Turkish Journal of Mathematics

Volume 48 | Number 1 Article 6

1-31-2024

Invariant symplectic forms on number fields

AHMAD RAFIQI
ha14mu@gmail.com

AYBERK ZEYTİN
ayberkz@gmail.com

Recommended Citation

RAFIQI, AHMAD and ZEYTİN, AYBERK (2024) "Invariant symplectic forms on number fields," *Turkish Journal of Mathematics*: Vol. 48: No. 1, Article 6. https://doi.org/10.55730/1300-0098.3491
Available at: https://journals.tubitak.gov.tr/math/vol48/iss1/6

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2024) 48: 53 – 61
© TÜBİTAK
doi:10.55730/1300-0098.3491

Invariant symplectic forms on number fields

Ahmad RAFIQI[1,*], **Ayberk ZEYTİN**[2]
[1]Department of Mathematics and Statistics, College of Arts and Sciences, American University of Sharjah, Sharjah, UAE
[2]Department of Mathematics, Faculty of Science, Galatasaray University, İstanbul, Turkiye

Received: 09.06.2023 • **Accepted/Published Online:** 08.12.2023 • **Final Version:** 31.01.2024

Abstract: We show that a number field of the form $\mathbb{Q}(\lambda)$ admits a symplectic form which is invariant under multiplication by λ if and only if the minimal polynomial of λ is palindromic of even degree. In particular, if λ is an algebraic integer, it is forced to be a unit. In the case when the minimal polynomial of λ is palindromic of degree $2d$, we show that there is a d-dimensional space of invariant symplectic forms on $\mathbb{Q}(\lambda)$.

Key words: Number field, symplectic form, invariance, palindromic

1. Motivation

Thurston [6] classified homotopy classes of homeomorphisms $f : S \to S$ of compact topological surfaces S into three types - periodic, reducible, and pseudo-Anosov. Each homeomorphism induces a map on the integral homology group $f_* : H_1(S;\mathbb{Z}) \circlearrowleft$. We will now assume our surfaces S to be orientable and without a boundary for clarity of exposition. Given $\alpha, \beta \in H_1(S;\mathbb{Z})$, the intersection form $\omega(\alpha,\beta)$ counts the minimum number of (signed) intersections of simple closed curves representing α with ones representing β. Since homeomorphisms preserve intersections, the form ω is invariant under the action f_* when f is orientation preserving. Similary $f_*(\omega) = -\omega$ when f is orientation reversing.

In the pseudo-Anosov case, there is a pair of singular foliations on S which are invariant under f up to homotopy. When these foliations are orientable, the largest eigenvalue $\lambda > 1$ of f_* is an important topological invariant of the mapping class known as the *stretch-factor* (or *dilatation*). Being an eigenvalue of an integer matrix, λ is an algebraic number. Moreover, $\log(\lambda)$ is the topological entropy of f [5], as well as the length of the geodesic corresponding to the mapping class $[f]$ in the moduli space of complete Riemannian metrics on S under the Teichmüller metric (see e.g. the book [3]).

It is an open problem to determine precisely what algebraic numbers arise in this way. Fried [2] showed that the largest eigenvalue λ of f_* is an algebraic unit in the ring of algebraic integers of the number field $\mathbb{Q}(\lambda)$. Moreover, he shows that λ is *biPerron*, meaning all its Galois conjugates μ_i satisfy $\{1/\lambda \leq |\mu_i| \leq \lambda\}$ with at most one conjugate on each boundary component. He also showed the characteristic polynomial of f_* is palindromic, meaning its coefficients are the same forwards and backwards. There are many related works,

*Correspondence: ha14mu@gmail.com
2020 *AMS Mathematics Subject Classification:* 11R04, 37B40, 37J11

most notably [5], which show that logarithms of *Perron* numbers (algebraic $\lambda > 1$ whose Galois conjugates μ_i satisfy $|\mu_i| < \lambda$), correspond exactly to the topological entropies of interval maps whose critical points have finite forward orbits. In particular, all Perron (and therefore biPerron) numbers are eigenvalues of integer *aperiodic* matrices, i.e. non-negative matrices with some strictly positive power; a converse to the classical Perron-Frobenius theorem originally due to Lind [4].

Now the action f_* on $H_1(S;\mathbb{R})$ is semi-conjugate to the automorphism of $\mathbb{Q}(\lambda)$ given by multiplication with λ. The semi-conjugacy $\pi : H_1(S;\mathbb{R}) \to \mathbb{Q}(\lambda)$ can be obtained by mapping each eigenvector of f_* corresponding to a Galois conjugate of λ to the corresponding eigenvector in $\mathbb{Q}(\lambda)$, and sending other eigenvectors (if any) to 0. If $1/\lambda$ is a Galois conjugate of λ, (which happens exactly when the minimal polynomial of λ is palindromic), the intersection form ω is non-degenerate when restricted to $\mathbb{Q}(\lambda)$, thus defining a symplectic form on $\mathbb{Q}(\lambda)$ invariant under multiplication by λ. The non-degeneracy of $\omega|_{\mathbb{Q}(\lambda)}$ when λ^{-1} is a conjugate can be seen by noting that for two eigenvectors v_μ, v_ν of f_*, corresponding to eigenvalues μ and ν respectively, $\omega(v_\mu, v_\nu) \neq 0$ if and only if $\mu = 1/\nu$ since,

$$\omega(\, v_\mu,\, v_\nu \,) = \omega(\, f_*\, v_\mu,\, f_*\, v_\nu \,) = \omega(\, \mu\, v_\mu,\, \nu\, v_\nu \,) = \mu\nu\, \omega(\, v_\mu,\, v_\nu \,). \tag{1.1}$$

The authors in [1] showed the following for all genus $g \geq 10$: among all biPerron numbers $\lambda \leq R$ of degree $d \leq 2g$ the proportion which corresponds to stretch-factors of pseudo-Anosov maps with orientable foliations on surfaces of genus g approaches 0 as $R \to \infty$. This suggests that stretch factors are sparse among biPerron numbers, or at least that the genus of the surfaces gets very large when trying to realize biPerron numbers of a given degree as stretch factors.

Our motivation for this paper was to rule out certain biPerron units λ from appearing as eigenvalues of surface automorphisms, so we sought a natural obstruction in the form of the following question: which number fields of the form $\mathbb{Q}(\lambda)$ admit symplectic forms that are invariant under multiplication by λ. Our work does not disprove the conjecture that all biPerron numbers are stretch factors. Rather, the answer to our question turns out to be that there is no such obstruction to the existence of a symplectic structure for number fields of biPerron numbers, since for any algebraic number λ,

Theorem 1.1 *$\mathbb{Q}(\lambda)$ admits a symplectic form invariant under multiplication by λ if and only if the minimal polynomial of λ is a palindromic polynomial of even degree.*

In section 2 we recall some definitions including symplectic forms, algebraic number fields, and describe what we mean by invariance. Section 3 proves the necessity of the minimal polynomial of λ to be palindromic for the existence of such forms. Section 4 is devoted to showing sufficience, and showing that the dimension of the space of invariant symplectic forms is half the degree of the minimal polynomial. In section 5, we provide some examples.

2. Introduction

A complex number λ is called *algebraic* if it is the root of a polynomial of the form

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \text{ with } a_i \in \mathbb{Q} \tag{2.1}$$

The unique polynomial of minimal degree is called the *minimal polynomial* of λ and the roots of this polynomial are the *Galois conjugates* of λ. If the coefficients a_i are integers, $a_i \in \mathbb{Z}$, then λ is called an *algebraic integer*. A polynomial $p(x)$ is called *palindromic* if $a_0 = 1$ and $a_i = a_{n-i}$ for all $1 \leq i \leq n-1$.

Suppose V is a finite-dimensional vector space defined over a field $\mathbb{F}$, or more generally a module over a ring. Then a *(linear) symplectic form* Θ on V is a non-degenerate alternating bilinear pairing. Namely, it is a map $\Theta : V \times V \to \mathbb{F}$, that satisfies the following three properties:

$$(i) \quad \Theta(u,v) = -\,\Theta(v,u), \ \text{ for all } u,v \in V \qquad \textit{(alternating)} \tag{2.2}$$

$$(ii) \quad \Theta(a\,u + b\,v, w) = a\,\Theta(u,w) + b\,\Theta(v,w), \quad \text{for all } u,v,w \in V \text{ and } a,b \in \mathbb{F} \qquad \textit{(bilinear)} \tag{2.3}$$

$$(iii) \quad \text{For any } u \in V \setminus \{0\}, \text{ there is some } v \in V \text{ such that } \Theta(u,v) \neq 0 \qquad \textit{(non-degenerate)} \tag{2.4}$$

In terms of a basis for V, bilinearity implies Θ can be written as $\Theta(u,v) = u^\top Q\,v$, where Q is a square matrix of the same size as the dimension of V. Θ is non-degenerate when $\det(Q) \neq 0$ and alternating when Q is skew-symmetric, i.e. when $Q^\top = -\,Q$.

In this paper, we will be concerned with number fields of the form $\mathbb{Q}(\lambda)$, where λ is an algebraic number. The *number field* $\mathbb{Q}(\lambda)$ is defined to be the minimal finite field extension of $\mathbb{Q}$ that contains λ. If the minimal polynomial of λ is of degree n, then $\mathbb{Q}(\lambda)$ is a $\mathbb{Q}$-vector space of dimension n, and a general element $\beta \in \mathbb{Q}(\lambda)$ is of the form

$$\beta = b_1 + b_2\lambda + \cdots + b_n\lambda^{n-1}, \quad \text{for unique } \ b_i \in \mathbb{Q}. \tag{2.5}$$

Multiplication with λ defines a field automorphism $(\cdot \times \lambda) : \mathbb{Q}(\lambda) \to \mathbb{Q}(\lambda)$ and thus an isomorphism of $\mathbb{Q}(\lambda)$ as a vector space. We wish to find symplectic forms Θ on $\mathbb{Q}(\lambda)$ which are invariant under this isomorphism.

Definition 2.1 *Let λ be an algebraic number. We say $\mathbb{Q}(\lambda)$ admits a **λ-invariant symplectic form** Θ if there exists a symplectic form $\Theta : \mathbb{Q}(\lambda) \times \mathbb{Q}(\lambda) \to \mathbb{Q}$ satisfying*

$$\Theta(u,v) = \Theta(\lambda\,u, \lambda\,v), \quad \textit{for all } \ u,v \in \mathbb{Q}(\lambda). \tag{2.6}$$

3. Necessity

If $\mathbb{Q}(\lambda)$ carries a symplectic form, it must be even-dimensional. This follows from the fact that an odd-dimensional skew-symmetric matrix has a determinant 0. Thus n must be an even integer.

Proposition 3.1 *If λ is an algebraic number such that $\mathbb{Q}(\lambda)$ admits a λ-invariant symplectic form, then the minimal polynomial of λ is palindromic, that is $a_0 = 1$ and $a_i = a_{n-i}$ for all $1 \leq i \leq n-1$.*

Proof Suppose $\mathbb{Q}(\lambda)$ admist a λ-invariant symplectic form Θ. If $u \in \mathbb{Q}(\lambda)$ is written as a column vector in terms of the standard basis $\{1, \lambda, \lambda^2, \cdots, \lambda^{n-1}\}$ of $\mathbb{Q}(\lambda)$, then $\lambda\,u$ is given by Cu, where C is the $n \times n$ companion matrix of $p(x)$ (equation 2.1):

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \tag{3.1}$$

On the other hand, Θ is given by $\Theta(u,v) = u^\top Q\, v$ where Q is the skew-symmetric and non-degenerate matrix

$$Q_{i\,j} = \Theta(\lambda^{i-1}, \lambda^{j-1}), \qquad 1 \le i, j \le n. \tag{3.2}$$

The λ-invaraiance relation $\Theta(u,v) = \Theta(\lambda\, u, \lambda\, v)$ for all $u, v \in \mathbb{Q}(\lambda)$ thus becomes

$$u^\top Q\, v = (Cu)^\top Q(Cv) = u^\top (C^\top QC)v. \tag{3.3}$$

So the form Θ on $\mathbb{Q}(\lambda)$ is λ-invariant exactly when

$$C^\top QC = Q. \tag{3.4}$$

If v is a λ-eigenvector of C, we see that $C^\top QCv = Qv$ implies Qv is a $\frac{1}{\lambda}$-eigenvector of $C^\top$. Thus $1/\lambda$ is an eigenvalue of $C^\top$ (and of C itself) and satisfies the minimal polynomial of λ:

$$p\left(\frac{1}{\lambda}\right) = \left(\frac{1}{\lambda}\right)^n + a_{n-1}\left(\frac{1}{\lambda}\right)^{n-1} + \cdots + a_1\left(\frac{1}{\lambda}\right) + a_0 = 0. \tag{3.5}$$

Multiplying throughout by λ^n and dividing by a_0, we get

$$\lambda^n + \frac{a_1}{a_0}\lambda^{n-1} + \cdots + \frac{a_{n/2}}{a_0}\lambda^{n/2} + \cdots + \frac{a_{n-1}}{a_0}\lambda + \frac{1}{a_0} = 0. \tag{3.6}$$

Comparing coefficients with the minimal polynomial $p(x)$ (2.1) of λ, which is unique, we see that $a_{n-i} = a_i/a_0$ for all i. In particular $a_{n/2} = a_{n/2}/a_0$, and $a_0 = 1/a_0 \implies a_0 = \pm 1$.

Note that $a_0 = -1$ contradicts the minimality of $p(x)$: If $a_0 = -1$ we get $a_{n/2} = 0$, and $a_{n-i} = -a_i$ for $1 \le i \le n-1$. But then $p(x)$ is of the form

$$p(x) = x^n - a_1 x^{n-1} - \cdots - a_{\frac{n}{2}-1} x^{\frac{n}{2}+1} + a_{\frac{n}{2}-1} x^{\frac{n}{2}-1} + \cdots + a_1 x - 1, \tag{3.7}$$

and $p(1) = 0$, so $p(x)$ is divisble by $x - 1$, and λ is the root of a monic polynomial of smaller degree over $\mathbb{Q}$, contradicting the minimality of $p(x)$.

Thus $a_0 = 1$ and for all $1 \le i \le n-1$ we have that $a_{n-i} = a_i$, so $p(x)$ is palindromic. □

As an immediate consequence of the proof, we obtain

Corollary 3.2 *If λ is an algebraic integer and $\mathbb{Q}(\lambda)$ admits a λ-invariant symplectic form, then λ is an algebraic unit.*

Proof If in the proof above the coefficients of $p(x)$ are integers, then $1/\lambda$ satisfying the same polynomial implies $1/\lambda$ is also an algebraic integer. □

4. Sufficience

In this section we show that if the minimal polynomial of λ is palindromic of even degree, then there exist λ-invariant symplectic forms on $\mathbb{Q}(\lambda)$. Thus supppose n is even and λ is an algebraic number whose minimal polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$, $a_i \in \mathbb{Q}$ is palindromic, namely $a_i = a_{n-i}$.

If Q is any non-degenerate, skew-symmetric matrix of size n, namely with $Q_{i\,j} = -Q_{j\,i}$ for $1 \leq i, j \leq n$, we can obtain a linear symplectic form on $\mathbb{Q}(\lambda)$ by simply defining $\Theta : \mathbb{Q}(\lambda) \times \mathbb{Q}(\lambda) \to \mathbb{Q}$ by setting $\Theta(u,\, v) = u^\top Q\, v$. Recall the entries of Q are related to Θ as

$$Q_{i,\,j} = \Theta(\lambda^{i-1}, \lambda^{j-1}), \qquad 1 \leq i, j \leq n. \tag{4.1}$$

But for the form Θ to be λ-invariant, we must have $\Theta(1, \lambda) = \Theta(\lambda, \lambda^2) = \cdots = \Theta(\lambda^{n-2}, \lambda^{n-1})$. So for instance, $Q_{1,\,2} = Q_{2,\,3} = \cdots = Q_{n-1,n} = x_1$ (say). Similarly, $Q_{1,\,3} = Q_{2,\,4} = \cdots = Q_{n-2,\,n} = x_2$ (say), etc. Thus if Θ is λ-invariant, all diagonals of Q parallel to the main diagonal (of zeros) must be constant, in other words, Q must take the following form:

$$Q = \begin{pmatrix} 0 & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} \\ -x_1 & 0 & x_1 & & & x_{n-2} \\ -x_2 & -x_1 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & x_1 & x_2 \\ -x_{n-2} & & & -x_1 & 0 & x_1 \\ -x_{n-1} & -x_{n-2} & \cdots & -x_2 & -x_1 & 0 \end{pmatrix} \tag{4.2}$$

Furthermore, Q must satisfy the relation $C^\top Q\, C = Q$ where C is the companion matrix (3.1) of the polynomial $p(x)$. We show that there exists an $n/2$-dimensional family of Q satisfying these conditions.

Theorem 4.1 *Let λ be an algebraic number whose minimal polynomial is palindromic of even degree n. Then there exists an $n/2$-dimensional family of λ-invariant (linear) symplectic forms on $\mathbb{Q}(\lambda)$.*

Proof To define a λ-invariant symplectic form, we need to find a matrix Q as in equation 4.2 above that satisfies the relation $C^\top Q\, C = Q$ where C is the companion matrix in 3.1. Computing, we see that $C^\top Q\, C$ equals

$$\begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -1 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \begin{pmatrix} 0 & x_1 & x_2 & \cdots & x_{n-1} \\ -x_1 & 0 & x_1 & & x_{n-2} \\ -x_2 & -x_1 & 0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & x_1 \\ -x_{n-1} & -x_{n-2} & \cdots & -x_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & x_1 & x_2 & \cdots & x_{n-2} & R_{n-1} \\ -x_1 & 0 & x_1 & & & R_{n-2} \\ -x_2 & -x_1 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & x_1 & R_2 \\ -x_{n-2} & & & -x_1 & 0 & R_1 \\ -R_{n-1} & -R_{n-2} & \cdots & -R_2 & -R_1 & 0 \end{pmatrix} \tag{4.3}$$

Note that the first $(n-1) \times (n-1)$ block of $C^\top Q\, C$ is the same as that of Q itself, and only the last row and column require careful computation. To save space, we have renamed the last column (and row) as R_i

and in order to satisfy $C^\top Q C = Q$ we must have $x_i = R_i$ for all $1 \le i \le n-1$. The R_i are computed below:

$$R_{n-1} = x_1 - a_2 x_1 - a_3 x_2 - \cdots - a_{n-1} x_{n-2} \tag{4.4}$$

$$\cdots$$

$$R_{n-j} = x_j + a_1 x_{j-1} + \cdots + a_{j-1} x_1 - a_{j+1} x_1 - \cdots - a_{n-1} x_{n-j-1} \tag{4.5}$$

$$\cdots$$

$$R_j = x_{n-j} + a_1 x_{n-j-1} + \cdots + a_{n-j-1} x_1 - a_{n-j+1} x_1 - \cdots - a_{n-1} x_{j-1} \tag{4.6}$$

$$\cdots$$

$$R_1 = x_{n-1} + a_1 x_{n-2} + \cdots + a_{n-2} x_1 \tag{4.7}$$

Note that for all $1 \le j \le n/2$, equations $x_j = R_j$ and $x_{n-j} = R_{n-j}$ are the same equation since all $a_i = a_{n-i}$ and $a_0 = 1$. Moreover, the equation corresponding to $j = n/2$,

$$x_{\frac{n}{2}} = x_{\frac{n}{2}} + a_1 x_{\frac{n}{2}-1} + \cdots + a_{\frac{n}{2}-1} x_1 - a_{\frac{n}{2}+1} x_1 - \cdots - a_{n-1} x_{\frac{n}{2}-1}, \tag{4.8}$$

simplifies to $x_{\frac{n}{2}} = x_{\frac{n}{2}}$, hence providing no information.

Thus there are only $\frac{n}{2}-1$ equations to satisfy $x_1 = R_1, \cdots, x_{\frac{n}{2}-1} = R_{\frac{n}{2}-1}$ and $n-1$ variables $x_1, ..., x_{n-1}$. These equations can be summarized by requiring that $(x_{n-1}, \cdots, x_1)^\top$ be in the kernel of the following matrix of dimension $(\frac{n}{2}-1, n-1)$:

$$\begin{pmatrix}
1 & a_1 & a_2 & \cdots & a_{\frac{n}{2}-2} & a_{\frac{n}{2}-1} & a_{\frac{n}{2}} & \cdots & a_{n-3} & a_{n-2}-1 \\
0 & 1 & a_1 & \cdots & a_{\frac{n}{2}-3} & a_{\frac{n}{2}-2} & a_{\frac{n}{2}-1} & \cdots & a_{n-4}-1 & a_{n-3}-a_{n-1} \\
0 & 0 & 1 & \cdots & a_{\frac{n}{2}-4} & a_{\frac{n}{2}-3} & a_{\frac{n}{2}-2} & \cdots & a_{n-5}-a_{n-1} & a_{n-4}-a_{n-2} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_1 & a_2 & a_3 & \cdots & a_{\frac{n}{2}} - a_{\frac{n}{2}+4} & a_{\frac{n}{2}+1} - a_{\frac{n}{2}+3} \\
0 & 0 & 0 & \cdots & 1 & a_1 & a_2-1 & \cdots & a_{\frac{n}{2}-1} - a_{\frac{n}{2}+3} & a_{\frac{n}{2}} - a_{\frac{n}{2}+2}
\end{pmatrix}
\begin{pmatrix} x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ \vdots \\ x_{\frac{n}{2}+1} \\ x_{\frac{n}{2}} \\ x_{\frac{n}{2}-1} \\ \vdots \\ x_2 \\ x_1 \end{pmatrix} = \mathbf{0} \tag{4.9}$$

This matrix is already in echelon form. If we let $x_1, \cdots, x_{\frac{n}{2}}$ be free variables, then from the linear system above one can uniquely solve for $x_{\frac{n}{2}+1}, \cdots, x_{n-1}$ in terms of these free variables. Thus we obtain an $\frac{n}{2}$ dimensional family of matrices $Q(x_1, \cdots, x_{\frac{n}{2}})$ that define λ-invariant forms on $\mathbb{Q}(\lambda)$.

$\det(Q(x_1, \cdots, x_{\frac{n}{2}})) = 0$ is a closed condition on the x_i, hence a dense open subset of this space of forms consists of symplectic forms. □

Combining Theorem 4.1 with Proposition 3.1, we have also proved Theorem 1.1.

Remark: Since the pivots in the linear system 4.9 are all equal to 1, there is no division involved in solving for $x_{\frac{n}{2}+1}, \cdots, x_{n-1}$ in terms of $x_1, \cdots, x_{\frac{n}{2}}$. Thus if λ is an algebraic integer, and if the $x_1, \cdots, x_{\frac{n}{2}}$ are chosen to be integers, then the λ-invariant symplectic forms abtained above are integral. That is, the forms Θ can be thought of as maps

$$\Theta : \mathbb{Z}[\lambda] \times \mathbb{Z}[\lambda] \to \mathbb{Z} \tag{4.10}$$

5. Examples

1. $\lambda = \sqrt{3} + \sqrt{2}$, has a palindromic minimal polynomial $x^4 - 10x^2 + 1$. Thus $\mathbb{Q}(\lambda)$ admits a 2-dimensional family of λ-invariant symplectic forms. With Q as in equation 4.2, x_1 and x_2 are free and $x_3 = 11x_1$. Thus in the standard basis $\{1, \lambda, \lambda^2, \lambda^3\}$ of $\mathbb{Q}(\lambda)$, a general λ-invariant form is given by

$$\Theta(u, v) \;=\; u^\top Q\, v \;=\; u^\top \begin{pmatrix} 0 & a & b & 11a \\ -a & 0 & a & b \\ -b & -a & 0 & a \\ -11a & -b & -a & 0 \end{pmatrix} v \tag{5.1}$$

where a, b are any rational numbers. Determinant of $Q = Q(a, b)$ is $(12a^2 - b^2)^2$, thus it defines a λ-invariant symplectic form whenever $b^2 \neq 12a^2$, hence for all $a, b \in \mathbb{Q}$.

2. $\mathbb{Q}(\lambda)$ with $\lambda = \sqrt{p} + \sqrt{q}$ for primes p, q, do not admit λ-invariant symplectic forms unless $(p, q) = (3, 2)$, since the minimal polynomial of λ is not palindromic:

$$x^4 - 2(p+q)\, x^2 + (p-q)^2. \tag{5.2}$$

However the above example works more generally when λ is of the form $\frac{\sqrt{p}+\sqrt{q}}{\sqrt{p-q}}$ for some primes $p > q$. The minimal polynomial of λ is

$$x^4 - 2\left(\frac{p+q}{p-q}\right) x^2 + 1 \tag{5.3}$$

and a general λ-invariant symplectic form is found with $Q = Q(a, b)$ of the form

$$Q(a, b) = \begin{pmatrix} 0 & a & b & \frac{3p+q}{p-q}a \\ -a & 0 & a & b \\ -b & -a & 0 & a \\ -\frac{3p+q}{p-q}a & -b & -a & 0 \end{pmatrix}. \tag{5.4}$$

Non-degeneracy is equivalent to $4\,p\,a^2 \neq (p-q)\, b^2$. And since p and $p-q$ are relatively prime, $Q(a, b)$ is again non-degenerate for all $(a, b) \in \mathbb{Q}^2$.

3. Take λ to be Lehmer's number, i.e. the largest root of the polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

Here $a_9 = a_1 = 1, a_8 = a_2 = 0$ and $a_7 = a_6 = a_5 = a_4 = a_3 = -1$. With Q as given in equation 4.2, we need to find $x_1, \cdots, x_9$ that satisfy the relations 4.9, that is:

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 & -1 & -2 & -2 \\ 0 & 0 & 1 & 1 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_9 \\ x_8 \\ x_7 \\ x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.5}$$

Solving this system, we let $x_1, ..., x_5$ be free variables and obtain

$$x_6 = x_2 + 2x_3 + x_4 - x_5 \tag{5.6}$$

$$x_7 = x_1 + x_2 + x_5 \tag{5.7}$$

$$x_8 = x_1 + x_2 + x_3 + x_4 \tag{5.8}$$

$$x_9 = x_2 + 2x_3 + x_4 \tag{5.9}$$

Letting $(x_1, ..., x_5) = (a, b, c, d, e)$ so $Q = Q(a, b, c, d, e)$, we see that a general λ-invariant form in the standard basis on $\mathbb{Q}(\lambda)$ takes the form:

$$\Theta(u, v) \;=\; u^\top Q\, v \;=\; u^\top \begin{pmatrix} A & B \\ -B^\top & A \end{pmatrix} v \tag{5.10}$$

where the matrices A, B are given below:

$$A \;=\; \begin{pmatrix} 0 & a & b & c & d \\ -a & 0 & a & b & c \\ -b & -a & 0 & a & b \\ -c & -b & -a & 0 & a \\ -d & -c & -b & -a & 0 \end{pmatrix} \tag{5.11}$$

$$B \;=\; \begin{pmatrix} e & b+2c+d-e & a+b+e & a+b+c+d & b+2c+d \\ d & e & b+2c+d-e & a+b+e & a+b+c+d \\ c & d & e & b+2c+d-e & a+b+e \\ b & c & d & e & b+2c+d-e \\ a & b & c & d & e \end{pmatrix} \tag{5.12}$$

The determinant of Q is the square of a homogeneous polynomial of degree 5 in a, b, c, d, e, which if not equal to 0, Q defines a λ-invariant symplectic form on $\mathbb{Q}(\lambda)$.

Acknowledgements

Author[1] would like to acknowledge the support of TÜBİTAK 2236 and Horizon 2020 Marie Skłodowska-Curie grant: 121C062 during part of this research project. Author[2] would like to acknowledge the support of TÜBİTAK 1001 grant : 119F405. We would also like to thank the referee for helpful comments.

References

[1] Baik H, Rafiqi A, Wu C. Is a typical bi-Perron algebraic unit a pseudo-Anosov dilatation?. Ergodic Theory and Dynamical Systems 2019; 39 (7): 1745-1750.

[2] Fried D. Growth rate of surface homeomorphisms and flow equivalence. Ergodic Theory and Dynamical Systems 1985 Dec; 5 (4): 539-563.

[3] Hubbard JH. Teichmüller theory and applications to geometry, topology, and dynamics. Matrix Editions 2016 Apr.

[4] Lind DA. The entropies of topological Markov shifts and a related class of algebraic integers. Ergodic Theory and Dynamical Systems 1984; 4 (2): 283-300.

[5] Thurston W. Entropy in dimension one. In: Frontiers in Complex Dynamics: In Celebration of John Milnor's 80th Birthday. Princeton: Princeton University Press 2014; p.339-384. https://doi.org/10.1515/9781400851317-016

[6] Thurston WP. On the geometry and dynamics of diffeomorphisms of surfaces. Bulletin of the American Mathematical Society 1988; 19 (2): 417-31.