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Combinatorial results for semigroups of orientation-preserving transformations

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Abstract: Let X_n denote the chain $\{1, 2, \ldots, n\}$ under its natural order. We denote the semigroups consisting of all order-preserving transformations and all orientation-preserving transformations on X_n by $\mathcal{O}_n$ and $\mathcal{OP}_n$, respectively. We denote by $E(U)$ the set of all idempotents of a subset U of a semigroup S. In this paper, we first determine the cardinalities of

$$\begin{aligned} E_r(\mathcal{O}_n) &= \{\alpha \in E(\mathcal{O}_n) : |\mathrm{im}(\alpha)| = |\mathrm{fix}(\alpha)| = r\}, \\ E_r^*(\mathcal{O}_n) &= \{\alpha \in E_r(\mathcal{O}_n) : 1, n \in \mathrm{fix}(\alpha)\}, \\ E_r(\mathcal{OP}_n) &= \{\alpha \in E(\mathcal{OP}_n) : |\mathrm{fix}(\alpha)| = r\}, \\ E_r^*(\mathcal{OP}_n) &= \{\alpha \in E_r(\mathcal{OP}_n) : n \in \mathrm{fix}(\alpha)\} \end{aligned}$$

$(1 \leq r \leq n)$ and then, by using these results, we determine the numbers of idempotents in $\mathcal{O}_n$ and $\mathcal{OP}_n$ by a new method. Let $\mathcal{OP}_n^-$ denote the semigroup of all orientation-preserving and order-decreasing transformations on X_n. Moreover, we determine the cardinalities of $\mathcal{OP}_n^-$, $\mathcal{OP}_{n,Y}^- = \{\alpha \in \mathcal{OP}_n^- : \mathrm{fix}(\alpha) = Y\}$ for any nonempty subset Y of X_n and $\mathcal{OP}_{n,r}^- = \{\alpha \in \mathcal{OP}_n^- : |\mathrm{fix}(\alpha)| = r\}$ for $1 \leq r \leq n$. Also, we determine the number of idempotents in $\mathcal{OP}_n^-$ and the number of nilpotents in $\mathcal{OP}_n^-$.

Key words: Order-preserving transformation, order-decreasing transformation, orientation-preserving transformation

1. Introduction

For $n \in \mathbb{Z}^+$, let $\mathcal{T}_n$ be the (full) transformation semigroup on the chain $X_n = \{1, 2, \ldots, n\}$ under its natural order. A transformation $\alpha \in \mathcal{T}_n$ is called *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X_n$, and *order-decreasing (order-increasing)* if $x\alpha \leq x$ $(x\alpha \geq x)$ for all $x \in X_n$. The subsemigroup of $\mathcal{T}_n$ consisting of all order-preserving transformations is denoted by $\mathcal{O}_n$, and the subsemigroup of $\mathcal{T}_n$ consisting of all order-decreasing (order-increasing) transformations is denoted by $\mathcal{D}_n$ $(\mathcal{D}_n^+)$. Also, the subsemigroup of $\mathcal{T}_n$ consisting of all order-preserving and order-decreasing (order-increasing) transformations is denoted by $\mathcal{C}_n$ $(\mathcal{C}_n^+)$ and called the Catalan monoid. A finite sequence $A = (a_1, a_2, \ldots, a_t)$ $(t \in \mathbb{Z}^+,\ a_1, \ldots, a_t \in X_n)$ is called *cyclic* if there exists no more than one subscript i such that $a_i > a_{i+1}$, and *anticyclic* if there exists no more than one subscript i such that $a_i < a_{i+1}$ where $a_{t+1} = a_1$. A transformation α in $\mathcal{T}_n$ is called *orientation-preserving* if the sequence $(1\alpha, 2\alpha, \ldots, n\alpha)$ is cyclic. The subsemigroup of $\mathcal{T}_n$ consisting of all orientation-preserving

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transformations is denoted by $\mathcal{OP}_n$. Moreover, the subsemigroup of $\mathcal{OP}_n$ consisting of all order-decreasing (order-increasing) transformations is denoted by $\mathcal{OP}_n^-$ ($\mathcal{OP}_n^+$). The *fix* and *image* sets of a transformation $\alpha \in \mathcal{T}_n$ are defined and denoted by

$$\mathrm{fix}(\alpha) = \{x \in X_n : x\alpha = x\} \text{ and } \mathrm{im}(\alpha) = \{x\alpha : x \in X_n\},$$

respectively. The set of all idempotents in any subset U of a semigroup S is denoted by $E(U)$, that is $E(U) = \{e \in U : e^2 = e\}$. It is clear that a transformation $\alpha \in \mathcal{T}_n$ is idempotent if and only if $\mathrm{fix}(\alpha) = \mathrm{im}(\alpha)$. The set of all nilpotents in a semigroup S with zero is denoted by $N(S)$, that is $N(S) = \{s \in S : s^m = 0, \text{ for some } m \in \mathbb{Z}^+\}$ where 0 denotes the zero element of S. For a nonempty subset A of a semigroup S, the smallest subsemigroup of S containing A is called the *subsemigroup generated by* A, and denoted by $\langle A \rangle$. If there exists a finite subset A of S such that $S = \langle A \rangle$, then S is called a *finitely generated semigroup*, and the *rank* of a finitely generated semigroup S is defined by $\mathrm{rank}(S) = \min\{\,|A| : \langle A \rangle = S\}$. Moreover, if $S = \langle A \rangle$ and $|A| = \mathrm{rank}(S)$, then A is called a *minimal generating set* of S. Similarly, the *idempotent rank* of a semigroup S is defined by $\mathrm{idrank}(S) = \min\{\,|A| : A \subseteq E(S) \text{ and } \langle A \rangle = S\}$. A minimal generating set (which is unique) and the rank of $N(\mathcal{C}_n)$, which is the nilpotent subsemigroup of $\mathcal{C}_n$, were determined in [6] and [14]. It is also clear that

$$\eta = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 1 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad \varepsilon = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n & \cdots & n \end{pmatrix},$$

are the zero elements of $\mathcal{OP}_n^-$ and $\mathcal{OP}_n^+$, respectively.

Some cardinalities of various kinds of transformation semigroups have been studied over a long period. Howie computed in [3] that the cardinality of $\mathcal{O}_n$ is $\binom{2n-1}{n-1}$, and Laradji and Umar computed in [9] that the cardinality of $\mathcal{C}_n$ is C_n, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. For $1 \leq r \leq n$, the numbers of elements in $\mathcal{O}_n$ and $\mathcal{C}_n$ with r fixed points are $\frac{r}{n}\binom{2n}{n+r}$ and $\frac{r}{2n-r}\binom{2n-r}{n}$, respectively (see, [5, 9]). In [1], the cardinalities of the sets $\mathcal{O}_{n,Y} = \{\alpha \in \mathcal{O}_n : \mathrm{fix}(\alpha) = Y\}$ and $\mathcal{C}_{n,Y} = \{\alpha \in \mathcal{C}_n : \mathrm{fix}(\alpha) = Y\}$ were computed for any nonempty subset Y of X_n. In [2], the set of all orientation-preserving transformations $\mathcal{OP}_n$ was considered and it is proven in [2, Theorem 2.2] that $\mathcal{OP}_n$ is a submonoid of $\mathcal{T}_n$ containing $\mathcal{O}_n$. Moreover, the authors of [2] proved that

$$\mathcal{OP}_n = \{a^k\alpha : 0 \leq k \leq n-1 \text{ and } \alpha \in \mathcal{O}_n\}$$

where $a = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}$, the n-cycle $(1\,2\,\cdots\,n)$, and that $|\mathcal{OP}_n| = n\binom{2n-1}{n-1} - n(n-1)$ in [2, Theorem 2.6 and Corollary 2.7], respectively. The semigroup of all orientation-preserving and order-increasing transformations $\mathcal{OP}_n^+$ was considered in [10] and the authors found a minimal (idempotent) generating set of $\mathcal{OP}_n^+$ in [10, Theorem 3.6] and [11]. We have not seen any information about the cardinality of $\mathcal{OP}_n^+$. For any $\alpha \in \mathcal{OP}_n^-$, consider the transformation $\hat{\alpha} : X_n \to X_n$, defined by $i\hat{\alpha} = n - (n-i+1)\alpha + 1$ for each $i = 1, 2, \ldots, n$. As defined and shown in [12, Lemma 1.1], the function $\theta : \mathcal{OP}_n^- \to \mathcal{OP}_n^+$, defined by $\alpha\theta = \hat{\alpha}$

for all $\alpha \in \mathcal{OP}_n^-$, is an isomorphism. Hence, we consider only the subsemigroup $\mathcal{OP}_n^-$, for $n \geq 2$. Let

$$\begin{aligned} E_r(\mathcal{O}_n) &= \{\alpha \in E(\mathcal{O}_n) : |\mathrm{im}(\alpha)| = |\mathrm{fix}(\alpha)| = r\}, \\ E_r^*(\mathcal{O}_n) &= \{\alpha \in E_r(\mathcal{O}_n) : 1, n \in \mathrm{fix}(\alpha)\}, \\ E_r(\mathcal{OP}_n) &= \{\alpha \in E(\mathcal{OP}_n) : |\mathrm{fix}(\alpha)| = r\} \text{ and} \\ E_r^*(\mathcal{OP}_n) &= \{\alpha \in E_r(\mathcal{OP}_n) : n \in \mathrm{fix}(\alpha)\}. \end{aligned}$$

Let f_n denote the nth Fibonacci number. For $1 \leq r \leq n$, the cardinality of $E_r(\mathcal{O}_n)$ is found in [9, Corollary 4.4]. Despite this fact, we first determine that $|E_r^*(\mathcal{O}_n)| = \binom{n+r-3}{2r-3}$ (for $2 \leq r \leq n-1$), and then we determine that $|E_r(\mathcal{O}_n)| = \binom{n+r-1}{2r-1}$, and conclude that $|E(\mathcal{O}_n)| = f_{2n}$. By using a similar method, we first find that

$$|E_r^*(\mathcal{OP}_n)| = \binom{n+r-1}{2r-1} \quad \text{and} \quad |E_r(\mathcal{OP}_n)| = \frac{n}{r}\binom{n+r-1}{2r-1}$$

for $2 \leq r \leq n$, then we conclude that $|E(\mathcal{OP}_n)| = f_{2n+1} + f_{2n-1} - n^2 + n - 2$ as in [2, Theorem 2.10]. In the last section, we show that

$$|\mathcal{OP}_n^-| = -n + 2 + \sum_{k=2}^{n} C_k \text{ and } |E(\mathcal{OP}_n^-)| = -n + 2^n$$

for all $n \geq 1$. It is shown in [9, Proposition 2.3] that $|N(\mathcal{C}_n)| = |\mathcal{C}_{n-1}| = C_{n-1}$, by using this result, we show that

$$|N(\mathcal{OP}_n^-)| = |\mathcal{OP}_{n-1}^-| = -n + 3 + \sum_{k=2}^{n-1} C_k$$

for all $n \geq 2$. In [1, 5, 9], the numbers of transformations in $\mathcal{O}_n$ and $\mathcal{C}_n$ with r fixed points were computed as $\frac{r}{n}\binom{2n}{n+r}$ and $\frac{r}{2n-r}\binom{2n-r}{n}$, respectively. By using a similar method as in [1, 13], the number of transformations in $\mathcal{OP}_n^-$ with r fixed points is computed as $\sum_{k=0}^{n-r} \frac{r}{2k+r}\binom{2k+r}{k}$ for $2 \leq r \leq n-1$.

2. Cardinalities related to $\mathcal{OP}_n$

We list some standard combinatorial results related to our studies. For natural numbers k and n, we have the following:

Result 1 *[8, Lemma 1.3].* $\sum_{i=0}^{n} \binom{k+i}{k} = \binom{n+k+1}{k+1}$.

Result 2 *[9, Corollary 4.5].* $\sum_{r=0}^{n} \binom{n+r}{2r} = f_{2n+1}$.

Result 3 *[9, Corollary 4.6].* $\sum_{r=1}^{n} \binom{n+r-1}{2r-1} = f_{2n}$.

Since $E_1(\mathcal{O}_n)$ consists of all the constant transformations in $\mathcal{O}_n$, and $E_n(\mathcal{O}_n)$ consists of only the identity, first we have $|E_1(\mathcal{O}_n)| = n$ and $|E_n(\mathcal{O}_n)| = 1$.

Proposition 1 *For $n \geq 3$, we have $|E_2^*(\mathcal{O}_n)| = n-1$ and $|E_2(\mathcal{O}_n)| = \binom{n+1}{3}$.*

Proof For any $i, j \in X_n$ with $i < j$, we first notice that there exist $j-i$ many idempotents in $E(\mathcal{O}_n)$ such that their image sets are the same and equal to $\{i, j\}$. Therefore, $|E_2^*(\mathcal{O}_n)| = n-1$, and moreover, it follows from Result 1 that

$$\begin{aligned} |E_2(\mathcal{O}_n)| &= \left| \bigcup_{1\leq i<j\leq n} \{\, \alpha \in E(\mathcal{O}_n) : \mathrm{im}(\alpha) = \{i,j\}\,\} \right| = \sum_{1\leq i<j\leq n} (j-i) \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \binom{j-i}{1} = \sum_{i=1}^{n-1} \sum_{j=0}^{n-i-1} \binom{1+j}{1} = \sum_{i=1}^{n-1} \binom{n-i+1}{2} \\ &= \sum_{i=0}^{n-2} \binom{2+i}{2} = \binom{n+1}{3}, \end{aligned}$$

as required.

Now recall that $\sum\limits_{k=1}^{n} k a_k = \sum\limits_{j=1}^{n} \sum\limits_{k=j}^{n} a_k$, which will be used in the proof of the following proposition.

Proposition 2 *For $2 \leq r \leq n-1$, we have $|E_r^*(\mathcal{O}_n)| = \binom{n+r-3}{2r-3}$.*

Proof We prove the claim by induction on r. For $r = 2$, the result follows from Proposition 1. Suppose that $2 \leq r \leq n-2$ and $\alpha \in E_{r+1}^*(\mathcal{O}_n)$. Then there exist $1 < i_1 < \cdots < i_{r-1} < n$ such that $\mathrm{fix}(\alpha) = \{1, i_1, \ldots, i_{r-1}, n\}$. If we define the following maps

$$\alpha_1 = \begin{pmatrix} 1 & 2 & \cdots & i_1 - 1 & i_1 \\ 1 & 2\alpha & \cdots & (i_1-1)\alpha & i_1 \end{pmatrix} \text{ and} \tag{1}$$

$$\alpha_2 = \begin{pmatrix} 1 & 2 & \cdots & n-i_1 & n-(i_1-1) \\ 1 & (i_1+1)\alpha-(i_1-1) & \cdots & (n-1)\alpha-(i_1-1) & n-(i_1-1) \end{pmatrix}, \tag{2}$$

then it is easy to see that α_1 and α_2 are two idempotents with the sets of fix points $\{1, i_1\}$ and $\{1, i_2 - i_1 + 1, \ldots, i_{r-1} - i_1 + 1, n - i_1 + 1\}$, respectively. Next let $i = i_1$ and consider the function

$$f : E_{r+1}^*(\mathcal{O}_n) \to \bigcup_{i=2}^{n-r+1} \left(E_2^*(\mathcal{O}_i) \times E_r^*(\mathcal{O}_{n-i+1}) \right)$$

which maps each $\alpha \in E_{r+1}^*(\mathcal{O}_n)$ to the ordered pair (α_1, α_2). For any $\alpha, \beta \in E_{r+1}^*(\mathcal{O}_n)$, if $\alpha f = \beta f$, then both $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, and so it follows from the definitions given in Equations 1 and 2 that $\alpha = \beta$. Moreover, for $(\gamma_1, \gamma_2) \in \bigcup\limits_{i=2}^{n-r+1} \left(E_2^*(\mathcal{O}_i) \times E_r^*(\mathcal{O}_{n-i+1}) \right)$, if we consider the following map

$$\gamma = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & 2+(i-1) & \cdots & n-i+(i-1) & n \\ 1 & 2\gamma_1 & \cdots & (i-1)\gamma_1 & i & 2\gamma_2+(i-1) & \cdots & (n-i)\gamma_2+(i-1) & n \end{pmatrix},$$

then it is easy to see that $\gamma \in E^*_{r+1}(\mathcal{O}_n)$ and $\gamma f = (\gamma_1, \gamma_2)$, and so f is a bijection. Thus, from Proposition 1 and the induction hypothesis, we have

$$\begin{aligned}
|E^*_{r+1}(\mathcal{O}_n)| &= \sum_{i=2}^{n-r+1} (i-1)\binom{n-i+1+r-3}{2r-3} = \sum_{i=1}^{n-r} i\binom{n-i+r-3}{2r-3} \\
&= \sum_{j=1}^{n-r}\sum_{i=j}^{n-r}\binom{n-i+r-3}{2r-3} = \sum_{j=1}^{n-r}\sum_{i=0}^{n-j-r}\binom{2r-3+i}{2r-3} \\
&= \sum_{j=1}^{n-r}\binom{n+r-2-j}{2r-2} = \sum_{j=0}^{n-r-1}\binom{2r-2+j}{2r-2} \\
&= \binom{n+r-2}{2r-1} = \binom{n+(r+1)-3}{2(r+1)-3},
\end{aligned}$$

as required.

In the above proof, $f : E^*_{r+1}(\mathcal{O}_n) \to \bigcup_{i=2}^{n-r+1} \big(E^*_2(\mathcal{O}_i) \times E^*_r(\mathcal{O}_{n-i+1})\big)$ is defined similar to the function defined in the proof of Lemma 7 in [1].

Theorem 1 *For $1 \leq r \leq n$, we have $|E_r(\mathcal{O}_n)| = \binom{n+r-1}{2r-1}$.*

Proof Since we know that $|E_1(\mathcal{O}_n)| = n$ and $|E_n(\mathcal{O}_n)| = 1$, we consider the case $2 \leq r \leq n-1$. If $\alpha \in E_r(\mathcal{O}_n)$ with $\text{fix}(\alpha) = \{i_1 < i_2 < \cdots < i_r\}$, then α has the following tabular form:

$$\alpha = \left(\begin{array}{ccccccc} 1 & \cdots & i_1 & \cdots & i_r & \cdots & n \\ i_1 & \cdots & i_1 & \cdots & i_r & \cdots & i_r \end{array}\right),$$

where $1 \leq i_1 \leq n-r+1$ and $i_1 + r - 1 \leq i_r \leq n$. Let $i = i_1$ and $j = i_r$. Then, since

$$|E_r(\mathcal{O}_n)| = \sum_{i=1}^{n-r+1}\sum_{j=i+r-1}^{n} |E^*_r(\mathcal{O}_{j-i+1})|,$$

it follows from Proposition 2 and Result 1 that

$$\begin{aligned}
|E_r(\mathcal{O}_n)| &= \sum_{i=1}^{n-r+1}\sum_{j=i+r-1}^{n}\binom{j-i+r-2}{2r-3} = \sum_{i=1}^{n-r+1}\sum_{j=0}^{n-i-r+1}\binom{2r-3+j}{2r-3} \\
&= \sum_{i=1}^{n-r+1}\binom{n-i+r-1}{2r-2} = \sum_{i=0}^{n-r}\binom{2r-2+i}{2r-2} = \binom{n+r-1}{2r-1},
\end{aligned}$$

as required.

Now we are able to give the result obtained in [3, Theorem 2.3] as an immediate result of Theorem 1 and Result 3:

Corollary 1 *For $n \geq 2$, we have $|E(\mathcal{O}_n)| = \sum_{r=1}^{n} \binom{n+r-1}{2r-1} = f_{2n}$.*

Now, we examine the cardinalities of $E_r^*(\mathcal{OP}_n)$ and $E_r(\mathcal{OP}_n)$. Similarly, it is clear that $|E_1^*(\mathcal{OP}_n)| = 1$, $|E_1(\mathcal{OP}_n)| = n$ and $|E_n^*(\mathcal{OP}_n)| = |E_n(\mathcal{OP}_n)| = 1$. For a nonconstant $\alpha \in \mathcal{OP}_n$, it is stated and proved in [2, Proposition 2.3] that α is order-preserving if and only if $1\alpha < n\alpha$.

Proposition 3 *For $2 \leq r \leq n$, we have $|E_r^*(\mathcal{OP}_n)| = \binom{n+r-1}{2r-1}$.*

Proof For $2 \leq r \leq n-1$, suppose that $\alpha \in E_r^*(\mathcal{OP}_n)$ with $\mathrm{fix}(\alpha) = \{i_1 < \cdots < i_{r-1} < n\}$. Then α has the following tabular form:

$$\alpha = \begin{pmatrix} 1 & \cdots & i_1-1 & i_1 & \cdots & i_{r-1} & \cdots & n-1 & n \\ 1\alpha & \cdots & (i_1-1)\alpha & i_1 & \cdots & i_{r-1} & \cdots & (n-1)\alpha & n \end{pmatrix}$$

where $1 \leq i_1 \leq n-r+1$. Now let

$$\begin{aligned} \alpha_1 &= \begin{pmatrix} n & 1 & \cdots & i_1-1 & i_1 \\ n & 1\alpha & \cdots & (i_1-1)\alpha & i_1 \end{pmatrix} \text{ and} \\ \alpha_2 &= \begin{pmatrix} i_1 & i_1+1 & \cdots & n-1 & n \\ i_1 & (i_1+1)\alpha & \cdots & (n-1)\alpha & n \end{pmatrix}. \end{aligned}$$

Since $i_1\alpha = i_1 < n = n\alpha$, it follows from [2, Proposition 2.3] that α_2 is an order-preserving idempotent on the set $\{i_1, i_1+1, \ldots, n\}$ with the standard order. Moreover, if we consider the set $\{n, 1, 2, \ldots, i_1\}$ with the order $n < 1 < 2 < \cdots < i_1$, then it is clear that α_1 is an order-preserving idempotent on the chain $\{n < 1 < 2 < \cdots < i_1\}$ with $\mathrm{fix}(\alpha_1) = \{n, i_1\}$. Next, let $i = i_1$ and $E(\mathcal{O}_{i+1})$ be the set of all order-preserving idempotents on the chain $\{n < 1 < 2 < \cdots < i\}$, and let $E^*(\mathcal{O}_{i+1}) = \{\, \alpha \in E(\mathcal{O}_{i+1}) : \mathrm{fix}(\alpha) = \{n, i\}\,\}$. Then consider the function

$$g : E_r^*(\mathcal{OP}_n) \to \bigcup_{i=1}^{n-r+1} \big(E^*(\mathcal{O}_{i+1}) \times E_r^*(\mathcal{O}_{n-i+1})\big)$$

defined by $g : \alpha \mapsto (\alpha_1, \alpha_2)$ for every $\alpha \in E_r^*(\mathcal{OP}_n)$. Similarly, g is also a bijection. Therefore, since $1 \leq i \leq n-r+1$, it follows from Propositions 1 and 2 that

$$\begin{aligned} |E_r^*(\mathcal{OP}_n)| &= \sum_{i=1}^{n-r+1} i \binom{n-r-i+1+2r-3}{2r-3} \\ &= \sum_{j=1}^{n-r+1} \sum_{i=j}^{n-r+1} \binom{n-r-i+1+2r-3}{2r-3} \\ &= \sum_{j=1}^{n-r+1} \sum_{i=0}^{n-r-j+1} \binom{i+2r-3}{2r-3} \\ &= \sum_{j=1}^{n-r+1} \binom{n-r-j+1+2r-2}{2r-2} = \sum_{j=0}^{n-r} \binom{j+2r-2}{2r-2} \\ &= \binom{n-r+2r-1}{2r-1} = \binom{n+r-1}{2r-1}, \end{aligned}$$

as required.

Theorem 2 *For $2 \le r \le n$, we have*

$$|E_r(\mathcal{OP}_n)| = \binom{n+r}{2r} + \binom{n+r-1}{2r} = \frac{n}{r}\binom{n+r-1}{2r-1}.$$

Proof Since $|E_n(\mathcal{OP}_n)| = 1$, we consider the case $2 \le r \le n-1$. If $\alpha \in E_r(\mathcal{OP}_n)$ with $\mathrm{fix}(\alpha) = \{i_1 < i_2 < \cdots < i_r\}$, then α has the following tabular form:

$$\alpha = \begin{pmatrix} 1 & \cdots & i_1 & \cdots & i_r & \cdots & n \\ 1\alpha & \cdots & i_1 & \cdots & i_r & \cdots & n\alpha \end{pmatrix},$$

where $1 \le i_1 \le n-r+1$ and $i_1 + r - 1 \le i_r \le n$. If we consider the following maps:

$$\alpha_1 = \begin{pmatrix} i_r & \cdots & n & 1 & \cdots & i_1 \\ i_r & \cdots & n\alpha & 1\alpha & \cdots & i_1 \end{pmatrix} \quad \text{and} \quad \alpha_2 = \begin{pmatrix} i_1 & \cdots & i_2 & \cdots & i_r \\ i_1 & \cdots & i_2 & \cdots & i_r \end{pmatrix},$$

then it is clear that α_2 is an order-preserving idempotent on the set $\{i_1, i_1+1, \ldots, i_r\}$ with the standard order. If we consider the set $\{i_r, \ldots, n, 1, \ldots, i_1\}$ with the order $i_r < \cdots < n < 1 < \cdots < i_1$, then α_1 is an order-preserving idempotent on the chain $\{i_r < \cdots < n < 1 < \cdots < i_1\}$ with $\mathrm{fix}(\alpha_1) = \{i_r, i_1\}$. We denote i_1 and i_r by i and j, respectively. Since there exist $(n-j+i)$ many order-preserving idempotent on the chain $\{j < \cdots < n < 1 < \cdots < i\}$ with $\mathrm{fix}(\alpha_1) = \{j, i\}$, similarly, it follows from Proposition 2 that

$$|E_r(\mathcal{OP}_n)| = \sum_{i=1}^{n-r+1} \sum_{j=i+r-1}^{n} (n-j+i)\binom{j-i+1+r-3}{2r-3}.$$

By replacing $j-i$ by j, we have

$$\begin{aligned}
|E_r(\mathcal{OP}_n)| &= \sum_{i=1}^{n-r+1}\sum_{j=r-1}^{n-i}(n-j)\binom{j+1+r-3}{2r-3} \\
&= \sum_{i=1}^{n-r+1}\left(n\sum_{j=r-1}^{n-i}\binom{j+1+r-3}{2r-3}-\sum_{j=r-1}^{n-i}j\binom{j+1+r-3}{2r-3}\right) \\
&= \sum_{i=1}^{n-r+1}\left(n\sum_{j=0}^{n-i-r+1}\binom{j+2r-3}{2r-3}-\sum_{j=0}^{n-i-r+1}(j+r-1)\binom{j+2r-3}{2r-3}\right) \\
&= \sum_{i=1}^{n-r+1}\left((n-r+1)\sum_{j=0}^{n-i-r+1}\binom{j+2r-3}{2r-3}-\sum_{j=1}^{n-i-r+1}j\binom{j+2r-3}{2r-3}\right) \\
&= \sum_{i=1}^{n-r+1}\Bigg((n-r+1)\binom{n-i-r+1+2r-2}{2r-2} \\
&\qquad -(n-i-r+1)\binom{n-i-r+1+2r-2}{2r-2}+\binom{n-i-r+1+2r-2}{2r-1}\Bigg) \\
&= \sum_{i=1}^{n-r+1}\left(i\binom{n-i-r+1+2r-2}{2r-2}+\binom{n-i-r+1+2r-2}{2r-1}\right) \\
&= \sum_{i=0}^{n-r}(n-r+1-i)\binom{i+2r-2}{2r-2}+\sum_{i=0}^{n-r-1}\binom{i+2r-1}{2r-1} \\
&= \binom{n+r-1}{2r-1}+\binom{n+r-1}{2r}+\binom{n+r-1}{2r} \\
&= \binom{n+r}{2r}+\binom{n+r-1}{2r}=\frac{n}{r}\binom{n+r-1}{2r-1},
\end{aligned}$$

as required.

It is shown in [2, Theorem 2.10] that $|E(\mathcal{OP}_n)|=f_{2n+1}+f_{2n-1}-n^2+n-2$. We also state and prove this result as a consequence of Theorem 2.

Corollary 2 *For* $n\geq 1$, $|E(\mathcal{OP}_n)|=\sum_{r=1}^{n}|E_r(\mathcal{OP}_n)|=f_{2n+1}+f_{2n-1}-n^2+n-2.$

Proof First recall that $|E_1(\mathcal{OP}_n)|=n$. Since $|E_r(\mathcal{OP}_n)|=\binom{n+r}{2r}+\binom{n+r-1}{2r}$ for every $2\leq r\leq n$, it follows from Result 2 that

$$\begin{aligned}
|E(\mathcal{OP}_n)| &= \sum_{r=1}^{n} |E_r(\mathcal{OP}_n)| = n + \sum_{r=2}^{n-1} |E_r(\mathcal{OP}_n)| + 1 \\
&= n + \sum_{r=2}^{n-1} \binom{n+r}{2r} + \sum_{r=2}^{n-1} \binom{n+r-1}{2r} + 1 \\
&= n + \left(\sum_{r=0}^{n} \binom{n+r}{2r} - \binom{n}{0} - \binom{n+1}{2} - \binom{2n}{2n} \right) \\
&\quad + \left(\sum_{r=0}^{n-1} \binom{n-1+r}{2r} - \binom{n-1}{0} - \binom{n}{2} \right) + 1 \\
&= n + f_{2n+1} - 1 - \frac{(n+1)n}{2} - 1 + f_{2(n-1)+1} - 1 - \frac{n(n-1)}{2} + 1 \\
&= f_{2n+1} + f_{2n-1} + n - \frac{n}{2}((n+1) + (n-1)) - 2 \\
&= f_{2n+1} + f_{2n-1} - n^2 + n - 2,
\end{aligned}$$

as required

3. Cardinalities related to $\mathcal{OP}_n^-$

In [2, Corollary 2.7], it is shown that $|\mathcal{OP}_n| = n\binom{2n-1}{n-1} - n^2 + n$. In [10], it is shown that $\mathcal{OP}_n^-$, the set of all orientation-preserving and order-decreasing transformations on the chain X_n, is a submonoid of $\mathcal{OP}_n$ containing the Catalan monoid $\mathcal{C}_n$. Next, we find the cardinalities of $\mathcal{OP}_n^-$ and $E(\mathcal{OP}_n^-)$ in the following theorem. Recall that $|\mathcal{C}_n| = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number, and that $|E(\mathcal{C}_n)| = 2^{n-1}$ (see, for examples [5, Theorems 3.1 and 3.19] and [7, Corollaries 3.9 and 3.11]).

Theorem 3 *For each $n \geq 1$, we have*

(*i*) $|\mathcal{OP}_n^-| = -n + 2 + \sum_{k=2}^{n} C_k$, *and*

(*ii*) $|E(\mathcal{OP}_n^-)| = -n + 2^n$.

Proof (*i*) Since $1\alpha = 1$ for all $\alpha \in \mathcal{OP}_n^-$, it is clear that $|\mathcal{OP}_1^-| = 1$ and $|\mathcal{OP}_2^-| = 2$. Suppose that $n \geq 3$. Then we show that for any $\alpha \in \mathcal{OP}_n^- \setminus \mathcal{C}_n$, there exists $2 \leq k \leq n-1$ such that $(k+1)\alpha = \cdots = n\alpha = 1$, that is α has the following tabular form:

$$\alpha = \left(\begin{array}{cccc|ccc} 1 & \cdots & k-1 & k & k+1 & \cdots & n \\ 1 & \cdots & (k-1)\alpha & k\alpha & 1 & \cdots & 1 \end{array} \right).$$

Since α is not constant, then it is different from the zero element of $\mathcal{C}_n$, and since α is not order-preserving, it follows from in [2, Proposition 2.3] that $n\alpha \leq 1\alpha = 1$, and so $n\alpha = 1$. Thus, there exists $2 \leq k \leq n-1$ such that $1 = (k+1)\alpha < k\alpha \neq 1$, as required. Moreover, it is clear that $\alpha_{|X_k} : X_k \to X_k$ is a nonconstant, order-preserving and order-decreasing transformation on the chain $X_k = \{1 < 2 < \cdots < k\}$. Therefore, we have

$$|\mathcal{OP}_n^-| = |\mathcal{C}_n| + |\mathcal{OP}_n^- \setminus \mathcal{C}_n| = C_n + \sum_{k=2}^{n-1}(C_k - 1) = -n + 2 + \sum_{k=2}^{n} C_k.$$

(ii) Similarly, we have

$$\begin{aligned} |E(\mathcal{OP}_n^-)| &= |E(\mathcal{C}_n)| + |E(\mathcal{OP}_n^- \setminus \mathcal{C}_n)| = 2^{n-1} + \sum_{k=2}^{n-1}\left(2^{k-1} - 1\right) \\ &= -n + 2 + \sum_{k=2}^{n} 2^{k-1} = -n + 2^n, \end{aligned}$$

as required.

Recall that the transformation η, which is defined by $x\eta = 1$ for all $x \in X_n$, is the zero element of $\mathcal{OP}_n^-$. Also recall that an element α of $\mathcal{OP}_n^-$ is nilpotent if $\alpha^k = \eta$ for some $k \geq 1$. Let $N(\mathcal{OP}_n^-)$ denotes the set of all nilpotent elements in $\mathcal{OP}_n^-$. Since $\mathcal{OP}_n^-$ is a subsemigroup of $\mathcal{D}_n$, from [12, Lemma 1.6], we have the following immediate lemma.

Lemma 1 *For $\alpha \in \mathcal{OP}_n^-$, α is nilpotent if and only if fix$(\alpha) = \{1\}$.*

In other words, we have $N(\mathcal{OP}_n^-) = \{\alpha \in \mathcal{OP}_n^- : \text{fix}(\alpha) = \{1\}\}$. In [12, Lemma 1.6], it is also shown that $N(\mathcal{D}_n)$ is an ideal of $\mathcal{D}_n$. Consequently, we have the following:

Lemma 2 *$N(\mathcal{OP}_n^-)$ is an ideal of $\mathcal{OP}_n^-$.*

From [9, Proposition 2.3] and [12, Lemma 4.2], it is known that $|N(\mathcal{C}_n)| = |\mathcal{C}_{n-1}| = C_{n-1}$ and $|N(\mathcal{D}_n)| = (n-1)!$, respectively. Now we have the following lemma.

Lemma 3 *For each $n \geq 2$, we have*

$$|N(\mathcal{OP}_n^-)| = |\mathcal{OP}_{n-1}^-| = -n + 3 + \sum_{k=2}^{n-1} C_k.$$

Proof For $\alpha \in N(\mathcal{OP}_n^-) \setminus N(\mathcal{C}_n)$, we know that there exists $2 \leq k \leq n-1$ such that $k\alpha \neq 1$ and $(k+1)\alpha = \cdots = n\alpha = 1$. Similarly, by defining $\alpha_{|X_k} : X_k \to X_k$ as in the proof Theorem 3 (i), we have

$$\begin{aligned} |N(\mathcal{OP}_n^-)| &= |N(\mathcal{C}_n)| + |N(\mathcal{OP}_n^- \setminus \mathcal{C}_n)| \\ &= C_{n-1} + \sum_{k=2}^{n-1}(C_{k-1} - 1) = -n + 3 + \sum_{k=2}^{n-1} C_k = |\mathcal{OP}_{n-1}^-|, \end{aligned}$$

as required.

For any $2 \leq r \leq n-1$, let $Y = \{m_1, m_2, \ldots, m_r\}$ with $1 = m_1 < m_2 < \cdots < m_r \leq n$ and let

$$\mathcal{C}_{n,Y} = \{\alpha \in \mathcal{C}_n : \text{fix}(\alpha) = Y\},\ \mathcal{C}_{n,r} = \{\alpha \in \mathcal{C}_n : |\text{fix}(\alpha)| = r\},$$

$$\mathcal{OP}^-_{n,Y} = \{\alpha \in \mathcal{OP}^-_n : \text{fix}(\alpha) = Y\} \text{ and } \mathcal{OP}^-_{n,r} = \{\alpha \in \mathcal{OP}^-_n : |\text{fix}(\alpha)| = r\}.$$

It is shown in [1, Theorem 10] that $|\mathcal{C}_{n,Y}| = \prod_{j=1}^{r} C_{s_j-1}$ where $s_j = m_{j+1} - m_j$ $(1 \leq j \leq r-1)$ and $s_r = n - m_r + 1$. It is also shown in [1, Theorem 11] that

$$|\mathcal{C}_{n,r}| = \frac{r}{2n-r} \binom{2n-r}{n}.$$

Next, we examine the cardinalities of $\mathcal{OP}^-_{n,Y}$ and $\mathcal{OP}^-_{n,r}$. From Lemma 1, we have $\mathcal{OP}^-_{n,\{1\}} = N(\mathcal{OP}^-_n)$, and so from Lemma 3, $|\mathcal{OP}^-_{n,\{1\}}| = |\mathcal{OP}^-_{n-1}|$. Now we suppose that $2 \leq |Y| \leq n-1$.

Lemma 4 *Let $Y = \{1, m_2, \ldots, m_r\}$ be a proper subset of X_n with $1 = m_1 < m_2 < \cdots < m_r$. Then*

$$|\mathcal{OP}^-_{n,Y}| = \sum_{k=m_r}^{n} |\mathcal{C}_{k,Y}| = \sum_{k=m_r}^{n} \left(\prod_{j=1}^{r} C_{s_j-1} \right)$$

where $s_j = m_{j+1} - m_j$ $(1 \leq j \leq r-1)$ and $s_r = k - m_r + 1$ $(m_r \leq k \leq n)$.

Proof Suppose that $\alpha \in \mathcal{OP}^-_n \setminus \mathcal{C}_n$. As stated in the proof of Theorem 3 (i), there exists $2 \leq k \leq n-1$ such that $k\alpha \neq 1$ and $(k+1)\alpha = \cdots = n\alpha = 1$, and so $Y \subseteq \{1, 2, \ldots, k\} = X_k$. Similarly, by defining $\alpha_{|X_k} : X_k \to X_k$ as in the proof of Theorem 3 (i), it follows from [1, Theorem 10] that

$$|\mathcal{OP}^-_{n,Y}| = |\mathcal{C}_{n,Y}| + \sum_{k=m_r}^{n-1} |\mathcal{C}_{k,Y}| = \sum_{k=m_r}^{n} |\mathcal{C}_{k,Y}| = \sum_{k=m_r}^{n} \left(\prod_{j=1}^{r} C_{s_j-1} \right)$$

where $s_j = m_{j+1} - m_j$ $(1 \leq j \leq r-1)$ and $s_r = k - m_r + 1$ $(m_r \leq k \leq n)$.

It follows from [1, Theorem 11] that $|\mathcal{C}_{k,r}| = \frac{r}{2k-r}\binom{2k-r}{k}$ for $2 \leq r \leq k$.

Theorem 4 *For $2 \leq r \leq n-1$, we have*

$$|\mathcal{OP}^-_{n,r}| = \sum_{k=0}^{n-r} \frac{r}{2k+r} \binom{2k+r}{k}.$$

Proof Similarly, from [1, Theorem 11], for each $2 \leq r \leq n-1$, we have

$$\begin{aligned} |\mathcal{OP}^-_{n,r}| &= |\mathcal{C}_{n,r}| + \sum_{k=r}^{n-1} |\mathcal{C}_{k,r}| = \sum_{k=r}^{n} |\mathcal{C}_{k,r}| \\ &= \sum_{k=r}^{n} \frac{r}{2k-r} \binom{2k-r}{k} = \sum_{k=0}^{n-r} \frac{r}{2k+r} \binom{2k+r}{k}, \end{aligned}$$

as required.

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