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On Lyapunov-type inequalities for five different types of higher order boundary value problems

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Abstract: In this paper, we establish the uniqueness and existence of the classical solution to higher-order boundary value problems. Then, we give a new Lyapunov-type inequalities for higher order boundary value problems. Our study is based on Green's functions corresponding to the five different types of two-point boundary value problems. In addition, some applications of the obtained inequalities are given.

Key words: Existence, uniqueness, Green's functions, Bernoulli polynomials, Lyapunov-type inequalities

1. Introduction

In this paper, we prove new Lyapunov-type inequalities for the higher-order differential equation of the form

$$y^{(n)} + p(x)\,y = 0, \tag{1.1}$$

where $n \in \mathbb{N}$, $p \in C([0,\infty),\mathbb{R})$ and y is a real solution of the Eq. (1.1) satisfying the two-point boundary conditions

$$\begin{array}{l} y^{(k)}(a) = y^{(k)}(b)\,;\ k = 0,1,...,n-1,\ \int_a^b y(\tau)\,d\tau = 0,\ \text{(Periodic (P))} \\ y^{(k)}(a) + y^{(k)}(b) = 0;\ k = 0,1,...,n-1,\ \text{(Anti-Periodic (AP))} \\ y^{(2k)}(a) = 0;\ k = 0,1,...,\left\lfloor \frac{n-1}{2} \right\rfloor,\ y^{(2k)}(b) = 0;\ k = 0,1,...,\left\lfloor \frac{n-2}{2} \right\rfloor,\ \text{(Dirichlet (D))} \\ y^{(2k+1)}(a) = 0;\ k = 0,1,...,\left\lfloor \frac{n-1}{2} \right\rfloor,\ y^{(2k+1)}(b) = 0;\ k = 0,1,...,\left\lfloor \frac{n-2}{2} \right\rfloor,\ \int_a^b y(\tau)\,d\tau = 0,\ \text{(Neumann (N))} \\ y^{(2k)}(a) = 0;\ k = 0,1,...,\left\lfloor \frac{n-1}{2} \right\rfloor,\ y^{(2k+1)}(b) = 0;\ k = 0,1,...,\left\lfloor \frac{n-2}{2} \right\rfloor,\ \text{(D-N (DN))} \end{array} \tag{1.2}$$

where $a, b \in \mathbb{R}$ with $a < b$, $y(x) \neq 0$, and $\lfloor x \rfloor = \max\{m \in \mathbb{Z} |\ m \leq x\}$.

In 1907, Lyapunov [27] obtained the following remarkable result: If $p \in C([0,\infty),\mathbb{R}^+)$ and y is a nontrivial solution of the Eq. (1.1) with $n = 2$ under the Dirichlet boundary conditions (1.2) with $n = 2$, then

$$\frac{4}{b-a} < \int_a^b p(\tau)\,d\tau. \tag{1.3}$$

Thus, this inequality provides a lower bound for the distance between the two consecutive zeros of y. The inequality (1.3) is the best possible in the sense that if the constant 4 in the left-hand side of (1.3) is replaced

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by any larger constant, then there exists an example of (1.1) with $n = 2$ for which (1.3) no longer holds ([25, p. 267]). This result has found many applications in areas, such as eigenvalue problems, stability, oscillation theory, disconjugate, etc.

Since the appearance of Lyapunov's fundamental paper, various proofs and generalizations or improvements have appeared in the literature. We should mention here that inequality (1.3) has been generalized to third order differential equations by Aktaş and Çakmak [5], Aktaş et al. [6]; to certain higher order differential equations by Agarwal and Özbekler [1–3], Beesack [10], Çakmak [16], Das and Vatsala [19, 20], Dhar and Kong [21], He and Tang [23], Pachpatte [31], Panigrahi [32], Parhi and Panigrahi [33], Yang et al. [39, 41], Yang and Lo [40], and Yang [42]; and to systems by Aktaş et al. [7, 9], Aktaş [8], Çakmak [12, 14, 15], Çakmak et al. [13], Çakmak and Tiryaki [17, 18], and Tiryaki et al. [34, 35].

In the literature, the authors have established some Lyapunov-type inequalities for higher-order differential equations as we stated above. By using Green's functions, Lyapunov-type inequalities can be found in Agarwal and Özbekler [1, 2], Aktaş et al. [4], Beesack [10], Das and Vatsala [19, 20], and Yang [42] for the same higher-order differential equations. Moreover, by using Green's functions corresponding to even order boundary value problems, Lyapunov-type inequalities for odd-order boundary value problems can be found in Aktaş et al. [6], and Dhar and Kong [21]. Here, we aim to obtain new Lyapunov-type inequalities for the same higher order problem, using the properties of Green's functions and Bernoulli polynomials.

Now, we give the definition and some properties of Bernoulli polynomials that are important for our results. We know that $B_n(x)$ is Bernoulli polynomial defined as

$$B_0(x) = 1,\ B_n'(x) = B_{n-1}(x),\ \int_0^1 B_n(\tau)\,d\tau = 0,\ 0 \le x \le 1,\ n \in \mathbb{N}. \tag{1.4}$$

It should be noted that the Bernoulli polynomial is defined differently by some authors. In fact, it is defined with the identity $B_n'(x) = nB_{n-1}(x)$. This is reflected in the fact that $B_n(x)$ is $\frac{1}{n!}$ multiple of the "original" one. We list up the properties of Bernoulli polynomial $B_n(x)$ which are required in this paper:

$$\begin{aligned}
B_j(1-x) &= (-1)^j B_j(x),\ j = 0, 1, ..., n, &(1.5)\\
B_{2j+1}(0) &= -\frac{1}{2}\ (j = 0),\ 0\ (j = 1, 2, ...), &(1.6)\\
\left(1-(-1)^j\right) B_j(0) &= 0\ (j = 0),\ -1\ (j = 1),\ 0\ (j = 2, 3, ...), &(1.7)\\
B_j\left(\frac{1}{2}\right) &= \left(2^{1-j}-1\right) B_j(0),\ j = 0, 1, ..., n. &(1.8)
\end{aligned}$$

The Fourier expansions of the Bernoulli polynomial are as follows ([22, pp. 37, 38]):

$$\begin{aligned}
B_{2n}(x) &= (-1)^{n-1}\frac{2}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^{2n}},\\
B_{2n-1}(x) &= (-1)^{n-1}\frac{2}{(2\pi)^{2n-1}}\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^{2n-1}},\\
B_n(x) &= -\frac{2}{(2\pi)^n}\sum_{k=1}^{\infty}\frac{\cos\left(2\pi kx-\frac{\pi n}{2}\right)}{k^n}
\end{aligned} \tag{1.9}$$

for $0 \leq x \leq 1$. It is easy to see that in (1.9), we get

$$\begin{array}{lcl} (-1)^{n-1} B_{2n}(0) & = & (-1)^{n-1} B_{2n}(1) = 2(2\pi)^{-2n} \zeta(2n) > 0, \\ (-1)^{n-1} B_{2n}(1/2) & = & 2(2\pi)^{-2n} \zeta(2n) \left(2^{1-2n} - 1\right) = (-1)^{n-1} B_{2n}(0) \left(2^{1-2n} - 1\right) < 0, \end{array} \tag{1.10}$$

$$(-1)^{n-1} B_{2n-1}(1/4) = -(-1)^{n-1} B_{2n-1}(3/4) = 2(2\pi)^{1-2n} \zeta_a(2n-1) < 0, \tag{1.11}$$

and

$$\begin{array}{lcl} (-1)^{n-1} B_n\left(\frac{m}{4}\right) & = & -B_n\left(1 - \frac{m}{4}\right) = \frac{2}{(2\pi)^n} \sum\limits_{k=1}^{\infty} \frac{\cos\left[\frac{(km+n)\pi}{2}\right]}{k^n} \\ & = & \frac{2}{(2\pi)^n} \sum\limits_{k=1}^{\infty} \frac{(-1)^{\left\lfloor \frac{km+n}{2} \right\rfloor}\left[1+(-1)^{km+n}\right]}{2k^n}, \\ -B_n\left[\mp\left(\frac{m}{4} - \frac{1}{2}\right)\right] & = & \frac{2}{(2\pi)^n} \sum\limits_{k=1}^{\infty} \frac{(-1)^k \cos\left[\frac{(km\pm n)\pi}{2}\right]}{k^n} \\ & = & \frac{2}{(2\pi)^n} \sum\limits_{k=1}^{\infty} \frac{(-1)^k (-1)^{\left\lfloor \frac{km\pm n}{2} \right\rfloor}\left[1+(-1)^{km\pm n}\right]}{2k^n}, \end{array} \tag{1.12}$$

where the Riemann zeta functions are

$$\zeta_a(n) = \sum_{k=1}^{\infty} \frac{(-1)^{\left\lfloor \frac{3k-1}{2} \right\rfloor}\left[1+(-1)^{k-1}\right]}{2k^n} \text{ and } \zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}, \ \mathcal{R}(n) > 1. \tag{1.13}$$

Note that when n is even, if we take $m = 0, 2, 4$ in (1.12), then they reduce to $B_{2n}(0)$, $B_{2n}(1/2)$, and $B_{2n}(1)$; whereas when n is odd, if we take $m = 1, 3$ in (1.12), they reduce to $B_{2n-1}\left(\frac{1}{4}\right)$ and $B_{2n-1}\left(\frac{3}{4}\right)$, respectively. It can be referred to Erdelyi [22, pp. 32-40]. For details about the Bernoulli polynomials and the Riemann zeta function are mentioned above.

Now, we state an important result for the even-order differential equations under five different types of boundary conditions. In 2011, by using the Sobolev inequalities, Watanabe et al. [36] obtained the following remarkable result for the even order differential equation

$$y^{(2n)} = g(x) \tag{1.14}$$

with the two-point boundary conditions

$$\begin{array}{ll} y^{(k)}(a) = y^{(k)}(b);\ k = 0, 1, ..., 2n-1,\ \int_a^b y(\tau)\, d\tau = 0, & \text{(P)} \\ y^{(k)}(a) + y^{(k)}(b) = 0;\ k = 0, 1, ..., 2n-1, & \text{(AP)} \\ y^{(2k)}(a) = y^{(2k)}(b) = 0;\ k = 0, 1, ..., n-1, & \text{(D)} \\ y^{(2k+1)}(a) = y^{(2k+1)}(b) = 0;\ k = 0, 1, ..., n-1,\ \int_a^b y(\tau)\, d\tau = 0, & \text{(N)} \\ y^{(2k)}(a) = y^{(2k+1)}(b) = 0;\ k = 0, 1, ..., n-1, & \text{(DN)} \end{array} \tag{1.15}$$

where $a, b \in \mathbb{R}$ with $a < b$ and $y(x) \neq 0$.

Theorem A ([36, 38]). *If $y(x)$ is a nontrivial solution of (1.14) satisfying one of the two-point boundary conditions (1.15), then*

$$y(x) = \int_a^b G_{2n}(\cdot; x, \tau)\, g(\tau)\, d\tau \tag{1.16}$$

holds, where

$$\begin{array}{lll} G_{2n}(\mathrm{P};x,\tau) & = & (-1)^{n-1} B_{2n}\left(\frac{|x-\tau|}{b-a}\right)(b-a)^{2n-1}, \\ G_{2n}(\mathrm{AP};x,\tau) & = & (-1)^{n-1}\left[B_{2n}\left(\frac{|x-\tau|}{2(b-a)}\right) - B_{2n}\left(\frac{1}{2}-\frac{|x-\tau|}{2(b-a)}\right)\right][2(b-a)]^{2n-1}, \\ G_{2n}(\mathrm{D};x,\tau) & = & (-1)^{n-1}\left[B_{2n}\left(\frac{|x-\tau|}{2(b-a)}\right) - B_{2n}\left(\frac{x+\tau-2a}{2(b-a)}\right)\right][2(b-a)]^{2n-1}, \\ G_{2n}(\mathrm{N};x,\tau) & = & (-1)^{n-1}\left[B_{2n}\left(\frac{|x-\tau|}{2(b-a)}\right) + B_{2n}\left(\frac{x+\tau-2a}{2(b-a)}\right)\right][2(b-a)]^{2n-1}, \\ G_{2n}(\mathrm{DN};x,\tau) & = & (-1)^{n-1}\left\{\left[B_{2n}\left(\frac{|x-\tau|}{4(b-a)}\right) - B_{2n}\left(\frac{1}{2}-\frac{|x-\tau|}{4(b-a)}\right)\right]\right. \\ & & \left. -\left[B_{2n}\left(\frac{x+\tau-2a}{4(b-a)}\right) - B_{2n}\left(\frac{1}{2}-\frac{x+\tau-2a}{4(b-a)}\right)\right]\right\}[4(b-a)]^{2n-1}, \end{array} \tag{1.17}$$

and $B_{2n}(x)$ is given in (1.4).

Theorem B ([36, Theorems 1.1 and 1.2]). *Let $n \in \mathbb{N}$ and $p \in C([0,\infty),\mathbb{R}^+)$. If the even-order differential equation (1.14) has a solution $y(x)$ satisfying one of the two-point boundary conditions (1.15), then the inequality*

$$\frac{1}{L(\cdot;2n,0)} \leq \int_a^b p(\tau)\,d\tau \tag{1.18}$$

holds, where

$$L(\cdot;2n,0) = \left\{\begin{array}{lll} \pi^{2n}\zeta^{-1}(2n)\left[2^{-1}(b-a)\right]^{1-2n} & = & L(\mathrm{P};2n,0), \\ \pi^{2n}\zeta^{-1}(2n)\left(2^{2n}-1\right)^{-1}\left[2^{-1}(b-a)\right]^{1-2n} & = & L(\mathrm{AP,D};2n,0), \\ 2^{-1}\pi^{2n}\zeta^{-1}(2n)(b-a)^{1-2n} & = & L(\mathrm{N};2n,0), \\ 2^{-1}\pi^{2n}\zeta^{-1}(2n)\left(2^{2n}-1\right)^{-1}(b-a)^{1-2n} & = & L(\mathrm{DN};2n,0), \end{array}\right. \tag{1.19}$$

and $\zeta(n)$ is given in (1.13).

Now, we give the best constants of Sobolev inequalities corresponding to various boundary value problems in the literature:

$$\begin{array}{lllr} S(\mathrm{P};n) & = & \left\{\begin{array}{lll} \|B_n(x)\|_{L^q(0,1)} & ; & n \text{ is odd}, \\ \|B_n(\alpha_0,x)\|_{L^q(0,1)} & ; & n \text{ is even}, \end{array}\right. & \begin{array}{r}[24]\\ [24]\end{array} \\ S(\mathrm{AP};n) & = & \frac{(b-a)^{n-1+\frac{1}{q}}}{2(n-1)!}\|E_{n-1}(x)\|_{L^q(0,1)}, & [26] \\ S(\mathrm{D};n) & = & 2^{2n-2}\|\delta_n(x)\|_{L^q(-1,1)} & [29, 30] \\ S(\mathrm{N};n) & = & \left\{\begin{array}{lll} 2^n\|B_n(x)\|_{L^q(0,1)} & ; & n \text{ is odd}, \\ 2^n\|B_n(\alpha_0,x)\|_{L^q(0,1)} & ; & n \text{ is even}, \end{array}\right. & \begin{array}{r}[28]\\ [28]\end{array} \\ S(\mathrm{DN};n) & = & \left\{\begin{array}{lll} 2^{2n-1}\|\gamma_n(x)\|_{L^q(0,1)} & ; & n \text{ is odd}, \\ 2\pi^{-2n}\zeta(2n)\left(2^{2n}-1\right)(b-a)^{2n-1} & ; & n \text{ is even}, \end{array}\right. & \begin{array}{r}[37]\\ [36]\end{array} \end{array} \tag{1.20}$$

where $E_{n-1}(x) = \frac{2^n}{n}\left[B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right]$ (Euler polynomial), α_0 is the unique solution to the equation $\int_0^\alpha \left[(-1)^{\left\lfloor\frac{n-1}{2}\right\rfloor}(B_n(x)-B_n(\alpha))\right]^{q-1}dx = \int_\alpha^{1/2}\left[(-1)^{\left\lfloor\frac{n}{2}\right\rfloor}(B_n(x)-B_n(\alpha))\right]^{q-1}dx$ in the interval $\left(0,\frac{1}{2}\right)$, $\|\cdot\|_{L^q}$ is the usual L^q norm, $\zeta(n)$ is given in (1.13), $q>1$, $\delta_n(x) = B_n\left(\frac{|x|}{4}\right) - B_n\left(\frac{2-x}{4}\right)$, and $\gamma_n(x) = (-1)^{n+1}B_n\left(\frac{1-x}{4}\right) + B_n\left(\frac{1+x}{4}\right)$. Note that most of the authors solve a specific problem on the interval $[0,1]$, but it can be simply extended to $[a,b]$.

There are various methods in the literature to obtain Lyapunov-type inequalities. Now, we state one of those methods that sheds light on our work. By taking the absolute value of both sides of the equation (1.16) with $g(x) = -p(x)\,y(x)$, choosing $x = x_1$ where $|y(x)|$ is maximized and cancelling out $|y(x_1)|$ on both sides of corresponding inequality, we have

$$1 < \max_{a \le x \le b} \int_a^b |G_{2n}(\cdot; x, \tau)|\, |p(\tau)|\, d\tau. \tag{1.21}$$

In this direction, when we look at the work done in the literature, we see that the best Lyapunov constant is obtained by taking the absolute maximum of the Green's functions. Yamagishi et al. [38] show that the absolute maximums of the Green's functions (1.17) are as follows:

$$\begin{aligned}
\max_{x,\tau\in[a,b]} |G_{2n}(\mathrm{P}; x, \tau)| &\le \max_{\tau\in[a,b]} |G_{2n}(\mathrm{P}; \tau, \tau)| = |B_{2n}(0)|\,(b-a)^{2n-1} \\
&= \pi^{-2n}\zeta(2n)\left[2^{-1}(b-a)\right]^{2n-1} = L(\mathrm{P}; 2n, 0), \\
\max_{x,\tau\in[a,b]} |G_{2n}(\mathrm{AP}; x, \tau)| &\le \max_{\tau\in[a,b]} |G_{2n}(\mathrm{AP}; \tau, \tau)| = |B_{2n}(0) - B_{2n}(1/2)|\,[2(b-a)]^{2n-1} \\
&= \pi^{-2n}\zeta(2n)\left(2^{2n}-1\right)\left[2^{-1}(b-a)\right]^{2n-1} = L(\mathrm{AP}; 2n, 0), \\
\max_{x,\tau\in[a,b]} |G_{2n}(\mathrm{D}; x, \tau)| &\le \max_{\tau\in[a,b]} |G_{2n}(\mathrm{D}; \tau, \tau)| = |G_{2n}(\mathrm{D}; (a+b)/2, (a+b)/2)| \\
&= |B_{2n}(0) - B_{2n}(1/2)|\,[2(b-a)]^{2n-1} = \pi^{-2n}\zeta(2n)\left(2^{2n}-1\right)\left[2^{-1}(b-a)\right]^{2n-1} \\
&= L(\mathrm{D}; 2n, 0), \\
\max_{x,\tau\in[a,b]} |G_{2n}(\mathrm{N}; x, \tau)| &\le \max_{\tau\in[a,b]} |G_{2n}(\mathrm{N}; \tau, \tau)| = |G_{2n}(\mathrm{N}; b, b)| = |B_{2n}(0) + B_{2n}(1)|\,[2(b-a)]^{2n-1} \\
&= 2\pi^{-2n}\zeta(2n)\,(b-a)^{2n-1} = L(\mathrm{N}; 2n, 0), \\
\max_{x,\tau\in[a,b]} |G_{2n}(\mathrm{DN}; x, \tau)| &\le \max_{\tau\in[a,b]} |G_{2n}(\mathrm{DN}; \tau, \tau)| = |G_{2n}(\mathrm{DN}; (a+b)/2, (a+b)/2)| \\
&= |B_{2n}(0) - B_{2n}(1/2)|\,[4(b-a)]^{2n-1} \\
&= 2\pi^{-2n}\zeta(2n)\left(2^{2n}-1\right)(b-a)^{2n-1} = L(\mathrm{DN}; 2n, 0),
\end{aligned} \tag{1.22}$$

where $\zeta(n)$ is given in (1.13). Note that the inequality (1.18) can be obtained by (1.10), (1.21), and (1.22). In this direction, we aim to obtain the new Lyapunov-type inequalities via the absolute maximums of the Green's functions in this paper.

Let $G_{2n-1}(\cdot; x, \tau)$ $(a < x, \tau < b)$ be Green functions defined by

$$\begin{aligned}
&G_{2n-1}(\mathrm{P}; x, \tau) = (-1)^{n-1}\, sgn(x-\tau)\, B_{2n-1}\left(\tfrac{|x-\tau|}{b-a}\right)(b-a)^{2(n-1)}, \\
&G_{2n-1}(\mathrm{AP}; x, \tau) = (-1)^{n-1}\, sgn(x-\tau)\left[B_{2n-1}\left(\tfrac{|x-\tau|}{2(b-a)}\right) + B_{2n-1}\left(\tfrac{1}{2} - \tfrac{|x-\tau|}{2(b-a)}\right)\right][2(b-a)]^{2(n-1)}, \\
&G_{2n-1}(\mathrm{D_a}; x, \tau) = (-1)^{n-1}\left[sgn(x-\tau)\, B_{2n-1}\left(\tfrac{|x-\tau|}{2(b-a)}\right) + B_{2n-1}\left(\tfrac{x+\tau-2a}{2(b-a)}\right)\right][2(b-a)]^{2(n-1)}, \\
&G_{2n-1}(\mathrm{N_a}; x, \tau) = (-1)^{n-1}\left[sgn(x-\tau)\, B_{2n-1}\left(\tfrac{|x-\tau|}{2(b-a)}\right) - B_{2n-1}\left(\tfrac{x+\tau-2a}{2(b-a)}\right)\right][2(b-a)]^{2(n-1)}, \\
&G_{2n-1}(\mathrm{DN_a}; x, \tau) = (-1)^{n-1}\left\{sgn(x-\tau)\left[B_{2n-1}\left(\tfrac{|x-\tau|}{4(b-a)}\right) + B_{2n-1}\left(\tfrac{1}{2} - \tfrac{|x-\tau|}{4(b-a)}\right)\right]\right. \\
&\left. + \left[B_{2n-1}\left(\tfrac{x+\tau-2a}{4(b-a)}\right) + B_{2n-1}\left(\tfrac{1}{2} - \tfrac{x+\tau-2a}{4(b-a)}\right)\right]\right\}[4(b-a)]^{2(n-1)},
\end{aligned} \tag{1.23}$$

where $B_{2n-1}(x)$ *is given in (1.4).* Thus, we obtain the Green's function (1.23) corresponding to the $(2n-1)$th order differential equation

$$y^{(2n-1)} = -g(x) \tag{1.24}$$

under the two-point boundary conditions

$$\begin{array}{lr} y^{(k)}(a)=y^{(k)}(b);\ k=0,1,...,2n-2,\ \int_a^b y(\tau)\,d\tau=0, & (\mathrm{P}) \\ y^{(k)}(a)+y^{(k)}(b)=0;\ k=0,1,...,2n-2, & (\mathrm{AP}) \\ y^{(2k)}(a)=0;\ k=0,1,...,n-1,\ y^{(2k)}(b)=0;\ k=0,1,...,n-2, & (\mathrm{D_a}) \\ y^{(2k+1)}(a)=0;\ k=0,1,...,n-1,\ y^{(2k+1)}(b)=0;\ k=0,1,...,n-2,\ \int_a^b y(\tau)\,d\tau=0, & (\mathrm{N_a}) \\ y^{(2k)}(a)=0,\ k=0,1,...,n-1,\ y^{(2k+1)}(b)=0;\ k=0,1,...,n-2 & (\mathrm{DN_a}) \end{array} \tag{1.25}$$

(see Theorem 2.1). Since only for $x<\tau$, it is sufficient to look at the absolute maximum of the Green's function $|G_{2n-1}(\cdot;x,\tau)|$, we have

$$\begin{aligned} &\max_{x,\tau\in[a,b]}|G_{2n-1}(\mathrm{P};x,\tau)|\le\max_{\tau\in[a,b]}|G_{2n-1}(\mathrm{P};\tau-(b-a)/4,\tau)|=|B_{2n-1}(1/4)|(b-a)^{2(n-1)} \\ &=\pi^{1-2n}|\zeta_a(2n-1)|\left[2^{-1}(b-a)\right]^{2(n-1)}=L(\mathrm{P};2n-1,1), \\ &\max_{x,\tau\in[a,b]}|G_{2n-1}(\mathrm{AP};x,\tau)|\le\max_{\tau\in[a,b]}|G_{2n-1}(\mathrm{AP};\tau-(b-a)/2,\tau)|=2|B_{2n-1}(1/4)|[2(b-a)]^{2(n-1)} \\ &=2\pi^{1-2n}|\zeta_a(2n-1)|(b-a)^{2(n-1)}=L(\mathrm{AP};2n-1,1), \\ &\max_{x,\tau\in[a,b]}|G_{2n-1}(\mathrm{D_a};x,\tau)|\le|G_{2n-1}(\mathrm{D_a};a,(a+b)/2)|=2|B_{2n-1}(1/4)|[2(b-a)]^{2(n-1)} \\ &=2\pi^{1-2n}|\zeta_a(2n-1)|(b-a)^{2(n-1)}=L(\mathrm{D_a};2n-1,1), \\ &\max_{x,\tau\in[a,b]}|G_{2n-1}(\mathrm{N_a};x,\tau)|\le|G_{2n-1}(\mathrm{N_a};(a+b)/2,b)|=2|B_{2n-1}(1/4)|[2(b-a)]^{2(n-1)} \\ &=2\pi^{1-2n}|\zeta_a(2n-1)|(b-a)^{2(n-1)}=L(\mathrm{N_a};2n-1,1), \\ &\max_{x,\tau\in[a,b]}|G_{2n-1}(\mathrm{DN_a};x,\tau)|\le|G_{2n-1}(\mathrm{DN_a};a,(a+b)/2)| \\ &=2|B_{2n-1}(1/8)+B_{2n-1}(3/8)|[4(b-a)]^{2(n-1)}=L(\mathrm{DN_a};2n-1,1), \end{aligned} \tag{1.26}$$

where $\zeta_a(n)$ is given in (1.13). Note that the values of $B_{2n-1}(1/8)$ and $B_{2n-1}(3/8)$ can be found by (1.9) in terms of Riemann zeta function.

On the other hand, let $G_n(\cdot;x,\tau)$ $(a<x,\tau<b)$ be Green functions defined by

$$\begin{aligned} &G_n(\mathrm{P};x,\tau)=(-1)^{\left\lfloor\frac{n-1}{2}\right\rfloor}[sgn(x-\tau)]^n B_n\left(\tfrac{|x-\tau|}{b-a}\right)(b-a)^{n-1}\ \ [24], \\ &G_n(\mathrm{AP};x,\tau)=(-1)^{\left\lfloor\frac{n-1}{2}\right\rfloor}[sgn(x-\tau)]^n\left[B_n\left(\tfrac{|x-\tau|}{2(b-a)}\right)-(-1)^n B_n\left(\tfrac12-\tfrac{|x-\tau|}{2(b-a)}\right)\right][2(b-a)]^{n-1}\ \ [26], \\ &G_n(\mathrm{D};x,\tau)=(-1)^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left[[sgn(x-\tau)]^n B_n\left(\tfrac{|x-\tau|}{2(b-a)}\right)-(-1)^n B_n\left(\tfrac{x+\tau-2a}{2(b-a)}\right)\right][2(b-a)]^{n-1}, \\ &G_n(\mathrm{N};x,\tau)=(-1)^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left[[sgn(x-\tau)]^n B_n\left(\tfrac{|x-\tau|}{2(b-a)}\right)+(-1)^n B_n\left(\tfrac{x+\tau-2a}{2(b-a)}\right)\right][2(b-a)]^{n-1}, \\ &G_n(\mathrm{DN};x,\tau)=(-1)^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left\{[sgn(x-\tau)]^n\left[B_n\left(\tfrac{|x-\tau|}{4(b-a)}\right)-(-1)^n B_n\left(\tfrac12-\tfrac{|x-\tau|}{4(b-a)}\right)\right]\right. \\ &\qquad\left.+(-1)^n\left[-B_n\left(\tfrac{x+\tau-2a}{4(b-a)}\right)+(-1)^n B_n\left(\tfrac12-\tfrac{x+\tau-2a}{4(b-a)}\right)\right]\right\}[4(b-a)]^{n-1}. \end{aligned} \tag{1.27}$$

where $B_n(x)$ *is given in (1.4).* Therefore, we also obtain the Green's function (1.27) corresponding to the nth boundary value problem

$$y^{(n)}=(-1)^n g(x) \tag{1.28}$$

under the two-point boundary conditions (1.2) (see Theorem 2.1). Since only for $x<\tau$, it is sufficient to look at the absolute maximum of the Green's function $|G_n(\cdot;x,\tau)|$, we have

$$\begin{aligned}
&\max_{x,\tau\in[a,b]}|G_n(\mathrm{P};x,\tau)| \le \max_{\tau\in[a,b]}\left|G_n\left(\mathrm{P};\tau-\tfrac{m(b-a)}{4},\tau\right)\right| = \left|B_n\left(\tfrac{m}{4}\right)\right|(b-a)^{n-1} = L(\mathrm{P};n,m),\\
&\max_{x,\tau\in[a,b]}|G_n(\mathrm{AP};x,\tau)| \le \max_{\tau\in[a,b]}\left|G_n\left(\mathrm{AP};\tau-\tfrac{m(b-a)}{2},\tau\right)\right|\\
&=\left|B_n\left(\tfrac{m}{4}\right)+(-1)^{n-1}B_n\left(\tfrac{2-m}{4}\right)\right|[2(b-a)]^{n-1} = L(\mathrm{AP};n,m),\\
&\max_{x,\tau\in[a,b]}|G_n(\mathrm{D};x,\tau)| \le \left|G_n\left(\mathrm{D};\tfrac{a+b}{2}-\tfrac{m(b-a)}{2},\tfrac{a+b}{2}\right)\right|\\
&=\left|B_n\left(\tfrac{m}{4}\right)+(-1)^{n-1}B_n\left(\tfrac{2-m}{4}\right)\right|[2(b-a)]^{n-1} = L(\mathrm{D};n,m),\\
&\max_{x,\tau\in[a,b]}|G_n(\mathrm{N};x,\tau)| \le \left|G_n\left(\mathrm{N};b-\tfrac{m(b-a)}{2},b\right)\right|\\
&=\left|B_n\left(\tfrac{m}{4}\right)-(-1)^{n-1}B_n\left(1-\tfrac{m}{4}\right)\right|[2(b-a)]^{n-1} = L(\mathrm{N};n,m),\\
&\max_{x,\tau\in[a,b]}|G_n(\mathrm{DN};x,\tau)| \le \left|G_n\left(\mathrm{DN};\tfrac{a+b}{2}-\tfrac{m(b-a)}{2},\tfrac{a+b}{2}\right)\right|\\
&=\Big|(-1)^n B_n\left(\tfrac{m}{8}\right)-B_n\left(\tfrac{1}{2}-\tfrac{m}{8}\right)\\
&-(-1)^n B_n\left(\tfrac{1}{4}-\tfrac{m}{8}\right)+B_n\left(\tfrac{1}{4}+\tfrac{m}{8}\right)\Big|[4(b-a)]^{n-1} = L(\mathrm{DN};n,m).
\end{aligned} \tag{1.29}$$

Note that when n is even (odd), if we take $m=0$ ($m=1$) in (1.29), then it reduces to (1.22) ((1.26)).

In this paper, we establish the uniqueness and existence of the classical solution to higher order boundary value problems. Then, we give new Lyapunov-type inequalities for higher order boundary value problems using the properties of Green's functions and Bernoulli polynomials.

2. Main results

Now, we give the uniqueness and existence of the classical solution to the boundary value problem (1.28) with (1.2). In fact, once Green's function is obtained, one can write down the solution of the problem (1.28) with (1.2) very easily using integral.

In the following result, the proofs of Green's functions $G_n(\mathrm{P};x,\tau)$ and $G_n(\mathrm{AP};x,\tau)$ in (1.27) are given the proof of Theorem 2.1 in Kametaka et al. [24] and Kiselak [26], respectively. Then, we give only the proof of Green's function $G_n(\mathrm{D};x,\tau)$ in (1.27). The proofs of Green's functions $G_n(\mathrm{N};x,\tau)$ and $G_n(\mathrm{DN};x,\tau)$ are similar to the proof of the Green's function $G_n(\mathrm{D};x,\tau)$ and hence are omitted.

Theorem 2.1 *Let g be a bounded continuous function. The nth order differential equation (1.28) has a nontrivial solution $y(x)$ satisfying one of the two-point boundary conditions (1.2) has one and only one classical solution $y(x)$ expressed as*

$$y(x)=\int_a^b G_n(\cdot;x,\tau)\,g(\tau)\,d\tau,\ a\le x\le b, \tag{2.1}$$

where Green function $G_n(\cdot;x,\tau)$ is given by (1.27).

Proof Let $y(x)$ be the function defined by the formula (2.1). We show that $y(x)$ satisfies Eq. (1.28) and the condition (1.2) (Dirichlet (D)), so that it is a solution of the equation Eq. (1.28).

The function $G_n(\mathrm{D};x,\tau)$ has continuous derivatives up to the $(n-2)$-th order inclusive, so we may differentiate with respect to x under the integral sign in (2.1) $(n-2)$ times. Hence

$$y^{(k)}(x)=\int_a^b \frac{\partial^k G_n(\mathrm{D};x,\tau)}{\partial x^k}g(\tau)\,d\tau \text{ for } k=1,2,...,(n-2). \tag{2.2}$$

Hence, the function $y(x)$ and its derivatives $y^{(k)}(x)$ up to the $(n-2)$-th order inclusive are continuous in the interval $[a,b]$.

The function $\frac{\partial^{n-1}G_n(\mathrm{D};x,\tau)}{\partial x^{n-1}}$ on the other hand, has a discontinuity at $x=\tau$. So, in calculating $y^{(n-1)}(x)$ and $y^{(n)}(x)$, we may not differentiate any more under the integral sign without preliminary manipulation. For this reason, we write formula (2.2) for $v=n-2$ in the following form:

$$y^{(n-2)}(x)=\int_a^x \frac{\partial^{n-2}G_n(\mathrm{D};x,\tau)}{\partial x^{n-2}}g(\tau)\,d\tau+\int_x^b \frac{\partial^{n-2}G_n(\mathrm{D};x,\tau)}{\partial x^{n-2}}g(\tau)\,d\tau. \tag{2.3}$$

In each of the intervals (a,x) and (x,b), the integrand and its derivative with respect to x are continuous; we therefore differentiate with respect to x under the integral sign and with respect to the upper (or lower) limit x, and obtain:

$$\begin{aligned} y^{(n-1)}(x) &= \int_a^x \frac{\partial^{n-1}G_n(\mathrm{D};x,\tau)}{\partial x^{n-1}}g(\tau)\,d\tau+\left[\frac{\partial^{n-2}G_n(\mathrm{D};x,\tau)}{\partial x^{n-2}}\right]\Bigg|_{\tau=x-0}g(x)\\ &\quad+\int_x^b \frac{\partial^{n-1}G_n(\mathrm{D};x,\tau)}{\partial x^{n-1}}g(\tau)\,d\tau-\left[\frac{\partial^{n-2}G_n(\mathrm{D};x,\tau)}{\partial x^{n-2}}\right]\Bigg|_{\tau=x+0}g(x). \end{aligned} \tag{2.4}$$

Since $\frac{\partial^{n-2}G_n(\mathrm{D};x,\tau)}{\partial x^{n-2}}$ is continuous at $\tau=x$, the two integrated terms cancel out, and there remains

$$y^{(n-1)}(x)=\int_a^x \frac{\partial^{n-1}G_n(\mathrm{D};x,\tau)}{\partial x^{n-1}}g(\tau)\,d\tau+\int_x^b \frac{\partial^{n-2}G_n(\mathrm{D};x,\tau)}{\partial x^{n-2}}g(\tau)\,d\tau, \tag{2.5}$$

i.e.

$$y^{(n-1)}(x)=\int_a^b \frac{\partial^{n-1}G_n(\mathrm{D};x,\tau)}{\partial x^{n-1}}g(\tau)\,d\tau. \tag{2.6}$$

By differentiating the formula (2.5) again we find, as above:

$$\begin{aligned} y^{(n)}(x) &= \int_a^x \frac{\partial^{n}G_n(\mathrm{D};x,\tau)}{\partial x^{n}}g(\tau)\,d\tau+\left[\frac{\partial^{n-1}G_n(\mathrm{D};x,\tau)}{\partial x^{n-1}}\right]\Bigg|_{\tau=x-0}g(x)\\ &\quad+\int_x^b \frac{\partial^{n-1}G_n(\mathrm{D};x,\tau)}{\partial x^{n-1}}g(\tau)\,d\tau-\left[\frac{\partial^{n-1}G_n(\mathrm{D};x,\tau)}{\partial x^{n-1}}\right]\Bigg|_{\tau=x+0}g(x). \end{aligned} \tag{2.7}$$

From the following Theorem 2.2 (g4),

$$\left[\frac{\partial^{n-1}G_n(\mathrm{D};x,\tau)}{\partial x^{n-1}}\right]\Bigg|_{\tau=x-0}-\left[\frac{\partial^{n-1}G_n(\mathrm{D};x,\tau)}{\partial x^{n-1}}\right]\Bigg|_{\tau=x+0}=(-1)^n \tag{2.8}$$

Hence (2.7) can be written in the form

$$y^{(n)}(x)=\int_a^b \frac{\partial^{n}G_n(\mathrm{D};x,\tau)}{\partial x^{n}}g(\tau)\,d\tau+(-1)^n g(x). \tag{2.9}$$

But the integral in the last formula vanishes; because, by hypothesis, the function $G_n(\mathrm{D};x,\tau)$, regarded as a function of x, satisfies the equation $y^{(n)}(x)=0$ in each of the intervals $[a,\tau)$ and $(\tau,b]$. Then, y satisfies the equation $y^{(n)}(x)=(-1)^n g(x)$.

Now, we show that $y(x)$ satisfies the condition (1.2) (Dirichlet (D)). The condition (1.2) contains merely the function values of $y(x)$ and its derivatives up to the order $(n-1)$ inclusive at the points $x=a$ and $x=b$. Then, we have

$$y^{(2k)}(a)=\int_a^b \frac{\partial^{2k} G_n(\mathrm{D};a,\tau)}{\partial x^{2k}} g(\tau)\,d\tau=0 \text{ and } y^{(2k)}(b)=\int_a^b \frac{\partial^{2k} G_n(\mathrm{D};b,\tau)}{\partial x^{2k}} g(\tau)\,d\tau=0 \tag{2.10}$$

for $k=0,1,...,\left\lfloor \frac{n-1}{2}\right\rfloor$. □

In the following result, the proofs of properties (g1)-(g6) of Green's functions $G_n(\mathrm{P};x,\tau)$ and $G_n(\mathrm{AP};x,\tau)$ are given the proofs of Lemmas 2.1 and 2.2 in Kametaka et al. [24] and Kiselak [26], respectively. Then, we give only the proof of properties of the Green's function under the Dirichlet boundary condition. The proofs of properties of Green's functions $G_n(\mathrm{N};x,\tau)$ and $G_n(\mathrm{DN};x,\tau)$ are similar to the proof of properties of the Green's function $G_n(\mathrm{D};x,\tau)$ and hence are omitted.

Theorem 2.2 *The Green function $G_n(\cdot;x,\tau)$ defined by (1.27) satisfies the following properties.*

(g1) $G_n(\cdot;x,\tau)=(-1)^n G_n(\cdot;\tau,x)$ *($\cdot$ =P,AP, $a<x,\tau<b$).*

(g2) $(-1)^n \dfrac{\partial^n G_n(\cdot;x,\tau)}{\partial x^n}=\begin{cases} 0 & (\cdot = AP,D,DN) \\ -1 & (\cdot = P,N,\ a<x,\tau<b,\ x\neq\tau). \end{cases}$

(g3) For $0\le k\le n-1$, we have

$$\left[\frac{\partial^k G_n(P;x,\tau)}{\partial x^k}\right]\Big|_{x=b} = \left[\frac{\partial^k G_n(P;x,\tau)}{\partial x^k}\right]\Big|_{x=a},$$
$$\left[\frac{\partial^k G_n(AP;x,\tau)}{\partial x^k}\right]\Big|_{x=b} = -\left[\frac{\partial^k G_n(AP;x,\tau)}{\partial x^k}\right]\Big|_{x=a}.$$

For $0\le k\le \left\lfloor\frac{n-1}{2}\right\rfloor$, we have

$$\left[\frac{\partial^{2k} G_n(D;x,\tau)}{\partial x^{2k}}\right]\Big|_{x=a,b}=0.$$

For $0\le k\le \left\lfloor\frac{n-2}{2}\right\rfloor$, we have

$$\left[\frac{\partial^{2k} G_n(N;x,\tau)}{\partial x^{2k}}\right]\Big|_{x=a,b}=0.$$

For $0\le k\le \left\lfloor\frac{n-1}{2}\right\rfloor$ and $0\le k\le \left\lfloor\frac{n-2}{2}\right\rfloor$, we have

$$\left[\frac{\partial^{2k} G_n(DN;x,\tau)}{\partial x^{2k}}\right]\Big|_{x=a}=0 \text{ and } \left[\frac{\partial^{2k+1} G_n(DN;x,\tau)}{\partial x^{2k+1}}\right]\Big|_{x=b}=0\ \ (a<\tau<b),$$

respectively.

(g4) $\left[\frac{\partial^k G_n(\cdot;x,\tau)}{\partial x^k}\right]\Big|_{\tau=x-0}-\left[\frac{\partial^k G_n(\cdot;x,\tau)}{\partial x^k}\right]\Big|_{\tau=x+0}$

$$=\begin{cases} 0 & (0\le k\le n-2) \\ (-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor} & (i=n-1)\ \ (\cdot = P,AP,D,N,DN,\ a<x<b). \end{cases}$$

$(g5)$ $\left[\frac{\partial^k G_n(\cdot;x,\tau)}{\partial x^k}\right]\Big|_{x=\tau+0} - \left[\frac{\partial^k G_n(\cdot;x,\tau)}{\partial x^k}\right]\Big|_{x=\tau-0}$

$$= \begin{cases} 0 & (0 \leq k \leq n-2) \\ (-1)^{\left\lfloor \frac{n+1}{2} \right\rfloor} & (i = n-1) \;\; (\cdot = P, AP, D, N, DN, \; a < \tau < b). \end{cases}$$

$(g6)$ $\int_a^b G_n(\cdot; x, \tau)\, dx = 0$ $(\cdot = P, N, \; a < \tau < b)$.

Proof Differentiating the Green's function $G_n(\mathrm{D}; x, \tau)$ $2k$ times with respect to x, we have

$$\frac{\partial^{2k} G_n(\mathrm{D};x,\tau)}{\partial x^{2k}} = (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left[[sgn\,(x-\tau)]^{n+2k} B_{n-2k}\left(\tfrac{|x-\tau|}{2(b-a)}\right) - (-1)^n B_{n-2k}\left(\tfrac{x+\tau-2a}{2(b-a)}\right) \right] [2\,(b-a)]^{n-2k-1}$$
$$\left(a < x, \tau < b, \; x \neq \tau, \; 0 \leq k \leq \left\lfloor \tfrac{n-1}{2} \right\rfloor\right). \tag{2.11}$$

(g2) follows (2.11). Substituting $x = a, b$ into (2.11), from (1.5), we have

$$\begin{aligned} \frac{\partial^{2k} G_n(\mathrm{D};x,\tau)}{\partial x^{2k}}\Big|_{x=b} &= (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left[[sgn\,(b-\tau)]^{n+2k} B_{n-2k}\left(\tfrac{|b-\tau|}{2(b-a)}\right) - (-1)^n B_{n-2k}\left(\tfrac{b+\tau-2a}{2(b-a)}\right) \right] [2\,(b-a)]^{n-2k-1} \\ &= (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left[B_{n-2k}\left(\tfrac{b-\tau}{2(b-a)}\right) - (-1)^{n-2k} B_{n-2k}\left(\tfrac{1}{2} + \tfrac{\tau-a}{2(b-a)}\right) \right] [2\,(b-a)]^{n-2k-1} \\ &= (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left[B_{n-2k}\left(\tfrac{b-\tau}{2(b-a)}\right) - B_{n-2k}\left(\tfrac{1}{2} - \tfrac{\tau-a}{2(b-a)}\right) \right] [2\,(b-a)]^{n-2k-1} \\ &= (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left[B_{n-2k}\left(\tfrac{b-\tau}{2(b-a)}\right) - B_{n-2k}\left(\tfrac{b-\tau}{2(b-a)}\right) \right] [2\,(b-a)]^{n-2k-1} = 0 \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \frac{\partial^{2k} G_n(\mathrm{D};x,\tau)}{\partial x^{2k}}\Big|_{x=a} &= (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left[[sgn\,(a-\tau)]^{n+2k} B_{n-2k}\left(\tfrac{|a-\tau|}{2(b-a)}\right) - (-1)^n B_{n-2k}\left(\tfrac{a+\tau-2a}{2(b-a)}\right) \right] [2\,(b-a)]^{n-2k-1} \\ &= (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left[B_{n-2k}\left(\tfrac{\tau-a}{2(b-a)}\right) - B_{n-2k}\left(\tfrac{\tau-a}{2(b-a)}\right) \right] [2\,(b-a)]^{n-2k-1} = 0 \\ &\left(a < \tau < b, \; x \neq \tau, \; 0 \leq k \leq \left\lfloor \tfrac{n-1}{2} \right\rfloor\right). \end{aligned} \tag{2.13}$$

So we have (g3). From (1.27), we have

$$\begin{aligned} & \left[\frac{\partial^k G_n(\mathrm{D};x,\tau)}{\partial x^k}\right]\Big|_{\tau=x-0} - \left[\frac{\partial^k G_n(\mathrm{D};x,\tau)}{\partial x^k}\right]\Big|_{\tau=x+0} \\ = \;& (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left[B_{n-k}(0) - (-1)^n B_{n-k}\left(\frac{x-a}{b-a}\right) \right] [2\,(b-a)]^{n-k-1} \\ & - (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left[(-1)^{n+k} B_{n-k}(0) - (-1)^n B_{n-k}\left(\frac{x-a}{b-a}\right) \right] [2\,(b-a)]^{n-k-1} \\ = \;& (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left(1 - (-1)^{n-k}\right) B_{n-k}(0)\, [2\,(b-a)]^{n-k-1} \\ & \left(a < \tau < b, \; x \neq \tau, \; 0 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor\right). \end{aligned} \tag{2.14}$$

From (1.7), we have (g4). (g5) is equivalent to (g4).

This completes the proof of Theorem 2.2. □

Now, we give the main results of this paper.

Theorem 2.3 *If the* $(2n-1)$*th order differential equation (1.24) has a nontrivial solution* $y(x)$ *satisfying one of the two-point boundary conditions (1.25), then the inequality*

$$1 < \int_a^b |G_{2n-1}(\cdot; x_1, \tau)| \, |p(\tau)| \, d\tau \tag{2.15}$$

holds, where $G_{2n-1}(\cdot; x, \tau)$ *is given in (1.23) and* $|y(x_1)| = \max\{|y(x)| : a \le x \le b\}$.

Proof Let $y(x)$ be a nontrivial solution of the problem (1.24) with one of the two-point boundary conditions (1.25). Pick $x_1 \in (a,b)$ so that $|y(x_1)| = \max\{|y(x)| : a \le x \le b\}$. From (1.16), $(2n-1)$ instead of $(2n)$, and (1.24), we obtain

$$|y(x_1)| \le \int_a^b |G_{2n-1}(\cdot; x_1, \tau)| \, |p(\tau)| \, |y(\tau)| \, d\tau \tag{2.16}$$

and hence

$$|y(x_1)| \le |y(x_1)| \int_a^b |G_{2n-1}(\cdot; x_1, \tau)| \, |p(\tau)| \, d\tau. \tag{2.17}$$

Dividing both sides by $y(x_1)$, we obtain the inequality (2.15). □

It is clear that from (1.26) and (2.15), we have the following result and hence the proof is omitted.

Theorem 2.4 *If the Eq. (1.24) has a nontrivial solution* $y(x)$ *satisfying one of the two-point boundary conditions (1.25), then the inequality*

$$\frac{1}{L(\cdot; 2n-1, 1)} < \int_a^b |p(\tau)| \, d\tau \tag{2.18}$$

holds, where $L(\cdot; 2n-1, 1)$ *is given in (1.26).*

Remark 2.5 *We believe that the Lyapunov-type inequality in Theorem 2.4 is the best possibility for the Eq. (1.24) under one of the boundary conditions (1.25) in the sense that one of the constants* $\{2^{2(1-n)}\pi^{1-2n}|\zeta_a(2n-1)|,$ $2\pi^{1-2n}|\zeta_a(2n-1)|,$ $2^{2n-1}\pi^{1-2n}|\zeta_a(2n-1)|\}$ *in the left hand side of the inequality (2.18) cannot be replaced by any larger constant.*

Now, we give another main result of this paper. Its proof is similar to the proof of Theorem 2.3 and hence the proof is omitted.

Theorem 2.6 *If the* n*th order differential equation (1.1) has a nontrivial solution* $y(x)$ *satisfying one of the two-point boundary conditions (1.2), then the inequality*

$$1 < \int_a^b |G_n(\cdot; x_1, \tau)| \, |p(\tau)| \, d\tau \tag{2.19}$$

holds, where $G_n(\cdot; x, \tau)$ *is given in (1.27) and* $|y(x_1)| = \max\{|y(x)| : a \le x \le b\}$.

It is easy to see that if it is replaced $(2n)$ and $(2n-1)$ by n in (2.19), then they reduce to (1.21) and (2.15), respectively. On the other hand, we have the following result from (1.29) and (2.19).

Theorem 2.7 *If the Eq. (1.1) has a nontrivial solution $y(x)$ satisfying one of the two-point boundary conditions (1.2), then the inequality*

$$\frac{1}{L(\cdot;n,m)} < \int_a^b |p(\tau)|\, d\tau \tag{2.20}$$

holds, where $L(\cdot;n,m)$ is given in (1.29).

Remark 2.8 *Note that when n is even (odd), if we take $m=0$ $(m=1)$ in (2.20), then it reduces to (1.18) ((2.18)). Therefore, our results improve and generalize the results in the literature.*

It is easy to see that from (1.9), we have $|B_n(x)| \le 2(2\pi)^{-n}\zeta(n)$. Thus, we also give the following result via (1.16), n instead of $(2n)$, and (1.27).

Theorem 2.9 *If the Eq. (1.1) has a nontrivial solution $y(x)$ satisfying one of the two-point boundary conditions (1.2), then the inequality*

$$\frac{1}{L(\cdot;n)} < \int_a^b |p(\tau)|\, d\tau \tag{2.21}$$

holds, where

$$L(\cdot;n) = \begin{cases} \pi^{-n}\zeta(n)\left[2^{-1}(b-a)\right]^{n-1} & = & L(P;n), \\ 2\pi^{-n}\zeta(n)\left[2^{-1}(b-a)\right]^{n-1} & = & L(AP,D,N;n), \\ 4\pi^{-n}\zeta(n)\left[2(b-a)\right]^{n-1} & = & L(DN;n). \end{cases} \tag{2.22}$$

We may adopt a different point of view and use (2.20) to obtain an extension of the following oscillation criterion due originally to Liapounoff (cf. [11]): If $y''(x)$ and $y''(x)\,y^{-1}(x)$ are continuous for $a \le x \le b$, with $y(a) = y(b) = 0$, then

$$\frac{4}{b-a} < \int_a^b \left|y''(\tau)\,y^{-1}(\tau)\right| d\tau. \tag{2.23}$$

Thus, (2.20) leads to the following extension: If $y^{(n)}(x)$ and $y^{(n)}(x)\,y^{-1}(x)$ are continuous for $a \le x \le b$, with (1.2), then

$$L(\cdot;n,m) < \int_a^b \left|y^{(n)}(\tau)\,y^{-1}(\tau)\right| d\tau, \tag{2.24}$$

where $L(\cdot;n,m)$ is given in (1.29).

Now, we give sufficient conditions for the nonexistence of nontrivial solutions of the boundary value problem (1.1) under one of the two-point boundary conditions (1.2) as immediate consequences of Theorem 2.7.

Corollary 2.10 *Assume that the inequality*

$$\frac{1}{L(\cdot;n,m)} \ge \int_a^b |p(\tau)|\, d\tau \tag{2.25}$$

holds, where $L(\cdot;n,m)$ *is given in (1.29). The problem (1.1) under one of the two-point boundary conditions (1.2) has no nontrivial solution.*

Next, the following result gives a sufficient condition for the uniqueness of the solution of the following nonhomogeneous boundary value problem

$$y^{(n)} + p(x)\,y = r(x) \tag{2.26}$$

under one of the two-point boundary conditions (1.2), where $n \in \mathbb{N},\ p \in C([0,\infty),\mathbb{R})$, and $r \in C([a,b],\mathbb{R})$ as an application of Lyapunov type inequality (2.20).

Theorem 2.11 *If the inequality (2.25) holds, then nonhomogenous boundary problem (2.26) under one of the two-point boundary conditions (1.2) has a unique solution.*

Proof To prove the uniqueness, it is sufficient to show that the homogeneous boundary value problem (2.26) has only trivial solution. Assume on the contrary that $y(x) \not\equiv 0$ is a solution of the homogeneous boundary value problem (2.26) under one of the two-point boundary conditions (1.2). Then, by using Lyapunov type inequality (2.20), we have

$$\frac{1}{L(\cdot;n,m)} < \int_a^b |p(\tau)|\,d\tau \tag{2.27}$$

which gives contradiction to (2.25). Therefore, the homogeneous boundary value problem (2.26) has only trivial solution. Because of the theory of boundary value problems, the nonhomogenous boundary problem (2.26) has a unique solution. □

As an example of Theorem 2.11, we consider the following nonhomogeneous boundary value problem

$$\begin{cases} y^{(n)} + y = 2\cos x,\ x \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right), \\ y\left(-\frac{\pi}{2}\right) = 0 = y\left(\frac{\pi}{2}\right). \end{cases} \tag{2.28}$$

Since the condition (2.25) is satisfied, the problem (2.28) has a unique solution. In particular, observe that $y(x) = \cos x$ is a solution of the problem (2.28) with $n = 4k,\ k \in \mathbb{Z}$. Note that the homogeneous boundary value problem corresponds to the problem (2.28) has only trivial solution.

In the following part, we apply the obtained Lyapunov-type inequality (2.20) to the eigenvalue problems associated with the nonlinear boundary value problem

$$y^{(n)} + \lambda h(x)\,y = 0 \tag{2.29}$$

under one of the two-point boundary conditions (1.2), where $h \in C([a,b],\mathbb{R})$ and $\lambda \in \mathbb{R}$ is an eigenvalue parameter. As direct consequences of Theorem 2.6, we obtain the following result.

Theorem 2.12 *Assume that* λ *is an eigenvalue of the boundary value problem (2.29). Then*

$$\frac{L(\cdot;n,m)}{\int_a^b |h(\tau)|\,d\tau} < |\lambda|, \tag{2.30}$$

where $L(\cdot;n,m)$ *is given in (1.29).*

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