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Extremal functions for a singular super-critical Trudinger-Moser inequality

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Abstract: In this paper, we deal with a singular super-critical Trudinger-Moser inequality on a unit ball of $\mathbb{R}^n$, $n \geq 3$. For any $p > 1$, we set

$$\lambda_p(\mathbb{B}) = \inf_{u \in W_0^{1,n}(\mathbb{B}), u \not\equiv 0} \frac{\int_{\mathbb{B}} |\nabla u|^n dx}{(\int_{\mathbb{B}} |u|^p dx)^{n/p}}$$

as an eigenvalue related to the n-Laplacian. Let $\mathscr{S}$ be a set of radially symmetric functions. Precisely, if $\beta \geq 0$ and $\alpha < (1+\frac{p}{n}\beta)^{n-1+n/p}\lambda_p(\mathbb{B})$, then there exists a positive constant ϵ_0 such that when $\lambda \leq 1+\epsilon_0$,

$$\sup_{u \in W_0^{1,n}(\mathbb{B}) \cap \mathscr{S}, \int_{\mathbb{B}} |\nabla u|^n dx - \alpha (\int_{\mathbb{B}} |u|^p |x|^{p\beta} dx)^{\frac{n}{p}} \leq 1} \int_{\mathbb{B}} |x|^{p\beta} \left(e^{\alpha_n (1+\frac{p}{n}\beta)|u|^{\frac{n}{n-1}}} - \lambda \sum_{k=0}^{m} \frac{|\alpha_n (1+\frac{p}{n}\beta) u^{\frac{n}{n-1}}|^k}{k!} \right) dx$$

is attained, where $\alpha_n = n\omega_{n-1}^{1/(n-1)}$, ω_{n-1} is the surface area of the unit ball in $\mathbb{R}^n$. The proof is based on the method of blow-up analysis. The case $\lambda = 0$ was studied by Yang-Zhu in [38]. de Figueiredo [11] considered the case $p = 2$, $\beta \geq 0$, and $\alpha = 0$ in two dimension. The case $\lambda = 0, p = n, -1 < \beta < 0$, and $\alpha = 0$ was considered by Adimurthi-Sandeep [1]. Our results extend those of the above cases.

Key words: Trudinger-Moser inequality, extremal functions, blow-up analysis

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $n \geq 3$, and $W_0^{1,n}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in the norm $\|u\|^n_{W_0^{1,n}(\Omega)} = \int_\Omega |\nabla u|^n dx$. The study of sharp constant for Trudinger-Moser inequality traces back to the 1960s and 1970s. In 1971, Moser [26] elegantly sharpened the results of Phohozaev [30] and Trudinger [33], then established the classical Trudinger-Moser inequality:

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_n = 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} dx < \infty \tag{1.1}$$

for any $\alpha \leq \alpha_n = n\omega_{n-1}^{1/(n-1)}$, where ω_{n-1} is the surface area of the unit ball in $\mathbb{R}^n$. Here and in the sequel, $\|\cdot\|_p$ denotes the L^p-norm with respect to the Lebesgue measure. Also, there are fruitful results in the literature dealing with the existence of extremal functions, such as Carleson-Chang [5], Flucher [15], and Lin [24].

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The extensions of (1.1) are numerous. Yang [35] proved singular versions of (1.1) for some subspaces of $W^{1,n}(\mathbb{R}^n)$ under the additional condition: $\nabla u_k(x) \to \nabla u(x)$. By using a symmetrization argument and a change of variables, Adimurthi and Sandeep [1] generalized (1.1) to the singular case:

$$\sup_{u\in W_0^{1,n}(\Omega),\|u\|_{W_0^{1,n}(\Omega)}\leq 1}\int_\Omega \frac{e^{\alpha_n\gamma|u|^{\frac{n}{n-1}}}}{|x|^{n\beta}}dx<\infty, \tag{1.2}$$

where $0\leq\beta<1$ and $0<\gamma\leq 1-\beta$. The inequality (1.2) was extended to the whole Euclidean space by Adimurthi-Yang [2]. Various extensions of the inequality (1.2) were obtained in [7, 28, 39, 40]. The problem on the existence of extremals for the singular Trudinger–Moser inequality was solved by Csató and Roy [7, 8], and by Csató, Roy and the author [6] in any dimension $n\geq 3$.

Trudinger-Moser inequalities were discussed in the unit ball as well. Let $\mathscr{S}$ be a set of all radially symmetric functions. In 1982, Ni [29] showed that Sobolev spaces of radially symmetric functions defined in the unit ball $\mathbb{B}\subset\mathbb{R}^n$, can be embedded into weighted Lebesgue spaces, i.e. $W_0^{1,n}(\mathbb{B})\cap\mathscr{S}$ can be embedded in $L^p(\mathbb{B},|x|^\alpha)$ with $\alpha>0$ and $p=\frac{2(n+\alpha)}{n-2}$ greater than $2^*=\frac{2n}{n-2}$. Based on the works of Bonheure et al. [3] and Calanchi [4], de Figueiredo [10, 11] proved that for any $\alpha\leq 4\pi(1+\gamma)$,

$$\sup_{u\in W_0^{1,2}(\mathbb{B})\cap\mathscr{S},\|u\|_{W_0^{1,2}(\mathbb{B})}\leq 1}\int_{\mathbb{B}} e^{\alpha u^2}|x|^{2\gamma}dx<\infty. \tag{1.3}$$

In [38], Yang-Zhu generalized (1.3) to a version involving $\lambda_p(\mathbb{B})$ in the unit ball: for any given $p>1$, if $\beta\geq 0$ and $\alpha<(1+\frac{p}{n}\beta)^{n-1+\frac{n}{p}}\lambda_p(\mathbb{B})$,

$$\sup_{u\in W_0^{1,n}(\mathbb{B})\cap\mathscr{S},\int_{\mathbb{B}}|\nabla u|^n dx-\alpha(\int_{\mathbb{B}}|u|^p|x|^{p\beta}dx)^{\frac{n}{p}}\leq 1}\int_{\mathbb{B}} e^{\gamma|u|^{\frac{n}{n-1}}}|x|^{p\beta}dx<\infty,\quad \gamma\leq\alpha_n(1+\frac{p\beta}{n}), \tag{1.4}$$

where $\lambda_p(\mathbb{B})=\inf_{u\in W_0^{1,n}(\mathbb{B}),u\not\equiv 0}\int_{\mathbb{B}}|\nabla u|^n dx/(\int_{\mathbb{B}}|u|^p dx)^{\frac{n}{p}}$ is an eigenvalue related to the n-Laplacian. Furthermore, the supremum in (1.4) can be attained. Nguyen [27] extended (1.4) to more general cases of the nonlinearity function F and the weight function h. In [9], de Figueiredo et al. gave a generalized result which states that

$$\sup_{u\in H_0^{1,n}(B_1(0)),\|\nabla u\|_{L^n(B_1(0))=1}}\int_{B_1(0)}(e^{\alpha_n|u|^{\frac{n}{n-1}}}-\lambda|u|^{\frac{n}{n-1}})dx$$

is attained for any $\lambda<\alpha_n$. In [22], Li proved a counter-example to the conjecture of de Figueiredo and Ruf in [9]:

$$f(\lambda)=I(M,\lambda,m)=\sup_{u\in H_0^{1,n}(M),\int_M|\nabla u|^n dV=1}\int_\Omega\left(e^{\alpha_n|u|^{\frac{n}{n-1}}}-\lambda\sum_{k=1}^m\frac{|\alpha_n u^{\frac{n}{n-1}}|^k}{k!}\right)dV \tag{1.5}$$

is continuous for a fixed integer m, where M is a compact manifold with boundary. Then he proved there is a constant $\lambda_0>1$ such that $I(M,\lambda,m)$ can be attained on $[0,\lambda_0]$.

In this paper, we consider a singular super-critical Trudinger-Moser inequality in the unit ball, which is a combination of (1.4) and (1.5). To state the main result of the paper, we introduce some relevant notations:

$$H_{\alpha,\beta}(u) = \|\nabla u\|_n^n - \alpha\|u\|_{p,\beta}^n = \int_{\mathbb{B}} |\nabla u|^n dx - \alpha \left(\int_{\mathbb{B}} |u|^p |x|^{p\beta} dx \right)^{\frac{n}{p}},$$

where $\|u\|_{p,\beta} = (\int_{\mathbb{B}} |u|^p|x|^{p\beta}dx)^{\frac{1}{p}}$. We use the symbol $\mathbb{B}_x(r)$ to represent a ball with x as the center and r as the radius. If $x = 0$, the symbol $\mathbb{B}(r)$ to represent a ball with 0 as the center and r as the radius. Then we state the following:

Theorem 1.1 *Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$, $n \geq 3$, for any $\beta \geq 0$ and $\alpha < (1+\frac{p}{n}\beta)^{n-1+\frac{n}{p}}\lambda_p(\mathbb{B})$, there exists a positive constant ϵ_0 such that if $\lambda \leq 1+\epsilon_0$, then*

$$\sup_{u\in\mathcal{H}} \int_{\mathbb{B}} |x|^{p\beta} \left(e^{\alpha_n(1+\frac{p}{n}\beta)|u|^{\frac{n}{n-1}}} - \lambda \sum_{k=0}^{m} \frac{|\alpha_n(1+\frac{p}{n}\beta)u^{\frac{n}{n-1}}|^k}{k!} \right) dx$$

can be attained, where $\mathcal{H} = \{u \in W_0^{1,n}(\mathbb{B}) \cap \mathscr{S} : H_{\alpha,\beta}(u) \leq 1\}$.

The case $\lambda = 0$ was studied by Yang-Zhu in [38]. de Figueiredo [11] considered the case $p = 2$, $\beta \geq 0$, and $\alpha = 0$ in two dimension. The case $\lambda = 0, p = n, -1 < \beta < 0$, and $\alpha = 0$ was considered by Adimurthi-Sandeep [1].

The remaining part of this paper is organized as follows: In section 2, we obtain the maximizer of the subcritical function. Section 3 provides the method of blow-up analysis, which was extensively used by [12, 13, 18–20, 36]. An upper bound of $\Lambda_{\lambda,\alpha_n}$ is derived in section 4. In section 5, we construct a sequence of functions which contradicts the upper bound.

2. The subcritical case

This section is devoted to the subcritical case of the singular Trudinger-Moser inequality. For the sake of simplicity, we define

$$\Lambda_{\lambda,\alpha_n}(u) = \sup_{u\in\mathcal{H}} \int_{\mathbb{B}} |x|^{p\beta} \left(e^{\alpha_n(1+\frac{p}{n}\beta)|u|^{\frac{n}{n-1}}} - \lambda \sum_{k=0}^{m} \frac{|\alpha_n(1+\frac{p}{n}\beta)u^{\frac{n}{n-1}}|^k}{k!} \right) dx,$$

and

$$\Lambda_{\lambda,n,\epsilon}(u) = \sup_{u\in\mathcal{H}} \int_{\mathbb{B}} |x|^{p\beta} \left(e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u|^{\frac{n}{n-1}}} - \lambda \sum_{k=0}^{m} \frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u^{\frac{n}{n-1}}|^k}{k!} \right) dx.$$

Then we have the following result:

Lemma 2.1 *For any $\epsilon > 0$, if $\alpha < (1+\frac{p}{n}\beta)^{n-1+\frac{n}{p}}\lambda_p(\mathbb{B})$, then there exists $u_\epsilon \in C^1(\overline{\mathbb{B}}) \cap W_0^{1,n}(\mathbb{B})$ with*

$$\int_{\mathbb{B}} |\nabla u_\epsilon|^n dx - \alpha \left(\int_{\mathbb{B}} |u_\epsilon|^p |x|^{p\beta} dx \right)^{\frac{n}{p}} = 1$$

such that

$$\int_{\mathbb{B}} |x|^{p\beta} \left(e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_\epsilon|^{\frac{n}{n-1}}} - \lambda \sum_{k=0}^{m} \frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_\epsilon^{\frac{n}{n-1}}|^k}{k!} \right) dx = \Lambda_{\lambda,n,\epsilon}(u). \tag{2.1}$$

Proof We take a sequence of decreasing radially symmetric functions $u_j \in W_0^{1,n}(\mathbb{B})$ such that $\|\nabla u_j\|_n^n - \alpha \|u_j\|_{p,\beta}^n = 1$ and

$$\lim_{j\to+\infty} \int_{\mathbb{B}} |x|^{p\beta} \left(e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_j|^{\frac{n}{n-1}}} - \lambda \sum_{k=0}^{m} \frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_j^{\frac{n}{n-1}}|^k}{k!} \right) dx = \Lambda_{\lambda,n,\epsilon}(u). \tag{2.2}$$

From the definition of $\lambda_p(\mathbb{B})$, we can get

$$1 \geq \|\nabla u_j\|_n^n - \alpha \|u_j\|_{p,\beta}^n \geq \left(1 - \frac{\alpha}{(1+\frac{p}{n}\beta)^{n-1+\frac{n}{p}}} \lambda_p(\mathbb{B}) \right) \|\nabla u_j\|_n^n.$$

Since $\alpha < (1+\frac{p}{n}\beta)^{n-1+\frac{n}{p}} \lambda_p(\mathbb{B})$, we obtain that u_j is bounded in $W_0^{1,n}(\mathbb{B})$, then we assume that

$$\begin{aligned} &u_j \rightharpoonup u_\epsilon \quad \text{weakly} \quad \text{in} \quad W_0^{1,n}(\mathbb{B}), \\ &u_j \to u_\epsilon \quad \text{strongly} \quad \text{in} \quad L^n(\mathbb{B}), \\ &u_j \to u_\epsilon \quad \text{a.e.} \quad \text{in} \quad \mathbb{B}. \end{aligned}$$

We claim that $u_\epsilon \not\equiv 0$. Suppose not, there holds $\|u_j\|_{W_0^{1,n}(\mathbb{B})} \leq 1 + o(1)$. Thus $e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_j|^{\frac{n}{n-1}}}$ converges to 1 in $L^1(\mathbb{B})$, which implies that $\Lambda_{\lambda,n,\epsilon}(u) = \int_{\mathbb{B}} |x|^{p\beta} dx$. But this is impossible. Therefore $u_\epsilon \not\equiv 0$. Then define a function sequence

$$v_j = \frac{u_j}{(1+\alpha(\int_{\mathbb{B}} |x|^{p\beta} u_j^p dx)^{\frac{n}{p}})^{1/n}}.$$

It follows that $\|v_j\|_{W_0^{1,n}(\mathbb{B})} \leq 1$ and v_j converges to $v_\epsilon = u_\epsilon/(1+\alpha(\int_{\mathbb{B}} |x|^{p\beta} u_\epsilon^p dx)^{\frac{n}{p}})^{1/n}$ weakly in $W_0^{1,n}(\mathbb{B})$. One can easily check that

$$\left(1 + \alpha (\int_{\mathbb{B}} |x|^{p\beta} u_\epsilon^p dx)^{\frac{n}{p}} \right) \left(1 - \|v_\epsilon\|_{W_0^{1,n}(\mathbb{B})}^n \right) = 1 - (\|\nabla u_\epsilon\|_n^n - \alpha \|u_\epsilon\|_{p,\beta}^n) < 1.$$

By a result of Lions [25], we can know $e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_j|^{\frac{n}{n-1}}}$ is bounded in $L^r(\mathbb{B})$ for some $r > 1$. Thus

$$\lim_{j\to+\infty} \int_{\mathbb{B}} e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_j|^{\frac{n}{n-1}}} dx = \int_{\mathbb{B}} e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_\epsilon|^{\frac{n}{n-1}}} dx.$$

Furthermore,

$$\int_{\mathbb{B}} \lambda \sum_{k=0}^{m} \frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_j^{\frac{n}{n-1}}|^k}{k!} dx - \int_{\mathbb{B}} \lambda \sum_{k=0}^{m} \frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_\epsilon^{\frac{n}{n-1}}|^k}{k!} dx = o_j(1).$$

Accordingly,

$$\begin{aligned}&\lim_{j\to+\infty}\int_{\mathbb{B}}|x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_j|^{\frac{n}{n-1}}}-\lambda\sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_j^{\frac{n}{n-1}}|^k}{k!}\right)dx\\&=\int_{\mathbb{B}}|x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_\epsilon|^{\frac{n}{n-1}}}-\lambda\sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_\epsilon^{\frac{n}{n-1}}|^k}{k!}\right)dx.\end{aligned}\tag{2.3}$$

Combining (2.2) and (2.3), we have

$$\int_{\mathbb{B}}|x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_\epsilon|^{\frac{n}{n-1}}}-\lambda\sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_\epsilon^{\frac{n}{n-1}}|^k}{k!}\right)dx=\Lambda_{\lambda,n,\epsilon}(u).$$

□

Furthermore, one can check that the corresponding Euler-Lagrange equation of u_ϵ is

$$\begin{cases}-\Delta_n u_\epsilon=\alpha|x|^{p\beta}\|u_\epsilon\|_p^{n-p}u_\epsilon^{p-1}+\frac{1}{\lambda_\epsilon}|x|^{p\beta}u_\epsilon^{\frac{1}{n-1}}e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_\epsilon^{\frac{n}{n-1}}}-\frac{\lambda}{\lambda_\epsilon}|x|^{p\beta}h_m'(u_\epsilon)\\ h_m(u_\epsilon)=\sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_\epsilon^{\frac{n}{n-1}}|^k}{k!}\\ \lambda_\epsilon=\int_{\mathbb{B}}|x|^{p\beta}(u_\epsilon^{\frac{n}{n-1}}e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_\epsilon^{\frac{n}{n-1}}}-\lambda u_\epsilon h_m'(u_\epsilon))dx.\end{cases}\tag{2.4}$$

According to the regularity theory for degenerate elliptic equations, see (Serrin [31], page 269, Theorem 8), (Tolksdorf [32], page 127, Theorem 1), and (Lieberman [23], page 1203, Theorem 1), we are able to attain $u_\epsilon\in C^1(\overline{\mathbb{B}})$. By the inequality $e^t\le 1+te^t$ and the definition of λ_ϵ, we can easily get $\liminf_{\epsilon\to0}\lambda_\epsilon>0$. From the equality (2.1), it is not difficult to see that

$$\lim_{\epsilon\to0}\int_{\mathbb{B}}|x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_\epsilon|^{\frac{n}{n-1}}}-\lambda\sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_\epsilon^{\frac{n}{n-1}}|^k}{k!}\right)dx=\Lambda_{\lambda,\alpha_n}(u).\tag{2.5}$$

Since $\int_{\mathbb{B}}|\nabla u_\epsilon|^n dx-\alpha(\int_{\mathbb{B}}|u_\epsilon|^p|x|^{p\beta}dx)^{\frac{n}{p}}=1$, without loss of generality, we can assume that u_ϵ converges to u_0 weakly in $W_0^{1,n}(\overline{\mathbb{B}})$, strongly in $L^s(\mathbb{B})$ for any $s>1$, and almost everywhere in $\mathbb{B}$. Let $c_\epsilon=u_\epsilon(0)=\max_{\mathbb{B}}u_\epsilon$. If c_ϵ is bounded, then applying the Lebesgue-dominated convergence theorem to (2.5), we know that u_0 is the desired extremal function for the supremum $\Lambda_{\lambda,\alpha_n}(u)$. In the following, we assume

$$c_\epsilon\to+\infty\quad\text{as}\quad\epsilon\to0.$$

The following concentration phenomenon is useful in our subsequent blow-up analysis:

Lemma 2.2 *Under the assumption that $c_\epsilon\to+\infty$, we have $u_0\equiv0$ and $|\nabla u_\epsilon|^n dx\rightharpoonup\delta_0$ in sense of measure, where δ_0 is the Dirac measure at 0.*

Proof Suppose $u_0\not\equiv0$, we can easily get $-\Delta_n u_\epsilon$ is bounded in $L^q(\Omega)$ for some $q>1$ provided that ϵ is sufficiently small. Applying the elliptic estimates to the Euler-Lagrange equation (2.4), one gets c_ϵ is bounded, which contradicts $c_\epsilon\to+\infty$. Therefore, $u_0\equiv0$.

Assume $|\nabla u_\epsilon|^n dx \rightharpoonup \mu$ in sense of measure. We can choose a cut-off function $\varphi \in C_0^1(\mathbb{B})$, which is supported in $\mathbb{B}(r_0) \subset \mathbb{B}$ and equals to 1 in $\mathbb{B}(r_0/2)$ for some small $r_0 > 0$. So

$$\int_{\mathbb{B}(r_0)} |\nabla(\varphi u_\epsilon)|^n dx \leq 1 - \eta$$

for some $\eta > 0$ provided that ϵ is sufficiently small. By the classical Trudinger-Moser inequality (1.1), we can know $e^{\alpha_\epsilon(\varphi u_\epsilon)^{\frac{n}{n-1}}}$ is bounded in $L^s(\Omega)$ for some $s > 1$. Then applying the elliptic estimates ([16], Chapter 9) to equation (2.4), we obtain $\|u_\epsilon\|_{W^{1,n}(\mathbb{B})} \leq C$, this together with the compact embedding theorem lead that u_ϵ is bounded in $L^\infty(\mathbb{B}(r_0/2))$, which contradicts the assumption that $c_\epsilon \to +\infty$. Therefore, $|\nabla u_\epsilon|^n dx \rightharpoonup \delta_0$. □

3. Blow-up analysis

In this section, we will use the method of blow-up analysis to investigate the asymptotic behaviour of u_ϵ near the blow-up point $x_0 = 0$. We set

$$r_\epsilon = \lambda_\epsilon^{\frac{1}{n}} c_\epsilon^{-\frac{1}{n-1}} e^{-\frac{\alpha_n(1+\frac{p}{n}\beta-\epsilon)}{n} c_\epsilon^{\frac{n}{n-1}}}.$$

By Lemma 2.2 and the classical Trudinger-Moser inequality (1.1), one can easily check that $\lim_{\epsilon\to 0} r_\epsilon^n e^{\delta c_\epsilon^{\frac{n}{n-1}}} = 0$ for any $0 < \delta < \alpha_n(1 + \frac{p}{n}\beta)$. Define two sequences of functions

$$\psi_\epsilon(x) = \frac{1}{c_\epsilon} u_\epsilon(r_\epsilon^{\frac{n}{n+p\beta}} x), \quad \varphi_\epsilon(x) = c_\epsilon^{\frac{1}{n-1}} (u_\epsilon(r_\epsilon^{\frac{n}{n+p\beta}} x) - c_\epsilon),$$

where ψ_ϵ and φ_ϵ are defined on $\mathbb{B}(r_\epsilon^{-1})$. By equation (2.4), we have

$$\begin{aligned} -\Delta_n \psi_\epsilon(x) = {} & c_\epsilon^{-n} \psi_\epsilon^{\frac{1}{n-1}} e^{\alpha_n(1+\frac{p\beta}{n}-\epsilon)(u_\epsilon^{\frac{n}{n-1}}(r_\epsilon^{\frac{n}{n+p\beta}}x) - c_\epsilon^{\frac{n}{n-1}})} |x|^{p\beta} \\ & + \alpha c_\epsilon^{p-n} r_\epsilon^n \|u_\epsilon\|_p^{n-p} \psi_\epsilon^{p-1} |x|^{p\beta} \\ & - \lambda c_\epsilon^{1-n} r_\epsilon^n c_\epsilon^{-\frac{n}{n-1}} e^{-\alpha_n(1+\frac{p\beta}{n}-\epsilon)c_\epsilon^{\frac{n}{n-1}}} |x|^{p\beta}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} -\Delta_n \varphi_\epsilon(x) = {} & \psi_\epsilon^{\frac{1}{n-1}} e^{\alpha_n(1+\frac{p\beta}{n}-\epsilon)(u_\epsilon^{\frac{n}{n-1}}(r_\epsilon^{\frac{n}{n+p\beta}}x) - c_\epsilon^{\frac{n}{n-1}})} |x|^{p\beta} \\ & + \alpha c_\epsilon^{p} r_\epsilon^n \|u_\epsilon\|_p^{n-p} \psi_\epsilon^{p-1} |x|^{p\beta} \\ & - \lambda c_\epsilon r_\epsilon^n c_\epsilon^{-\frac{n}{n-1}} e^{-\alpha_n(1+\frac{p\beta}{n}-\epsilon)c_\epsilon^{\frac{n}{n-1}}} |x|^{p\beta}. \end{aligned} \tag{3.2}$$

Since u_ϵ is bounded in $L^p(\mathbb{B})$, we have

$$\left(\int_{\mathbb{B}(r_\epsilon^{-1})} (c_\epsilon^{p-n} r_\epsilon^n \|u_\epsilon\|_p^{n-p} \psi_\epsilon^{p-1} |x|^{p\beta})^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} = c_\epsilon^{1-n} r_\epsilon^{\frac{n}{p}} \|u_\epsilon\|_p^{n-1} |x|^{p\beta} \to 0.$$

Then we can get $\Delta_n \psi_\epsilon(x)$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{B}(r_\epsilon^{-1}))$. Applying the standard elliptic regularity theory [32] to (3.1), we obtain $\psi_\epsilon \to \psi$ in $C^0_{\text{loc}}(\mathbb{R}^n)$. When $1 < p \leq n$, one can easily see that

$$\alpha c_\epsilon^p r_\epsilon^n \|u_\epsilon\|_p^{n-p} \psi_\epsilon^{p-1} |x|^{p\beta} \to 0$$

uniformly in $x \in \mathbb{B}(r_\epsilon^{-1})$ as $\epsilon \to 0$. When $p > n$, we have that for any $R > 0$ and sufficiently small ϵ,

$$\|u_\epsilon\|_p^{n-p} = \left(\int_{\mathbb{B}} u_\epsilon^p dx\right)^{\frac{n}{p}-1} \leq \left(\int_{\mathbb{B}(Rr_\epsilon)} u_\epsilon^p dx\right)^{\frac{n}{p}-1} = c_\epsilon^{n-p} r_\epsilon^{\frac{n^2}{p}-n} \left(\int_{\mathbb{B}(R)} \psi_\epsilon^p dx\right)^{\frac{n}{p}-1}.$$

Then we have

$$\|u_\epsilon\|_p^{n-p} \leq 2c_\epsilon^{n-p} r_\epsilon^{\frac{n^2}{p}-n} \left(\int_{\mathbb{B}(R)} \psi^p dx\right)^{\frac{n}{p}-1}.$$

In view of $\lim_{\epsilon\to 0} r_\epsilon^n e^{\delta c_\epsilon^{\frac{n}{n-1}}} = 0$ for any $0 < \delta < \alpha_n(1 + \frac{p}{n}\beta)$, we can obtain

$$\alpha c_\epsilon^p r_\epsilon^n \|u_\epsilon\|_p^{n-p} \psi_\epsilon^{p-1} |x|^{p\beta} \leq 2c_\epsilon^n r_\epsilon^{\frac{n^2}{p}} \left(\int_{\mathbb{B}(R)} \psi^p dx\right)^{\frac{n}{p}-1} \to 0.$$

It follows that $\Delta_n \psi_\epsilon$ is bounded in $L^\infty(\mathbb{B}(R))$. According to the regularity theory [32], we conclude that $\psi_\epsilon \to \psi$ in $C^1(\mathbb{B}(R/2))$. Therefore, $\psi_\epsilon \to \psi$ in $C^1_{\text{loc}}(\mathbb{R}^n)$. Hence ψ satisfies $-\Delta_n \psi(x) = 0$ in $\mathbb{R}^n$. Obviously we have $0 \leq \psi(x) \leq \psi(0) = 1$, so Liouville type theorem implies that $\psi = 1$.

Applying the standard elliptic regularity theory [32] to (3.2), then by the similar argument, we have for any $p > 1$, $\varphi_\epsilon \to \varphi$ in $C^1_{\text{loc}}(\mathbb{R}^n)$. In this situation, we have

$$\begin{aligned} u_\epsilon(r_\epsilon^{\frac{n}{n+p\beta}} x)^{\frac{n}{n-1}} - c_\epsilon^{\frac{n}{n-1}} &= c_\epsilon^{\frac{n}{n-1}} \left((1 + \frac{\varphi_\epsilon}{c_\epsilon^{\frac{n}{n-1}}})^{\frac{n}{n-1}} - 1 \right) \\ &= \frac{n}{n-1}\varphi_\epsilon + c_\epsilon^{\frac{n}{n-1}} o\left(\frac{\varphi_\epsilon}{c_\epsilon^{\frac{n}{n-1}}}\right) \\ &= \frac{n}{n-1}\varphi + o(1). \end{aligned}$$

Hence $\varphi(x)$ is the distributional solution of the equation

$$-\Delta_n \varphi(x) = |x|^{p\beta} e^{\frac{n}{n-1}\alpha_n(1+\frac{p\beta}{n})\varphi(x)} \quad \text{in} \quad \mathbb{R}^n.$$

We make the change of variable $y = r_\epsilon^{\frac{n}{n+p\beta}} x$ with $|x| \leq R$, then for any fixed $R > 1$, there holds $|y| \leq 2Rr_\epsilon^{\frac{n}{n+p\beta}}$. We also have

$$\int_{\mathbb{B}(R)} |x|^{p\beta} e^{\frac{n}{n-1}\alpha_n(1+\frac{p\beta}{n})\varphi} dx = \lim_{\epsilon\to 0} \int_{\mathbb{B}(R)} |x|^{p\beta}\, e^{\alpha_n(1+\frac{p\beta}{n}-\epsilon)(u_\epsilon^{\frac{n}{n-1}}(r_\epsilon^{\frac{n}{n+p\beta}} x) - c_\epsilon^{\frac{n}{n-1}})} dx \leq 1.$$

In viewing of [14], it is not hard to see that

$$\varphi(x) = -\frac{n-1}{\alpha_n(1+\frac{p\beta}{n})} \ln\left(1 + \left(\frac{\omega_{n-1}}{n+p\beta}\right)^{\frac{1}{n-1}} |x|^{\frac{n+p\beta}{n-1}}\right).$$

In particular,

$$\int_{\mathbb{B}} e^{\frac{n}{n-1}\alpha_n(1+\frac{p\beta}{n})\varphi} |x|^{p\beta} dx = 1. \tag{3.3}$$

Define $u_{\epsilon,\delta} = \min\{u_\epsilon, \delta c_\epsilon\}$ for any real number $0 < \delta < 1$. In the same way as [21, 34], we have the following lemma:

Lemma 3.1 *There holds*

$$\lim_{\epsilon\to 0}\int_{\mathbb{B}} |\nabla u_{\epsilon,\delta}|^n dx = \delta.$$

Proof We have by the equation (2.4) and the divergence theorem,

$$\begin{aligned}\int_{\mathbb{B}} |\nabla u_{\epsilon,\delta}|^n dx &= -\int_{\mathbb{B}} u_{\epsilon,\delta}(\Delta_n u_\epsilon)dx \\ &= \int_{\mathbb{B}} u_{\epsilon,\delta}\left(\alpha|x|^{p\beta}\|u_\epsilon\|_p^{n-p}u_\epsilon^{p-1} + \frac{1}{\lambda_\epsilon}|x|^{p\beta}u_\epsilon^{\frac{1}{n-1}}e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_\epsilon^{\frac{n}{n-1}}} - \frac{\lambda}{\lambda_\epsilon}|x|^{p\beta}h'_m(u_\epsilon)\right)dx \\ &\geq \int_{\mathbb{B}(Rr_\epsilon^{\frac{n}{n+p\beta}})}(\delta c_\epsilon + o_\epsilon(1))\left(\frac{1}{\lambda_\epsilon}|x|^{p\beta}u_\epsilon^{\frac{1}{n-1}}e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_\epsilon^{\frac{n}{n-1}}}\right)dx + o(1).\end{aligned}$$

By making the change of variable $x = r_\epsilon^{\frac{n}{n+p\beta}}y$, we get

$$\int_{\mathbb{B}} |\nabla u_{\epsilon,\delta}|^n dx \geq \delta(1+o_\epsilon(1))\int_{\mathbb{B}(R)} e^{\alpha_n(1+\frac{p\beta}{n}-\epsilon)(u_\epsilon(r_\epsilon^{\frac{n}{n+p\beta}}y)-c_\epsilon^{\frac{n}{n-1}})}|y|^{p\beta}dy,$$

which yields

$$\liminf_{\epsilon\to 0}\int_{\mathbb{B}} |\nabla u_{\epsilon,\delta}|^n dx \geq \delta\int_{\mathbb{B}(R)} e^{\frac{n}{n-1}\alpha_n(1+\frac{p\beta}{n})\varphi(y)}|y|^{p\beta}dy.$$

Letting $R\to+\infty$ and by equation (3.3), we obtain

$$\liminf_{\epsilon\to 0}\int_{\mathbb{B}} |\nabla u_{\epsilon,\delta}|^n dx \geq \delta.$$

By the same argument, we establish that

$$\int_{\mathbb{B}} |\nabla(u_\epsilon - u_{\epsilon,\delta})|^n dx \geq 1-\delta.$$

Since

$$\int_{\mathbb{B}} |\nabla u_{\epsilon,\delta}|^n dx + \int_{\mathbb{B}} |\nabla(u_\epsilon - u_{\epsilon,\delta})|^n dx = 1,$$

we get the result. □

The following lemma is used in proving the existence of extremal functions of the Trudinger-Moser inequality. Due to it providing the asymptotic behavior of u_ϵ, we include it here.

Lemma 3.2 *There holds*

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{\mathbb{B}} |x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_\epsilon|^{\frac{n}{n-1}}} - \lambda\sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_\epsilon^{\frac{n}{n-1}}|^k}{k!}\right)dx \\ &\leq \int_{\mathbb{B}} |x|^{p\beta}dx + \limsup_{\epsilon\to 0}\frac{\lambda_\epsilon}{c_\epsilon^{\frac{n}{n-1}}}.\end{aligned} \tag{3.4}$$

The proof is similar to the proof of Lemma 4.8 in [34], so we omit here. It follows from Lemma 3.2 that

$$\lim_{\epsilon\to 0}\frac{c_\epsilon^{\frac{n}{n-1}}}{\lambda_\epsilon}=0. \tag{3.5}$$

In order to investigate the convergence behaviour of u_ϵ away from the blow-up point, we need the following lemma:

Lemma 3.3 *For any $\varphi\in C(\overline{\mathbb{B}})$, we have*

$$\lim_{\epsilon\to 0}\int_{\mathbb{B}}\frac{1}{\lambda_\epsilon}c_\epsilon u_\epsilon^{\frac{1}{n-1}}e^{\alpha_n(1+\frac{p\beta}{n}-\epsilon)u_\epsilon^{\frac{n}{n-1}}}|x|^{p\beta}\varphi(x)dx=\varphi(0).$$

Proof We divide $\mathbb{B}$ into three parts as follows:

$$\mathbb{B}=\Big(\{u_\epsilon>\delta c_\epsilon\}\setminus\mathbb{B}(Rr_\epsilon^{\frac{n}{n+p\beta}})\Big)\cup\{u_\epsilon\le\delta c_\epsilon\}\cup\mathbb{B}(Rr_\epsilon^{\frac{n}{n+p\beta}}),$$

where $\delta\in(0,1)$. Denote the integrals on the above three domains by I_1, I_2 and I_3 respectively. Letting $\mathbb{B}(Rr_\epsilon^{\frac{n}{n+p\beta}})\subset\{u_\epsilon>\delta c_\epsilon\}$, we have

$$\begin{aligned}|I_1|&\le\sup_{\mathbb{B}}|\varphi|\frac{1}{\lambda_\epsilon}\left(\int_{\{u_\epsilon>\delta c_\epsilon\}}c_\epsilon u_\epsilon^{\frac{1}{n-1}}e^{\alpha_n(1+\frac{p\beta}{n}-\epsilon)u_\epsilon^{\frac{n}{n-1}}}|x|^{p\beta}dx-\int_{\mathbb{B}_{Rr_\epsilon^{\frac{n}{n+p\beta}}}(x_\epsilon)}c_\epsilon u_\epsilon^{\frac{1}{n-1}}e^{\alpha_n(1+\frac{p\beta}{n}-\epsilon)u_\epsilon^{\frac{n}{n-1}}}|x|^{p\beta}dx\right)\\&\le\sup_{\mathbb{B}}|\varphi|\left(\frac{1}{\delta}-\int_{\mathbb{B}(R)}e^{\alpha_n(1+\frac{p\beta}{n}-\epsilon)(u_\epsilon^{\frac{n}{n-1}}(r_\epsilon^{\frac{n}{n+p\beta}}x)-c_\epsilon^{\frac{n}{n-1}})}|x|^{p\beta}\,dx\right)\\&=\sup_{\mathbb{B}}|\varphi|\left(\frac{1}{\delta}-\int_{\mathbb{B}(R)}e^{\frac{n}{n-1}\alpha_n(1+\frac{p\beta}{n})\varphi}|x|^{p\beta}dx+o(1)\right)\to 0.\end{aligned}$$

Recalling the definition of $u_{\epsilon,\delta}$, we obtain

$$\begin{aligned}|I_2|&\le\sup_{\mathbb{B}}|\varphi|\frac{c_\epsilon}{\lambda_\epsilon}\int_{\{u_\epsilon\le\delta c_\epsilon\}}u_\epsilon^{\frac{1}{n-1}}e^{\alpha_n(1+\frac{p\beta}{n}-\epsilon)u_\epsilon^{\frac{n}{n-1}}}|x|^{p\beta}dx\\&\le\sup_{\mathbb{B}}|\varphi|\frac{c_\epsilon}{\lambda_\epsilon}\int_{\mathbb{B}}u_{\epsilon,\delta}^{\frac{1}{n-1}}e^{\alpha_n(1+\frac{p\beta}{n}-\epsilon)u_{\epsilon,\delta}^{\frac{n}{n-1}}}|x|^{p\beta}dx.\end{aligned}$$

From Lemma 3.1 and (3.5), we conclude that $I_2\to 0$. Finally, making the change of variable $y=r_\epsilon^{\frac{n}{n+p\beta}}x$, we get

$$\begin{aligned}I_3&=\int_{\mathbb{B}(Rr_\epsilon^{\frac{n}{n+p\beta}})}\frac{1}{\lambda_\epsilon}c_\epsilon u_\epsilon(y)^{\frac{1}{n-1}}e^{\alpha_n(1+\frac{p\beta}{n}-\epsilon)u_\epsilon(y)^{\frac{n}{n-1}}}|y|^{p\beta}dy\\&=(1+o_\epsilon(1))\int_{\mathbb{B}(R)}\varphi(r_\epsilon^{\frac{n}{n+p\beta}}x)e^{\alpha_n(1+\frac{p\beta}{n}-\epsilon)(u_\epsilon^{\frac{n}{n-1}}(r_\epsilon^{\frac{n}{n+p\beta}}x)-c_\epsilon^{\frac{n}{n-1}})}|x|^{p\beta}\,dx\\&=(\varphi(0)+o_\epsilon(1))\left(\int_{\mathbb{B}(R)}e^{\frac{n}{n-1}\alpha_n(1+\frac{p\beta}{n}-\epsilon)\varphi}|x|^{p\beta}dx+o_\epsilon(1)\right).\end{aligned}$$

Letting $\epsilon \to 0$, we have $I_3 \to \varphi(0)$. Combining all the above three estimates, we conclude the result. □

The following statement is similar to Lemma 3.10 in [41]:

Lemma 3.4 *If $f \in L^1(\mathbb{B})$, and $u \in C^1(\overline{\mathbb{B}}) \cap H_0^{1,n}(\mathbb{B})$ satisfies the following equation*

$$-\Delta_n u = f + \alpha \|u\|_p^{n-p} u^{p-1},$$

where $\alpha < (1+\frac{p}{n}\beta)^{n-1+\frac{n}{p}} \lambda_p(\mathbb{B})$ is a constant. Then for any $1 < s < n$, we have $\|\nabla u\|_s \le C\|f\|_1$ for some constant C depending only on p, s, α, n, $\lambda_p(\mathbb{B})$.

We omit the proof here. The interested readers can refer to [34] and its corrigendum in [34] to get the detailed process of argumentation. Using Lemma 3.4, we can prove the following:

Lemma 3.5 *For any $1 < s < n$, $c_\epsilon^{\frac{1}{n-1}} u_\epsilon$ is bounded in $H_0^{1,s}(\mathbb{B})$.*

Proof We denote $\omega_\epsilon = c_\epsilon^{\frac{1}{n-1}} u_\epsilon$, then it is easy to verify that

$$-\Delta_n \omega_\epsilon = \alpha |x|^{p\beta} \|\omega_\epsilon\|_p^{n-p} \omega_\epsilon^{p-1} + \frac{1}{\lambda_\epsilon} |x|^{p\beta} c_\epsilon u_\epsilon^{\frac{1}{n-1}} e^{\alpha_n (1+\frac{p}{n}\beta - \epsilon) u_\epsilon^{\frac{n}{n-1}}} - \frac{\lambda}{\lambda_\epsilon} |x|^{p\beta} h_m'(\omega_\epsilon) c_\epsilon. \tag{3.6}$$

We assert that $\|\omega_\epsilon\|_p$ is bounded. Suppose not, we can assume that $\|\omega_\epsilon\|_p \to +\infty$ as $\epsilon \to 0$. Letting $\tilde{\omega}_\epsilon = \omega_\epsilon / \|\omega_\epsilon\|_p$, we have $\|\tilde{\omega}_\epsilon\|_p = 1$ and

$$-\Delta_n \tilde{\omega}_\epsilon = \alpha |x|^{p\beta} \tilde{\omega}_\epsilon^{p-1} + \frac{\frac{1}{\lambda_\epsilon} |x|^{p\beta} c_\epsilon u_\epsilon^{\frac{1}{n-1}} e^{\alpha_n (1+\frac{p}{n}\beta - \epsilon) u_\epsilon^{\frac{n}{n-1}}}}{\|\omega_\epsilon\|_p^{n-1}} + o(1). \tag{3.7}$$

It can be deduced from (3.7) that $\Delta_n \tilde{\omega}_\epsilon$ is bounded in $L^1(\mathbb{B})$. By Lemma 3.4, we get $\tilde{\omega}_\epsilon$ is bounded in $H_0^{1,s}(\mathbb{B})$ for any $1 < s < n$. Assume $\tilde{\omega}_\epsilon \rightharpoonup \tilde{\omega}$ weakly in $H_0^{1,s}(\mathbb{B})$ for any $1 < s < n$, and $\tilde{\omega}_\epsilon \to \tilde{\omega}$ strongly in $L^p(\mathbb{B})$. Testing (3.6) with $\varphi \in C_0^1(\mathbb{B})$ and letting $\epsilon \to 0$, we obtain

$$\int_{\mathbb{B}} \nabla \varphi \nabla \tilde{\omega} dx = \alpha \int_{\mathbb{B}} \varphi |x|^{p\beta} \tilde{\omega}^{p-1} dx. \tag{3.8}$$

One can derive from (3.8) that $\omega \equiv 0$, which contradicts the fact that $\|\tilde{\omega}\|_p = 1$. Hence $\|\omega_\epsilon\|_p$ is bounded. Again by using Lemma 3.4, we complete the proof. □

The following lemma reveals how u_ϵ converges away from $x_0 = 0$:

Lemma 3.6 *$c_\epsilon^{\frac{1}{n-1}} u_\epsilon \rightharpoonup G_\alpha$ weakly in $H^{1,s}(\mathbb{B})$ for any $1 < s < n$, where G_α is a Green function satisfying*

$$\begin{cases} -\Delta_n G_\alpha - \alpha \|G_\alpha\|_p^{n-p} G_\alpha^{p-1} |x|^{p\beta} = \delta_0 & \text{in } \mathbb{B} \\ G_\alpha = 0 & \text{on } \partial\mathbb{B}. \end{cases} \tag{3.9}$$

Furthermore, $c_\epsilon^{\frac{1}{n-1}} u_\epsilon \to G_\alpha$ in $C^1(\overline{\mathbb{B}'})$ for any domain $\mathbb{B}' \subset\subset \overline{\mathbb{B}} \setminus \{0\}$.

Proof Assume $c_\epsilon^{\frac{1}{n-1}} u_\epsilon \rightharpoonup G_\alpha$ weakly in $H^{1,s}(\mathbb{B})$. Testing equation (2.4) with $\varphi \in C_0^\infty(\Omega)$, we have

$$-\int_{\mathbb{B}} \varphi \Delta_n \omega_\epsilon dx = \int_{\mathbb{B}} \left(\varphi \frac{1}{\lambda_\epsilon} |x|^{p\beta} c_\epsilon u_\epsilon^{\frac{1}{n-1}} e^{\alpha_n (1+\frac{p}{n}\beta - \epsilon) u_\epsilon^{\frac{n}{n-1}}} + \alpha \|\omega_\epsilon\|_p^{n-p} \int_{\mathbb{B}} \varphi \omega_\epsilon^{p-1} |x|^{p\beta} \right) dx + o(1)$$
$$\to \varphi(0) + \alpha \|G_\alpha\|_p^{n-p} G_\alpha^{p-1} |x|^{p\beta}.$$

Hence

$$\int_\Omega \nabla \varphi |\nabla G_\alpha|^{n-2} \nabla G_\alpha dx = \varphi(0) + \alpha \|G_\alpha\|_p^{n-p} G_\alpha^{p-1} |x|^{p\beta}.$$

Then there holds

$$-\Delta_n G_\alpha = \delta_0 + \alpha \|G_\alpha\|_p^{n-p} G_\alpha^{p-1} |x|^{p\beta}.$$

The usual elliptic estimates give the second assertion of Lemma 3.6. □

According to Kichenassamy and Veron [17], G_α can be represented by

$$G_\alpha(x) = -\frac{n}{\alpha_n} \ln |x| + A_\alpha + \psi_\alpha(x),$$

where A_α is a constant, $\psi_\alpha(x) \in C^\nu(\mathbb{B})$ for some $0 < \nu < 1$ and $\psi_\alpha(0) = 0$.

4. The estimate of upper bound

In this section, we use the capacity estimate, which was inspired by [28, 37], to derive an upper bound of $\Lambda_{\lambda,\alpha_n}$. Taking $R > 0$ and $\delta > 0$ small enough such that $\mathbb{B}(2\delta) \subset \mathbb{B}$, for $a, b \in \mathbb{R}$, we define the function space

$$W_\epsilon(a,b) = \left\{ u \in W^{1,n}(\mathbb{B}(\delta) \setminus \mathbb{B}(R r_\epsilon^{\frac{n}{n+p\beta}})) : u|_{\partial \mathbb{B}(\delta)} = a, u|_{\partial \mathbb{B}(R r_\epsilon^{\frac{n}{n+p\beta}})} = b \right\}.$$

Let

$$i_\epsilon = \inf_{\partial \mathbb{B}(R r_\epsilon^{\frac{n}{n+p\beta}})} u_\epsilon, \qquad s_\epsilon = \sup_{\partial \mathbb{B}(\delta)} u_\epsilon.$$

It follows from (3.8) and Lemma 3.6 that

$$i_\epsilon = c_\epsilon + \frac{1}{c_\epsilon^{\frac{1}{n-1}}} \left(-\frac{n}{\alpha_n} \ln R - \frac{1}{\alpha_n (1 + \frac{p\beta}{n})} \ln \frac{\omega_{n-1}}{n + p\beta} + o(1) \right), \tag{4.1}$$

and

$$s_\epsilon = c_\epsilon^{-\frac{1}{n-1}} \left(-\frac{n}{\alpha_n} \ln \delta + A_\alpha + o(1) \right). \tag{4.2}$$

Therefore, $i_\epsilon > s_\epsilon$. It is not hard to see that

$$\inf_{u \in W_\epsilon(a,b)} \int_{\mathbb{B}(\delta) \setminus \mathbb{B}(R r_\epsilon^{\frac{n}{n+p\beta}})} |\nabla u|^n dx$$

is attained by a function $h(x)$ satisfying

$$\begin{cases} -\Delta_n h(x) = 0 \quad \text{in} \quad \mathbb{B}(\delta) \setminus \overline{\mathbb{B}(\mathrm{Rr}_\epsilon^{\frac{n}{n+p\beta}})} \\ h|_{\partial\mathbb{B}(\delta)} = s_\epsilon \\ h|_{\partial\mathbb{B}(Rr_\epsilon^{\frac{n}{n+p\beta}})} = i_\epsilon. \end{cases}$$

By the uniqueness of the solution, we obtain

$$h(x) = \frac{s_\epsilon(\ln|x| - \ln(Rr_\epsilon^{\frac{n}{n+p\beta}})) + i_\epsilon(\ln\delta - \ln|x|)}{\ln\delta - \ln(Rr_\epsilon^{\frac{n}{n+p\beta}})},$$

and hence

$$\int_{\mathbb{B}(\delta)\setminus\mathbb{B}(Rr_\epsilon^{\frac{n}{n+p\beta}})} |\nabla h|^n dx = \frac{\omega_{n-1}(i_\epsilon - s_\epsilon)^n}{(\ln\delta - \ln(Rr_\epsilon^{\frac{n}{n+p\beta}}))^{n-1}}. \tag{4.3}$$

Defining $\tilde{u}_\epsilon = \max\{s_\epsilon, \min\{u_\epsilon, i_\epsilon\}\}$, one gets $\tilde{u}_\epsilon \in W_\epsilon(s_\epsilon, i_\epsilon)$ and $|\nabla\tilde{u}_\epsilon| \leq |\nabla u_\epsilon|$ a.e. in $\mathbb{B}(\delta) \setminus \mathbb{B}(Rr_\epsilon^{\frac{n}{n+p\beta}})$. Then we have

$$\begin{aligned}\int_{\mathbb{B}(\delta)\setminus\mathbb{B}(Rr_\epsilon^{\frac{n}{n+p\beta}})} |\nabla h|^n dx &\leq \int_{\mathbb{B}(\delta)\setminus\mathbb{B}(Rr_\epsilon^{\frac{n}{n+p\beta}})} |\nabla\tilde{u}_\epsilon|^n dx \\ &\leq \int_{\mathbb{B}(\delta)\setminus\mathbb{B}(Rr_\epsilon^{\frac{n}{n+p\beta}})} |\nabla u_\epsilon|^n dx \\ &= 1 + \alpha\|u\|_{p,\beta}^n - \int_{\mathbb{B}(Rr_\epsilon^{\frac{n}{n+p\beta}})} |\nabla u_\epsilon|^n dx - \int_{\mathbb{B}\setminus\mathbb{B}(\delta)} |\nabla u_\epsilon|^n dx.\end{aligned}$$

We next estimate two integrals on the right-hand side of the above equation. We have

$$\begin{aligned}\int_{\mathbb{B}(Rr_\epsilon^{\frac{n}{n+p\beta}})} |\nabla u_\epsilon|^n dx &= c_\epsilon^{-\frac{n}{n-1}} \int_{\mathbb{B}(R)} |\nabla\varphi_\epsilon|^n dx \\ &= c_\epsilon^{-\frac{n}{n-1}} \left(\int_{\mathbb{B}(R)} |\nabla\varphi_0|^n dx + o_\epsilon(1)\right) \\ &= c_\epsilon^{-\frac{n}{n-1}} \left(\frac{n}{\alpha_n}\ln R + \frac{1}{\alpha_n(1+\frac{p\beta}{n})}\ln\frac{\omega_{n-1}}{n+p\beta}\right. \\ &\qquad \left. - \frac{n-1}{\alpha_n(1+\frac{p\beta}{n})}\sum_{k=1}^{n-1}\frac{1}{k} + o(1)\right).\end{aligned} \tag{4.4}$$

Since $\|u_\epsilon\|_{p,\beta}^n = c_\epsilon^{-\frac{n}{n-1}}(\|G_\alpha\|_{p,\beta}^n + o(1))$, integrating by parts with Lemma 3.6 leads to

$$\begin{aligned}\int_{\mathbb{B}\setminus\mathbb{B}(\delta)} |\nabla u_\epsilon|^n dx &= c_\epsilon^{-\frac{n}{n-1}} \left(\int_{\mathbb{B}\setminus\mathbb{B}_\delta(0)} |\nabla G_\alpha|^n dx + o(1) \right) \\ &= c_\epsilon^{-\frac{n}{n-1}} \left(\int_{\mathbb{B}\setminus\mathbb{B}(\delta)} (-\Delta_n G_\alpha) G_\alpha dx + \int_{\partial\mathbb{B}(\delta)} |\nabla G_\alpha|^{n-2} \nabla G_\alpha \cdot \frac{\partial G_\alpha}{\partial \nu} ds + o(1) \right) \\ &= c_\epsilon^{-\frac{n}{n-1}} \left(\alpha \|G_\alpha\|_p^{n-p} G_\alpha^{p-1} |x|^{p\beta} - \frac{n}{\alpha_n} \ln \delta + A_\alpha + o(1) \right).\end{aligned} \tag{4.5}$$

Combining (4.3), (4.4), and (4.5) together, we obtain

$$\begin{aligned}\frac{\omega_{n-1}^{\frac{1}{n-1}} (i_\epsilon - s_\epsilon)^{\frac{n}{n-1}}}{\ln \frac{\delta}{R} - \frac{1}{n+p\beta} \ln r_\epsilon^n} &\leq \Bigg(1 + c_\epsilon^{-\frac{n}{n-1}} \Bigg(\frac{n}{\alpha_n} \ln \frac{\delta}{R} - \frac{1}{\alpha_n (1 + \frac{p\beta}{n})} \ln \frac{\omega_{n-1}}{n+p\beta} \\ &\quad + \frac{n-1}{\alpha_n(1+\frac{p\beta}{n})} \sum_{k=1}^{n-1} \frac{1}{k} - A_\alpha + o(1) \Bigg) \Bigg)^{\frac{1}{n-1}} \\ &\leq 1 + \frac{1}{n-1} c_\epsilon^{-\frac{n}{n-1}} \Bigg(\frac{n}{\alpha_n} \ln \frac{\delta}{R} - \frac{1}{\alpha_n (1 + \frac{p\beta}{n})} \ln \frac{\omega_{n-1}}{n+p\beta} \\ &\quad + \frac{n-1}{\alpha_n(1+\frac{p\beta}{n})} \sum_{k=1}^{n-1} \frac{1}{k} - A_\alpha + o(1) \Bigg).\end{aligned} \tag{4.6}$$

From the definition of r_ϵ, we get

$$\ln \frac{\delta}{R} - \frac{1}{n+p\beta} \ln r_\epsilon^n = \ln \frac{\delta}{R} - \frac{1}{n+p\beta} \ln \frac{\lambda_\epsilon}{c_\epsilon^{\frac{n}{n-1}}} + \frac{\alpha_n (1 + \frac{p\beta}{n}) c_\epsilon^{\frac{n}{n-1}}}{n+p\beta}. \tag{4.7}$$

It follows from (4.1) and (4.2) that

$$\begin{aligned}(i_\epsilon - s_\epsilon)^{\frac{n}{n-1}} &= c_\epsilon^{\frac{n}{n-1}} \left(1 + c_\epsilon^{-\frac{n}{n-1}} \left(\frac{n}{\alpha_n} \ln \frac{\delta}{R} - \frac{1}{\alpha_n(1+\frac{p\beta}{n})} \ln \frac{\omega_{n-1}}{n+p\beta} - A_\alpha + o(1) \right) \right)^{\frac{n}{n-1}} \\ &\geq c_\epsilon^{\frac{n}{n-1}} + \frac{n}{n-1} \left(\frac{n}{\alpha_n} \ln \frac{\delta}{R} - \frac{1}{\alpha_n(1+\frac{p\beta}{n})} \ln \frac{\omega_{n-1}}{n+p\beta} - A_\alpha + o(1) \right).\end{aligned} \tag{4.8}$$

Denoting $b = \frac{1}{n-1} c_\epsilon^{-\frac{n}{n-1}} \left(\frac{n}{\alpha_n} \ln \frac{\delta}{R} - \frac{1}{\alpha_n(1+\frac{p\beta}{n})} \ln \frac{\omega_{n-1}}{n+p\beta} + \frac{n-1}{\alpha_n(1+\frac{p\beta}{n})} \sum_{k=1}^{n-1} \frac{1}{k} - A_\alpha + o(1) \right)$, we can obtain $b \to 0$. Then putting (4.6), (4.7), and (4.8) together, we have

$$\begin{aligned}
(1+b) \ln \frac{\lambda_\epsilon}{c_\epsilon^{\frac{n}{n-1}}} \le & -\epsilon c_\epsilon^{\frac{n}{n-1}} + \left(\alpha_n (1+\frac{p\beta}{n}) b - \frac{\epsilon}{n-1} \right) \frac{n}{\alpha_n} \ln \frac{\delta}{R} + \frac{\alpha_n(1+\frac{p\beta}{n} - \epsilon)}{\alpha_n(1+\frac{p\beta}{n})} \sum_{k=1}^{n-1} \frac{1}{k} \\
& + \left(1 + \frac{\epsilon}{\alpha_n(1+\frac{p\beta}{n})(n-1)} \right) \left(\ln \frac{\omega_{n-1}}{n+p\beta} + \alpha_n(1+\frac{p\beta}{n}) A_\alpha \right) + o(1) \\
\le & \left(\alpha_n (1+\frac{p\beta}{n}) b - \frac{\epsilon}{n-1} \right) \frac{n}{\alpha_n} \ln \frac{\delta}{R} + \frac{\alpha_n(1+\frac{p\beta}{n} - \epsilon)}{\alpha_n(1+\frac{p\beta}{n})} \sum_{k=1}^{n-1} \frac{1}{k} \\
& + \left(1 + \frac{\epsilon}{\alpha_n(1+\frac{p\beta}{n})(n-1)} \right) \left(\ln \frac{\omega_{n-1}}{n+p\beta} + \alpha_n(1+\frac{p\beta}{n}) A_\alpha \right) + o(1),
\end{aligned}$$

which implies that

$$\limsup_{\epsilon \to 0} \ln \frac{\lambda_\epsilon}{c_\epsilon^{\frac{n}{n-1}}} \le \ln \frac{\omega_{n-1}}{n+p\beta} + \alpha_n(1+\frac{p\beta}{n}) A_\alpha + \sum_{k=1}^{n-1} \frac{1}{k}.$$

Therefore, we conclude by (3.4),

$$\begin{aligned}
\Lambda_{\lambda,\alpha_n} &= \lim_{\epsilon \to 0} \int_{\mathbb{B}} |x|^{p\beta} \left(e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_\epsilon|^{\frac{n}{n-1}}} - \lambda \sum_{k=0}^{m} \frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon) u_\epsilon^{\frac{n}{n-1}}|^k}{k!} \right) dx \\
&\le \int_{\mathbb{B}} |x|^{p\beta} dx + \frac{\omega_{n-1}}{n+p\beta} e^{\alpha_n(1+\frac{p\beta}{n})A_\alpha + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}}.
\end{aligned} \tag{4.9}$$

5. The existence result

In this section, we will construct a blow-up sequence $\varphi_\epsilon(x) \in \mathcal{H}$ such that when ϵ is small enough, there holds

$$\begin{aligned}
& \int_{\mathbb{B}} |x|^{p\beta} \left(e^{\alpha_n(1+\frac{p}{n}\beta)|\varphi_\epsilon|^{\frac{n}{n-1}}} - \lambda \sum_{k=0}^{m} \frac{|\alpha_n(1+\frac{p}{n}\beta) \varphi_\epsilon^{\frac{n}{n-1}}|^k}{k!} \right) dx \\
> & \int_{\mathbb{B}} |x|^{p\beta} dx + \frac{\omega_{n-1}}{n+p\beta} e^{\alpha_n(1+\frac{p\beta}{n})A_\alpha + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}}.
\end{aligned}$$

We first establish several properties of G_α as following:

Lemma 5.1 *Let G_α be the n-Green function in the above (3.9).*

(a) The sets $\{G_\alpha > t\}$ form a sequence of approximately small balls of radii $\rho_t = e^{\omega_{n-1}^{\frac{1}{n-1}}(A_\alpha - t)}$. In other words, $B_{\rho_t - r_t}(p) \subset \{G_\alpha > t\} \subset B_{\rho_t + r_t}(p)$, with $r_t/\rho_t \to 0$ as $t \to +\infty$. In particular, $\lim_{t \to +\infty} e^{\alpha_n(1+\frac{p\beta}{n})t} \int_{G_\alpha > t} |x|^{p\beta} dx = \frac{\omega_{n-1}}{n+p\beta} e^{\alpha_n(1+\frac{p\beta}{n})A_\alpha}$.

(b) $\int_{G_\alpha < t} |\nabla G_\alpha|^n dx = t + \alpha \|G_\alpha\|_p^{n-p} G_\alpha^{p-1} |x|^{p\beta} + O(t^{n-1} e^{-\alpha_n(1+\frac{p\beta}{n}t)})$ as $t \to +\infty$.

(c) $\int_{G_\alpha=t}|\nabla G_\alpha|^{n-1}dx = 1+O(t^{n-1}e^{-\alpha_n(1+\frac{p\beta}{n})t})$ as $t\to+\infty$.

(d) $\int_{G_\alpha=t}\frac{|x|^{p\beta}}{|\nabla G_\alpha|}ds \geq \omega_{n-1}^{\frac{n}{n-1}}e^{\alpha_n(1+\frac{p\beta}{n})(A_\alpha-t)}(1+O(t^{n-1}e^{-\alpha_n(1+\frac{p\beta}{n})t}))$ as $t\to+\infty$.

The proof is similar to [34] so we omit the process of proof here. Then we take

$$f_\epsilon(t)=\begin{cases} c+c^{-\frac{1}{n-1}}\left(-\frac{n-1}{\alpha_n(1+\frac{p\beta}{n})}\ln(1+(\frac{\omega_{n-1}}{n+p\beta})^{\frac{1}{n-1}}\epsilon^{-\frac{n+p\beta}{n-1}}e^{-\frac{\alpha_n(1+\frac{p\beta}{n})}{n-1}t})+b\right) & \text{for} \quad t\geq t_\epsilon \\ c^{-\frac{1}{n-1}}t & \text{for} \quad t<t_\epsilon, \end{cases}$$

with $t_\epsilon=\frac{n}{\alpha_n}\ln\frac{1}{R\epsilon}$, R, b, and c are constants to be chosen later such that $R\to+\infty$ and $R\epsilon\to 0$ as $\epsilon\to 0$. Let G_α be as above. Set

$$\varphi_\epsilon(x)=f_\epsilon(G_\alpha(x)).$$

To ensure $\varphi_\epsilon\in H_0^{1,n}(\mathbb{B})$, we assume

$$c+c^{-\frac{1}{n-1}}\left(-\frac{n-1}{\alpha_n(1+\frac{p\beta}{n})}\ln(1+(\frac{\omega_{n-1}}{n+p\beta})^{\frac{1}{n-1}}\epsilon^{-\frac{n+p\beta}{n-1}}e^{-\frac{\alpha_n(1+\frac{p\beta}{n})}{n-1}t_\epsilon})+b\right)=c^{-\frac{1}{n-1}}t_\epsilon. \tag{5.1}$$

We have by Lemma 5.1(b),

$$\int_{G_\alpha<t_\epsilon}|\nabla\varphi_\epsilon|^n dx=c^{-\frac{n}{n-1}}\left(\frac{n}{\alpha_n}\ln\frac{1}{R\epsilon}+\alpha\|G_\alpha\|_{p,\beta}^n+O((R\epsilon)^{n+p\beta}(\ln\frac{1}{R\epsilon})^{n-1}\right).$$

An elementary calculation shows

$$\begin{aligned}\int_{t_\epsilon}^{+\infty}|f_\epsilon'(t)|^n dt &= c^{-\frac{n}{n-1}}\int_{t_\epsilon}^{+\infty}\left(\frac{(\frac{\omega_{n-1}}{n+p\beta})^{\frac{1}{n-1}}\epsilon^{-\frac{n+p\beta}{n-1}}e^{-\frac{\alpha_n(1+\frac{p\beta}{n})}{n-1}t}}{1+(\frac{\omega_{n-1}}{n+p\beta})^{\frac{1}{n-1}}\epsilon^{-\frac{n+p\beta}{n-1}}e^{-\frac{\alpha_n(1+\frac{p\beta}{n})}{n-1}t}}\right)^n dt \\ &= \frac{n-1}{\alpha_n(1+\frac{p\beta}{n})}c^{-\frac{n}{n-1}}\int_0^{(\frac{\omega_{n-1}}{n+p\beta})^{\frac{1}{n-1}}R^{\frac{n+p\beta}{n-1}}}\frac{s^{n-1}}{(1+s)^n}ds \\ &= c^{-\frac{n}{n-1}}\left(\frac{n}{\alpha_n}\ln R+\frac{1}{\alpha_n(1+\frac{p\beta}{n})}\ln\frac{\omega_{n-1}}{n+p\beta}-\frac{n-1}{\alpha_n(1+\frac{p\beta}{n})}\sum_{k=1}^{n-1}\frac{1}{k}+O(R^{-\frac{n+p\beta}{n-1}})\right).\end{aligned}$$

Hence we have by Lemma 5.1(c),

$$\begin{aligned}\int_{G_\alpha>t_\epsilon}|\nabla\varphi_\epsilon|^n dx &= \int_{t_\epsilon}^{+\infty}|f_\epsilon'(t)|^n\left(\int_{G_\alpha=t}|\nabla G_\alpha|^n\frac{1}{|\nabla G_\alpha|}ds\right)dt \\ &= c^{-\frac{n}{n-1}}\left(\frac{n}{\alpha_n}\ln R+\frac{1}{\alpha_n(1+\frac{p\beta}{n})}\ln\frac{\omega_{n-1}}{n+p\beta}-\frac{n-1}{\alpha_n(1+\frac{p\beta}{n})}\sum_{k=1}^{n-1}\frac{1}{k}+O(R^{-\frac{n+p\beta}{n-1}})\right).\end{aligned}$$

Therefore,

$$\int_{\mathbb{B}} |\nabla \varphi_\epsilon|^n dx = c^{-\frac{n}{n-1}} \left(\frac{n}{\alpha_n} \ln \epsilon + \frac{1}{\alpha_n(1+\frac{p\beta}{n})} \ln \frac{\omega_{n-1}}{n+p\beta} - \frac{n-1}{\alpha_n(1+\frac{p\beta}{n})} \sum_{k=1}^{n-1} \frac{1}{k} + \alpha \|G_\alpha\|_{p,\beta}^n + O(R^{-\frac{n+p\beta}{n-1}}) \right).$$

Since $\|\varphi_\epsilon\|_{p,\,\beta}^n = c^{-\frac{n}{n-1}}(\|G_\alpha\|_{p,\beta}^n + O(R^{-\frac{n+p\beta}{n-1}}))$, then we have

$$c^{\frac{n}{n-1}} = -\frac{n}{\alpha_n} \ln \epsilon + \frac{1}{\alpha_n(1+\frac{p\beta}{n})} \ln \frac{\omega_{n-1}}{n+p\beta} - \frac{n-1}{\alpha_n(1+\frac{p\beta}{n})} \sum_{k=1}^{n-1} \frac{1}{k} + O(R^{-\frac{n+p\beta}{n-1}}). \tag{5.2}$$

Combining (5.1) and (5.2), one gets

$$b = \frac{n-1}{\alpha_n(1+\frac{p\beta}{n})} \sum_{k=1}^{n-1} \frac{1}{k} + O(R^{-\frac{n}{n-1}}) + O\left((R\epsilon)^n \ln^n \frac{1}{R\epsilon} \ln R \right).$$

For $t \geq t_\epsilon$, one can check that

$$f_\epsilon(t)^{\frac{n}{n-1}} \geq c^{\frac{n}{n-1}} + \frac{n}{n-1} b - \frac{n}{\alpha_n(1+\frac{p\beta}{n})} \ln \left(1 + (\frac{\omega_{n-1}}{n+p\beta})^{\frac{1}{n-1}} \epsilon^{-\frac{n+p\beta}{n-1}} e^{-\frac{\alpha_n(1+\frac{p\beta}{n})}{n-1} t} \right).$$

Hence we have by Lemma 5.1(d),

$$\begin{aligned}
\int_{G_\alpha \geq t_\epsilon} e^{\alpha_n(1+\frac{p\beta}{n})|\varphi_\epsilon|^{\frac{n}{n-1}}} |x|^{p\beta} dx &= \int_{t_\epsilon}^{+\infty} e^{\alpha_n(1+\frac{p\beta}{n})|f_\epsilon(t)|^{\frac{n}{n-1}}} \left(\int_{G_\alpha = t} \frac{|x|^{p\beta}}{|\nabla G_\alpha|} ds \right) dt \\
&\geq (n-1) e^{\alpha_n(1+\frac{p\beta}{n})(A_\alpha + c^{\frac{n}{n-1}} + \frac{n}{n-1} b)} \epsilon^{n+p\beta} (1 + O(t_\epsilon^{n-1} e^{-\alpha_n(1+\frac{p\beta}{n}) t_\epsilon})) \\
&\quad \times \int_0^{(\frac{\omega_{n-1}}{n+p\beta})^{\frac{1}{n-1}} R^{\frac{n+p\beta}{n-1}}} \frac{s^{n-2}}{(1+s)^n} ds \\
&\geq \frac{\omega_{n-1}}{n+p\beta} e^{\alpha_n(1+\frac{p\beta}{n}) A_\alpha + \sum_{k=1}^{n-1} \frac{1}{k}} + O(R^{-\frac{n+p\beta}{n-1}}).
\end{aligned}$$

Since $\frac{\ln R}{c^{\frac{n}{n-1}}} \to 0$, we can obtain $\frac{c}{2} < \varphi_\epsilon|_{G_\alpha > t} < 2c$. Then we have

$$\int_{G_\alpha > t_\epsilon} \sum_{k=0}^{m} \frac{(\alpha_n(1+\frac{p\beta}{n})|\varphi_\epsilon|^{\frac{n}{n-1}})^k}{k!} = O(c^{\frac{mn}{n-1}} \epsilon^2 R^2).$$

Moreover, we get

$$\begin{aligned}
\int_{G_\alpha < t_\epsilon} |x|^{p\beta} (e^{\alpha_n(1+\frac{p\beta}{n})|\varphi_\epsilon|^{\frac{n}{n-1}}} - h_m(\varphi_\epsilon)) dx &\geq \int_{\mathbb{B}} |x|^{p\beta} dx - \int_{G_\alpha \geq t_\epsilon} |x|^{p\beta} dx \\
&\quad + \int_{G_\alpha < t_\epsilon} \frac{(\alpha_n(1+\frac{p\beta}{n}))^{m+1}}{(m+1)!} \left| \frac{G_\alpha}{c^{\frac{1}{n-1}}} \right|^{\frac{n(m+1)}{n-1}} dx.
\end{aligned}$$

Combining the above two estimates, we obtain

$$\int_{\mathbb{B}} |x|^{p\beta}(e^{\alpha_n(1+\frac{p\beta}{n})|\varphi_\epsilon|^{\frac{n}{n-1}}} - h_m(\varphi_\epsilon))dx \geq \int_{\mathbb{B}} |x|^{p\beta}dx + \frac{\omega_{n-1}}{n+p\beta}e^{\alpha_n(1+\frac{p\beta}{n})A_\alpha + \sum_{k=1}^{n-1}\frac{1}{k}}$$
$$+ c^{-\frac{n(m+1)}{(n-1)^2}}\left(\int_{G_\alpha < t_\epsilon} \frac{|\alpha_n(1+\frac{p\beta}{n})G_\alpha^{\frac{n}{n-1}}|^{m+1}}{(m+1)!}dx\right.$$
$$\left. + O(c^{\frac{n(m+1)}{(n-1)^2}}R^2\epsilon^2) + O(c^{\frac{n(m+1)}{(n+1)^2}}R^{-\frac{n}{n-1}})\right).$$

Letting $R = (-\ln\epsilon)^{m+1}$, we immediately have

$$\int_{\mathbb{B}} |x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta)|\varphi_\epsilon|^{\frac{n}{n-1}}} - \sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta)\varphi_\epsilon^{\frac{n}{n-1}}|^k}{k!}\right)dx$$
$$> \int_{\mathbb{B}} |x|^{p\beta}dx + \frac{\omega_{n-1}}{n+p\beta}e^{\alpha_n(1+\frac{p\beta}{n})A_\alpha + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}}.$$

For any $\lambda \leq 1$, we have

$$\begin{aligned}
&\int_{\mathbb{B}} |x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta)|\varphi_\epsilon|^{\frac{n}{n-1}}} - \lambda\sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta)\varphi_\epsilon^{\frac{n}{n-1}}|^k}{k!}\right)dx \\
\geq &\int_{\mathbb{B}} |x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta)|\varphi_\epsilon|^{\frac{n}{n-1}}} - \sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta)\varphi_\epsilon^{\frac{n}{n-1}}|^k}{k!}\right)dx \\
> &\int_{\mathbb{B}} |x|^{p\beta}dx + \frac{\omega_{n-1}}{n+p\beta}e^{\alpha_n(1+\frac{p\beta}{n})A_\alpha + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}}.
\end{aligned} \tag{5.3}$$

The contradiction between (4.9) and (5.3) implies that c_ϵ is bounded and Theorem 1 follows when $\lambda \leq 1$. In the following, we consider the situation when $\lambda \in (1, 1+\epsilon_0)$, ϵ_0 is a constant. First we claim that $\Lambda_{\lambda,\alpha_n}$ is continuous with respect to λ at $\lambda = 1$. It is clearly that there exists u_1 such that

$$\Lambda_{1,\alpha_n} = \int_{\mathbb{B}} |x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_1|^{\frac{n}{n-1}}} - \sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_1^{\frac{n}{n-1}}|^k}{k!}\right)dx.$$

Since $\int_{\mathbb{B}} |x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_1|^{\frac{n}{n-1}}} - \lambda\sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_1^{\frac{n}{n-1}}|^k}{k!}\right)dx$ is continuous with respect to λ at $\lambda = 1$, for any $\delta > 0$, there exists $\epsilon_1 > 0$ such that

$$\left|\int_{\mathbb{B}} |x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_1|^{\frac{n}{n-1}}} - \lambda\sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_1^{\frac{n}{n-1}}|^k}{k!}\right)dx - \Lambda_{1,\alpha_n}\right| < \delta,$$

where $1 < \lambda < 1+\epsilon_1$, then

$$\Lambda_{1,\alpha_n} - \delta < \int_{\mathbb{B}} |x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta-\epsilon)|u_1|^{\frac{n}{n-1}}} - \lambda\sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta-\epsilon)u_1^{\frac{n}{n-1}}|^k}{k!}\right)dx < \Lambda_{1,\alpha_n} + \delta. \tag{5.4}$$

Moreover, $\Lambda_{\lambda,\alpha_n}$ is monotonically decreasing with respect to λ. Thus for any $1<\lambda<1+\epsilon_1$, we have

$$\Lambda_{1,\alpha_n}-\delta<\Lambda_{\lambda,\alpha_n}\le\Lambda_{1,\alpha_n}.$$

So our claim is true. If the extremal function of $\Lambda_{\lambda,\alpha_n}$ does not exist when $1<\lambda<1+\epsilon_0$, then similar to the proof of the above, we can derive

$$\Lambda_{\lambda,\alpha_n}\le\int_{\mathbb{B}}|x|^{p\beta}dx+\frac{\omega_{n-1}}{n+p\beta}e^{\alpha_n(1+\frac{p\beta}{n})A_\alpha+1+\frac{1}{2}+\cdots+\frac{1}{n-1}},$$

but we found that $\int_{\mathbb{B}}|x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta)|\varphi_\epsilon|^{\frac{n}{n-1}}}-\lambda\sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta)\varphi_\epsilon^{\frac{n}{n-1}}|^k}{k!}\right)dx$ is continuous with respect to λ at $\lambda=1$, so there exists a constant $\epsilon_2>0$ such that

$$\begin{aligned}&\int_{\mathbb{B}}|x|^{p\beta}\left(e^{\alpha_n(1+\frac{p}{n}\beta)|\varphi_\epsilon|^{\frac{n}{n-1}}}-\lambda\sum_{k=0}^{m}\frac{|\alpha_n(1+\frac{p}{n}\beta)\varphi_\epsilon^{\frac{n}{n-1}}|^k}{k!}\right)dx\\&>\int_{\mathbb{B}}|x|^{p\beta}dx+\frac{\omega_{n-1}}{n+p\beta}e^{\alpha_n(1+\frac{p\beta}{n})A_\alpha+1+\frac{1}{2}+\cdots+\frac{1}{n-1}}\end{aligned}$$

for any $\lambda\in(1,1+\epsilon_2)$, which contradicts with (5.4). Thus $\Lambda_{\lambda,\alpha_n}$ can be attained if $\lambda\le1+\epsilon_0$.

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