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Fusion in supersolvable Hall subgroups

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Abstract: Let H be a supersolvable Hall π-subgroup of a finite group G. We prove that G has a normal π-complement if and only if H controls G-fusion in H.

Key words: Control of fusion, supersolvable Hall subgroups, normal π-complement

1. Introduction

All groups considered in this article are finite. Let G be a group and K be a subgroup of G. A subgroup H of G that contains K is said to control G-fusion in K if whenever two elements of K are conjugate by an element of G, they are also conjugate by an element of H.

The concept of control of fusion especially produces interesting results when K is a Sylow subgroup (refer to [2, Chapter 5] for relevant theorems in this context). Although some of the fusion results concerning Sylow subgroups could be extended to Hall subgroups, it is important to note that many of these results do not hold true, as evidenced by specific counterexamples. The following is one of such theorems:

Theorem 1 *[2, Theorem 5.25] Let p be a prime and P be a Sylow p-subgroup of a group G. Then G has a normal p-complement if and only if P controls G-fusion in P.*

Let π be a set of primes. Recall that G has a normal π-complement if G has a normal Hall π'-subgroup N. If π consists of single prime, that is, if $\pi = \{p\}$, we say that G has a normal p-complement (or alternatively G is p-nilpotent).

Consequently, the above theorem says that G has a normal subgroup N such that $G = NP$ and $N \cap P = 1$ if and only if P controls G-fusion in P. However, the conclusion of the above theorem is not correct for Hall subgroups in general as the following example illustrates.

Example 1 *Let $G = S_5$, the symmetric group on 5-letters and $H = S_4$. First notice that H is a Hall $\{2,3\}$-subgroup of G. Moreover, H-controls G-fusion in H. To see this, notice that they are both symmetric groups, and so two elements $x, y \in H$ are conjugate (in H or G) if and only if they have the same cycle type. However, $G = S_5$ obviously do not have a normal $\{2,3\}$-complement.*

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After seeing that the answer is not affirmative for general Hall subgroups, it is reasonable to investigate the case for some particular classes of Hall subgroups. In this article, we shall consider groups having supersolvable Hall π-subgroups. Here is our main theorem:

Theorem 2 *Let H be a supersolvable Hall π-subgroup of a group G. Then G has a normal π-complement if and only if H controls G-fusion in H.*

One can wonder whether the condition "supersolvability" can be replaced by "solvability" in the main theorem yet Example 1 also shows that this is not possible (in this example, notice that $H = S_4$, which is a solvable group).

Consequently, our main theorem generalizes Theorem 1 to groups having supersolvable Hall subgroups, and this is the best possible in the sense that we have a counterexample for groups having solvable Hall subgroups.

2. The proof of main theorem

Recall that a group G is said to be **supersolvable** if there is a normal series (each N_i is normal in G)

$$1 = N_0 \leq N_1 \leq \ldots \leq N_{n-1} \leq N_n = G$$

of G such that each N_{i+1}/N_i is cyclic. We remind our reader that this definition easily yields that any minimal normal subgroup of a supersolvable group must be of prime order.

Let p be a prime. We say that G is **p-closed** if G has a normal Sylow p-subgroup. Let $P \in Syl_p(G)$ (we do not assume that P is normal in G now) and $W \leq P$. We say that W is **weakly closed** in P with respect to G if whenever $W^g \subseteq P$ for some $g \in G$, the equality $W = W^g$ holds. We also note that $\pi(G)$ denotes the set of primes dividing the order of G for any group G.

2.1. Some lemmas

We now state some lemmas used in the proof of the main theorem.

The following is a well-known fact about supersolvable groups. For the sake of completeness, we provide a proof.

Lemma 1 *Let G be a supersolvable group and q be the largest prime dividing the order of G. Then G is q-closed.*

Proof We proceed by induction on the order of G. Let M be a minimal normal subgroup of G and $Q \in Syl_q(G)$. Note that $|M| = r$ where r is prime as G is supersolvable. G/M has a normal Sylow q-subgroup by the inductive argument and so MQ is normal in G. If $r = q$, then $M \subseteq Q$, and hence $MQ = Q \lhd G$. Then assume $r \neq q$. As q is the largest prime dividing the order G, we have $r < q$ and $r \not\equiv 1 \ mod\ q$. Then Sylow theorems easily yield that $Q \lhd MQ$, and so Q is characteristic in $MQ \lhd G$, which causes Q to be normal in G as desired.

The following lemma remarkably simplifies our arguments.

Lemma 2 *Let* $P \in Syl_p(G)$*. If every characteristic subgroup of* P *is weakly closed in* P *with respect to* G*, then* $N_G(P)$ *controls* G*-fusion in* P*.*

Proof It directly follows as a corollary from the main theorem of [1].

Lemma 3 *Let* H *be a subgroup of a group* G *such that*

$$G = \bigcup_{g \in G} H^g.$$

Then $H = G$*.*

We skip the proof of the above lemma as it is a classical exercise in elementary group theory.

2.2. Proof of Theorem 2

Let us first prove the easy direction. Fix $\pi = \pi(H)$. Suppose G has a normal π-complement, say M. Then $G = HM$ and $H \cap M = 1$. Let $x, x^g \in H$ for some $g \in G$. Then we can decompose $g = hm$ for some $h \in H$ and $m \in M$. Notice that

$$x^g M = m^{-1} x^h m M = x^h M,$$

and so $(x^g)^{-1} x^h \in M \cap H = 1$. Hence, we obtain that $x^g = x^h$, that is, H controls G-fusion in H as desired, which completes the proof one direction. We remark that we do not use the supersolvability of H for this direction.

Now suppose that H is a supersolvable Hall π-subgroup of G such that H controls G-fusion in H. We claim that G has a normal π-complement. We proceed by induction on the order of G and $|\pi|$. If $|\pi| = 1$, then H is a Sylow subgroup and we are done by Theorem 1 in this case. Thus, we may assume $|\pi| > 1$.

Let $r = max(\pi(H))$ and $\sigma = \pi(H) - \{r\}$ (note that σ is not empty). We see that H is r-closed by Lemma 1, and so we can write

$$H = RK$$

where R is a normal Sylow r-subgroup of H and K is Hall σ-subgroup of H by the Schur-Zassenhaus theorem. Notice that K is also a Hall subgroup of G. Moreover, K controls H fusion in K as K has a normal σ-complement R in H. On the other hand, H controls G-fusion in H by hypothesis, and so K controls G-fusion in K. Since $|\sigma| = |\pi(K)| < |\pi| = |\pi(H)|$, we obtain that G has a normal σ-complement N by the inductive argument, and so we have

$$G = NK, \; N \cap K = 1, and \; R \subseteq N.$$

Now we follow three steps to complete the proof:

Step 1: $N_N(R)$ controls N-fusion in R.

Let W be a characteristic subgroup of R. We observe that W is normal in H as $R \trianglelefteq H$. Suppose that $W^g \subseteq R$ for some $g \in G$. Let $x \in W$, then $x^g \in R$. Moreover, H controls G-fusion in R, and so there exists $h \in H$ such that $x^g = x^h$. On the other hand, H normalizes W, which yields that $x^g = x^h \in W$, and hence $W{=}W^g$. Consequently, each characteristic subgroup of R is weakly closed in R with respect to G as W is arbitrary. In particular, they are also weakly closed in R with respect to N. It follows that $N_N(R)$ controls N-fusion in R by Lemma 2.

Step 2: R is normal in G.

First assume that $N_G(R) < G$. Notice that $H \subseteq N_G(R)$ and H also controls $N_G(R)$ fusion in H, and so we get that $N_G(R)$ has a normal π-complement by induction applied to $N_G(R)$. Thus, $N_N(R) = N_G(R) \cap N$ has a normal π-complement as well. It means that $N_N(R)$ has an r-complement, that is, $N_N(R)$ is r-nilpotent as $\pi(H) \cap \pi(N) = r$. Thus, R controls $N_N(R)$ fusion in R. On the other hand, $N_N(R)$ controls N-fusion in R by the previous step, and so R controls N-fusion in R. It follows that N has a normal r-complement by Theorem 1. Consequently, N has a normal Hall r'-subgroup M, and so $M \lhd G$ and M is a Hall π'-subgroup of G, which completes the proof for this case. Hence, we may assume that R is normal in G.

Step 3: G has a normal π-complement.

By Schur-Zassenhaus theorem, N has a Hall r'-subgroup M. We note that M is also a Hall π'-subgroup of G. Notice that any $M^* \in Hall_{\pi'}(G)$, must lie in N, and hence there exists an $n \in N$ such that $M^* = M^n$ by the conjugacy part of the Schur-Zassenhaus theorem. Since we can decompose n by $n = my$ for some $m \in M$ and $y \in R$, we obtain that

$$M^* = M^n = M^{my} = M^y, \tag{1}$$

that is, any two Hall π'-subgroup of G is R-conjugate.

Now pick arbitrary $x \in R$. Since H controls G-fusion in R, we see that H acts transitively on $x^G \subseteq R$, and hence we have

$$G = HC_G(x).$$

We can easily argue that $C_G(x)$ has a Hall π'-subgroup M^* (notice that $C_N(x) = N \cap C_G(x)$ is r-closed and Schur-Zassenhaus theorem can be applied). On the other hand, the equality $G = HC_G(x)$ shows that $|M|$ divides $|C_G(x)|$, that is, $M^* \in Hall_{\pi'}(G)$. Then we may choose $y \in R$ such that $M^* = M^y$ by 1. Consequently, for all $x \in R$, there exists $y \in R$ such that

$$M \subseteq C_G(x^{y^{-1}}),$$

that is, when we consider the conjugation action of M on R, M fixes an element from each R-conjugacy classes of R. It follows that

$$R = \bigcup_{y \in R} C_R(M)^y,$$

and so we obtain that $R = C_R(M)$ by Lemma 3. It follows that $[M, R] = 1$ and $M \trianglelefteq G$, and hence M is the desired normal Hall π'-subgroup of G.

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