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Globally generated vector bundles on the del Pezzo threefold of degree 6 with Picard number 2

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Abstract: We classify globally generated vector bundles on a general hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$ embedded by the Segre embedding, considering small first Chern classes $c_1 = (1,1)$ and $c_1 = (2,1)$.

Key words: Vector bundles, globally generated, curves in projective spaces

1. Introduction

Let $\mathbb{P}^n$ be the n-dimensional projective space over an algebraically closed field k of characteristic 0. Globally generated vector bundles on projective varieties are a significant topic in classical algebraic geometry. However, the classification of globally generated vector bundles with small first Chern classes, especially for projective spaces $\mathbb{P}^n$, has been undertaken only relatively recently by several authors, such as [1, 11, 14, 15]. Following these developments, Ballico, Huh, and Malaspina further investigated globally generated vector bundles on various projective varieties, including smooth quadric threefolds [4], complete intersection Calabi-Yau threefolds [5], Segre threefolds $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ [6] and $\mathbb{P}^1 \times \mathbb{P}^2$ [7]. Additionally, Ballico studied the case of $\mathbb{P}^3$ blown up at finitely many points [8].

Many of these are Fano threefolds, which are smooth projective threefolds X with ample $\omega_X^\vee$. The greatest positive integer r such that $\omega_X^\vee \cong \mathcal{H}^r$ for some ample $\mathcal{H} \in \operatorname{Pic}(X)$ is called the index of X. The index of X falls within the range $1 \le r \le 4$. Moreover, when $r = 4$, X is a projective space $\mathbb{P}^3$, and when $r = 3$, it becomes a quadric threefold. Fano threefolds with $r = 2$ are called del Pezzo. They are completely classified and fall into a very short list (see, e.g., [10]). In particular, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a del Pezzo threefold of degree 6, and the blow-up of $\mathbb{P}^3$ at a point is the del Pezzo threefold of degree 7. Thus, it is natural to explore the classification of globally generated vector bundles on other del Pezzo threefolds, such as cubic threefolds, as well.

In this article, we focus on a general hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$ embedded by the Segre embedding, which is the other example of a del Pezzo threefold of degree 6. Let X be a linear section of $\mathbb{P}^2 \times \mathbb{P}^2$ embedded by the Segre embedding. Our main result is to classify globally generated vector bundles on X with small first Chern classes, up to a trivial factor. When $c_1 = (1,1)$ or $c_1 = (2,1)$, we have the following result.

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Theorem 1.1 *Let $\mathcal{E}$ be a globally generated vector bundle of rank $r \geq 2$ on X with the first Chern class $c_1 = (1,1)$ or $(2,1)$ and the second Chern class $c_2 = (e_1, e_2)$. If $\mathcal{E}$ has no trivial factor, then $(e_1, e_2; r)$ is one of the following:*

$c_1 = (1,1)$*:*

$$(1,1;2), \quad (1,2;3), \quad (2,1;3), \quad (2,2;3 \leq r \leq 4), \quad (3,3;3 \leq r \leq 7).$$

$c_1 = (2,1)$*:*

$$\begin{aligned}&(1,2;2), \quad (1,3;3), \quad (2,2;2), \quad (2,c;3 \leq r \leq c) \text{ with } 3 \leq c \leq 6,\\ &(3,c;3 \leq r \leq c+1) \text{ with } 2 \leq c \leq 6, \quad (4,c;3 \leq r \leq c+3) \text{ with } 5 \leq c \leq 6,\\ &(5,8;3 \leq r \leq 14).\end{aligned}$$

Note that if C is the associated curve to $\mathcal{E}$ via the Hartshorne-Serre correspondence, we denote its second Chern class as $c_2 = C = (e_1, e_2)$, where (e_1, e_2) represents the bidegree of C (see the next section).

2. Preliminaries

The general references for this section are [6, 7, 9]. As in the Introduction, let X be a general hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$ embedded by the Segre embedding. There is the two projections $\pi_i : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$, $i = 1, 2$. The morphisms π_i induce maps $p_i : X \to \mathbb{P}^2$ by restriction, and such maps are isomorphic to the canonical map $\mathbb{P}(\Omega^1_{\mathbb{P}^2}(2)) \to \mathbb{P}^2$. Thinking of the second copy of $\mathbb{P}^2$ as the dual of the first one, then X can also be viewed naturally as the flag variety of pairs point-line in $\mathbb{P}^2$. We denote by $A(X)$ the Chow ring of X. Let h_i, $i = 1, 2$ be the classes of $p_i^* \mathcal{O}_{\mathbb{P}^2}(1)$ in $A(X)$ respectively. We have an isomorphism

$$A(X) \cong \mathbb{Z}[h_1, h_2]/(h_1^2 - h_1 h_2 + h_2^2, h_1^3, h_2^3).$$

We may identify $A^1(X) \cong \mathbb{Z}^2$ by $a_1 h_1 + a_2 h_2 \mapsto (a_1, a_2)$, $A^2(X) \cong \mathbb{Z}^2$ by $e_1 h_2^2 + e_2 h_1^2 \mapsto (e_1, e_2)$, and $A^3(X) \cong \mathbb{Z}$ by $c h_1^2 h_2 \mapsto c$.

For a curve C, write $C = C_1 \sqcup \cdots \sqcup C_s$ with $s \geq 1$ and $C_1, \ldots, C_s$ the connected components of C. Set

$$e_1 := \deg(\mathcal{O}_C(1,0)), e_2 := \deg(\mathcal{O}_C(0,1))$$

and call (e_1, e_2) the bidegree of C. We also set $\deg C := e_1 + e_2$ and call it the degree of C.

Let $\mathcal{E}$ be a globally generated vector bundle of rank r on X with Chern classes $c_1 = (a_1, a_2), c_2 = (e_1, e_2)$. Then it fits into the exact sequence

$$0 \to \mathcal{O}_X^{r-1} \to \mathcal{E} \to \mathcal{I}_{C|X}(a_1, a_2) \to 0, \tag{2.1}$$

where C is a smooth curve on X by [11, Section 2. G]. The associated curve C has bidegree (e_1, e_2).

For the construction of vector bundles, we use [2, Theorem 1.1.] for $L := \mathcal{O}_X(c_1)$. Note that for a smooth curve C, $\bigwedge_C := \bigwedge^2 N \otimes L^\vee|_C \cong \omega_C \otimes \omega_X^\vee \otimes L^\vee|_C$. For the existence of a globally generated vector bundle of rank r, $\bigwedge_C$ must be globally generated (and trivial if $r = 2$), and an $(r-1)$-dimensional vector subspace of $h^0(\bigwedge_C)$ corresponds to a globally generated vector bundle of rank r without trivial factor, i.e. with no factor isomorphic to $\mathcal{O}_X$ (cf. [2], [5, Theorem 2.8.]). The vector bundle $\mathcal{E}$ constructed from C is globally generated if and only if $\mathcal{I}_{C|X}(c_1)$ is globally generated.

Proposition 2.1 ([13, Proposition 1]) *Let $\mathcal{E}$ be a globally generated vector bundle of rank r on a reduced irreducible projective variety V over k such that $h^0(\mathcal{E}(-c_1)) \neq 0$. Then we have*

$$\mathcal{E} \cong \mathcal{O}_V^{r-1} \oplus \mathcal{O}_V(c_1).$$

According to this proposition, when $c_1 = 0$, the only globally generated vector bundle of rank r on X is $\mathcal{O}_X^r$.

Proposition 2.2 *Let $\mathcal{E}$ be a globally generated vector bundle of rank at least 2 on X with $c_1 = (a, 0), a > 0$. Then there exists a globally generated vector bundle $\mathcal{F}$ on $\mathbb{P}^2$ such that $\mathcal{E} = p_1^*\mathcal{F}$.*

Proof Let $\mathcal{E}$ be a globally generated vector bundle with $c_1 = (a, 0)$, and let $F = p_1^{-1}(p) \cong \mathbb{P}^1$ be a fiber of a point $p \in \mathbb{P}^2$. Then the restricted bundle $\mathcal{E}|_F$ is a globally generated vector bundle on $\mathbb{P}^1$ with $c_1 = 0$, hence $\mathcal{E}|_F \cong \mathcal{O}_{\mathbb{P}^1}^r$ by [12, Chapter 2, Lemma 1.3.3.]. The base-change theorem [12, page 11] implies $p_{1*}\mathcal{E}$ is locally free of rank r. Let $\mathcal{K}$ be the kernel of the evaluation map $\mathcal{O}_X^n \to \mathcal{E}$, where $n = h^0(\mathcal{E})$. Since the map $h^0(\mathcal{O}_F^n) \to h^0(\mathcal{E}|_F)$ is surjective, we have $h^1(\mathcal{K}|_F) = 0$, and so $R^1p_{1*}(\mathcal{K}) = 0$ by the base-change theorem. Therefore $\mathcal{O}_{\mathbb{P}^2}^n \to p_{1*}\mathcal{E}$ is surjective and so it follows that $p_{1*}\mathcal{E}$ is a globally generated vector bundle on $\mathbb{P}^2$.

The natural map $p_1^*p_{1*}\mathcal{E} \to \mathcal{E}$ is surjective since $\mathcal{E}$ is globally generated. Furthermore, as it is a surjective morphism between two vector bundles of the same rank, it is indeed an isomorphism. Thus, the proof is complete. □

Let $\mathcal{E}$ be a globally generated vector bundle with $c_1 = (a, 0)$ or $(0, a), a \geq 0$. By the above proposition, in these cases, the classification of globally generated vector bundles on X is reduced to the classification of globally generated vector bundles on $\mathbb{P}^2$.

Remark 2.3 *(cf. [15, Lemma 2]) If $\mathcal{E}$ is a globally generated vector bundle with Chern classes (c_1, c_2), then the dual of the kernel of the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E}$ is also a globally generated vector bundle with Chern classes $(c_1, c_1^2 - c_2)$.*

Example 2.4 *Let Y be a smooth complete intersection of two divisors in $|\mathcal{O}_X(a, b)|$ with $a > 0, b > 0$. Then $\mathcal{I}_{Y|X}(a, b)$ is globally generated. $Y = (ah_1 + bh_2)(ah_1 + bh_2) = (b^2 + 2ab)h_2^2 + (a^2 + 2ab)h_1^2$, hence its bidegree is $(e_1, e_2) = (b^2 + 2ab, a^2 + 2ab)$. The adjunction formula gives $\omega_Y \cong \mathcal{O}_Y(2a - 2, 2b - 2)$, and so we have*

$$\begin{aligned} 2p_a(Y) - 2 &= (ah_1 + bh_2)(ah_1 + bh_2)((2a - 2)h_1 + (2b - 2)h_2) \\ &= 6a^2b + 6ab^2 - 8ab - 2a^2 - 2b^2. \end{aligned}$$

We have $\bigwedge_Y = \mathcal{O}_Y(a, b)$, and this is globally generated and nontrivial. By the Hartshorne-Serre correspondence, this case gives a globally generated bundle of rank r with no trivial factor if and only if $3 \leq r \leq h^0(\bigwedge_Y) + 1$. The corresponding bundle in Remark 2.3 is a globally generated bundle with $c_2 = 0$, hence $\mathcal{O}_X(c_1) \oplus \mathcal{O}_X^n$ with its associated curve $C = \emptyset$.

Remark 2.5 *(cf. [6, Remark 2.4, 2.7.], [7, Remark 2.7.]) Assume that $\mathcal{I}_{C|X}(a, b)$ is globally generated. Let Y be a complete intersection of two general divisors in $|\mathcal{I}_{C|X}(a, b)|$. By the Bertini theorem, we have $Y = C \cup D$ with either $D = \emptyset$ or D a reduced curve containing no component of C and smooth outside $C \cap D$. Each* *connected component C_i of C appears with multiplicity one in Y because, affixing points $p_i \in C_i$ for every i, we can find a divisor $T \in |\mathcal{I}_{C|X}(a,b)|$ not containing the tangent line of C_i at p_i. Furthermore, Y is also connected since we have $h^0(\mathcal{O}_Y) = 1$ by vanishing of cohomologies and a Koszul complex standard exact sequence.*

Lemma 2.6 *Let $\mathcal{E}$ be a globally generated vector bundle with $c_1 = (a,1), a > 0$ and C be its associated curve. Then the map $p_1|_C : C \to \mathbb{P}^2$ is an embedding. In particular, $s = 1$ and C is isomorphic to a smooth plane curve of degree e_1.*

Proof Let $F = p_1^{-1}(p) \cong \mathbb{P}^1$ be a fiber of a point $p \in \mathbb{P}^2$ with class h_1^2. Since $\mathcal{O}_N(2-a,1)$ has degree 1 and F is rational, $\bigwedge_F = \omega_F(2-a,1)$ is not globally generated. Hence F cannot be a connected component of C.

Since $\mathcal{I}_{C|X}(a,1)$ is globally generated, we have $\deg(F \cap C) \leq 1$ for all $p \in \mathbb{P}^2$ and so $p_1|_C : C \to \mathbb{P}^2$ is an embedding. C is isomorphic to a smooth plane curve of degree e_1. Since each plane curve is connected, we have $s = 1$. □

Remark 2.7 *If $\mathcal{I}_{C|X}(a,b)$ is globally generated, then the curve C is contained in the complete intersection of two hypersurfaces of type (a,b). As a result, we have $e_1 \leq 2ab + b^2, e_2 \leq a^2 + 2ab$.*

3. Case of $c_1 = (1,1)$

Let C be an associated curve of a globally generated vector bundle with $c_1 = (1,1)$ and no trivial factor. As in the previous subsection, if $\omega_C(1,1)$ is globally generated, there exists a globally generated bundle of rank r for $3 \leq r \leq h^0(\omega_C(1,1)) + 1$, and r can be 2 if and only if $\omega_C(1,1) \cong \mathcal{O}_C$. In any case, since $\bigwedge_C = \omega_C(1,1)$ must be globally generated, no component of C is a line.

Assume that C is a complete intersection curve of two general divisors in $|\mathcal{O}_X(1,1)|$, with $(e_1, e_2) = (3,3)$. Then $\bigwedge_C = \mathcal{O}_C(1,1)$. Since C is elliptic and $\mathcal{O}_C(1,1)$ is a line bundle of degree 6, as stated in Example 2.4, C gives a globally generated bundle of rank r if and only if $3 \leq r \leq 7$.

In the following, we assume $C \neq \emptyset$. By symmetry, we also assume $e_1 \leq e_2$. According to Remark 2.7, we have $e_1 \leq 3$ and $e_2 \leq 3$. Moreover, as stated in Remark 2.3, we can assume $e_1 \leq 1$ in order to classify globally generated vector bundles. If $e_1 = 0$, it implies that C is a disjoint union of e_2 fibers of p_1. However, we have excluded these cases, so we must have $e_1 \neq 0$ (and similarly, $e_2 \neq 0$). Additionally, based on Remark 2.3, we can also exclude the cases $(e_1, e_2) = (1,3)$ and $(e_1, e_2) = (2,3)$.

Proposition 3.1 *Let $\mathcal{E}$ be a globally generated vector bundle of rank $r \geq 2$ on X with $c_1 = (1,1)$ and no trivial factor. If the associated curve C has bidegree $(1,1)$, then we have $\mathcal{E} \cong \mathcal{O}_X(1,0) \oplus \mathcal{O}_X(0,1)$.*

Proof By Lemma 2.6, C is isomorphic to a line. Since C is connected and rational, we have $\bigwedge_C = \omega_C(1,1) \cong \mathcal{O}_C$ and $h^0(\mathcal{O}_C) = 1$, and so this curve gives a rank 2 vector bundle $\mathcal{E}$. Since $h^0(\mathcal{O}_C(1,0)) = 2$ and $h^0(\mathcal{O}_X(1,0)) = 3$ by [9, Proposition 2.5.], so $h^0(\mathcal{I}_{C|X}(1,0)) > 0$. Since $h^1(\mathcal{O}_X(0,-1)) = 0$, by the exact sequence (2.1), we have $h^0(\mathcal{E}(0,-1)) > 0$ and so there is a non-zero map $m : \mathcal{O}_X(0,1) \to \mathcal{E}$. We have $h^0(\mathcal{E}(-1,-1)) = h^0(\mathcal{E}(0,-2)) = 0$ since $h^0(\mathcal{I}_{C|X}) = h^0(\mathcal{I}_{C|X}(1,-1)) = 0$, and so $\mathcal{O}_X(0,1)$ is saturated in $\mathcal{E}$. Hence the cokernel of m is torsion free, i.e. $\operatorname{coker}(m) \cong \mathcal{I}_T(1,0)$ with either $T = \emptyset$ or T a locally complete intersection curve. Since $c_2(\mathcal{O}_X(1,0)\oplus\mathcal{O}_X(0,1)) = h_2^2 + h_1^2 = C$, we have $T=\emptyset$. Since $h^1(\mathcal{O}_X(-1,1)) = 0$, this implies $\mathcal{E}\cong\mathcal{O}_X(1,0)\oplus\mathcal{O}_X(0,1)$. □

Proposition 3.2 *Let $\mathcal{E}$ be a globally generated vector bundle of rank $r\geq 2$ on X with $c_1=(1,1)$ and no trivial factor. If the associated curve C has bidegree $(1,2)$, then we have $\mathcal{E}\cong p_1^*(\mathcal{T}_{\mathbb{P}^2}(-1))\oplus\mathcal{O}_X(0,1)$.*

Proof By Lemma 2.6, C is isomorphic to a line. Since C is connected and rational, we have $\bigwedge_C = \omega_C(1,1)\cong\mathcal{O}_{\mathbb{P}^1}(1)$ and $h^0(\mathcal{O}_{\mathbb{P}^1}(1)) = 2$, and so this curve gives a rank 3 vector bundle $\mathcal{E}$. Since $h^0(\mathcal{O}_C(1,0)) = 2$ and $h^0(\mathcal{O}_X(1,0)) = 3$, so $h^0(\mathcal{I}_{C|X}(1,0)) > 0$. Since $h^1(\mathcal{O}_X(0,-1)) = 0$, by the exact sequence (2.1), we have $h^0(\mathcal{E}(0,-1)) > 0$ and so there is a non-zero map $m:\mathcal{O}_X(0,1)\to\mathcal{E}$. By the same discussion as in the previous proposition, $\mathcal{F} := \operatorname{coker}(m)$ is torsion free, and $c_1(\mathcal{F}) = (1,0)$.

Since C is rational and of degree 3, $H^0(\mathcal{O}_X(1,1))\to H^0(\mathcal{O}_C(1,1))$ is surjective. Hence, $h^0(\mathcal{I}_C(1,1)) = 4$, and thus $h^0(\mathcal{E}) = 6$ by (2.1). It follows that $h^0(\mathcal{F}) = 3$. Since $\mathcal{F}$ is globally generated, there exists a surjective map $\mathcal{O}_X^3\to\mathcal{F}\to 0$. By [12, Chapter 2, Lemma 1.1.16.], the kernel is reflexive of rank 1, hence a line bundle $\mathcal{O}_X(-1,0)$ by [12, Chapter 2, Lemma 1.1.15.].

A map $v:\mathcal{O}_X(-1,0)\to\mathcal{O}_X^3$ is defined by three elements v_1,v_2,v_3 of $H^0(\mathcal{O}_X(1,0))$. Then, the map v is induced by a map $v':\mathcal{O}_{\mathbb{P}^2}(-1)\to\mathcal{O}_{\mathbb{P}^2}^3$ and the corresponding three elements v_i''s of $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$. The v_i's have no common zero if and only if the v_i''s have no common zero. Hence, $\operatorname{coker}(v)$ is locally free if and only if $\operatorname{coker}(v')$ is locally free.

If the v_i's are linearly dependent, by changing a basis of $\mathcal{O}_X^3$, we may assume $v_3 = 0$. Then $\mathcal{F}$ has a trivial factor, and the composed map $\mathcal{E}\to\mathcal{F}\to\mathcal{O}_X$ is surjective. Since $\mathcal{E}$ is globally generated, this implies that $\mathcal{E}$ has a trivial factor, leading to a contradiction. Hence the v_i's are linearly independent and so these elements span $H^0(\mathcal{O}_X(1,0))$. Then, the corresponding map v' is defined by a basis of $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, hence $\operatorname{coker}(v')$ is locally free. This is the twisted Euler sequence $0\to\mathcal{O}_{\mathbb{P}^2}(-1)\to\mathcal{O}_{\mathbb{P}^2}^3\to\mathcal{T}_{\mathbb{P}^2}(-1)\to 0$, and so $\mathcal{F}\cong p_1^*(\mathcal{T}_{\mathbb{P}^2}(-1))$. By the exact sequence $0\to\mathcal{O}_X(-2,1)\to\mathcal{O}_X(-1,1)^3\to p_1^*(\mathcal{T}_{\mathbb{P}^2}(-2))(0,1)\to 0$ on X and [9, Proposition 2.5.], we have $h^1(p_1^*(\mathcal{T}_{\mathbb{P}^2}(-1))^\vee(0,1)) = h^1(p_1^*(\mathcal{T}_{\mathbb{P}^2}(-2))(0,1)) = 0$. Therefore, the exact sequence $0\to\mathcal{O}_X(0,1)\to\mathcal{E}\to p_1^*(\mathcal{T}_{\mathbb{P}^2}(-1))\to 0$ splits and we have $\mathcal{E}\cong p_1^*(\mathcal{T}_{\mathbb{P}^2}(-1))\oplus\mathcal{O}_X(0,1)$.

The associated curve of $p_1^*(\mathcal{T}_{\mathbb{P}^2}(-1))\oplus\mathcal{O}_X(0,1)$ has bidegree $(1,2)$, because $(1+h_1+h_1^2)(1+h_2) = 1+h_1+h_2+h_2^2+2h_1^2$. □

Proposition 3.3 *Let $\mathcal{E}$ be a globally generated vector bundle of rank $r\geq 2$ on X with $c_1=(1,1)$ and no trivial factor. If C is the associated curve, then C is connected, and $(e_1,e_2;r)$ is as follows (we assume $e_1\leq e_2$ by symmetry):*

$$(1,1;2),\quad (1,2;3),\quad (2,2;3\leq r\leq 4),\quad (3,3;3\leq r\leq 7).$$

Proof The remaining case is $(e_1,e_2) = (2,2)$ and this case occurs by Proposition 3.1 and Remark 2.3. The associated curve C is isomorphic to a conic, hence it is connected and rational. Since $h^0(\bigwedge_C) = h^0(\mathcal{O}_{\mathbb{P}^1}(2)) = 3$, this case gives a rank r globally generated bundle if and only if $3\leq r\leq 4$. □

4. Case of $c_1 = (2,1)$

Let C be an associated curve of a globally generated vector bundle with $c_1 = (2,1)$ and no trivial factor.

Assume that C is a complete intersection curve of two general divisors in $|\mathcal{O}_X(2,1)|$, with $(e_1, e_2) = (5,8)$. Since the genus of C is 6 and $\mathcal{O}_C(0,1)$ is a line bundle of degree 8, $h^0(\bigwedge_C) = 13$. As in Example 2.4, C gives a globally generated bundle of rank r if and only if $3 \le r \le 14$.

Now, let us assume that $C \neq \emptyset$. According to Remark 2.7, we have $e_1 \le 5$ and $e_2 \le 8$. Additionally, based on Remark 2.3, we can assume $e_1 \le 2$ in order to classify globally generated vector bundles. If $e_1 = 0$, then C would be a union of fibers of p_2. However, Lemma 2.6 states that a fiber cannot be a connected component of C, which leads to a contradiction. Therefore, we can assume $e_1 = 1, 2$. Note that once the bidegree of C is given, we can determine the genus of C and $h^0(\bigwedge_C)$ since C is isomorphic to a plane curve in any case.

Proposition 4.1 *Let $\mathcal{E}$ be a globally generated vector bundle of rank $r \ge 2$ on X with $c_1 = (2,1)$ and no trivial factor. If the associated curve C has bidegree $(1,2)$, then we have $\mathcal{E} \cong \mathcal{O}_X(1,0) \oplus \mathcal{O}_X(1,1)$.*

Proof The proof is almost identical to that of Proposition 3.1. □

Proposition 4.2 *Let $\mathcal{E}$ be a globally generated vector bundle of rank $r \ge 2$ on X with $c_1 = (2,1)$ and no trivial factor. If the associated curve C has bidegree $(2,2)$, then we have $\mathcal{E} \cong \mathcal{O}_X(2,0) \oplus \mathcal{O}_X(0,1)$ or $\mathcal{E} \cong p_2^*(\mathcal{T}_{\mathbb{P}^2}(-1))(1,0)$.*

Proof As in the previous proposition, we will follow the same argument as in Proposition 3.1. Since C is connected and rational, we have $\bigwedge_C = \omega_C(0,1) \cong \mathcal{O}_C$ and $h^0(\mathcal{O}_C) = 1$, and so this curve gives a rank 2 vector bundle $\mathcal{E}$. Since $h^0(\mathcal{O}_C(2,0)) = 5$ and $h^0(\mathcal{O}_X(2,0)) = 6$, so $h^0(\mathcal{I}_{C|X}(2,0)) > 0$. Using the fact that $h^1(\mathcal{O}_X(0,-1)) = 0$, from the exact sequence (2.1), we obtain $h^0(\mathcal{E}(0,-1)) > 0$, which implies the existence of a non-zero map $m : \mathcal{O}_X(0,1) \to \mathcal{E}$.

We observe that $h^0(\mathcal{E}(0,-2)) = 0$ since $h^0(\mathcal{I}_{C|X}(2,-1)) = 0$. Moreover, by Lemma 2.6, $p_1(C)$ is a conic, implying that $h^0(\mathcal{I}_{C|X}(1,0)) = 0$. Consequently, we have $h^0(\mathcal{E}(-1,-1)) = 0$, which implies that $\mathcal{O}_X(0,1)$ is saturated in $\mathcal{E}$. As a result, the cokernel of m is torsion free, denoted as coker$(m) \cong \mathcal{I}_T(2,0)$ with either $T = \emptyset$ or T a locally complete intersection curve. Since $c_2(\mathcal{O}_X(2,0) \oplus \mathcal{O}_X(0,1)) = 2h_2^2 + 2h_1^2 = C$, we deduce that $T = \emptyset$.

Considering $h^1(\mathcal{O}_X(-2,1)) = 1$, $\mathcal{E}$ can be either $\mathcal{E} \cong \mathcal{O}_X(2,0) \oplus \mathcal{O}_X(0,1)$ or the unique non-split extension $0 \to \mathcal{O}_X(0,1) \to \mathcal{E} \to \mathcal{O}_X(2,0) \to 0$. By applying Example 6.1 in [9], we can conclude that $\mathcal{E} \cong p_2^*(\mathcal{T}_{\mathbb{P}^2}(-1))(1,0)$. □

Proposition 4.3 *Let $\mathcal{E}$ be a globally generated vector bundle of rank $r \ge 2$ on X with $c_1 = (2,1)$ and no trivial factor. If C is the associated curve and we assume $e_1 \le 2$, then C is connected and $(e_1, e_2; r)$ is as follows:*

$$(1,2;2), \quad (1,3;3), \quad (2,2;2), \quad (2,c;3 \le r \le c) \text{ with } 3 \le c \le 6.$$

Proof Let C be an associated curve of a globally generated vector bundle with $c_1 = (2,1)$ and no trivial factor.

Assume first that $e_1 = 1$. By Lemma 2.6, $p_1|_C : C \to \mathbb{P}^2$ is an embedding. Therefore $p_1(C)$ is a line. In particular, $(e_1, e_2) = (1,0), (1,1)$ cannot occur as an associated curve since the degree of $\bigwedge_C = \omega_C(0,1)$ is negative. For a line $\mathbb{P}^1$ in $\mathbb{P}^2$, $\Omega^1_{\mathbb{P}^2}(2)|_{\mathbb{P}^1}$ splits as a direct sum of line bundles by the theorem of Grothendieck. By the twisted Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}^3_{\mathbb{P}^2} \to \mathcal{T}_{\mathbb{P}^2}(-1) \to 0, \tag{4.1}$$

we have $\Omega^1_{\mathbb{P}^2}(2)|_{\mathbb{P}^1} \cong \mathcal{T}_{\mathbb{P}^2}(-1)|_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Hence C is contained in a smooth Hirzebruch surface $F_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}) \to \mathbb{P}^1$ in X.

The Picard group $\mathrm{Pic}(F_1) = \mathbb{Z}s \oplus \mathbb{Z}f$ is generated by the class of a fiber f of the projection $F_1 \to \mathbb{P}^1$ and the class of a section s with self-intersection $s^2 = -1$. By construction, the class of f in $A(X)$ is h_1^2. To obtain the class of s, we use the adjunction formula. Since $F_1 \in |\mathcal{O}_X(1,0)|$, $\omega_{F_1} = \mathcal{O}_{F_1}(-2s - 3f) \cong \mathcal{O}_{F_1}(-1,-2)$. We have $F_1 \cdot -h_1 = -h_1 \cdot h_1 = -f$, and so $F_1 \cdot -2h_2 = -2h_1^2 - 2h_2^2 = -2s - 2f$. Hence we have $s = h_2^2$.

Since $\mathcal{I}_{F_1|X} \cong \mathcal{O}_X(-1,0)$, $\mathcal{I}_{C|X}(2,1)$ is globally generated if and only if $\mathcal{I}_{C|F_1}(2,1)$ is globally generated. We see $C = s + e_2 f$ and $\mathcal{I}_{C|F_1}(2,1) = \mathcal{O}_{F_1}((3-e_2)f)$, so it follows that $e_2 \leq 3$.

Next, let C be a rational curve in $\mathbb{P}^2$ of degree d, and let $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ be a parametrization of C (cf. [3]). By pulling back the twisted exact sequence (4.1), we have

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \varphi^*\mathcal{O}_{\mathbb{P}^2}(1)^3 \to \varphi^*\mathcal{T}_{\mathbb{P}^2} \to 0.$$

By Grothendieck's theorem, the vector bundle $\varphi^*\mathcal{T}_{\mathbb{P}^2}$ splits into a direct sum of line bundles. Since there exists a surjection $\mathcal{O}_{\mathbb{P}^1}(d)^3 \to \varphi^*\mathcal{T}_{\mathbb{P}^2}$, we can write the decomposition of $\varphi^*\mathcal{T}_{\mathbb{P}^2}$ as

$$\varphi^*\mathcal{T}_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^1}(d + d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d + d_2), \tag{4.2}$$

where $d_1 + d_2 = d$ and $0 \leq d_1 \leq d_2$.

Assuming that $d_1 = 0$, it follows that the three elements of $H^0(\varphi^*\mathcal{O}_{\mathbb{P}^2}(1))$ defining the map $\mathcal{O}_{\mathbb{P}^1} \to \varphi^*\mathcal{O}_{\mathbb{P}^2}(1)^3$ are linearly dependent. This implies that C is a degenerate curve, specifically a line. Therefore, if C is not a line, we must have $1 \leq d_1 \leq d_2$.

Now, assume that $e_1 = 2$. By Lemma 2.6, $p_1(C)$ is a conic in $\mathbb{P}^2$. As in the previous case, $(e_1, e_2) = (2,0), (2,1)$ cannot occur as an associated curve since the degree of $\bigwedge_C = \omega_C(0,1)$ is negative. If the rational curve $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ in (4.2) is a conic, then in particular

$$\varphi^*\mathcal{T}_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(3).$$

Thus, C is contained in a smooth surface $F_0 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(3)) = \mathbb{P}^1 \times \mathbb{P}^1$ in X. Let f_1, f_2 be the generators of the Picard group $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}$. By construction, a line with class h_1^2 is in the class of fiber in $\mathbb{P}^1 \times \mathbb{P}^1$, say f_1. Since $F_0 \in |\mathcal{O}_X(2,0)|$, $\omega_{F_0} = \mathcal{O}_{F_0}(-2f_1 - 2f_2) \cong \mathcal{O}_{F_0}(0,-2)$. We have $F_0 \cdot h_1 = 2h_1 \cdot h_1 = 2f_1$ and, by $F_0 \cdot -2h_2 = -4h_1^2 - 4h_2^2 = -2f_1 - 2f_2$, we have $f_2 = h_1^2 + 2h_2^2$.

Since $\mathcal{I}_{F_0|X} \cong \mathcal{O}_X(-2,0)$, $\mathcal{I}_{C|X}(2,1)$ is globally generated if and only if $\mathcal{I}_{C|F_0}(2,1)$ is globally generated. We observe that $C = e_2 h_1^2 + 2h_2^2 = (e_2 - 1)f_1 + f_2$, and thus $\mathcal{I}_{C|F_0}(2,1) = \mathcal{O}_{F_0}((6-e_2)f_1)$. Consequently, we conclude that $e_2 \leq 6$. □

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Conflict of interest

The author declares that there is no conflict of interest.

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