepsilon)=\frac{\sqrt{p_{2,31}q_{2,31}}}{\mu_2(\mu_3^2-\mu_1^2)}$, $\xi=\pm\frac{\sqrt{q_{2,31}}e_1+\sqrt{p_{2,31}}e_3}{\mu_3^2-\mu_1^2}$;
- $\lambda_1>\lambda_2>\lambda_3$, $\sigma_{31}>0$, $p_{2,31},q_{2,31}>0$, $e_1(\varepsilon)=e_3(\varepsilon)=0$, $e_2(\varepsilon)=-\frac{\sqrt{p_{2,31}q_{2,31}}}{\mu_2(\mu_3^2-\mu_1^2)}$, $\xi=\pm\frac{\sqrt{q_{2,31}}e_1-\sqrt{p_{2,31}}e_3}{\mu_3^2-\mu_1^2}$;
- $\lambda_1>\lambda_2>\lambda_3$, $\sigma_{12}>0$, $p_{3,12},q_{3,12}>0$, $e_1(\varepsilon)=e_2(\varepsilon)=0$, $e_3(\varepsilon)=\frac{\sqrt{p_{3,12}q_{3,12}}}{\mu_3(\mu_2^2-\mu_1^2)}$, $\xi=\pm\frac{\sqrt{p_{3,12}}e_1+\sqrt{q_{3,12}}e_2}{\mu_2^2-\mu_1^2}$;
- $\lambda_1>\lambda_2>\lambda_3$, $\sigma_{12}>0$, $p_{3,12},q_{3,12}>0$, $e_1(\varepsilon)=e_2(\varepsilon)=0$, $e_3(\varepsilon)=-\frac{\sqrt{p_{3,12}q_{3,12}}}{\mu_3(\mu_2^2-\mu_1^2)}$, $\xi=\pm\frac{\sqrt{p_{3,12}}e_1-\sqrt{q_{3,12}}e_2}{\mu_2^2-\mu_1^2}$;

where $p_{i,jk}=(\mu_i^2-\mu_j^2)(\mu_k^2-\mu_j^2)-2\mu_i\mu_j\mu_k(\mu_k-\mu_j)$, $q_{i,jk}=(\mu_k^2-\mu_i^2)(\mu_k^2-\mu_j^2)+2\mu_i\mu_j\mu_k(\mu_k-\mu_j)$, $i,j,k=1,2,3$.

Proof According to Table 1, we have $\lambda_1>0$, $\lambda_2>0$, $\lambda_3>0$, then $\mu_2>0$, $\mu_3>0$, $\mu_1\leq\mu_2\leq\mu_3$. Note that, using (5.2), we have $||A_\xi||\neq 0$. Consider the systems (5.3) and (5.4).

Let $\lambda_1=\lambda_2=\lambda_3$, that is, $\mu_1=\mu_2=\mu_3\neq 0$. Then $e_1(\varepsilon)=e_2(\varepsilon)=e_3(\varepsilon)=0$ and $\xi\in\mathbb{S}$.

Let $\lambda_1>\lambda_2=\lambda_3$, that is $\mu_1<\mu_2=\mu_3$. Denote $\mu=\mu_2=\mu_3$. Using the system (5.4), we have

$$e_1(\varepsilon)=0,\quad e_2(\varepsilon)=\frac{2\xi^3\xi^1(\mu-\mu_1)\mu^2}{||A_\xi||^2},\quad e_3(\varepsilon)=\frac{2\xi^1\xi^2(\mu_1-\mu)\mu^2}{||A_\xi||^2}.$$

Hence, $e_2(\varepsilon)=const$ and $e_3(\varepsilon)=const$. Using the system (5.3), we have

$$||A_\xi||^2(\xi^1)^2\xi^3=\frac{2\mu^3}{\mu_1+\mu}(\xi^1)^2\xi^3,$$
$$||A_\xi||^2(\xi^1)^2\xi^2=\frac{2\mu^3}{\mu_1+\mu}(\xi^1)^2\xi^2.$$

If $\xi\in\{\pm e_1\}\cup(\mathbb{S}\cap\{e_2,e_3\}_{\mathbb{R}})$, then $e_1(\varepsilon)=e_2(\varepsilon)=e_3(\varepsilon)=0$. Let $\xi^1\neq 0$ and $(\xi^2)^2+(\xi^3)^2\neq 0$, then

$$||A_\xi||^2=\frac{2\mu^3}{\mu_1+\mu}.$$

On the other hand, using (5.2), we have $||A_\xi||^2=\mu_1^2+\mu^2+(\mu^2-\mu_1^2)(\xi^1)^2$. Therefore, if $\mu_1>0$ and $\mu^4-2\mu_1\mu^3+\mu_1^4>0$, then

$$\xi^1=\pm\frac{\sqrt{\mu^4-2\mu_1\mu^3+\mu_1^4}}{\mu^2-\mu_1^2},$$

$$\xi^2=\mp\frac{e_3(\varepsilon)\mu\sqrt{\mu^4-2\mu_1\mu^3+\mu_1^4}}{\mu^4-2\mu_1\mu^3+\mu_1^4}\quad \xi^3=\pm\frac{e_2(\varepsilon)\mu\sqrt{\mu^4-2\mu_1\mu^3+\mu_1^4}}{\mu^4-2\mu_1\mu^3+\mu_1^4},$$

where

$$e_2(\varepsilon)^2+e_3(\varepsilon)^2=\frac{2\mu_1(\mu-\mu_1)(\mu^4-2\mu_1\mu^3+\mu_1^4)}{(\mu^2-\mu_1^2)^2}.$$

Let $\lambda_1=\lambda_2>\lambda_3$, that is, $\mu_1=\mu_2<\mu_3$. Denote $\mu=\mu_1=\mu_2>0$. In similar way, we get that if $\xi\in\{\pm e_3\}\cup(\mathbb{S}\cap\{e_1,e_2\}_{\mathbb{R}})$, then $e_1(\varepsilon)=e_2(\varepsilon)=e_3(\varepsilon)=0$. Also, if $\xi^3\neq 0$ and $(\xi^1)^2+(\xi^2)^2\neq 0$, then $||A_\xi||^2=\frac{2\mu^3}{\mu_3+\mu}$. However, on the other hand, using (5.2), we have $||A_\xi||^2=(\mu_3^2-\mu^2)((\xi^1)^2+(\xi^2)^2)+2\mu^2$. Therefore,

$$(\xi^1)^2+(\xi^2)^2=-\frac{2\mu^2\mu_3(\mu_3-\mu)}{(\mu_3^2-\mu^2)^2}<0,$$

but it is impossible.

Let $\lambda_1>\lambda_2>\lambda_3$, that is, $\mu_1<\mu_2<\mu_3$. Using the system (5.4), we have

$$e_1(\varepsilon)=\frac{2\xi^2\xi^3(\mu_2-\mu_3)\sigma_{23}}{||A_\xi||^2},$$

$$e_2(\varepsilon) = \frac{2\xi^3\xi^1(\mu_3-\mu_1)\sigma_{31}}{||A_\xi||^2}, \quad e_3(\varepsilon) = \frac{2\xi^1\xi^2(\mu_1-\mu_2)\sigma_{12}}{||A_\xi||^2}.$$

Hence, $e_1(\varepsilon) = const$, $e_2(\varepsilon) = const$ and $e_3(\varepsilon) = const$. If $\xi \in \{\pm e_1, \pm e_2, \pm e_3\}$, then $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$. Let $\xi^1 = 0$ and $\xi^2, \xi^3 \neq 0$, then $e_2(\varepsilon) = e_3(\varepsilon) = 0$ and, using the system (5.3), we have

$$||A_\xi||^2 = \frac{2\mu_1\sigma_{23}}{\mu_2+\mu_3}.$$

Hence, if $\mu_1\sigma_{23} > 0$, then

$$e_1(\varepsilon) = \frac{\mu_2^2-\mu_3^2}{\mu_1}\xi^2\xi^3.$$

On the other hand, using (5.2), we have $||A_\xi||^2 = (\mu_1^2+\mu_3^2)(\xi^2)^2 + (\mu_1^2+\mu_2^2)(\xi^3)^2$. Denote $p_{1,23} = (\mu_1^2-\mu_2^2)(\mu_3^2-\mu_2^2) - 2\mu_1\mu_2\mu_3(\mu_3-\mu_2)$ and $q_{1,23} = (\mu_3^2-\mu_1^2)(\mu_3^2-\mu_2^2) + 2\mu_1\mu_2\mu_3(\mu_3-\mu_2)$. Therefore, if $p_{1,23}, q_{1,23} > 0$, then

$$e_1(\varepsilon) = \pm\frac{\sqrt{p_{1,23}q_{1,23}}}{\mu_1(\mu_3^2-\mu_2^2)},$$

$$\xi \in \left\{\frac{\sqrt{p_{1,23}}e_2 \pm \sqrt{q_{1,23}}e_3}{\mu_3^2-\mu_2^2}, -\frac{\sqrt{p_{1,23}}e_2 \pm \sqrt{q_{1,23}}e_3}{\mu_3^2-\mu_2^2}\right\}.$$

Note that we can obtain similar results for $\xi^2 = 0$, $\xi^1, \xi^3 \neq 0$ and $\xi^3 = 0$, $\xi^1, \xi^2 \neq 0$. Now let $\xi^1, \xi^2, \xi^3 \neq 0$. Using the system (5.3), we have

$$\begin{cases} ((\mu_1^2-\mu_2^2)(\xi^2)^2 + (\mu_1^2-\mu_3^2)(\xi^3)^2)||A_\xi||^2 = 2(\xi^2)^2\mu_3(\mu_1-\mu_2)\sigma_{12} - 2(\xi^3)^2\mu_2(\mu_3-\mu_1)\sigma_{31} \\ ((\mu_2^2-\mu_3^2)(\xi^3)^2 + (\mu_2^2-\mu_1^2)(\xi^1)^2)||A_\xi||^2 = 2(\xi^3)^2\mu_1(\mu_2-\mu_3)\sigma_{23} - 2(\xi^1)^2\mu_3(\mu_1-\mu_2)\sigma_{12} \\ ((\mu_3^2-\mu_1^2)(\xi^1)^2 + (\mu_3^2-\mu_2^2)(\xi^2)^2)||A_\xi||^2 = 2(\xi^1)^2\mu_2(\mu_3-\mu_1)\sigma_{31} - 2(\xi^2)^2\mu_1(\mu_2-\mu_3)\sigma_{23} \end{cases}.$$

Note that

$$\mu_1(\mu_2-\mu_3)\sigma_{23} + \mu_2(\mu_3-\mu_1)\sigma_{31} + \mu_3(\mu_1-\mu_2)\sigma_{12} = 0, \tag{5.5}$$

because of $\sigma_{31} - \sigma_{12} = 2\mu_1(\mu_2-\mu_3)$, $\sigma_{12} - \sigma_{23} = 2\mu_2(\mu_3-\mu_1)$, $\sigma_{23} - \sigma_{31} = 2\mu_3(\mu_1-\mu_2)$. Multiply the third row by -1 and add to the second row of the system. Hence, using (5.5), we get $||A_\xi||^2 = \frac{2\mu_1\sigma_{23}}{\mu_2+\mu_3}$. However, in the similar way we can get $||A_\xi||^2 = \frac{2\mu_2\sigma_{31}}{\mu_3+\mu_1}$ and $||A_\xi||^2 = \frac{2\mu_3\sigma_{12}}{\mu_1+\mu_2}$. Then $\frac{\mu_1\sigma_{23}}{\mu_2+\mu_3} = \frac{\mu_2\sigma_{31}}{\mu_3+\mu_1} = \frac{\mu_3\sigma_{12}}{\mu_1+\mu_2}$, that is $\mu_1^2 - \frac{\mu_1\mu_2\mu_3}{\mu_2+\mu_3} = \mu_2^2 - \frac{\mu_1\mu_2\mu_3}{\mu_3+\mu_1} = \mu_3^2 - \frac{\mu_1\mu_2\mu_3}{\mu_1+\mu_2}$. Therefore,

$$\mu_1\mu_2\mu_3 = (\mu_2+\mu_3)(\mu_3+\mu_1)(\mu_1+\mu_2),$$

that is,

$$||A_\xi||^2 = -2(\mu_2\mu_3+\mu_3\mu_1+\mu_1\mu_2), \text{ where } \mu_2\mu_3+\mu_3\mu_1+\mu_1\mu_2 < 0.$$

Note that if $\mu_1 \leq 0$, then $\mu_1\mu_2\mu_3 \leq 0$, but $(\mu_2+\mu_3)(\mu_3+\mu_1)(\mu_1+\mu_2) > 0$. If $\mu_1 > 0$, then $\mu_2\mu_3+\mu_3\mu_1+\mu_1\mu_2 > 0$. Thus, if $\xi^1, \xi^2, \xi^3 \neq 0$, then there are no solutions of the systems. □

Theorem 5.4 *Let the groups $SL(2,\mathbb{R})$ and $O(1,2)$ equip with vertical rescaled metric $G^{0,\varepsilon}$ on the unit tangent bundle T_1G, where $G = SL(2,\mathbb{R})$ or $O(1,2)$ with left-invariant metric g. Then the groups $SL(2,\mathbb{R})$ and $O(1,2)$ admit left-invariant harmonic unit vector fields ξ which determine harmonic maps $\xi\colon (G,g) \to (T_1G, G^{0,\varepsilon})$ with respect to the orthonormal frame of the Lie algebra of a Lie group G satisfying (5.1) if and only if one of the following cases is held.*

- $\lambda_1 = \lambda_2$, $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$, $\xi \in \{\pm e_3\} \cup (\mathbb{S} \cap \{e_1, e_2\}_{\mathbb{R}})$;
- $\lambda_1 > \lambda_2$, $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$, $\xi \in \{\pm e_1, \pm e_2, \pm e_3\}$;
- $\lambda_1 > \lambda_2$, $\sigma_{23} < 0$, $p_{1,23}, q_{1,23} > 0$, $e_2(\varepsilon) = e_3(\varepsilon) = 0$, $e_1(\varepsilon) = \frac{\sqrt{p_{1,23}q_{1,23}}}{\mu_1(\mu_3^2-\mu_2^2)}$, $\xi = \pm\frac{\sqrt{p_{1,23}}e_2+\sqrt{q_{1,23}}e_3}{\mu_3^2-\mu_2^2}$;
- $\lambda_1 > \lambda_2$, $\sigma_{23} < 0$, $p_{1,23}, q_{1,23} > 0$, $e_2(\varepsilon) = e_3(\varepsilon) = 0$, $e_1(\varepsilon) = -\frac{\sqrt{p_{1,23}q_{1,23}}}{\mu_1(\mu_3^2-\mu_2^2)}$, $\xi = \pm\frac{\sqrt{p_{1,23}}e_2-\sqrt{q_{1,23}}e_3}{\mu_3^2-\mu_2^2}$;
- $\lambda_1 > \lambda_2$, $\mu_2\sigma_{31} > 0$, $p_{2,31}, q_{2,31} > 0$, $e_1(\varepsilon) = e_3(\varepsilon) = 0$, $e_2(\varepsilon) = \frac{\sqrt{p_{2,31}q_{2,31}}}{\mu_2(\mu_3^2-\mu_1^2)}$, $\xi = \pm\frac{\sqrt{q_{2,31}}e_1+\sqrt{p_{2,31}}e_3}{\mu_3^2-\mu_1^2}$;
- $\lambda_1 > \lambda_2$, $\mu_2\sigma_{31} > 0$, $p_{2,31}, q_{2,31} > 0$, $e_1(\varepsilon) = e_3(\varepsilon) = 0$, $e_2(\varepsilon) = -\frac{\sqrt{p_{2,31}q_{2,31}}}{\mu_2(\mu_3^2-\mu_1^2)}$, $\xi = \pm\frac{\sqrt{q_{2,31}}e_1-\sqrt{p_{2,31}}e_3}{\mu_3^2-\mu_1^2}$;
- $\lambda_1 > \lambda_2$, $\sigma_{12} < 0$, $p_{3,12}, q_{3,12} > 0$, $e_1(\varepsilon) = e_2(\varepsilon) = 0$, $e_3(\varepsilon) = \frac{\sqrt{p_{3,12}q_{3,12}}}{\mu_3(\mu_2^2-\mu_1^2)}$, $\xi = \pm\frac{\sqrt{p_{3,12}}e_1+\sqrt{q_{3,12}}e_2}{\mu_2^2-\mu_1^2}$;
- $\lambda_1 > \lambda_2$, $\sigma_{12} < 0$, $p_{3,12}, q_{3,12} > 0$, $e_1(\varepsilon) = e_2(\varepsilon) = 0$, $e_3(\varepsilon) = -\frac{\sqrt{p_{3,12}q_{3,12}}}{\mu_3(\mu_2^2-\mu_1^2)}$, $\xi = \pm\frac{\sqrt{p_{3,12}}e_1-\sqrt{q_{3,12}}e_2}{\mu_2^2-\mu_1^2}$;
- $\lambda_1 > \lambda_2$, $\mu_2 > 0$, $e_1(\varepsilon) = const$, $e_2(\varepsilon) = const$, $e_3(\varepsilon) = const$,

$$\begin{cases} \mu_1^2(\gamma_2\gamma_3)^2 + \mu_2^2(\gamma_3\gamma_1)^2 + \mu_3^2(\gamma_1\gamma_2)^2 = (\mu_1+\mu_2+\mu_3)^2\gamma_1\gamma_2\gamma_3 \\ (\gamma_2\gamma_3)^2 + (\gamma_3\gamma_1)^2 + (\gamma_1\gamma_2)^2 = \gamma_1\gamma_2\gamma_3 \end{cases},$$

$\xi = \xi^1 e_1 + \xi^2 e_2 + \xi^3 e_3$, *where* $(\xi^1)^2 = \frac{\gamma_2\gamma_3}{\gamma_1}$, $(\xi^2)^2 = \frac{\gamma_3\gamma_1}{\gamma_2}$, $(\xi^3)^2 = \frac{\gamma_1\gamma_2}{\gamma_3}$, $\gamma_1 = \frac{\mu_1 e_1(\varepsilon)}{\mu_2^2-\mu_3^2}$, $\gamma_2 = \frac{\mu_2 e_2(\varepsilon)}{\mu_3^2-\mu_1^2}$, $\gamma_3 = \frac{\mu_3 e_3(\varepsilon)}{\mu_1^2-\mu_2^2}$;

where $p_{i,jk} = (\mu_i^2 - \mu_j^2)(\mu_k^2 - \mu_j^2) - 2\mu_i\mu_j\mu_k(\mu_k - \mu_j)$, $q_{i,jk} = (\mu_k^2 - \mu_i^2)(\mu_k^2 - \mu_j^2) + 2\mu_i\mu_j\mu_k(\mu_k - \mu_j)$.

Proof According to Table 1, we have $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, then $\mu_1 < 0$, $\mu_3 > 0$, $\mu_1 \le \mu_2 < \mu_3$. Note that, using (5.2), we have $||A_\xi|| \neq 0$. Consider the systems (5.3) and (5.4).

Let $\lambda_1 = \lambda_2$, that is, $\mu_1 = \mu_2 < \mu_3$. In similar way, as in the Theorem 5.3, we get $\xi \in \{\pm e_3\} \cup (\mathbb{S} \cap \{e_1, e_2\}_{\mathbb{R}})$ and $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$.

Let $\lambda_1 > \lambda_2$, that is, $\mu_1 < \mu_2 < \mu_3$. Therefore, we can obtain similar results for $\xi^i = 0$, $\xi^j, \xi^k \neq 0$, $i,j,k = 1,2,3$, as in the Theorem 5.3. Let $\xi^1, \xi^2, \xi^3 \neq 0$. In similar way, as in the Theorem 5.3, we get

$$||A_\xi||^2 = \frac{2\mu_1\sigma_{23}}{\mu_2+\mu_3} = \frac{2\mu_2\sigma_{31}}{\mu_3+\mu_1} = \frac{2\mu_3\sigma_{12}}{\mu_1+\mu_2},$$

that is $||A_\xi||^2 = -2(\mu_2\mu_3 + \mu_3\mu_1 + \mu_1\mu_2)$, where $\mu_2\mu_3 + \mu_3\mu_1 + \mu_1\mu_2 < 0$, and $\mu_1\mu_2\mu_3 = (\mu_2+\mu_3)(\mu_3+\mu_1)(\mu_1+\mu_2)$. If $\mu_2 \le 0$, then $\mu_1\mu_2\mu_3 \ge 0$, but $(\mu_2+\mu_3)(\mu_3+\mu_1)(\mu_1+\mu_2) < 0$, hence there are no solutions of the systems. Let $\mu_2 > 0$. Using (5.2) and $(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = 1$, we can get

$$||A_\xi||^2 = \mu_1^2 + \mu_2^2 + \mu_3^2 - \mu_1^2(\xi^1)^2 - \mu_2^2(\xi^2)^2 - \mu_3^2(\xi^3)^2.$$

Therefore, because of $(\mu_1+\mu_2+\mu_3)^2 = \mu_1^2+\mu_2^2+\mu_3^2+2(\mu_2\mu_3+\mu_3\mu_1+\mu_1\mu_2)$, we have $\mu_1^2(\xi^1)^2+\mu_2^2(\xi^2)^2+\mu_3^2(\xi^3)^2 = (\mu_1 + \mu_2 + \mu_3)^2$. Because of $\mu_2^2 < (\mu_1 + \mu_2 + \mu_3)^2 < \mu_3^2$, there are vector fields ξ such as

$$\begin{cases} \mu_1^2(\xi^1)^2 + \mu_2^2(\xi^2)^2 + \mu_3^2(\xi^3)^2 = (\mu_1 + \mu_2 + \mu_3)^2 \\ (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = 1 \end{cases}. \tag{5.6}$$

Using the system (5.4), we have $e_1(\varepsilon) = const$, $e_2(\varepsilon) = const$, $e_3(\varepsilon) = const$ and

$$e_1(\varepsilon) = \frac{\mu_2^2 - \mu_3^2}{\mu_1}\xi^2\xi^3, \quad e_2(\varepsilon) = \frac{\mu_3^2 - \mu_1^2}{\mu_2}\xi^3\xi^1, \quad e_3(\varepsilon) = \frac{\mu_1^2 - \mu_2^2}{\mu_3}\xi^1\xi^2.$$

Denote

$$\gamma_1 = \frac{\mu_1 e_1(\varepsilon)}{\mu_2^2 - \mu_3^2}, \quad \gamma_2 = \frac{\mu_2 e_2(\varepsilon)}{\mu_3^2 - \mu_1^2}, \quad \gamma_3 = \frac{\mu_3 e_3(\varepsilon)}{\mu_1^2 - \mu_2^2}.$$

Then $\xi^2\xi^3 = \gamma_1$, $\xi^3\xi^1 = \gamma_2$, $\xi^1\xi^2 = \gamma_3$, that is, $(\xi^1)^2 = \frac{\gamma_2\gamma_3}{\gamma_1}$, $(\xi^2)^2 = \frac{\gamma_3\gamma_1}{\gamma_2}$, $(\xi^3)^2 = \frac{\gamma_1\gamma_2}{\gamma_3}$. Therefore, using (5.6), we get

$$\begin{cases} \mu_1^2(\gamma_2\gamma_3)^2 + \mu_2^2(\gamma_3\gamma_1)^2 + \mu_3^2(\gamma_1\gamma_2)^2 = (\mu_1 + \mu_2 + \mu_3)^2\gamma_1\gamma_2\gamma_3 \\ (\gamma_2\gamma_3)^2 + (\gamma_3\gamma_1)^2 + (\gamma_1\gamma_2)^2 = \gamma_1\gamma_2\gamma_3 \end{cases}.$$

This completes the proof of the theorem. □

Theorem 5.5 *Let the group $E(2)$ equip with vertical rescaled metric $G^{0,\varepsilon}$ on the unit tangent bundle T_1G, where $G = E(2)$ with left-invariant metric g. Then the group $E(2)$ admits left-invariant harmonic unit vector fields ξ which determine harmonic maps $\xi\colon (G, g) \to (T_1G, G^{0,\varepsilon})$ with respect to orthonormal frame of the Lie algebra of a Lie group G satisfying (5.1) if and only if one of the following cases is held.*

- $\lambda_1 = \lambda_2$, ε *is any smooth function,* $\xi = \pm e_3$;
- $\lambda_1 = \lambda_2$, $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$, $\xi \in \{\pm e_3\} \cup (\mathbb{S} \cap \{e_1, e_2\}_{\mathbb{R}})$;
- $\lambda_1 > \lambda_2$, $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$, $\xi \in \{\pm e_1, \pm e_2, \pm e_3\}$;
- $\lambda_1 > \lambda_2$, $q_{1,23} > 0$, $e_2(\varepsilon) = e_3(\varepsilon) = 0$, $e_1(\varepsilon) = \frac{\sqrt{p_{1,23}q_{1,23}}}{\mu_1(\mu_3^2-\mu_2^2)}$, $\xi = \pm\frac{\sqrt{p_{1,23}}e_2+\sqrt{q_{1,23}}e_3}{\mu_3^2-\mu_2^2}$;
- $\lambda_1 > \lambda_2$, $q_{1,23} > 0$, $e_2(\varepsilon) = e_3(\varepsilon) = 0$, $e_1(\varepsilon) = -\frac{\sqrt{p_{1,23}q_{1,23}}}{\mu_1(\mu_3^2-\mu_2^2)}$, $\xi = \pm\frac{\sqrt{p_{1,23}}e_2-\sqrt{q_{1,23}}e_3}{\mu_3^2-\mu_2^2}$;
- $\lambda_1 > \lambda_2$, $p_{2,31} > 0$, $e_1(\varepsilon) = e_3(\varepsilon) = 0$, $e_2(\varepsilon) = \frac{\sqrt{p_{2,31}q_{2,31}}}{\mu_2(\mu_3^2-\mu_1^2)}$, $\xi = \pm\frac{\sqrt{q_{2,31}}e_1+\sqrt{p_{2,31}}e_3}{\mu_3^2-\mu_1^2}$;
- $\lambda_1 > \lambda_2$, $p_{2,31} > 0$, $e_1(\varepsilon) = e_3(\varepsilon) = 0$, $e_2(\varepsilon) = -\frac{\sqrt{p_{2,31}q_{2,31}}}{\mu_2(\mu_3^2-\mu_1^2)}$, $\xi = \pm\frac{\sqrt{q_{2,31}}e_1-\sqrt{p_{2,31}}e_3}{\mu_3^2-\mu_1^2}$;

where $p_{i,jk} = (\mu_i^2-\mu_j^2)(\mu_k^2-\mu_j^2)-2\mu_i\mu_j\mu_k(\mu_k-\mu_j)$, $q_{i,jk} = (\mu_k^2-\mu_i^2)(\mu_k^2-\mu_j^2)+2\mu_i\mu_j\mu_k(\mu_k-\mu_j)$, $i, j, k = 1, 2, 3$.

Proof According to Table 1, we have $\lambda_1, \lambda_2 > 0$, $\lambda_3 = 0$, then $\mu_1 = -\mu_2$, $\mu_3 > 0$ $\mu_3 \neq \mu_1$, $\mu_3 \neq \mu_2$. Consider the systems (5.3) and (5.4).

Let $\lambda_1 = \lambda_2$, that is, $\mu_1 = \mu_2 = 0$. Then $||A_\xi||^2 = \mu_3^2((\xi^1)^2 + (\xi^2)^2)$ and

$$\begin{cases} \xi^2 e_3(\varepsilon) = -\mu_3(\xi^3)^2\xi^1 \\ \xi^1 e_3(\varepsilon) = \mu_3(\xi^3)^2\xi^2 \\ ((\xi^1)^2 + (\xi^2)^2)\xi^3 = 0 \end{cases}, \begin{cases} ||A_\xi||^2 e_1(\varepsilon) = 0 \\ ||A_\xi||^2 e_2(\varepsilon) = 0 \\ ||A_\xi||^2 e_3(\varepsilon) = 0 \end{cases}.$$

If $\xi \in \{\pm e_3\}$, then $||A_\xi||^2 = 0$ and ε is any smooth function. If $\xi \in \mathbb{S} \cap \{e_1, e_2\}_{\mathbb{R}}$, then $||A_\xi||^2 = \mu_3^2 \neq 0$ and $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$.

Let $\lambda_1 > \lambda_2$, that is, $\mu_1 < \mu_2$, $\mu_2 = -\mu_1 > 0$. Note that $||A_\xi||^2 \neq 0$. Denote $\mu = \mu_2$, then $\mu_1 = -\mu$. If $\xi \in \{\pm e_1, \pm e_2, \pm e_3\}$, then $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$. It is easy to verify that if $\xi^3 = 0$ and $\xi^1, \xi^2 \neq 0$, then there are no solutions. Note that we can obtain similar results for $\xi^1 = 0$, $\xi^2, \xi^3 \neq 0$ and $\xi^2 = 0$, $\xi^1, \xi^3 \neq 0$ as in the Theorem 5.3. Let $\xi^1, \xi^2, \xi^3 \neq 0$, then in similar way as in the Theorem 5.3, we can get

$$\begin{cases} (\mu^2 - \mu_3^2)(\xi^3)^2||A_\xi||^2 = 2(\xi^2)^2\mu_3(\mu_1 - \mu_2)\sigma_{12} - 2(\xi^3)^2\mu_2(\mu_3 - \mu_1)\sigma_{31} \\ (\mu^2 - \mu_3^2)(\xi^3)^2||A_\xi||^2 = 2(\xi^3)^2\mu_1(\mu_2 - \mu_3)\sigma_{23} - 2(\xi^1)^2\mu_3(\mu_1 - \mu_2)\sigma_{12} \end{cases}.$$

Multiply the second row by -1 and add to the first row of the system. Hence, using (5.5), we get $\mu_3(\mu_1 - \mu_2)\sigma_{12} = 0$. However, $\mu_3(\mu_1 - \mu_2)\sigma_{12} = -2\mu^3\mu_3 \neq 0$. Thus, if $\xi^1, \xi^2, \xi^3 \neq 0$, then there are no solutions. □

Theorem 5.6 *Let the group $E(1,1)$ equip with vertical rescaled metric $G^{0,\varepsilon}$ on the unit tangent bundle T_1G, where $G = E(1,1)$ with left-invariant metric g. Then the group $E(1,1)$ admits left-invariant harmonic unit vector fields ξ which determine harmonic maps $\xi\colon (G,g) \to (T_1G, G^{0,\varepsilon})$ with respect to orthonormal frame of the Lie algebra of a Lie group G satisfying (5.1) if and only if one of the following cases is held.*

- $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$, $\xi \in \{\pm e_1, \pm e_2, \pm e_3\}$;
- $\mu_2 = 0$, $e_1(\varepsilon) = e_3(\varepsilon) = 0$, $e_2(\varepsilon) = const$, $0 < e_2(\varepsilon)^2 \leq 4\mu^2$, $\xi = \xi^1 e_1 + \xi^3 e_3$, *where* $(\xi^1)^2 = \frac{2\mu \pm \sqrt{4\mu^2 - e_2(\varepsilon)^2}}{4\mu}$, $(\xi^3)^2 = \frac{2\mu \mp \sqrt{4\mu^2 - e_2(\varepsilon)^2}}{4\mu}$;
- $\frac{1}{2}\mu_1 < \mu_2 < 0$, $p_{1,23} > 0$, $e_2(\varepsilon) = e_3(\varepsilon) = 0$, $e_1(\varepsilon) = \frac{\sqrt{p_{1,23}q_{1,23}}}{\mu_1(\mu_3^2 - \mu_2^2)}$, $\xi = \pm\frac{\sqrt{p_{1,23}}e_2 + \sqrt{q_{1,23}}e_3}{\mu_3^2 - \mu_2^2}$;
- $\frac{1}{2}\mu_1 < \mu_2 < 0$, $p_{1,23} > 0$, $e_2(\varepsilon) = e_3(\varepsilon) = 0$, $e_1(\varepsilon) = -\frac{\sqrt{p_{1,23}q_{1,23}}}{\mu_1(\mu_3^2 - \mu_2^2)}$, $\xi = \pm\frac{\sqrt{p_{1,23}}e_2 - \sqrt{q_{1,23}}e_3}{\mu_3^2 - \mu_2^2}$;
- $0 < \mu_2 < \frac{1}{2}\mu_3$, $q_{3,12} > 0$, $e_1(\varepsilon) = e_2(\varepsilon) = 0$, $e_3(\varepsilon) = \frac{\sqrt{p_{3,12}q_{3,12}}}{\mu_3(\mu_2^2 - \mu_1^2)}$, $\xi = \pm\frac{\sqrt{p_{3,12}}e_1 + \sqrt{q_{3,12}}e_2}{\mu_2^2 - \mu_1^2}$;
- $0 < \mu_2 < \frac{1}{2}\mu_3$, $q_{3,12} > 0$, $e_1(\varepsilon) = e_2(\varepsilon) = 0$, $e_3(\varepsilon) = -\frac{\sqrt{p_{3,12}q_{3,12}}}{\mu_3(\mu_2^2 - \mu_1^2)}$, $\xi = \pm\frac{\sqrt{p_{3,12}}e_1 - \sqrt{q_{3,12}}e_2}{\mu_2^2 - \mu_1^2}$;

where $p_{i,jk} = (\mu_i^2 - \mu_j^2)(\mu_k^2 - \mu_j^2) - 2\mu_i\mu_j\mu_k(\mu_k - \mu_j)$, $q_{i,jk} = (\mu_k^2 - \mu_i^2)(\mu_k^2 - \mu_j^2) + 2\mu_i\mu_j\mu_k(\mu_k - \mu_j)$, $i, j, k = 1, 2, 3$.

Proof According to Table 1, we have $\lambda_1 > 0$, $\lambda_2 = 0$, $\lambda_3 < 0$ then $\mu_3 = -\mu_1 > 0$, $\mu_2 \neq \mu_1$, $\mu_2 \neq \mu_3$ and $||A_\xi|| \neq 0$. Consider the systems (5.3) and (5.4). Denote $\mu = \mu_3$, then $\mu_1 = -\mu$. If $\xi \in \{\pm e_1, \pm e_2, \pm e_3\}$, then $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$. Let $\xi^2 = 0$, $\xi^1, \xi^3 \neq 0$, then

$$e_1(\varepsilon) = e_3(\varepsilon) = 0, \quad \mu_2 e_2(\varepsilon) = 0, \quad e_2(\varepsilon) = \frac{4\mu^3}{||A_\xi||^2}\xi^3\xi^1.$$

If $e_2(\varepsilon) = 0$, then there are no solutions. If $\mu_2 = 0$, then $||A_\xi||^2 = \mu^2$ and $e_2(\varepsilon) = 4\mu\xi^3\xi^1$, that is $e_2(\varepsilon) = const$. Therefore, if $0 < e_2(\varepsilon)^2 \leq 4\mu^2$, then

$$(\xi^1)^2 = \frac{2\mu \pm \sqrt{4\mu^2 - e_2(\varepsilon)^2}}{4\mu}, \quad (\xi^3)^2 = \frac{2\mu \mp \sqrt{4\mu^2 - e_2(\varepsilon)^2}}{4\mu}.$$

Note that we can obtain similar results for $\xi^1 = 0$, $\xi^2, \xi^3 \neq 0$ and $\xi^3 = 0$, $\xi^1, \xi^2 \neq 0$ as in the Theorem 5.3. Let $\xi^1, \xi^2, \xi^3 \neq 0$, then in similar way as in the Theorem 5.5, there are no solutions, if $\mu_2 \neq 0$. Also, if $\mu_2 = 0$, then we get $(\xi^2)^2 = 1$, $\xi^1 = \xi^3 = 0$, but it is impossible. Thus, if $\xi^1, \xi^2, \xi^3 \neq 0$, then there are no solutions. □

Theorem 5.7 *Let the Heisenberg group equip with vertical rescaled metric $G^{0,\varepsilon}$ on the unit tangent bundle T_1G, where G is Heisenberg group with left-invariant metric g. Then the Heisenberg group admits left-invariant harmonic unit vector fields ξ which determine harmonic maps $\xi\colon (G,g) \to (T_1G, G^{0,\varepsilon})$ with respect to orthonormal frame of the Lie algebra of a Lie group G satisfying (5.1) if and only if $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$, $\xi \in \{\pm e_1\} \cup (\mathbb{S} \cap \{e_2, e_3\}_{\mathbb{R}})$.*

Proof According to Table 1, we have $\lambda_1 > 0$, $\lambda_2 = \lambda_3 = 0$, then $-\mu_1 = \mu_2 = \mu_3 > 0$ and $||A_\xi|| \neq 0$. Therefore, using the systems (5.3) and (5.4), we get $e_1(\varepsilon) = e_2(\varepsilon) = e_3(\varepsilon) = 0$, $\xi \in \{\pm e_1\} \cup (\mathbb{S} \cap \{e_2, e_3\}_{\mathbb{R}})$. □

Theorem 5.8 *Let the group $\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ equip with vertical rescaled metric $G^{0,\varepsilon}$ on the unit tangent bundle T_1G, where $G = \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ with left-invariant metric g. Then the group $\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ admits left-invariant harmonic unit vector fields ξ which determine harmonic maps $\xi\colon (G,g) \to (T_1G, G^{0,\varepsilon})$ with respect to orthonormal frame of the Lie algebra of a Lie group G satisfying (5.1) if and only if $\varepsilon(x)$ is any smooth function and $\xi \in \mathbb{S}$.*

Proof According to Table 1, we have $\lambda_1 = \lambda_2 = \lambda_3 = 0$, then $\mu_1 = \mu_2 = \mu_3 = 0$ and $||A_\xi|| = 0$. Using the systems (5.3) and (5.4), we have $\varepsilon(x)$ is any smooth function and $\xi \in \mathbb{S}$. □

Using Corollary 3.5, we obtain the following result.

Corollary 5.9 *Let the groups $E(2)$ and $\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ equip with vertical rescaled metric $G^{0,\varepsilon}$ on the unit tangent bundle T_1G, where $G = E(2)$ or $\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ with left-invariant metric g. Then the groups $\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ and $E(2)$ admit left-invariant harmonic unit vector fields ξ which determine harmonic maps $\xi\colon (G,g) \to (T_1G, G^{0,\varepsilon})$ regardless of the deformation function $\varepsilon(x)$.*

Example 5.10 *Consider the Euclidean motion group $E(2)$ given explicitly by*

$$E(2) = \left\{ \begin{pmatrix} \cos x^3 & -\sin x^3 & x^1 \\ \sin x^3 & \cos x^3 & x^2 \\ 0 & 0 & 1 \end{pmatrix} \middle| x^1, x^2 \in \mathbb{R}, x^3 \in \mathbb{S}^1 \right\}.$$

Consider $\lambda_1 = \frac{5}{2}$, $\lambda_2 = \frac{3}{2}$, $\lambda_3 = 0$ *and*

$$e_1 = \frac{\sqrt{10}}{5}(\cos x^3 \frac{\partial}{\partial x^1} + \sin x^3 \frac{\partial}{\partial x^2}), \quad e_2 = \frac{\sqrt{6}}{3}(-\sin x^3 \frac{\partial}{\partial x^1} + \cos x^3 \frac{\partial}{\partial x^2}), \quad e_3 = \frac{\sqrt{15}}{2}\frac{\partial}{\partial x^3}.$$

Therefore, e_1, e_2, e_3 *satisfy* (5.1) *and* $\mu_1 = -\frac{1}{2}$, $\mu_2 = \frac{1}{2}$, $\mu_3 = 2$, $p_{2,31} = \frac{185}{16} > 0$, $q_{2,31} = \frac{5}{2}$. *Find* ε *such as* $e_1(\varepsilon) = e_3(\varepsilon) = 0$, $e_2(\varepsilon) = \frac{\sqrt{74}}{6}$, *that is*

$$\begin{cases} \cos x^3 \frac{\partial \varepsilon}{\partial x^1} + \sin x^3 \frac{\partial \varepsilon}{\partial x^2} = 0 \\ \frac{\sqrt{6}}{3}(-\sin x^3 \frac{\partial \varepsilon}{\partial x^1} + \cos x^3 \frac{\partial \varepsilon}{\partial x^2}) = \frac{\sqrt{74}}{6} \\ \frac{\partial \varepsilon}{\partial x^3} = 0 \end{cases}.$$

The solution of the system is $\varepsilon(x^1, x^2) = \frac{\sqrt{111}}{6}(-x^1 \sin x^3 + x^2 \cos x^3) + C$. *Then, using Theorem* 5.5*, left-invariant unit vector fields* $\xi = \pm\frac{1}{15}(2\sqrt{10}e_1 + \sqrt{185}e_3)$ *are harmonic and determine harmonic maps* $\xi\colon (G, g) \to (T_1G, G^{0,\varepsilon})$.

Similarly, we can construct examples for all other cases with Theorem 5.3–5.8.

Table 1. Three-dimensional unimodular Lie groups

Signs of $\lambda_1, \lambda_2, \lambda_3$	**Associated Lie group** G
$+,+,+$	$SU(2)$ or $SO(3)$
$+,+,-$	$SL(2,\mathbb{R})$ or $O(1,2)$
$+,+,0$	$E(2)$
$+,0,-$	$E(1,1)$
$+,0,0$	Heisenberg group
$0,0,0$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$

Table 2. The sets of left-invariant harmonic unit vector fields and harmonic maps $(G,g) \to (T_1G, G^{0,0})$. [12]

Lie group G	**conditions for** λ_i	**vector fields**	**maps** $(G,g)\to(T_1G,G^{0,0})$
$SU(2)$ or $SO(3)$	$\lambda_1=\lambda_2=\lambda_3$	$\mathbb{S}$	$\mathbb{S}$
	$\lambda_1>\lambda_2=\lambda_3$	$\{\pm e_1\}\cup(\mathbb{S}\cap\{e_2,e_3\}_{\mathbb{R}})$	$\{\pm e_1\}\cup(\mathbb{S}\cap\{e_2,e_3\}_{\mathbb{R}})$
	$\lambda_1=\lambda_2>\lambda_3$	$\{\pm e_3\}\cup(\mathbb{S}\cap\{e_1,e_2\}_{\mathbb{R}})$	$\{\pm e_3\}\cup(\mathbb{S}\cap\{e_1,e_2\}_{\mathbb{R}})$
	$\lambda_1>\lambda_2>\lambda_3$	$\{\pm e_1,\pm e_2,\pm e_3\}$	$\{\pm e_1,\pm e_2,\pm e_3\}$
$SL(2,\mathbb{R})$ or $O(1,2)$	$\lambda_1=\lambda_2$	$\{\pm e_3\}\cup(\mathbb{S}\cap\{e_1,e_2\}_{\mathbb{R}})$	$\{\pm e_3\}\cup(\mathbb{S}\cap\{e_1,e_2\}_{\mathbb{R}})$
	$\lambda_1>\lambda_2$	$\{\pm e_1,\pm e_2,\pm e_3\}$	$\{\pm e_1,\pm e_2,\pm e_3\}$
$E(2)$	$\lambda_1=\lambda_2$	$\{\pm e_3\}\cup(\mathbb{S}\cap\{e_1,e_2\}_{\mathbb{R}})$	$\{\pm e_3\}\cup(\mathbb{S}\cap\{e_1,e_2\}_{\mathbb{R}})$
	$\lambda_1>\lambda_2$	$\{\pm e_3\}\cup(\mathbb{S}\cap\{e_1,e_2\}_{\mathbb{R}})$	$\{\pm e_1,\pm e_2,\pm e_3\}$
$E(1,1)$		$\{\pm e_2\}\cup(\mathbb{S}\cap\{e_1,e_3\}_{\mathbb{R}})$	$\{\pm e_1,\pm e_2,\pm e_3\}$
Heisenberg group		$\mathbb{S}$	$\{\pm e_1\}\cup(\mathbb{S}\cap\{e_2,e_3\}_{\mathbb{R}})$
$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$		$\mathbb{S}$	$\mathbb{S}$

Acknowledgment

The author thanks Alexander YAMPOLSKY for posing a problem, for useful advice concerning the research, and for invaluable support in difficult times. The author thanks the anonymous reviewers for their careful reading of the manuscript and helpful comments, which have improved the quality of the manuscript. The research was supported by the Akhiezer Foundation.

References

[1] Abbassi MTK, Sarih M. On Riemannian g-natural metrics of the form $ag^s + bg^h + cg^v$ on the Tangent Bundle of a Riemannian Manifold (M, g). Mediterranean Journal of Mathematics 2005; 2 (1): 19-43. https://doi.org/10.1007/s00009-005-0028-8

[2] Belarbi L, El Hendi H. Geometry of Twisted Sasaki Metric. Journal of Geometry and Symmetry in Physics 2019; 53: 1-19. https://doi.org/10.7546/jgsp-53-2019-1-19

[3] Belarbi L, El Hendi H, Latti F. On the geometry of the tangent bundle with vertical rescaled generalized Cheeger-Gromoll metric. Bulletin of the Transilvania University of Brasov 2020; 12:247-264. https://doi.org/10.31926/but.mif.2019.12.61.2.5

[4] Boeckx E, Vanhecke L. Characteristic reflections on unit tangent sphere bundles. Houston Journal of Mathematics 1997; 23: 427-448.

[5] Boeckx E, Vanhecke L. Harmonic and minimal vector fields on tangent and unit tangent bundles. Differential Geometry and its Applications 2000; 13 (1): 77-93. https://doi.org/10.1016/S0926-2245(00)00021-8

[6] Boeckx E, Vanhecke L. Unit Tangent Sphere Bundles with Constant Scalar Curvature. Czechoslovak Mathematical Journal 2001; 51 (3): 523-544. https://doi.org/10.1023/A:1013779805244

[7] Cheeger J, Gromoll D. On the structure of complete manifolds of nonnegative curvature. Annals of Mathematics 1972; 96: 413-443.

[8] Dida HM, Hathout F, Azzouz A. On the geometry of the tangent bundle with vertical rescaled metric. Communications Faculty Of Science University of Ankara Series A1 Mathematics and Statistics 2019; 68: 222-235. https://doi.org/10.31801/cfsuasmas.443735

[9] Djaa NE, Gezer A, Karaca K. F-geodesics on the second order tangent bundle over a Riemannian manifold. Filomat 2023; 37 (8): 2561-2576.

[10] Djaa NE, Zagane A. On the geodesics of deformed Sasaki metric. Turkish Journal of Mathematics 2022; 46 (6): 2121-2140.

[11] Eells J, Lemaire L. A Report on Harmonic Maps. Bulletin of London Mathematical Society 1978; 10: 1-68.

[12] González-Dávila JC, Vanhecke L. Invariant harmonic unit vector fields on Lie groups. Bollettino dell'Unione Matematica Italiana 2002; 5-B: 377-403.

[13] Kowalski O. Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold. Journal für die reine und angewandte Mathematik 1971; 250: 124-129.

[14] Lotarets L. Geodesics of fiberwise cigar soliton deformation of the Sasaki metric. Turkish Journal of Mathematics 2022; 46: 130-144.

[15] Milnor J. Curvatures of left invariant metrics on Lie groups. Advances in Mathematics 1976; 21: 293-329.

[16] Musso E, Tricerri F. Riemannian metrics on tangent bundles. Annali di Matematica Pura ed Applicata 1988; 150 (4): 1-19. https://doi.org/10.1007/BF01761461

[17] Sasaki S. On the differential geometry of tangent bundles of Riemannian manifolds. Tohoku Mathematical Journal 1958; 10 (3): 338-354. https://doi.org/10.2748/tmj/1178244668

[18] Sasaki S. On the differential geometry of tangent bundles of Riemannian manifolds II. Tokyo Journal of Mathematics 1962; 14 (2): 146-155.

[19] Wiegmink G. Total bending of vector fields on Riemannian manifolds. Mathematische Annalen 1995; 303: 325-344.

[20] Yampolsky A. On geodesics of tangent bundle with fiberwise deformed Sasaki metric over Kahlerian manifold. Journal of Mathematical Physics, Analysis, Geometry 2012; 8 (2): 177-189.

[21] Yampolsky A. On special types of minimal and totally geodesic unit vector fields. Proceedings of the Seventh International Conference in Geometry, Integrability and Quantization; Varna, Bulgaria; 2005. pp. 292-306.

[22] Zagane A, Djaa M. Geometry of Mus-Sasaki metric. Communications in Mathematics 2018; 26 (2): 113-126. https://doi.org/10.2478/cm-2018-0008