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Combinatorial results for semigroups of orientation-preserving transformations

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Abstract: Let $X_n$ denote the chain $\{1, 2, \ldots, n\}$ under its natural order. We denote the semigroups consisting of all order-preserving transformations and all orientation-preserving transformations on $X_n$ by $\mathcal{O}_n$ and $\mathcal{OP}_n$, respectively. We denote by $E(U)$ the set of all idempotents of a subset $U$ of a semigroup $S$. In this paper, we first determine the cardinalities of

\[
\begin{align*}
E_r(\mathcal{O}_n) &= \{ \alpha \in E(\mathcal{O}_n) : |\text{im}(\alpha)| = |\text{fix}(\alpha)| = r \}, \\
E_r(\mathcal{OP}_n) &= \{ \alpha \in E(\mathcal{OP}_n) : |\text{fix}(\alpha)| = r \}, \\
E_r^*(\mathcal{OP}_n) &= \{ \alpha \in E(\mathcal{OP}_n) : n \in \text{fix}(\alpha) \}, \\
E_r(\mathcal{OP}_n) &= \{ \alpha \in E_r(\mathcal{OP}_n) : n \in \text{fix}(\alpha) \}
\end{align*}
\]

(1 $\leq$ r $\leq$ n) and then, by using these results, we determine the numbers of idempotents in $\mathcal{O}_n$ and $\mathcal{OP}_n$ by a new method. Let $\mathcal{OP}_n^-$ denote the semigroup of all orientation-preserving and order-decreasing transformations on $X_n$. Moreover, we determine the cardinalities of $\mathcal{OP}_n^-$, $\mathcal{OP}_n^-, Y = \{ \alpha \in \mathcal{OP}_n^- : \text{fix}(\alpha) = Y \}$ for any nonempty subset $Y$ of $X_n$ and $\mathcal{OP}_n^-, Y = \{ \alpha \in \mathcal{OP}_n^- : |\text{fix}(\alpha)| = r \}$ for 1 $\leq$ r $\leq$ n. Also, we determine the number of idempotents in $\mathcal{OP}_n^-$ and the number of nilpotents in $\mathcal{OP}_n^-$. Key words: Order-preserving transformation, order-decreasing transformation, orientation-preserving transformation

1. Introduction

For $n \in \mathbb{Z}^+$, let $\mathcal{T}_n$ be the (full) transformation semigroup on the chain $X_n = \{1, 2, \ldots, n\}$ under its natural order. A transformation $\alpha \in \mathcal{T}_n$ is called order-preserving if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X_n$, and order-decreasing (order-increasing) if $x\alpha \leq x$ ($x\alpha \geq x$) for all $x \in X_n$. The subsemigroup of $\mathcal{T}_n$ consisting of all order-preserving transformations is denoted by $\mathcal{O}_n$, and the subsemigroup of $\mathcal{T}_n$ consisting of all order-decreasing (order-increasing) transformations is denoted by $\mathcal{D}_n$ ($\mathcal{D}_n^+$). Also, the subsemigroup of $\mathcal{T}_n$ consisting of all order-preserving and order-decreasing (order-increasing) transformations is denoted by $\mathcal{C}_n$ ($\mathcal{C}_n^+$) and called the Catalan monoid. A finite sequence $A = (a_1, a_2, \ldots, a_t)$ ($t \in \mathbb{Z}^+$, $a_1, \ldots, a_t \in X_n$) is called cyclic if there exists no more than one subscript $i$ such that $a_i > a_{i+1},$ and anticyclic if there exists no more than one subscript $i$ such that $a_i < a_{i+1}$ where $a_{t+1} = a_1$. A transformation $\alpha$ in $\mathcal{T}_n$ is called orientation-preserving if the sequence $(1\alpha, 2\alpha, \ldots, n\alpha)$ is cyclic. The subsemigroup of $\mathcal{T}_n$ consisting of all orientation-preserving

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transformations is denoted by $\mathcal{OP}_n$. Moreover, the subsemigroup of $\mathcal{OP}_n$ consisting of all order-decreasing (order-increasing) transformations is denoted by $\mathcal{OP}_n^−$ ($\mathcal{OP}_n^+$). The fix and image sets of a transformation $\alpha \in \mathcal{T}_n$ are defined and denoted by

$$\text{fix}(\alpha) = \{x \in X_n : x\alpha = x\} \text{ and } \text{im}(\alpha) = \{x\alpha : x \in X_n\},$$

respectively. The set of all idempotents in any subset $U$ of a semigroup $S$ is denoted by $E(U)$, that is $E(U) = \{e \in U : e^2 = e\}$. It is clear that a transformation $\alpha \in \mathcal{T}_n$ is idempotent if and only if $\text{fix}(\alpha) = \text{im}(\alpha)$. The set of all nilpotents in a semigroup $S$ with zero is denoted by $N(S)$, that is $N(S) = \{s \in S : s^m = 0, \text{ for some } m \in \mathbb{Z}^+\}$ where 0 denotes the zero element of $S$. For a nonempty subset $A$ of a semigroup $S$, the smallest subsemigroup of $S$ containing $A$ is called the subsemigroup generated by $A$, and denoted by $\langle A \rangle$. If there exists a finite subset $A$ of $S$ such that $S = \langle A \rangle$, then $S$ is called a finitely generated semigroup, and the rank of a finitely generated semigroup $S$ is defined by $\text{rank}(S) = \min \{ |A| : \langle A \rangle = S \}$. Moreover, if $S = \langle A \rangle$ and $|A| = \text{rank}(S)$, then $A$ is called a minimal generating set of $S$. Similarly, the idempotent rank of a semigroup $S$ is defined by $\text{idrank}(S) = \min \{ |A| : A \subseteq E(S) \text{ and } \langle A \rangle = S \}$. A minimal generating set (which is unique) and the rank of $N(C_n)$, which is the nilpotent subsemigroup of $C_n$, were determined in [6] and [14]. It is also clear that

$$\eta = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 1 & \cdots & 1 \end{pmatrix} \text{ and } \varepsilon = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n & \cdots & n \end{pmatrix},$$

are the zero elements of $\mathcal{OP}_n^−$ and $\mathcal{OP}_n^+$, respectively.

Some cardinalities of various kinds of transformation semigroups have been studied over a long period. Howie computed in [3] that the cardinality of $O_n$ is $\binom{2n-1}{n-1}$, and Laradji and Umar computed in [9] that the cardinality of $C_n$ is $C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the $n$th Catalan number. For $1 \leq r \leq n$, the numbers of elements in $O_n$ and $C_n$ with $r$ fixed points are $\frac{r}{n}\binom{2n}{n+r}$ and $\frac{r}{2n-r}\binom{2n}{n-r}$, respectively (see, [5, 9]). In [1], the cardinalities of the sets $O_n, Y = \{\alpha \in O_n : \text{fix}(\alpha) = Y\}$ and $C_n, Y = \{\alpha \in C_n : \text{fix}(\alpha) = Y\}$ were computed for any nonempty subset $Y$ of $X_n$. In [2], the set of all orientation-preserving transformations $\mathcal{OP}_n$ was considered and it is proven in [2, Theorem 2.2] that $\mathcal{OP}_n$ is a submonoid of $\mathcal{T}_n$ containing $O_n$. Moreover, the authors of [2] proved that

$$\mathcal{OP}_n = \{a^k\alpha : 0 \leq k \leq n-1 \text{ and } \alpha \in O_n\}$$

where $a = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}$, the $n$-cycle $(1\ 2\ \cdots\ n)$, and that $|\mathcal{OP}_n| = n\binom{2n-1}{n-1} - n(n-1)$ in [2, Theorem 2.6 and Corollary 2.7], respectively. The semigroup of all orientation-preserving and order-increasing transformations $\mathcal{OP}_n^+$ was considered in [10] and the authors found a minimal (idempotent) generating set of $\mathcal{OP}_n^+$ in [10, Theorem 3.6] and [11]. We have not seen any information about the cardinality of $\mathcal{OP}_n^+$. For any $\alpha \in \mathcal{OP}_n^−$, consider the transformation $\hat{\alpha} : X_n \rightarrow X_n$, defined by $i\hat{\alpha} = n - (n - i + 1)\alpha + 1$ for each $i = 1, 2, \ldots, n$. As defined and shown in [12, Lemma 1.1], the function $\theta : \mathcal{OP}_n^− \rightarrow \mathcal{OP}_n^+$, defined by $\alpha\theta = \hat{\alpha}$
for all $\alpha \in \mathcal{OP}_n^-$, is an isomorphism. Hence, we consider only the subsemigroup $\mathcal{OP}_n^-$ for $n \geq 2$. Let
\[
E_r(\mathcal{O}_n) = \{ \alpha \in E(\mathcal{O}_n) : |\text{im}(\alpha)| = |\text{fix}(\alpha)| = r \},
\]
\[
E_r^*(\mathcal{O}_n) = \{ \alpha \in E_r(\mathcal{O}_n) : 1, n \in \text{fix}(\alpha) \},
\]
\[
E_r(\mathcal{OP}_n) = \{ \alpha \in E(\mathcal{OP}_n) : |\text{fix}(\alpha)| = r \}
\]
and
\[
E_r^*(\mathcal{OP}_n) = \{ \alpha \in E_r(\mathcal{OP}_n) : n \in \text{fix}(\alpha) \}.
\]
Let $f_n$ denote the $n$th Fibonacci number. For $1 \leq r \leq n$, the cardinality of $E_r(\mathcal{O}_n)$ is found in [9, Corollary 4.4]. Despite this fact, we first determine that $|E_r(\mathcal{O}_n)| = \binom{n+r-3}{2r-3}$ (for $2 \leq r \leq n-1$), and then we determine that $|E_r(\mathcal{O}_n)| = \binom{n+r-1}{2r-1}$, and conclude that $|E(\mathcal{O}_n)| = f_{2n}$. By using a similar method, we first find that
\[
|E_r^*(\mathcal{O}_n)| = \binom{n+r-1}{2r-1}
\]
and
\[
|E_r(\mathcal{OP}_n)| = \frac{n}{r} \binom{n+r-1}{2r-1}
\]
for $2 \leq r \leq n$, then we conclude that $|E(\mathcal{OP}_n)| = f_{2n+1} + f_{2n-1} - n^2 + n - 2$ as in [2, Theorem 2.10]. In the last section, we show that
\[
|\mathcal{OP}_n^-| = -n + 2 + \sum_{k=2}^{n} C_k \quad \text{and} \quad |E(\mathcal{OP}_n^-)| = -n + 2^n
\]
for all $n \geq 1$. It is shown in [9, Proposition 2.3] that $|\mathcal{C}_n| = |\mathcal{C}_{n-1}| = C_{n-1}$, by using this result, we show that
\[
|N(\mathcal{OP}_n^-)| = |\mathcal{OP}_n^-| = -n + 3 + \sum_{k=2}^{n-1} C_k
\]
for all $n \geq 2$. In [1, 5, 9], the numbers of transformations in $\mathcal{O}_n$ and $\mathcal{C}_n$ with $r$ fixed points were computed as $\frac{r}{n} \binom{2n}{n+r}$ and $\frac{r}{2n-r} \binom{2n-r}{n}$, respectively. By using a similar method as in [1, 13], the number of transformations in $\mathcal{OP}_n^-$ with $r$ fixed points is computed as
\[
\sum_{k=0}^{n-r} \frac{r}{2k+r} \binom{2k+r}{k}
\]
for $2 \leq r \leq n-1$.

2. Cardinalities related to $\mathcal{OP}_n$

We list some standard combinatorial results related to our studies. For natural numbers $k$ and $n$, we have the following:

**Result 1** [8, Lemma 1.3]. \[
\sum_{i=0}^{n} \binom{k+i}{k} = \binom{n+k+1}{k+1}.
\]

**Result 2** [9, Corollary 4.5]. \[
\sum_{r=0}^{n} \binom{n+r}{2r} = f_{2n+1}.
\]

**Result 3** [9, Corollary 4.6]. \[
\sum_{r=1}^{n} \binom{n+r-1}{2r-1} = f_{2n}.
\]

Since $E_1(\mathcal{O}_n)$ consists of all the constant transformations in $\mathcal{O}_n$, and $E_n(\mathcal{O}_n)$ consists of only the identity, first we have $|E_1(\mathcal{O}_n)| = n$ and $|E_n(\mathcal{O}_n)| = 1$. 108
Proposition 1 For \( n \geq 3 \), we have \( |E_2^*(\mathcal{O}_n)| = n - 1 \) and \( |E_2(\mathcal{O}_n)| = \binom{n+1}{3} \).

Proof For any \( i, j \in X_n \) with \( i < j \), we first notice that there exist \( j - i \) many idempotents in \( E(\mathcal{O}_n) \) such that their image sets are the same and equal to \( \{i, j\} \). Therefore, \( |E_2^*(\mathcal{O}_n)| = n - 1 \), and moreover, it follows from Result 1 that

\[
|E_2(\mathcal{O}_n)| = \left| \bigcup_{1 \leq i < j \leq n} \{ \alpha \in E(\mathcal{O}_n) : \text{im}(\alpha) = \{i, j\} \} \right| = \sum_{1 \leq i < j \leq n} (j - i)
\]

\[
= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (j - i) = \sum_{i=1}^{n-1} \sum_{j=0}^{n-i-1} \binom{1+j}{1} = \sum_{i=1}^{n-1} \binom{n-i+1}{2}
\]

\[
= \sum_{i=0}^{n-2} \left( \frac{2+i}{2} \right) = \binom{n+1}{3},
\]
as required.

Now recall that \( \sum_{k=1}^{n} ka_k = \sum_{j=1}^{n} \sum_{k=j}^{n} a_k \), which will be used in the proof of the following proposition.

Proposition 2 For \( 2 \leq r \leq n - 1 \), we have \( |E_r^*(\mathcal{O}_n)| = \binom{n+r-3}{2r-3} \).

Proof We prove the claim by induction on \( r \). For \( r = 2 \), the result follows from Proposition 1. Suppose that \( 2 \leq r \leq n - 2 \) and \( \alpha \in E_{r+1}^*(\mathcal{O}_n) \). Then there exist \( 1 < i_1 < \cdots < i_{r-1} < n \) such that \( \text{fix}(\alpha) = \{1, i_1, \ldots, i_{r-1}, n\} \). If we define the following maps

\[
\alpha_1 = \begin{pmatrix} 1 & 2 & \cdots & i_1 - 1 & i_1 \\ 1 & 2\alpha & \cdots & (i_1 - 1)\alpha & i_1 \end{pmatrix} \quad \text{and}
\]

\[
\alpha_2 = \begin{pmatrix} 1 & 2 & \cdots & n - i_1 & n - (i_1 - 1) \\ 1 & (i_1 + 1)\alpha - (i_1 - 1) & \cdots & (n-1)\alpha - (i_1 - 1) & n - (i_1 - 1) \end{pmatrix},
\]

then it is easy to see that \( \alpha_1 \) and \( \alpha_2 \) are two idempotents with the sets of fix points \( \{1, i_1\} \) and \( \{1, i_2 - i_1 + 1, \ldots, i_{r-1} - i_1 + 1, n - i_1 + 1\} \), respectively. Next let \( i = i_1 \) and consider the function

\[
f : E_{r+1}^*(\mathcal{O}_n) \to \bigcup_{i=2}^{n-r+1} (E_2^*(\mathcal{O}_i) \times E_r^*(\mathcal{O}_{n-i+1}))
\]

which maps each \( \alpha \in E_{r+1}^*(\mathcal{O}_n) \) to the ordered pair \( (\alpha_1, \alpha_2) \). For any \( \alpha, \beta \in E_{r+1}^*(\mathcal{O}_n) \), if \( \alpha f = \beta f \), then both \( \alpha_1 = \beta_1 \) and \( \alpha_2 = \beta_2 \), and so it follows from the definitions given in Equations 1 and 2 that \( \alpha = \beta \). Moreover, for \( (\gamma_1, \gamma_2) \in \bigcup_{i=2}^{n-r+1} (E_2^*(\mathcal{O}_i) \times E_r^*(\mathcal{O}_{n-i+1})) \), if we consider the following map

\[
\gamma = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & 2+i & \cdots & n-i+(i-1) & n \\ 1 & 2\gamma_1 & \cdots & (i-1)\gamma_1 & i & 2\gamma_2+(i-1) & \cdots & (n-i)\gamma_2+(i-1) & n \end{pmatrix},
\]

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then it is easy to see that $\gamma \in E^*_{r+1}(O_n)$ and $\gamma f = (\gamma_1, \gamma_2)$, and so $f$ is a bijection. Thus, from Proposition 1 and the induction hypothesis, we have

$$|E^*_{r+1}(O_n)| = \sum_{i=2}^{n-r+1} (i-1) \left( \frac{n-i+1+r-3}{2r-3} \right) = \sum_{i=1}^{n-r} \left( \frac{n-i+r-3}{2r-3} \right)$$

$$= \sum_{j=1}^{n-r} \sum_{i=j}^{n-r} \left( \frac{n-i+r-3}{2r-3} \right) = \sum_{j=1}^{n-r} \sum_{i=0}^{n-j-r} \left( \frac{2r-3+i}{2r-3} \right)$$

$$= \sum_{j=1}^{n-r} \left( \frac{n+r-2-j}{2r-2} \right) = \sum_{j=0}^{n-r-1} \left( \frac{2r-2+j}{2r-2} \right)$$

$$= \left( \frac{n+r-2}{2r-1} \right) = \left( \frac{n+(r+1)-3}{2(r+1)-3} \right),$$

as required.

In the above proof, $f : E^*_{r+1}(O_n) \to \bigcup_{i=2}^{n-r+1} (E^*_2(O_i) \times E^*_r(O_{n-i+1}))$ is defined similar to the function defined in the proof of Lemma 7 in [1].

**Theorem 1** For $1 \leq r \leq n$, we have $|E_r(O_n)| = \binom{n+r-1}{2r-1}$.

**Proof** Since we know that $|E_1(O_n)| = n$ and $|E_n(O_n)| = 1$, we consider the case $2 \leq r \leq n-1$. If $\alpha \in E_r(O_n)$ with $\text{fix}(\alpha) = \{i_1 < i_2 < \cdots < i_r\}$, then $\alpha$ has the following tabular form:

$$\alpha = \left( \begin{array}{cccc} 1 & \cdots & i_1 & \cdots \ i_r & \cdots & n \\ i_1 & \cdots & i_1 & \cdots & i_r & \cdots & i_r \end{array} \right),$$

where $1 \leq i_1 \leq n-r+1$ and $i_1 + r - 1 \leq i_r \leq n$. Let $i = i_1$ and $j = i_r$. Then, since

$$|E_r(O_n)| = \sum_{i=1}^{n-r+1} \sum_{j=i+1}^{n-r} |E^*_r(O_{r-i+1})|,$$

it follows from Proposition 2 and Result 1 that

$$|E_r(O_n)| = \sum_{i=1}^{n-r+1} \sum_{j=i+1}^{n-r+1} \left( \frac{j-i+r-2}{2r-3} \right) = \sum_{i=1}^{n-r+1} \sum_{j=0}^{n-i-r-1} \left( \frac{2r-3+j}{2r-3} \right)$$

$$= \sum_{i=1}^{n-r+1} \left( \frac{n-i+r-1}{2r-2} \right) = \sum_{i=0}^{n-r-1} \left( \frac{2r-2+i}{2r-2} \right) = \left( \frac{n+r-1}{2r-1} \right),$$

as required.

Now we are able to give the result obtained in [3, Theorem 2.3] as an immediate result of Theorem 1 and Result 3:

**Corollary 1** For $n \geq 2$, we have $|E(O_n)| = \sum_{r=1}^{n} \binom{n+r-1}{2r-1} = f_{2n}$. 

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Now, we examine the cardinalities of $E^*_t(\mathcal{OP}_n)$ and $E_r(\mathcal{OP}_n)$. Similarly, it is clear that $|E^*_1(\mathcal{OP}_n)| = 1$, $|E_1(\mathcal{OP}_n)| = n$ and $|E^*_n(\mathcal{OP}_n)| = |E_n(\mathcal{OP}_n)| = 1$. For a nonconstant $\alpha \in \mathcal{OP}_n$, it is stated and proved in [2, Proposition 2.3] that $\alpha$ is order-preserving if and only if $1 \alpha < n \alpha$.

**Proposition 3** For $2 \leq r \leq n$, we have $|E^*_r(\mathcal{OP}_n)| = \binom{n+r-1}{2r-1}$.

**Proof** For $2 \leq r \leq n - 1$, suppose that $\alpha \in E^*_r(\mathcal{OP}_n)$ with $\text{fix}(\alpha) = \{i_1 < \cdots < i_{r-1} < n\}$. Then $\alpha$ has the following tabular form:

$$
\alpha = \begin{pmatrix}
1 & \cdots & i_1 - 1 & i_1 & \cdots & n - 1 & n \\
1 \alpha & \cdots & (i_1 - 1) \alpha & i_1 & \cdots & (n - 1) \alpha & n
\end{pmatrix}
$$

where $1 \leq i_1 \leq n - r + 1$. Now let

$$
\alpha_1 = \begin{pmatrix}
n & 1 & \cdots & i_1 - 1 & i_1 \\
n & 1 \alpha & \cdots & (i_1 - 1) \alpha & i_1
\end{pmatrix}
$$

and

$$
\alpha_2 = \begin{pmatrix}
i_1 & i_1 + 1 & \cdots & n - 1 & n \\
i_1 & (i_1 + 1) \alpha & \cdots & (n - 1) \alpha & n
\end{pmatrix}.
$$

Since $i_1 \alpha = i_1 < n = n \alpha$, it follows from [2, Proposition 2.3] that $\alpha_2$ is an order-preserving idempotent on the set $\{i_1, i_1 + 1, \ldots, n\}$ with the standard order. Moreover, if we consider the set $\{n, 1, 2, \ldots, i_1\}$ with the order $n < 1 < 2 < \cdots < i_1$, then it is clear that $\alpha_1$ is an order-preserving idempotent on the chain $\{n < 1 < 2 < \cdots < i_1\}$ with $\text{fix}(\alpha_1) = \{n, i_1\}$. Next, let $i = i_1$ and $E(\mathcal{O}_{i+1})$ be the set of all order-preserving idempotents on the chain $\{n < 1 < 2 < \cdots < i\}$, and let $E^*_r(\mathcal{O}_{i+1}) = \{\alpha \in E(\mathcal{O}_{i+1}) : \text{fix}(\alpha) = \{n, i\}\}$. Then consider the function

$$
g : E^*_r(\mathcal{OP}_n) \rightarrow \bigcup_{i=1}^{n-r+1} (E^*_r(\mathcal{O}_{i+1}) \times E^*_r(\mathcal{O}_{n-i+1}))
$$

defined by $g : \alpha \mapsto (\alpha_1, \alpha_2)$ for every $\alpha \in E^*_r(\mathcal{OP}_n)$. Similarly, $g$ is also a bijection. Therefore, since $1 \leq i \leq n - r + 1$, it follows from Propositions 1 and 2 that

$$
|E^*_r(\mathcal{OP}_n)| = \sum_{i=1}^{n-r+1} i \binom{n-r-i+1+2r-3}{2r-3} = \sum_{j=1}^{n-r+1} \sum_{i=j}^{n-r+1} \binom{n-r-i+1+2r-3}{2r-3} = \sum_{j=1}^{n-r+1} \sum_{i=0}^{n-r-j+1} \binom{i+2r-3}{2r-3} = \sum_{j=1}^{n-r+1} \binom{n-r-j+1+2r-2}{2r-2} = \sum_{j=0}^{n-r} \binom{j+2r-2}{2r-2} = \binom{n-r+2r-1}{2r-1} = \binom{n+r-1}{2r-1},
$$

as required.
Theorem 2 For $2 \leq r \leq n$, we have

$$|E_r(\mathcal{OP}_n)| = \binom{n+r}{2r} + \binom{n+r-1}{2r} = \frac{n}{r} \binom{n+r-1}{2r-1}.$$  

Proof Since $|E_n(\mathcal{OP}_n)| = 1$, we consider the case $2 \leq r \leq n - 1$. If $\alpha \in E_r(\mathcal{OP}_n)$ with $\text{fix}(\alpha) = \{i_1 < i_2 < \cdots < i_r\}$, then $\alpha$ has the following tabular form:

$$\alpha = \begin{pmatrix} 1 & \cdots & i_1 & \cdots & i_r & \cdots & n \\ 1\alpha & \cdots & i_1 & \cdots & i_r & \cdots & n\alpha \end{pmatrix},$$

where $1 \leq i_1 \leq n - r + 1$ and $i_1 + r - 1 \leq i_r \leq n$. If we consider the following maps:

$$\alpha_1 = \begin{pmatrix} i_r & \cdots & n & 1 & \cdots & i_1 \\ i_r & \cdots & n\alpha & 1\alpha & \cdots & i_1 \end{pmatrix} \quad \text{and} \quad \alpha_2 = \begin{pmatrix} i_1 & \cdots & i_2 & \cdots & i_r \\ i_1 & \cdots & i_2 & \cdots & i_r \end{pmatrix},$$

then it is clear that $\alpha_2$ is an order-preserving idempotent on the set $\{i_1, i_1 + 1, \ldots, i_r\}$ with the standard order. If we consider the set $\{i_r, \ldots, n, 1, \ldots, i_1\}$ with the order $i_r < \cdots < n < 1 < \cdots < i_1$, then $\alpha_1$ is an order-preserving idempotent on the chain $\{i_r, \cdots, n, 1, \cdots, i_1\}$ with $\text{fix}(\alpha_1) = \{i_r, i_1\}$. We denote $i_1$ and $i_r$ by $i$ and $j$, respectively. Since there exist $(n - j + i)$ many order-preserving idempotent on the chain $\{j < \cdots < n < 1 < \cdots < i\}$ with $\text{fix}(\alpha_1) = \{j, i\}$, similarly, it follows from Proposition 2 that

$$|E_r(\mathcal{OP}_n)| = \sum_{i=1}^{n-r+1} \sum_{j=i+r-1}^{n} (n - j + i) \binom{j - i + 1 + r - 3}{2r - 3}.$$
By replacing $j - i$ by $j$, we have

$$|E_r(\mathcal{OP}_n)| = \sum_{i=1}^{n-r+1} \sum_{j=r-1}^{n-i} (n-j) \left( \frac{j+1+r-3}{2r-3} \right)$$

$$= \sum_{i=1}^{n-r+1} \left( n \sum_{j=r-1}^{n-i} \left( \frac{j+1+r-3}{2r-3} \right) - \sum_{j=r-1}^{n-i} j \left( \frac{j+1+r-3}{2r-3} \right) \right)$$

$$= \sum_{i=1}^{n-r+1} \left( n \sum_{j=0}^{n-i-r+1} \left( \frac{j+2r-3}{2r-3} \right) - \sum_{j=0}^{n-i-r+1} (j+r-1) \left( \frac{j+2r-3}{2r-3} \right) \right)$$

$$= \sum_{i=1}^{n-r+1} \left( (n-r+1) \sum_{j=0}^{n-i-r+1} \left( \frac{j+2r-3}{2r-3} \right) - \sum_{j=1}^{n-i-r+1} j \left( \frac{j+2r-3}{2r-3} \right) \right)$$

$$= \sum_{i=1}^{n-r+1} \left( \frac{n-i-r+1+2r-2}{2r-2} \right)$$

$$-(n-i-r+1) \left( \frac{n-i-r+1+2r-2}{2r-2} \right) + \left( \frac{n-i-r+1+2r-2}{2r-1} \right)$$

$$= \sum_{i=1}^{n-r} \left( \frac{i+2r-2}{2r-2} \right) + \sum_{i=0}^{n-r-1} \left( \frac{i+2r-1}{2r-1} \right)$$

$$= \frac{n+r-1}{2r-1} + \frac{n+r-1}{2r} + \frac{n+r-1}{2r}$$

$$= \frac{n+r}{2r} + \frac{n+r-1}{2r} = \frac{n}{2r} \left( n+r-1 \right),$$

as required.

It is shown in [2, Theorem 2.10] that $|E(\mathcal{OP}_n)| = f_{2n+1} + f_{2n-1} - n^2 + n - 2$. We also state and prove this result as a consequence of Theorem 2.

**Corollary 2** For $n \geq 1$, $|E(\mathcal{OP}_n)| = \sum_{r=1}^{n} |E_r(\mathcal{OP}_n)| = f_{2n+1} + f_{2n-1} - n^2 + n - 2$.

**Proof** First recall that $|E_1(\mathcal{OP}_n)| = n$. Since $|E_r(\mathcal{OP}_n)| = \binom{n+r}{2r} + \binom{n+r-1}{2r}$ for every $2 \leq r \leq n$, it follows
from Result 2 that

\[ |E(OP_n)| = \sum_{r=1}^{n} |E_r(OP_n)| = n + \sum_{r=2}^{n} |E_r(OP_n)| + 1 \]

\[ = n + \sum_{r=2}^{n-1} \left( \frac{n + r}{2r} \right) + \sum_{r=2}^{n-1} \left( \frac{n + r - 1}{2r} \right) + 1 \]

\[ = n + \left( \sum_{r=0}^{n} \left( \frac{n + r}{2r} \right) - \left( \frac{n}{0} \right) - \left( \frac{n + 1}{2} \right) - \left( \frac{2n}{2n} \right) \right) \]

\[ + \sum_{r=0}^{n-1} \left( \frac{n - 1 + r}{2r} \right) - \left( \frac{n - 1}{0} \right) - \left( \frac{n}{2} \right) + 1 \]

\[ = n + f_{2n+1} - 1 - \frac{(n + 1)n}{2} - 1 + f_{2(n-1)+1} - 1 - \frac{n(n - 1)}{2} + 1 \]

\[ = f_{2n+1} + f_{2n-1} + n - \frac{n}{2}((n + 1) + (n - 1)) - 2 \]

\[ = f_{2n+1} + f_{2n-1} - n^2 + n - 2, \]

as required

3. Cardinalities related to \( OP_n^- \)

In [2, Corollary 2.7], it is shown that \( |OP_n| = n \left( \binom{2n-1}{n-1} \right) - n^2 + n \). In [10], it is shown that \( OP_n^- \), the set of all orientation-preserving and order-decreasing transformations on the chain \( X_n \), is a submonoid of \( OP_n \) containing the Catalan monoid \( C_n \). Next, we find the cardinalities of \( OP_n^- \) and \( E(OP_n^-) \) in the following theorem. Recall that \( |C_n| = C_n \), where \( C_n = \frac{1}{n+1} \left( \binom{2n}{n} \right) \) is the \( n \)th Catalan number, and that \( |E(C_n)| = 2^{n-1} \) (see, for examples [5, Theorems 3.1 and 3.19] and [7, Corollaries 3.9 and 3.11]).

**Theorem 3** For each \( n \geq 1 \), we have

(i) \( |OP_n^-| = -n + 2 + \sum_{k=2}^{n} C_k \), and

(ii) \( |E(OP_n^-)| = -n + 2^n \).

**Proof** (i) Since \( 1\alpha = 1 \) for all \( \alpha \in OP_n^- \), it is clear that \( |OP_1^-| = 1 \) and \( |OP_2^-| = 2 \). Suppose that \( n \geq 3 \). Then we show that for any \( \alpha \in OP_n^- \setminus C_n \), there exists \( 2 \leq k \leq n - 1 \) such that \( (k + 1)\alpha = \cdots = n\alpha = 1 \), that is \( \alpha \) has the following tabular form:

\[
\begin{array}{cccccccc}
1 & \cdots & k - 1 & k & k + 1 & \cdots & n \\
1 & \cdots & (k - 1)\alpha & k\alpha & 1 & \cdots & 1 \\
\end{array}
\]

Since \( \alpha \) is not constant, then it is different from the zero element of \( C_n \), and since \( \alpha \) is not order-preserving, it follows from in [2, Proposition 2.3] that \( n\alpha \leq 1\alpha = 1 \), and so \( n\alpha = 1 \). Thus, there exists \( 2 \leq k \leq n - 1 \) such that \( 1 = (k + 1)\alpha < k\alpha \neq 1 \), as required. Moreover, it is clear that \( \alpha|_{X_k} : X_k \rightarrow X_k \) is a nonconstant,
order-preserving and order-decreasing transformation on the chain $X_k = \{1 < 2 \leq \cdots < k\}$. Therefore, we have

$$|\mathcal{OP}_n^-| = |\mathcal{C}_n| + |\mathcal{OP}_n^- \setminus \mathcal{C}_n| = -n + 2 + \sum_{k=2}^{n-1} C_k.$$  

(ii) Similarly, we have

$$|E(\mathcal{OP}_n^-)| = |E(\mathcal{C}_n)| + |E(\mathcal{OP}_n^- \setminus \mathcal{C}_n)| = 2^{n-1} + \sum_{k=2}^{n-1} (2^{k-1} - 1)$$  

$$= -n + 2 + \sum_{k=2}^{n} 2^{k-1} = -n + 2^n,$$

as required.

Recall that the transformation $\eta$, which is defined by $x\eta = 1$ for all $x \in X_n$, is the zero element of $\mathcal{OP}_n^-$. Also recall that an element $\alpha$ of $\mathcal{OP}_n^-$ is nilpotent if $\alpha^k = \eta$ for some $k \geq 1$. Let $N(\mathcal{OP}_n^-)$ denotes the set of all nilpotent elements in $\mathcal{OP}_n^-$. Since $\mathcal{OP}_n^-$ is a subsemigroup of $\mathcal{D}_n$, from [12, Lemma 1.6], we have the following immediate lemma.

**Lemma 1** For $\alpha \in \mathcal{OP}_n^-$, $\alpha$ is nilpotent if and only if $\text{fix}(\alpha) = \{1\}$.

In other words, we have $N(\mathcal{OP}_n^-) = \{\alpha \in \mathcal{OP}_n^- : \text{fix}(\alpha) = \{1\}\}$. In [12, Lemma 1.6], it is also shown that $N(\mathcal{D}_n)$ is an ideal of $\mathcal{D}_n$. Consequently, we have the following:

**Lemma 2** $N(\mathcal{OP}_n^-)$ is an ideal of $\mathcal{OP}_n^-$.  

From [9, Proposition 2.3] and [12, Lemma 4.2], it is known that $|N(\mathcal{C}_n)| = |\mathcal{C}_{n-1}| = C_{n-1}$ and $|N(\mathcal{D}_n)| = (n-1)!$, respectively. Now we have the following lemma.

**Lemma 3** For each $n \geq 2$, we have

$$|N(\mathcal{OP}_n^-)| = |\mathcal{OP}_{n-1}^-| = -n + 3 + \sum_{k=2}^{n-1} C_k.$$  

**Proof** For $\alpha \in N(\mathcal{OP}_n^-) \setminus N(\mathcal{C}_n)$, we know that there exists $2 \leq k \leq n - 1$ such that $k\alpha \neq 1$ and $(k+1)\alpha = \cdots = n\alpha = 1$. Similarly, by defining $\alpha|_{X_k} : X_k \rightarrow X_k$ as in the proof Theorem 3 (i), we have

$$|N(\mathcal{OP}_n^-)| = |N(\mathcal{C}_n)| + |N(\mathcal{OP}_n^+ \setminus \mathcal{C}_n)|$$  

$$= C_{n-1} + \sum_{k=2}^{n-1} (C_{k-1} - 1) = -n + 3 + \sum_{k=2}^{n-1} C_k = |\mathcal{OP}_{n-1}^-|,$$

as required.
For any \(2 \leq r \leq n-1\), let \(Y = \{m_1, m_2, \ldots, m_r\}\) with \(1 = m_1 < m_2 < \cdots < m_r \leq n\) and let
\[
\mathcal{C}_{n,Y} = \{\alpha \in \mathcal{C}_n : \text{fix}(\alpha) = Y\}, \quad \mathcal{C}_{n,r} = \{\alpha \in \mathcal{C}_n : |\text{fix}(\alpha)| = r\},
\]
\[
\mathcal{OP}_{n,Y} = \{\alpha \in \mathcal{OP}_n^- : \text{fix}(\alpha) = Y\} \quad \text{and} \quad \mathcal{OP}_{n,r} = \{\alpha \in \mathcal{OP}_n^- : |\text{fix}(\alpha)| = r\}.
\]
It is shown in [1, Theorem 10] that \(|\mathcal{C}_{n,Y}| = \prod_{j=1}^r C_{s_j-1}\) where \(s_j = m_{j+1} - m_j \quad (1 \leq j \leq r-1)\) and \(s_r = n-m_r+1\).

It is also shown in [1, Theorem 11] that
\[
|\mathcal{C}_{n,r}| = \frac{r}{2n-r} \left(\frac{2n-r}{n}\right).
\]

Next, we examine the cardinalities of \(\mathcal{OP}_{n,Y}^{-}\) and \(\mathcal{OP}_{n,r}^{-}\). From Lemma 1, we have \(|\mathcal{OP}_{n,(1)}^{-} = N(\mathcal{OP}_{n}^{-})\), and so from Lemma 3, \(|\mathcal{OP}_{n,(1)}^{-} = |\mathcal{OP}_{n-1}^{-}|\). Now we suppose that \(2 \leq |Y| \leq n-1\).

**Lemma 4** Let \(Y = \{1, m_2, \ldots, m_r\}\) be a proper subset of \(X_n\) with \(1 = m_1 < m_2 < \cdots < m_r\). Then
\[
|\mathcal{OP}_{n,Y}^{-} = \sum_{k=m_r}^n |\mathcal{C}_{k,Y}| = \sum_{k=m_r}^n \left(\prod_{j=1}^r C_{s_j-1}\right)
\]
where \(s_j = m_{j+1} - m_j \quad (1 \leq j \leq r-1)\) and \(s_r = k-m_r+1 \quad (m_r \leq k \leq n)\).

**Proof** Suppose that \(\alpha \in \mathcal{OP}_{n,Y}^{-} \setminus \mathcal{C}_n\). As stated in the proof of Theorem 3 (i), there exists \(2 \leq k \leq n-1\) such that \(k\alpha \neq 1\) and \((k+1)\alpha = \cdots = n\alpha = 1\), and so \(Y \subseteq \{1, 2, \ldots, k\} = X_k\). Similarly, by defining \(\alpha_{|X_k} : X_k \rightarrow X_k\) as in the proof of Theorem 3 (i), it follows from [1, Theorem 10] that
\[
|\mathcal{OP}_{n,Y}^{-} = \mathcal{C}_{n,Y} + \sum_{k=m_r}^{n-1} |\mathcal{C}_{k,Y}| = \sum_{k=m_r}^n |\mathcal{C}_{k,Y}| = \sum_{k=m_r}^n \left(\prod_{j=1}^r C_{s_j-1}\right)
\]
where \(s_j = m_{j+1} - m_j \quad (1 \leq j \leq r-1)\) and \(s_r = k-m_r+1 \quad (m_r \leq k \leq n)\).

It follows from [1, Theorem 11] that \(|\mathcal{C}_{k,r}| = \frac{r}{2k-r} \left(\frac{2k-r}{k}\right)\) for \(2 \leq r \leq k\).

**Theorem 4** For \(2 \leq r \leq n-1\), we have
\[
|\mathcal{OP}_{n,r}^{-} = \sum_{k=r}^{n-r} \frac{r}{2k-r} \left(\frac{2k+r}{k}\right).
\]

**Proof** Similarly, from [1, Theorem 11], for each \(2 \leq r \leq n-1\), we have
\[
|\mathcal{OP}_{n,r}^{-} = \mathcal{C}_{n,r} + \sum_{k=r}^{n-1} |\mathcal{C}_{k,r}|
\]
\[
= \sum_{k=r}^{n} \frac{r}{2k-r} \left(\frac{2k-r}{k}\right) = \sum_{k=0}^{n-r} \frac{r}{2k+r} \left(\frac{2k+r}{k}\right),
\]
as required.
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References


