Turkish Journal of Mathematics

Volume 48 | Number 1 Article 5

1-31-2024

On the qualitative analysis of nonlinear q-fractional delay descriptor systems

ABDULLAH YİĞİT
a-yigit63@hotmail.com

Recommended Citation

YİĞİT, ABDULLAH (2024) "On the qualitative analysis of nonlinear q-fractional delay descriptor systems," *Turkish Journal of Mathematics*: Vol. 48: No. 1, Article 5. https://doi.org/10.55730/1300-0098.3490
Available at: https://journals.tubitak.gov.tr/math/vol48/iss1/5

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2024) 48: 34 – 52
© TÜBİTAK
doi:10.55730/1300-0098.3490

On the qualitative analysis of nonlinear q-fractional delay descriptor systems

Abdullah YİĞİT*
Mesut ÖZATA Anatolian High School, Van, Turkiye

Received: 08.06.2023 • **Accepted/Published Online:** 27.11.2023 • **Final Version:** 31.01.2024

Abstract: In this manuscript, we obtain some sufficient conditions for a nonlinear q fractional integro singular system with constant delays to be asymptotically admissible and a nonlinear q fractional non-singular system to be asymptotically stable. We use Lyapunov-Krasovskii functionals and some inequalities to obtain these conditions. At the same time, we present some numerical examples that confirm the sufficient conditions we obtained theoretically, with their annotated solutions and graphs.

Key words: Asymptotically admissible, constant delays, descriptor systems, initial function, qualitative analysis

1. Introduction

The development and progress of science is the result of the curiosity of scientists. As a matter of fact, the emergence of the subject of fractional calculus was due to the curious questions and discussions of the two scientists. According to the information you have received from history and academic studies, the beginning of the subject of fractional calculus started with the question of the applicability of fractional calculus, which L' Hospital asked in a letter sent to his friend Leibnitz. The studies in this area, which were started by L' Hospital in 1695, have expanded and reached the present day. Fractional calculus, which has a wide application area today, is being handled by many scientists in various studies in mathematics and other sciences and its popularity is increasing. Fractional and non-fractional studies and references in these studies can be made available to curious researchers as follows (see [1, 28]).

According to the information we have taken from the relevant literature, fractional calculus or fractional differential equations can be descriptive of natural events better than ordinary differential equations. For this reason, it has been extensively discussed by many well-known scientists around the world (see [21]). Some studies on this topic, which is attracting the attention of an increasing number of researchers, can be presented as below for those interested.

The q-fractional calculus was first discussed by Agarwal [1] and Al-salam [3] in 1969. In these studies, some q-fractional derivatives and q-fractional integrals were brought to the attention of researchers. Alkhazzan et al., in [2], discussed a new class of nonlinear fractional stochastic differential equations with fractional integrals and discussed existence, uniqueness, continuity of solutions and Ulam-Hyers stability with the help of Banach contraction theorem. They supported their work with an example. In [4] , the author investigated the asymptotic stability of Riemann-Liouville fractional neutral systems with variable delays by using the Lyapunov-Krasovskii

*Correspondence: a-yigit63@hotmail.com
2010 *AMS Mathematics Subject Classification:* 34A08, 34D05, 34K20

functional. Altun and Tunç [5], discussed the asymptotic stability of a nonlinear fractional-order system with multiple variable delays. The authors proved a new result on the subject by means of Lyapunov-Krasovskii functional. In [6], Altun proved the asymptoc stability of fractional neutral-type neural networks with Riemann-Liouville derivative. Atici and Eloe, in [8], discussed the function between the study of the fractional q-calculus and the fractional q-calculus on the time scale. Chartbupapan et al., in [9], studied the asymptotic stability of nonlinear fractional Riemann-Liouville differential equations with fixed delay and gave examples supporting the conditions they obtained. Jarad et al., in [11], studied the stability, uniform stability, and asymptotic stability conditions in the framework of non-autonomous Caputo fractional derivative with the help of the Lyapunov direct method. Koca and Demirci, derived some sufficient criteria for the local stability of nonlinear q-fractional dynamical systems in [14]. In 2017, Liu et al. [15], derived two asymptotic stability results for Riemann-Liouville fractional singular systems with multiple time varying delays. Lu et al. established numerous sufficient stability conditions for asymptotic stability of fractional nonlinear neutral singular systems and Mittag-Leffler stability with the help of the Lyapunov direct method in [16]. Mahdi and Khudair, developed the delta q-Mittag-Leffler stability conditions for nonlinear q-fractional dynamical systems on the time scale in [17]. Tunç and Tunç studied the qualitative behavior of the solutions of Caputo proportional derivatives of delayed integro-differential equations in [19]. In 2023, Xiao-Chung et al. [22], derived two delay-dependent and order-dependent conditions for asymptotic stability of Riemann-Liouville fractional systems. In [25], the authors investigated the asymptotic stability of zero solution of a nonlinear fractional neutral system with unbounded delay. Yiğit et al. studied the asymptotic stability of delayed fractional singular systems in [26]. Zhang et al. studied the global asymptotic stability of delayed fractional Riemann-Liouville neural networks in [27]. In [28], Zhai and Ren studied the existence and uniqueness of solutions of fractional q-difference equations with nonlinear three-point boundary conditions. In addition, references can be made to numerous books on these topics. Examples of some of these books include q-fractional calculations [7, 12], fractional calculations [13, 18] and singular systems [10, 23, 24]. Inspired by the studies summarized above and references in these studies (especially motivated by articles [20] and [21] and the references in these articles), we considered non-linear mixed-delay q-fractional integro neutral systems. We obtained asymptotic admissibility and asymptotic stability conditions for these systems. For this, we have used the Lyapunov direct method, some inequalities, and lemmas. We have given four examples with their simulations and solutions showing the practical applicability of the conditions we obtained theoretically. We believe that our study will add a new perspective to the literature.

2. Basic informations

Now, we give some basic definitions and lemmas that should be known before describing our systems and going into the details of our work.

Definition 2.1 ([21]). For $q \in (0,1)$, the equation $T_q = \{q^n : n \in Z\} \cup \{0\}$ is defined as time scale, here Z is the set of all integers.

In order to learn more about quantum theory, we can present some studies and references in these studies to the attention of researchers as (see [7, 8, 11, 12, 17, 21]).

Lemma 2.1([21]). Assume that $x(t) \in R^n$ be a vector of q-fractional differentiable function and $S \in R^{n\times n}, S = S^T > 0$, is a constant matrix. Then,

$$\nabla_q^\alpha (x^T(t)Sx(t)) \leq 2x^T(t)S\nabla_q^\alpha x(t), 0 < \alpha < 1, \forall t \in T_q, t > 0,$$

is satisfied.

Lemma 2.2([21]). Assume that $x(t):[0,\rho]\to R^n$ be a vector function such that the terms $\int_0^\rho x^T(s)Sx(s)\nabla_q s, \int_0^\rho x(s)\nabla_q s$ are well defined and $S\in R^{n\times n}, S=S^T>0,$ is a constant matrix. Then,

$$(\int_0^\rho x(s)\nabla_q s)^T S(\int_0^\rho x(s)\nabla_q s)\le \rho\int_0^\rho x^T(s)Sx(s)\nabla_q s,$$

is satisfied.

Lemma 2.3([20]). Assume that $S=S^T>0,$ and $\Delta,\Gamma,S\in R^{n\times n},$ we get

$$\Delta^T\Gamma+\Gamma^T\Delta\le\Delta^T S\Delta+\Gamma^T S^{-1}\Gamma.$$

Especially, when $S=\varpi I,$ we have

$$\Delta^T\Gamma+\Gamma^T\Delta\le\varpi\Delta^T\Delta+\frac{1}{\varpi}\Gamma^T\Gamma,$$

where, ϖ is a positive number and $I\in R^{n\times n},$ is identity matrix.

3. Main results

In this part of our work, we first describe a new q-fractional nonlinear singular system with mixed delays. With the help of the Lyapunov method, some basic information, and inequalities, we obtain sufficient conditions to show that this system is asymptotically admissible and give two examples that support these conditions. Then we describe a new nonsingular q-fractional nonlinear system. We derive sufficient conditions to prove the asymptotic stability of this system using the Lyapunov method and give two examples.

Now, we introduce a non-linear q-fractional delay neutral singular system as:

$$\begin{aligned}A\nabla_q^\alpha x(t)-B\nabla_q^\alpha x(t-k)&=Dx(t)+Ux(t-k)+L\int_{t-\rho}^t h(t-s)x(s)\nabla_q s\\&+H(x(t-k))+G(x(t-\rho)),t\ge 0,\end{aligned}\tag{3.1}$$

with the initial value condition as:

$$I_q^{1-\alpha}x(t)=\varphi(t),t\in[-\tau_0,t_0]_q,\tau_0=max\{k,\rho\},\alpha\in(0,1),$$

for $0<\alpha<1,$ the state vector $x(t)\in R^n,$ and $A,B,D,U,L\in R^{n\times n}$ are five constant system matrices with $||B||<1,$ $A\in R^{n\times n}$ is singular and $rankA=r\le n,n\ge 1.$ Here $[-\tau_0,t_0]_q=[-\tau_0,t_0]\cap T_q$ and $\varphi(t)\in C([-\tau_0,t_0],R^n)$ is the initial function with $\tau_0=max\{k,\rho\}.$ The values k,ρ are real positive numbers and $h(s)=diag(h_1(s),h_2(s),...,h_n(s))$ is a nonnegative real-valued function matrix satisfying $\int_0^\rho h_j(\theta)\nabla_q\theta=1,j=1,2,...,n.$ The nonlinear terms $H(.)$ and $G(.)$ satisfying

$$H^T(x(t))H(x(t))\le a^2x^T(t)x(t),\tag{3.2}$$

$$H^T(x(t-k))H(x(t-k))\le b^2x^T(t-k)x(t-k),\tag{3.3}$$

$$G^T(x(t))G(x(t))\le c^2x^T(t)x(t),\tag{3.4}$$

$$G^T(x(t-\rho))G(x(t-\rho))\le d^2x^T(t-\rho)x(t-\rho),\tag{3.5}$$

here a, b, c, d are given real numbers.

Theorem 3.1. The q-fractional system (3.1) is asymptotically admissible, if there exist $k > 0, m > 0, \epsilon > 0, \lambda > 0, \rho > 0, ||B|| < 1$ and a, b, c, d given any constants, some positive definite symmetric matrices $P, Q, K, R, S_i, (i = 1, 2, 3, 4)$ and a matrix S of suitable dimensions such that the following LMI holds:

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & \Gamma_{14} & \Gamma_{15} & P + D^T(S_1 + kS_2) & P + D^T(S_1 + kS_2) \\ * & \Gamma_{22} & 0 & U^T(S_1 + kS_2)L & \Gamma_{25} & U^T(S_1 + kS_2) & U^T(S_1 + kS_2) \\ * & * & \Gamma_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Gamma_{44} & \Gamma_{45} & L^T(S_1 + kS_2) & L^T(S_1 + kS_2) \\ * & * & * & * & \Gamma_{55} & \Gamma_{56} & \Gamma_{57} \\ * & * & * & * & * & \Gamma_{66} & S_1 + kS_2 \\ * & * & * & * & * & * & \Gamma_{77} \end{bmatrix} < 0, \tag{3.6}$$

where

$$\begin{aligned} \Gamma_{11} =& D^T P + PD + Q + K + R + a^2 S_3 + c^2 S_4 + D^T(S_1 + kS_2)D, \\ \Gamma_{12} =& PU + D^T(S_1 + kS_2)U, \Gamma_{14} = PL + D^T(S_1 + kS_2)L, \\ \Gamma_{15} =& PB + D^T(S_1 + kS_2)B + D^T Z S^T, \\ \Gamma_{22} =& -Q + \epsilon b^2 I + U^T(S_1 + kS_2)U, \Gamma_{25} = U^T(S_1 + kS_2)B + U^T Z S^T, \\ \Gamma_{33} =& -K + \lambda d^2 I, \Gamma_{44} = L^T(S_1 + kS_2)L - R, \\ \Gamma_{45} =& L^T(S_1 + kS_2)B + L^T Z S^T, \\ \Gamma_{55} =& -A^T - A - 2mI - m^2 S_1 + B^T(S_1 + kS_2)B + B^T Z S^T + S Z^T B, \\ \Gamma_{56} =& B^T(S_1 + kS_2) + S Z^T, \Gamma_{57} = B^T(S_1 + kS_2) + S Z^T, \\ \Gamma_{66} =& S_1 + kS_2 - S_3 - \epsilon I, \Gamma_{77} = S_1 + kS_2 - S_4 - \lambda I. \end{aligned}$$

where $Z \in R^{n\times(n-r)}$ is any matrix satisfying $A^T Z = 0$ and I is identity matrix with appropriate dimension.

Proof . Let us first show that the system (3.1) is impulse-free and regular (see [23]). Because of $rankA = r \le n$, there exist two inverse matrices N and T such that

$$\overline{A} = NAT = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \overline{D} = NDT = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}, \overline{P} = N^{-T} PT = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}. \tag{3.7}$$

From Γ_{11}, (3.7) and some mathematical calculations, we get $D_4^T P_4 + P_4 D_4 < 0$. This proves that D_4 is regular. Therefore, the pair (A, D) is regular and impulse-free (see Dai [10]). According to Wu et al. [20], the system (3.1) is regular and impulse free.

Now, let us select a new LKF as:

$$
\begin{aligned}
V(t) =& I_q^{1-\alpha}(x^T(t)PAx(t)) + \int_{t-k}^{t} x^T(s)Qx(s)\nabla_q s + \int_{t-\rho}^{t} x^T(s)Kx(s)\nabla_q s \\
&+ \sum_{j=1}^{n} r_j \int_0^{\rho}\int_{t-s}^{t} h_j(s)x_j^2(\theta)\nabla_q\theta\nabla_q s + \int_{t-k}^{t} \nabla_q^{\alpha}x^T(s)A^TS_1A\nabla_q^{\alpha}x(s)\nabla_q s \\
&+ \int_{t-k}^{t}\int_0^{t} \nabla_q^{\alpha}x^T(s)A^TS_2A\nabla_q^{\alpha}x(s)\nabla_q s\nabla_q\theta + \int_{t-k}^{t} H^T(x(s))S_3H(x(s))\nabla_q s \\
&+ \int_{t-\rho}^{t} G^T(x(s))S_4G(x(s))\nabla_q s.
\end{aligned}
$$

Clearly, $V(t) > 0$. According to the Lemma 2.1, Lemma 2.2 and $\nabla_q^{\alpha} I_q^{\beta} f(t) = \nabla_q^{\alpha-\beta} f(t), \alpha > \beta \geq 0$ (see [21]), If we take the q-derivative of $V(t)$ along the trajectories of the system (3.1), we get the following inequality as:

$$
\begin{aligned}
\nabla_q V(t) \leq& 2x^T(t)PA\nabla_q^{\alpha}x(t) + x^T(t)Qx(t) - x^T(t-k)Qx(t-k) \\
&+ x^T(t)Kx(t) - x^T(t-\rho)Kx(t-\rho) + \sum_{j=1}^{n} r_j x_j^2(t) \\
&- \sum_{j=1}^{n} r_j \int_0^{\rho} h_j(s)x_j^2(t-s)\nabla_q s + \nabla_q^{\alpha}x^T(t)A^TS_1A\nabla_q^{\alpha}x(t) \\
&- \nabla_q^{\alpha}x^T(t-k)A^TS_1A\nabla_q^{\alpha}x(t-k) + k\nabla_q^{\alpha}x^T(t)A^TS_2A\nabla_q^{\alpha}x(t) \\
&- \int_{t-k}^{t} \nabla_q^{\alpha}x^T(s)A^TS_2A\nabla_q^{\alpha}x(s)\nabla_q s + H^T(x(t))S_3H(x(t)) \\
&- H^T(x(t-k))S_3H(x(t-k)) + G^T(x(t))S_4G(x(t)) \\
&- G^T(x(t-\rho))S_4G(x(t-\rho)). \qquad (3.8)
\end{aligned}
$$

According to the previously given Lemmas and information, let us point out that for some terms of inequality (3.8), we can write the following relationships, respectively.

$$
\begin{aligned}
2x^T(t)PA\nabla_q^{\alpha}x(t) =& 2x^T(t)P[Dx(t) + Ux(t-k) + L\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s \\
&+ H(x(t-k)) + G(x(t-\rho)) + B\nabla_q^{\alpha}x(t-k)] \\
=& x^T(t)[PD + D^TP]x(t) + 2x^T(t)PUx(t-k) \\
&+ 2x^T(t)PL\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s + 2x^T(t)PH(x(t-k)) \\
&+ 2x^T(t)PG(x(t-\rho)) + 2x^T(t)PB\nabla_q^{\alpha}x(t-k), \qquad (3.9)
\end{aligned}
$$

$$\sum_{j=1}^{n} r_j x_j^2(t) - \sum_{j=1}^{n} r_j \int_0^{\rho} h_j(s) x_j^2(t-s)\nabla_q s \leq x^T(t)Rx(t)$$

$$- (\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)^T R(\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s), \tag{3.10}$$

$$\nabla_q^{\alpha} x^T(t)A^T S_1 A\nabla_q^{\alpha} x(t) + k\nabla_q^{\alpha} x^T(t)A^T S_2 A\nabla_q^{\alpha} x(t) = [Dx(t) + Ux(t-k)$$

$$+ L\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s + H(x(t-k)) + G(x(t-\rho)) + B\nabla_q^{\alpha} x(t-k)]^T$$

$$\times (S_1 + kS_2)[Dx(t) + Ux(t-k) + L\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s$$

$$+ H(x(t-k)) + G(x(t-\rho)) + B\nabla_q^{\alpha} x(t-k)]$$

$$= x^T(t)D^T[S_1 + kS_2]Dx(t) + x^T(t)D^T[S_1 + kS_2]Ux(t-k) + x^T(t)D^T[S_1 + kS_2]$$

$$\times L(\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s) + x^T(t)D^T[S_1 + kS_2]H(x(t-k))$$

$$+ x^T(t)D^T[S_1 + kS_2]G(x(t-\rho)) + x^T(t)D^T[S_1 + kS_2]B\nabla_q^{\alpha} x(t-k)$$

$$+ x^T(t-k)U^T[S_1 + kS_2]Dx(t) + x^T(t-k)U^T[S_1 + kS_2]Ux(t-k)$$

$$+ x^T(t-k)U^T[S_1 + kS_2]L(\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s) + x^T(t-k)U^T$$

$$\times [S_1 + kS_2]H(x(t-k)) + x^T(t-k)U^T[S_1 + kS_2]G(x(t-\rho))$$

$$+ x^T(t-k)U^T[S_1 + kS_2]B\nabla_q^{\alpha} x(t-k) + (\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)^T$$

$$\times L^T[S_1 + kS_2]Dx(t) + (\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)^T L^T[S_1 + kS_2]Ux(t-k)$$

$$+ (\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)^T L^T[S_1 + kS_2]L(\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)$$

$$+ (\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)^T L^T[S_1 + kS_2]H(x(t-k))$$

$$+ (\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)^T L^T[S_1 + kS_2]G(x(t-\rho))$$

$$+ (\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)^T L^T[S_1 + kS_2]B\nabla_q^{\alpha} x(t-k)$$

$$+ H^T(x(t-k))[S_1 + kS_2]Dx(t) + H^T(x(t-k))[S_1 + kS_2]$$

$$\times Ux(t-k) + H^T(x(t-k))[S_1 + kS_2]L(\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)$$

$$+ H^T(x(t-k))[S_1 + kS_2]H(x(t-k)) + H^T(x(t-k))[S_1 + kS_2]$$

$$
\begin{aligned}
&\times G(x(t-\rho)) + H^T(x(t-k))[S_1+kS_2]B\nabla_q^\alpha x(t-k)\\
&+ G^T(x(t-\rho))[S_1+kS_2]Dx(t) + G^T(x(t-\rho))[S_1+kS_2]\\
&\times Ux(t-k) + G^T(x(t-\rho))[S_1+kS_2]L(\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)\\
&+ G^T(x(t-\rho))[S_1+kS_2]H(x(t-k)) + G^T(x(t-\rho))[S_1+kS_2]\\
&\times G(x(t-\rho)) + G^T(x(t-\rho))[S_1+kS_2]B\nabla_q^\alpha x(t-k)\\
&+ (\nabla_q^\alpha x(t-k))^T B^T[S_1+kS_2]Dx(t) + (\nabla_q^\alpha x(t-k))^T B^T[S_1+kS_2]\\
&\times Ux(t-k) + (\nabla_q^\alpha x(t-k))^T B^T[S_1+kS_2]L(\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)\\
&+ (\nabla_q^\alpha x(t-k))^T B^T[S_1+kS_2]H(x(t-k)) + (\nabla_q^\alpha x(t-k))^T B^T\\
&\times [S_1+kS_2]G(x(t-\rho)) + (\nabla_q^\alpha x(t-k))^T B^T[S_1+kS_2]\\
&\times B(\nabla_q^\alpha x(t-k)). \qquad (3.11)
\end{aligned}
$$

Furthermore, let us point out that $A^TZ=0,$ we can deduce

$$2\nabla_q^\alpha x^T(t)A^TZS^T\nabla_q^\alpha x(t-k)=0, \qquad (3.12)$$

From (3.3) and (3.5), we deduce

$$
\begin{aligned}
0 \le&\epsilon b^2 x^T(t-k)x(t-k) - \epsilon H^T(x(t-k))H(x(t-k)) \qquad (3.13)\\
0 \le&\lambda d^2 x^T(t-\rho)x(t-\rho) - \lambda G^T(x(t-\rho))G(x(t-\rho)) \qquad (3.14)
\end{aligned}
$$

here ϵ,λ are positive numbers. From Lemma 2.3, we can get

$$-\nabla_q^\alpha x^T(t-k)A^TS_1A\nabla_q^\alpha x(t-k) \le -\nabla_q^\alpha x^T(t-k)[A^T+A-S_1^{-1}]\nabla_q^\alpha x(t-k).$$

Let us point out that the relationship $(I+mS_1)^TS_1^{-1}(I+mS_1)>0$ implies $-S_1^{-1}<2mI+m^2S_1$. From here, we can obtain

$$-\nabla_q^\alpha x^T(t-k)A^TS_1A\nabla_q^\alpha x(t-k) \le -\nabla_q^\alpha x^T(t-k)[A^T+A+2mI+m^2S_1]\nabla_q^\alpha x(t-k). \qquad (3.15)$$

If the relationships (3.9)-(3.15) are substituted in (3.8), the following inequality is obtained.

$$\nabla_q V(t) \le \mu^T(t)\Gamma\mu(t),$$

here the matrix Γ is defined with (3.6) and

$$\mu^T(t) = \begin{bmatrix} F_1(t) & F_2(t) & F_3(t) & F_4(t) & F_5(t) & F_6(t) & F_7(t) \end{bmatrix},$$

where

$$
\begin{aligned}
&F_1(t)=x^T(t), F_2(t)=x^T(t-k), F_3(t)=x^T(t-\rho), F_4(t)=(\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)^T\\
&F_5(t)=(\nabla_q^\alpha x(t-k))^T, F_6(t)=H^T(x(t-k)), F_7(t)=G^T(x(t-\rho)).
\end{aligned}
$$

Clearly, for $\mu(t) \neq 0, \nabla_q V(t) < 0$. Since the conditions discussed in this Theorem are satisfied, the trivial solution of the nonlinear q-fractional neutral delay singular system (3.1) is asymptotically admissible.

Now, let us make the following definition for convenience. Let us define

$$M(t) = Ax(t) - Bx(t-k).$$

Then, rewrite the system (3.1) in the following new system

$$\nabla_q^\alpha M(t) = Dx(t) + Ux(t-k) + L\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s + H(x(t-k)) + G(x(t-\rho)), t \geq 0. \tag{3.16}$$

Theorem 3.2. The q-fractional system (3.1) is asymptotically admissible, if there exist a, b, c, d given any constants, $k > 0, m > 0, \epsilon > 0, \lambda > 0, \rho > 0, ||B|| < 1$ and some positive definite symmetric matrices $P, Q, K, R, S_i, N_i, (i = 1, 2, 3)$ such that the following LMI holds:

$$\Pi = \begin{bmatrix} -N_1 - N_1^T & \Pi_{12} & \Pi_{13} & 0 & PL & P & P \\ * & \Pi_{22} & \Pi_{23} & 0 & kD^T S_1 L & kD^T S_1 & kD^T S_1 \\ * & * & \Pi_{33} & 0 & kU^T S_1 L & kU^T S_1 & kU^T S_1 \\ * & * & * & -K + \lambda d^2 I & 0 & 0 & 0 \\ * & * & * & * & -R + kL^T S_1 L & kL^T S_1 & kL^T S_1 \\ * & * & * & * & * & \Pi_{66} & kS_1 \\ * & * & * & * & * & * & \Pi_{77} \end{bmatrix} < 0, \tag{3.17}$$

where

$$\begin{aligned}
\Pi_{12} =& PD + N_1 A + N_2^T A, \Pi_{13} = PU - N_1 B - B^T N_3 B, \\
\Pi_{22} =& -2A^T - 2A - 4mI - 2m^2 N_2 + K + Q + R + kD^T S_1 D + a^2 S_2 + c^2 S_3, \\
\Pi_{23} =& A^T N_2 B + A^T N_3^T B + kD^T S_1 U, \\
\Pi_{33} =& -2B^T N_3 B - Q + kU^T S_1 U + \epsilon b^2 I, \\
\Pi_{66} =& -S_2 + kS_1 - \epsilon I, \Pi_{77} = -S_3 + kS_1 - \lambda I.
\end{aligned}$$

where I is the identity matrix with appropriate dimension.

Proof . It has been proved in the previous theorem that the system (3.1) is impulse-free and regular. Therefore, it will suffice to show that the system (3.1) is asymptotically stable. For this purpose, let us define a new LKF below:

$$\begin{aligned}
V(t) =& I_q^{1-\alpha}(M^T(t)PM(t)) + \int_{t-\rho}^{t} x^T(s)Kx(s)\nabla_q s + \int_{t-k}^{t} x^T(s)Qx(s)\nabla_q s \\
&+ \sum_{j=1}^{n} r_j \int_0^\rho \int_{t-s}^{t} h_j(s)x_j^2(\theta)\nabla_q\theta\nabla_q s + \int_{t-k}^{t}\int_0^t \nabla_q^\alpha M^T(s)S_1\nabla_q^\alpha M(s)\nabla_q s\nabla_q\theta \\
&+ \int_{t-k}^{t} H^T(x(s))S_2H(x(s))\nabla_q s + \int_{t-\rho}^{t} G^T(x(s))S_3G(x(s))\nabla_q s.
\end{aligned}$$

Clearly, $V(t) > 0$. According to the Lemma 2.1, Lemma 2.2 and $\nabla_q^{\alpha} I_q^{\beta} f(t) = \nabla_q^{\alpha-\beta} f(t), \alpha > \beta \geq 0$ (see [21]), If we take the q-derivative of $V(t)$ along the trajectories of the system (3.1), we get the following inequality as:

$$\begin{aligned}
\nabla_q V(t) \leq & 2M^T(t)P[Dx(t) + Ux(t-k) + L\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s + H(x(t-k)) \\
& + G(x(t-\rho))] + 2M^T(t)N_1[-M(t) + Ax(t) - Bx(t-k)] \\
& - 2x^T(t)A^T N_2[-M(t) + Ax(t) - Bx(t-k)] \\
& + 2x^T(t-k)B^T N_3[-M(t) + Ax(t) - Bx(t-k)] \\
& + x^T(t)Qx(t) - x^T(t-k)Qx(t-k) + x^T(t)Kx(t) - x^T(t-\rho)Kx(t-\rho) \\
& + \sum_{j=1}^{n} r_j x_j^2(t) - \sum_{j=1}^{n} r_j \int_0^{\rho} h_j(s)x_i^2(t-s)\nabla_q s \\
& + k[Dx(t) + Ux(t-k) + L\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s + H(x(t-k)) \\
& + G(x(t-\rho))]S_1[Dx(t) + Ux(t-k) + L\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s \\
& + H(x(t-k)) + G(x(t-\rho))] + a^2 x^T(t)S_2 x(t) \\
& - H^T(x(t-k))S_2 H(x(t-k)) + c^2 x^T(t)S_3 x(t) \\
& - G^T(x(t-\rho))S_3 G(x(t-\rho)). \qquad (3.18)
\end{aligned}$$

From (3.3) and (3.5), we deduce

$$0 \leq \epsilon b^2 x^T(t-k)x(t-k) - \epsilon H^T(x(t-k))H(x(t-k)) \qquad (3.19)$$

$$0 \leq \lambda d^2 x^T(t-\rho)x(t-\rho) - \lambda G^T(x(t-\rho))G(x(t-\rho)) \qquad (3.20)$$

here ϵ, λ are positive numbers. From Lemma 2.3, we get

$$-x^T(t)A^T N_2 Ax(t) \leq -x^T(t)[A^T + A - N_2^{-1}]x(t).$$

Let us point out that the relationship $(I + mN_2)^T N_2^{-1}(I + mN_2) > 0$ implies $-N_2^{-1} < 2mI + m^2 N_2$. From here, we can obtain

$$-x^T(t)A^T N_2 Ax(t) \leq -x^T(t)[A^T + A + 2mI + m^2 N_2]x(t). \qquad (3.21)$$

If calculations similar to the mathematical operations made in the previous theorem are applied to some terms in the (3.18) inequality, and the relationships (3.19)-(3.21) are substituted in the (3.18) inequality, the following inequality is obtained.

$$\nabla_q V(t) \leq \xi^T(t)\Pi\xi(t),$$

here the matrix Π is defined with (3.17) and

$$\xi^T(t) = \begin{bmatrix} F_1(t) & F_2(t) & F_3(t) & F_4(t) & F_5(t) & F_6(t) & F_7(t) \end{bmatrix},$$

where

$$F_1(t) = M^T(t), F_2(t) = x^T(t), F_3(t) = x^T(t-k), F_4(t) = x^T(t-\rho)$$
$$F_5(t) = (\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)^T, F_6(t) = H^T(x(t-k)), F_7(t) = G^T(x(t-\rho)).$$

Clearly, for $\xi(t) \neq 0, \nabla_q V(t) < 0$. Since the conditions discussed in this Theorem are satisfied, the trivial solution of the nonlinear q-fractional neutral delay singular system (3.1) is asymptotically admissible.

Remark 3.1. When $A = I$, system (3.1) reduces to the following non-linear delay neutral nonsingular system as:

$$\nabla_q^\alpha x(t) - B\nabla_q^\alpha x(t-k) = Dx(t) + Ux(t-k) + L\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s$$
$$+ H(x(t-k)) + G(x(t-\rho)), t \geq 0, \tag{3.22}$$

with the initial value condition as:

$$I_q^{1-\alpha} x(t) = \varphi(t), t \in [-\tau_0, t_0]_q, \tau_0 = max\{k, \rho\}, \alpha \in (0,1),$$

for $0 < \alpha < 1$, the state vector $x(t) \in R^n$, and $B, D, U, L \in R^{n\times n}$ are some constant system matrices with $||B|| < 1$, Here $[-\tau_0, t_0]_q = [-\tau_0, t_0] \cap T_q$ and $\varphi(t) \in C([-\tau_0, t_0], R^n)$ is the initial function with $\tau_0 = max\{k, \rho\}$. The values τ, ρ are real positive numbers. Furthermore, the definitions of the integral term and the nonlinear terms in the system (3.22) are as given in system (3.1).

Theorem 3.3. The q-fractional system (3.22) is asymptotically stable, if there exist a, b, c, d given any constants, $k > 0, \epsilon > 0, \lambda > 0, \rho > 0, ||B|| < 1$ and some positive definite symmetric matrices $P, Q, K, R, S_i, (i = 1, 2, 3, 4)$, such that the following LMI holds:

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & 0 & \Sigma_{14} & \Sigma_{15} & P + D^T(S_1 + kS_2) & P + D^T(S_1 + kS_2) \\ * & \Sigma_{22} & 0 & U^T(S_1 + kS_2)L & \Sigma_{25} & U^T(S_1 + kS_2) & U^T(S_1 + kS_2) \\ * & * & \Sigma_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Sigma_{44} & \Sigma_{45} & L^T(S_1 + kS_2) & L^T(S_1 + kS_2) \\ * & * & * & * & \Sigma_{55} & \Sigma_{56} & \Sigma_{57} \\ * & * & * & * & * & \Sigma_{66} & S_1 + kS_2 \\ * & * & * & * & * & * & \Sigma_{77} \end{bmatrix} < 0, \tag{3.23}$$

where

$$
\begin{aligned}
\Sigma_{11} =& PD + D^T P + Q + K + R + a^2 S_3 + c^2 S_4 + D^T (S_1 + kS_2)D, \\
\Sigma_{12} =& PU + D^T (S_1 + kS_2)U, \Sigma_{14} = PL + D^T (S_1 + kS_2)L, \\
\Sigma_{15} =& PB + D^T (S_1 + kS_2)B, \\
\Sigma_{22} =& - Q + \epsilon b^2 I + U^T (S_1 + kS_2)U, \Sigma_{25} = U^T (S_1 + kS_2)B, \\
\Sigma_{33} =& - K + \lambda d^2 I, \Sigma_{44} = L^T (S_1 + kS_2)L - R, \\
\Sigma_{45} =& L^T (S_1 + kS_2)B, \Sigma_{55} = -S_1 + B^T (S_1 + kS_2)B, \\
\Sigma_{56} =& B^T (S_1 + kS_2), \Sigma_{57} = B^T (S_1 + kS_2), \\
\Sigma_{66} =& S_1 + kS_2 - S_3 - \epsilon I, \Sigma_{77} = S_1 + kS_2 - S_4 - \lambda I.
\end{aligned}
$$

where I is the identity matrix with appropriate dimension.

Proof. Let us choose the following new LKF, which can clearly be seen to be positive definite:

$$
\begin{aligned}
V(t) =& I_q^{1-\alpha}(x^T(t)Px(t)) + \int_{t-k}^{t} x^T(s)Qx(s)\nabla_q s + \int_{t-\rho}^{t} x^T(s)Kx(s)\nabla_q s \\
&+ \sum_{j=1}^{n} r_j \int_0^{\rho} \int_{t-s}^{t} h_j(s)x_j^2(\theta)\nabla_q\theta\nabla_q s + \int_{t-k}^{t} \nabla_q^{\alpha} x^T(s)S_1\nabla_q^{\alpha} x(s)\nabla_q s \\
&+ \int_{t-k}^{t} \int_0^{t} \nabla_q^{\alpha} x^T(s)S_2\nabla_q^{\alpha} x(s)\nabla_q s\nabla_q\theta + \int_{t-k}^{t} H^T(x(s))S_3H(x(s))\nabla_q s \\
&+ \int_{t-\rho}^{t} G^T(x(s))S_4G(x(s))\nabla_q s.
\end{aligned}
$$

If the derivative of the LKF selected above is taken according to the system (3.22) and the mathematical operations made in the proof of Theorem 3.1 are applied, the following inequality is obtained.

$$\nabla_q V(t) \leq \vartheta^T(t)\Sigma\vartheta(t),$$

here the matrix Σ is defined with (3.23) and

$$\vartheta^T(t) = \begin{bmatrix} F_1(t) & F_2(t) & F_3(t) & F_4(t) & F_5(t) & F_6(t) & F_7(t) \end{bmatrix},$$

where

$$
\begin{aligned}
&F_1(t) = x^T(t), F_2(t) = x^T(t-k), F_3(t) = x^T(t-\rho), F_4(t) = (\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)^T \\
&F_5(t) = (\nabla_q^{\alpha} x(t-k))^T, F_6(t) = H^T(x(t-k)), F_7(t) = G^T(x(t-\rho)).
\end{aligned}
$$

Clearly, for $\vartheta(t) \neq 0, \nabla_q V(t) < 0$. Since the conditions discussed in this Theorem are satisfied, the trivial solution of the nonlinear q-fractional neutral delay nonsingular system (3.22) is asymptotically stable.

Theorem 3.4. The q-fractional system (3.22) is asymptotically stable, if there exist a, b, c, d given any constants, $k > 0, \epsilon > 0, \lambda > 0, \rho > 0, ||B|| < 1$ and some positive definite symmetric matrices $P, Q, K, R, S_i, (i = 1, 2, 3)$ such that the following LMI holds:

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 & PL + kD^T S_1 L & P + kD^T S_1 & P + kD^T S_1 \\ * & \Lambda_{22} & 0 & \Lambda_{24} & -B^T P + kU^T S_1 & -B^T P + kU^T S_1 \\ * & * & -K + \lambda d^2 I & 0 & 0 & 0 \\ * & * & * & -R + kL^T S_1 L & kL^T S_1 & kL^T S_1 \\ * & * & * & * & \Lambda_{55} & kS_1 \\ * & * & * & * & * & \Lambda_{66} \end{bmatrix} < 0, \quad (3.24)$$

where

$$\begin{aligned} \Lambda_{11} =& PD + D^T P + Q + K + R + kD^T S_1 D + a^2 S_2 + c^2 S_3, \\ \Lambda_{12} =& PU - D^T PB + kD^T S_1 U, \Lambda_{22} = -B^T PU - U^T PB - Q + kU^T S_1 U + \epsilon b^2 I, \\ \Lambda_{24} =& - B^T PL + kU^T S_1 L, \Lambda_{55} = -S_2 + kS_1 - \epsilon I, \Lambda_{66} = -S_3 + kS_1 - \lambda I. \end{aligned}$$

Proof. Let us choose the following new LKF, which can clearly be seen to be positive definite:

$$\begin{aligned} V(t) =& I_q^{1-\alpha}[x(t) - Bx(t-k)]^T P[x(t) - Bx(t-k)] + \int_{t-\rho}^{t} x^T(s)Kx(s)\nabla_q s \\ &+ \int_{t-k}^{t} x^T(s)Qx(s)\nabla_q s + \sum_{j=1}^{n} r_j \int_0^{\rho} \int_{t-s}^{t} h_j(s)x_j^2(\theta)\nabla_q\theta\nabla_q s \\ &+ \int_{t-k}^{t} \int_0^{t} \nabla_q^{\alpha}[x(s) - Bx(s-k)]^T S_1 \nabla_q^{\alpha}[x(s) - Bx(s-k)]\nabla_q s\nabla_q\theta \\ &+ \int_{t-k}^{t} H^T(x(s))S_2 H(x(s))\nabla_q s + \int_{t-\rho}^{t} G^T(x(s))S_3 G(x(s))\nabla_q s. \end{aligned}$$

If the derivative of the LKF selected above is taken according to the system (3.22) and the mathematical operations made in the proof of Theorem 3.2 are applied, the following inequality is obtained.

$$\nabla_q V(t) \leq \Xi^T(t)\Lambda\Xi(t),$$

here the matrix Λ is defined with (3.24) and

$$\Xi^T(t) = \begin{bmatrix} F_1(t) & F_2(t) & F_3(t) & F_4(t) & F_5(t) & F_6(t) \end{bmatrix},$$

where

$$\begin{aligned} &F_1(t) = x^T(t), F_2(t) = x^T(t-k), F_3(t) = x^T(t-\rho), F_4(t) = (\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s)^T \\ &F_5(t) = H^T(x(t-k)), F_6(t) = G^T(x(t-\rho)). \end{aligned}$$

Clearly, for $\Xi(t) \neq 0, \nabla_q V(t) < 0$. Since the conditions discussed in this Theorem are satisfied, the trivial solution of the nonlinear q-fractional neutral delay nonsingular system (3.22) is asymptotically stable.

Remark 3.2. When $H(x(t-k)) = G(x(t-\rho)) = 0$, system (3.22) reduces to the Riemann-Liouville q-fractional neutral systems with mixed delays in [21]. Additionally, if the integral term along with the non-linear terms is taken to be zero, then the system 3.22 is reduced to the system of Liu et al., (see [15]). This explanation is detailed by Yang et al., (see [Yang et al., [21]; Remark 1]). Thus, our study generalizes and extends some results found in the literature.

4. Numerical applications

In this part of our study, we give some numerical examples embodying the conditions we obtained theoretically in the previous part, together with their solutions and graphics.

Example 4.1. Let us define the below nonlinear q-fractional neutral delay singular system as:

$$\begin{aligned} A\nabla_q^\alpha x(t) - B\nabla_q^\alpha x(t-k) &= Dx(t) + Ux(t-k) + L\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s \\ &\quad + H(x(t-k)) + G(x(t-\rho)), \end{aligned} \tag{4.1}$$

for $0 < \alpha < 1, 0 < q < 1, x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T, k = 0.5, \rho = 0.8, m = 1, a = 0.2, b = 0.3, c = 0.4, d = 0.6, \epsilon = 1.5, \lambda = 2$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} -6 & 0 \\ 0 & -10 \end{bmatrix}, U = \begin{bmatrix} 0.03 & 0 \\ 0 & -0.12 \end{bmatrix},$$
$$L = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.032 \end{bmatrix}, B = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.4 \end{bmatrix}.$$

According to Dai (see [10]), Xu and Lam (see [23]), and Wu et al. (see [20]), it is clear that the system (4.1) is regular and impulse-free. Also, the integral term in the system (4.1) is accepted as defined in the system (3.1).

Now, we select

$$P = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, Q = \begin{bmatrix} 2 & 0 \\ 0 & 0.2 \end{bmatrix}, R = \begin{bmatrix} 2 & 0 \\ 0 & 0.05 \end{bmatrix}, K = \begin{bmatrix} 3 & 0 \\ 0 & 0.8 \end{bmatrix},$$
$$S_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.03 \end{bmatrix}, S_2 = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.02 \end{bmatrix}, S_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$S_4 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, S = \begin{bmatrix} 0.5 \\ 0.05 \end{bmatrix}, Z = \begin{bmatrix} 0 \\ -1.2 \end{bmatrix}.$$

If we calculate some mathematical operations with the help of MATLAB-Software, then we can obtain $\Gamma < 0$. Because of the all eigenvalues of Γ are $-36.2382, -10.4257, -3.4660, -2.2800, -2.2012, -2.0264, -2.0002, -1.9809, -1.8647, -1.7430, -0.7865, -0.0800, -0.0626, -0.0498$. Thus, all conditions of Theorem 3.1 are satisfied. In view of Theorem 3.1, the zero solution of the nonlinear q-fractional delay neutral singular system (4.1) is asymptotically admissible. Moreover, the graph showing the numerical simulation of the system (4.1) is given in Figure 1.

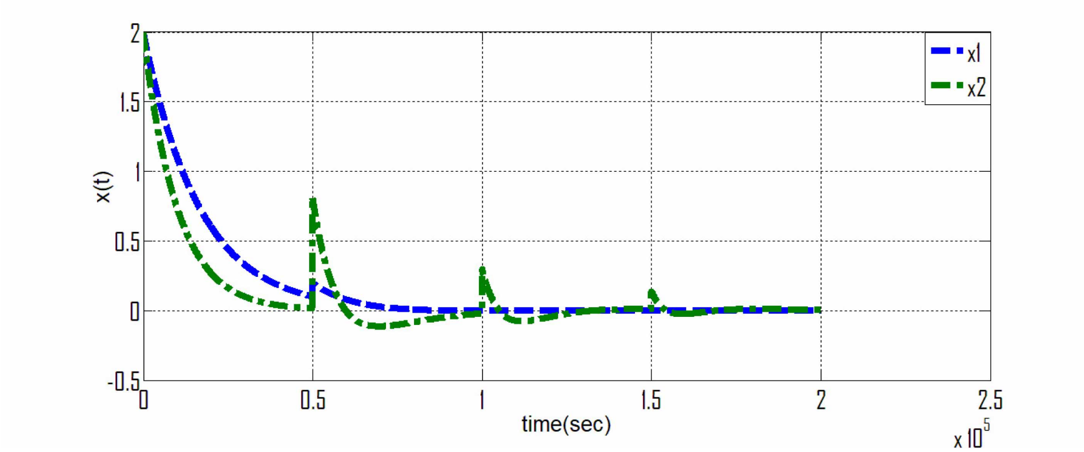

Figure 1. Numerical simulation of system (4.1).

Example 4.2. Let us define the below nonlinear q-fractional neutral delay singular system as:

$$A\nabla_q^\alpha x(t) - B\nabla_q^\alpha x(t-k) = Dx(t) + Ux(t-k) + L\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s$$
$$+ H(x(t-k)) + G(x(t-\rho)), \tag{4.2}$$

for $0<\alpha<1, 0<q<1, x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T, k=0.5, \rho=0.8, m=1, a=0.2, b=0.3, c=0.4, d=0.6, \epsilon = 1.5, \lambda=2$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.3 \end{bmatrix}, U = \begin{bmatrix} 0.3 & 0 \\ 0 & -0.2 \end{bmatrix},$$
$$L = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.32 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}.$$

According to Dai (see [10]), Xu and Lam (see [23]) and Wu et al. (see [20]), it is clear that the system (4.2) is regular and impulse-free. Also, the integral term in the system (4.2) is accepted as defined in system (3.1).

Now, we select

$$P = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix}, Q = \begin{bmatrix} 2 & 0 \\ 0 & 0.2 \end{bmatrix}, R = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.05 \end{bmatrix}, K = \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix},$$
$$S_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.03 \end{bmatrix}, S_2 = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.02 \end{bmatrix}, S_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$N_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, N_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, N_3 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.05 \end{bmatrix}.$$

If we calculate some mathematical operations with the help of MATLAB-Software, then we can obtain $\Pi < 0$. Because of the all eigenvalues of Π are $-5.2300, -3.4691, -2.5061, -2.0987, -1.9411, -1.5472, -1.3107, -0.5960, -0.2800, -0.2427, -0.1617, -0.0800, -0.0561, -0.0122$. Thus, all conditions of Theorem 3.2 are satisfied. In view of Theorem 3.2, the zero solution of the nonlinear q-fractional delay neutral singular system (4.2) is asymptotically admissible. Moreover, the graph showing the numerical simulation of the system (4.2) is as given in Figure 2.

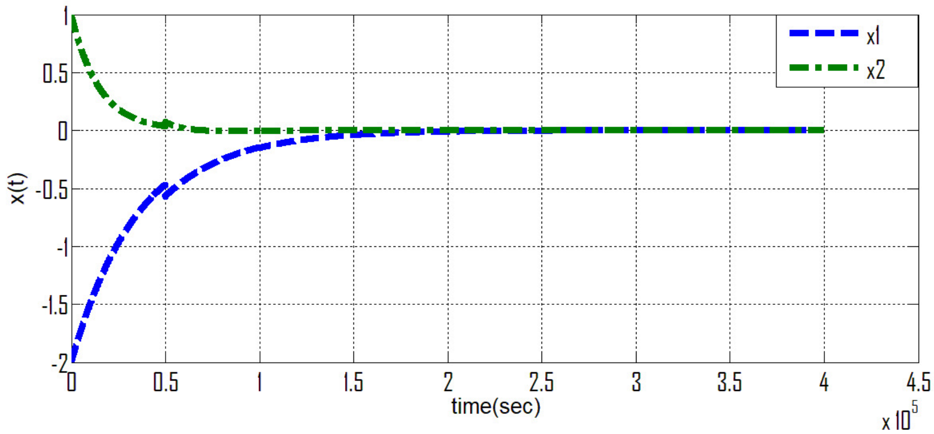

Figure 2. Numerical simulation of the system (4.2).

Example 4.3. Let us define the below nonlinear q-fractional neutral delay system as:

$$\nabla_q^\alpha x(t) - B\nabla_q^\alpha x(t-k) = Dx(t) + Ux(t-k) + L\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s \\ + H(x(t-k)) + G(x(t-\rho)), \tag{4.3}$$

for $0<\alpha<1, 0<q<1, x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^T, k=0.5, \rho=0.6, a=0.2, b=0.3, c=0.4, d=0.6, \epsilon = 1.5, \lambda = 2,$

$$D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -8 \end{bmatrix}, U = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & -0.4 \end{bmatrix},$$

$$L = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.32 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}.$$

Also, the integral term in the system (4.3) is accepted as defined in the system (3.1).

Now, we select

$$P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, R = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.06 \end{bmatrix},$$

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}, S_1 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.04 \end{bmatrix}, S_2 = \begin{bmatrix} 0.03 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.01 \end{bmatrix},$$

$$S_3 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} S_4 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}.$$

If we calculate some mathematical operations with the help of MATLAB-Software, then we can obtain $\Sigma < 0$. Because of the all eigenvalues of Σ are $-27.8226, -12.6951, -9.6130, -2.2865, -2.2228, -2.0002, -1.8478, -1.8378, -1.6232, -0.9093, -0.2800, -0.2800, -0.2410, -0.2210, -0.0800, -0.0631, -0.0545, -0.0443, -0.0171, -0.0154, -0.0037$. Thus, all conditions of Theorem 3.3 are satisfied. In view of Theorem 3.3, the zero solution of the nonlinear q-fractional delay neutral system (4.3) is asymptotically stable. Moreover, the graph showing the numerical simulation of the system (4.3) is given in Figure 3.

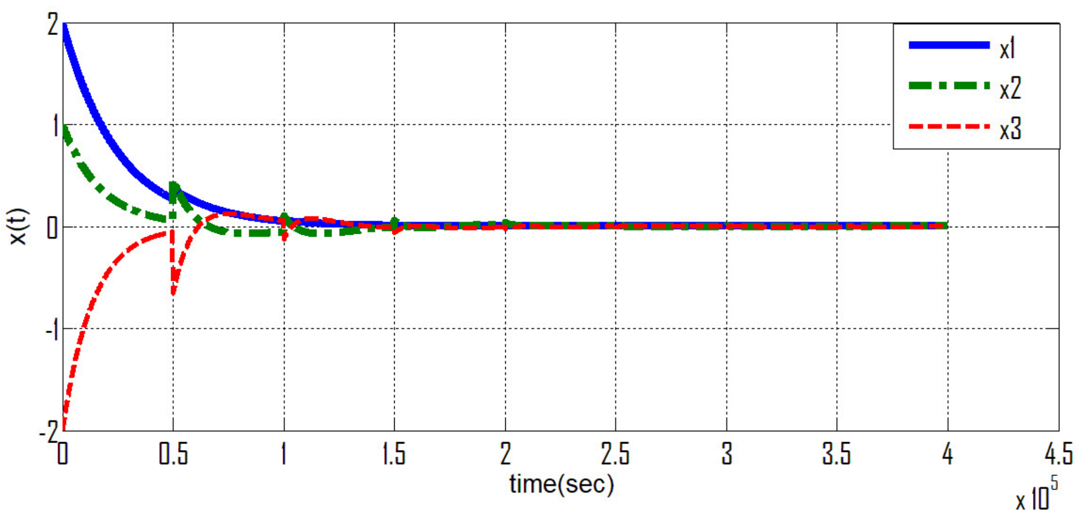

Figure 3. Numerical simulation of the system (4.3).

Example 4.4. Let us define the below nonlinear q-fractional nonsingular neutral delay system as:

$$\nabla_q^\alpha x(t) - B\nabla_q^\alpha x(t-k) = Dx(t) + Ux(t-k) + L\int_{t-\rho}^{t} h(t-s)x(s)\nabla_q s$$
$$+ H(x(t-k)) + G(x(t-\rho)), \tag{4.4}$$

for $0 < \alpha < 1, 0 < q < 1, x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^T, k = 0.4, \rho = 0.6, a = 0.12, b = 0.13, c = 0.5, d = 0.4, \epsilon = 1.5, \lambda = 2,$

$$D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -3 \end{bmatrix}, U = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.15 \end{bmatrix},$$

$$L = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.32 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}, B = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.06 & 0 \\ 0 & 0 & 0.04 \end{bmatrix}.$$

Also, the integral term in the system (4.4) is accepted as defined in the system (3.1).

Now, we select

$$P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, R = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.06 \end{bmatrix},$$

$$K = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}, S_1 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, S_2 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.4 \end{bmatrix},$$

$$S_3 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.08 \end{bmatrix}.$$

If we calculate some mathematical operations with the help of MATLAB-Software, then we can obtain $\Lambda < 0$. Because of the all eigenvalues of Λ are $-21.0050, -9.8250, -3.3461, -2.6800, -2.4009, -2.0102, -2.0070, -1.9913, -1.9187, -1.6049, -1.5721, -0.9193, -0.5800, -0.4800, -0.2688, -0.1521, -0.0276, -0.0275$. Thus, all conditions of Theorem 3.4 are satisfied. In view of Theorem 3.4, the zero solution of the nonlinear q-fractional delay neutral system (4.4) is asymptotically stable. Moreover, the graph showing the numerical simulation of the system (4.4) is given in Figure 4.

5. Conclusion

We have found some sufficient conditions for two different q fractional neutral systems with delay. First, we have found sufficient conditions for the asymptotic admissibility of a nonlinear delayed q fractional neutral singular system. Then, we have found sufficient criteria for the asymptotic stability of a nonsingular q-fractional nonlinear delayed neutral system. We have used the Lyapunov-Krasovskii functional, the matrix inequality, and various lemmas to obtain these conditions. We have shown some numerical examples demonstrating the applicability of these finally obtained conditions. We believe that our study is interesting and up-to-date enough to make a comprehensive contribution to the relevant literature. In our next study, we intend to consider the system as a variable delay.

Acknowledgment

The author would like to thank the editors and the referees for carefully reading the paper and for their comments which have helped to greatly improve the paper.

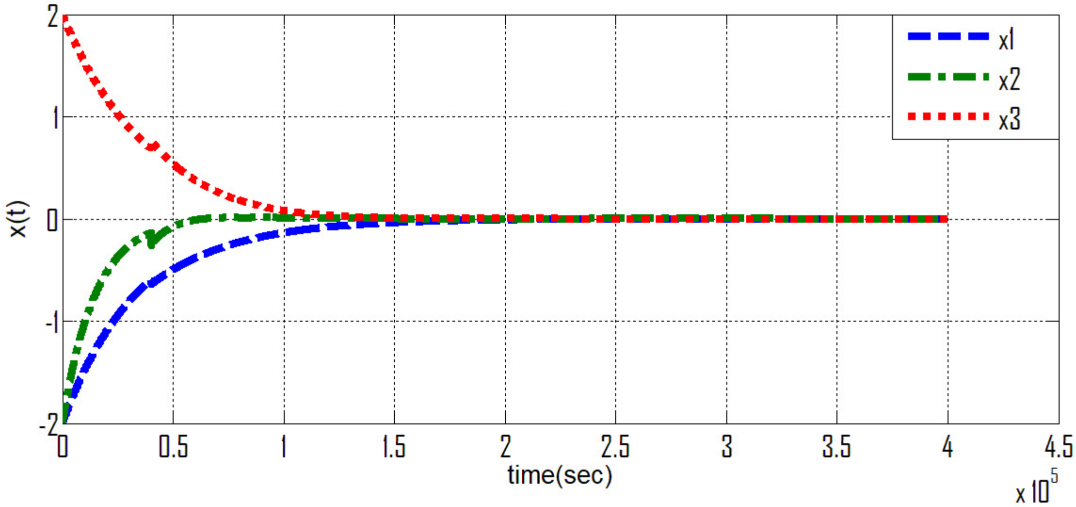

Figure 4. Numerical simulation of the system (4.4).

References

[1] Agarwal RP. Certain fractional q-integrals and q-derivatives. Mathematical Proceedings of the Cambridge Philosophical Society 1969; 66 (2): 365-370. https://doi.org/10.1017/S0305004100045060

[2] Alkhazzan A, Wang J, Tunç C, Ding X, Yuan Z, Nie Y. On Existence and Continuity Results of Solution for Multi-time Scale Fractional Stochastic Differential Equation. Qualitative Theory of Dynamical Systems 2023; 22 (49): 1-23. https://doi.org/10.1007/s12346-023-00750-x

[3] Al-Salam WA. Some fractional q-integrals and q-derivatives. Proceedings of the Edinburgh Mathematical Society 1969; 15 (2): 135-140. https://doi.org/10.1017/S0013091500011469

[4] Altun Y. Further results on the asymptotic stability of Riemann-Liouville fractional neutral systems with variable delays. Advances in Difference Equations 2019; 437: 1-13. https://doi.org/10.1186/s13662-019-2366-z

[5] Altun Y, Tunç C. On the asymptotic stability of a nonlinear fractional-order system with multiple variable delays. Applications and Applied Mathematics: An International Journal (AAM) 2020; 15 (1): 458-468.

[6] Altun Y. LMI based approach to asymptotically stability analysis for fractional neutral-type neural networks with Riemann-Liouville derivative. Nonlinear Studies 2022; 29 (2): 635-647.

[7] Annaby MH, Mansour ZS. Q-fractional Calculus and Equations. New York, Springer-Heidelberg, 2012.

[8] Atici FM, Eloe PW. Fractional q-calculus on a time scale. Journal of Nonlinear Mathematical Physics 2007; 14 (3): 341-352. https://doi.org/10.2991/jnmp.2007.14.3.4

[9] Chartbupapan W, Bagdasar O, Mukdasai K. A Novel Delay-Dependent Asymptotic Stability Conditions for Differential and Riemann-Liouville Fractional Differential Neutral Systems with Constant Delays and Nonlinear Perturbation. Mathematics 2020; 8 (1): 1-10. https://doi.org/10.3390/math8010082

[10] Dai L. Singular control systems. Lecture Notes in Control and Information Sciences, 118. Springer-Verlag, Berlin, 1989.

[11] Jarad F, Abdeljawad T, Baleanu D. Stability of q-fractional non-autonomous systems. Nonlinear Analysis: Real World Applications 2013; 14 (1): 780-784. https://doi.org/10.1016/j.nonrwa.2012.08.001

[12] Kac V, Cheung P. Quantum calculus. New York, Springer-Verlag, 2002.

[13] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Application of Fractional Differential Equations. Elsevier, New York, USA, 2006.

[14] Koca I, Demirci E. On local asymptotic stability of q-fractional nonlinear dynamical systems. Applications and Applied Mathematics: An International Journal (AAM) 2016; 11 (1): 174-183. https://digitalcommons.pvamu.edu/aam/vol11/iss1/10

[15] Liu S, Zhou X. F, Li X, Jiang W. Asymptotical stability of Riemann–Liouville fractional singular systems with multiple time-varying delays. Applied Mathematics Letters 2017; 65: 32-39.

[16] Lu YF, Wu RC, Qin ZQ. Asymptotic stability of nonlinear fractional neutral singular systems. Journal of Applied Mathematics and Computing 2014; 45: 351-364. https://doi.org/10.1007/s12190-013-0726-5

[17] Mahdi NK, Khudair AR. Stability of nonlinear q-fractional dynamical systems on time scale. Partial Differential Equations in Applied Mathematics 2023; 7: Article 100496. https://doi.org/10.1016/j.padiff.2023.100496

[18] Podlubny I. Fractional Differential Equations. Academic Press, New York, USA, 1999.

[19] Tunç C, Tunç O. Solution estimates to Caputo proportional fractional derivative delay integro–differential equations. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 2023; 117 (12). https://doi.org/10.1007/s13398-022-01345-y

[20] Wu Q, Song Q, Hu B, Zhao Z, Liu Y, Alsaadi FE. Robust stability of uncertain fractional order singular systems with neutral and time-varying delays. Neurocomputing 2020; 401: 145-152. https://doi.org/10.1016/j.neucom.2020.03.015

[21] Yang W, Alsaedi A, Hayat T, Fardoun HM. Asymptotical stability analysis of Riemann-Liouville q-fractional neutral systems with mixed delays. Mathematical Methods in the Applied Sciences 2019; 42 (14): 4876-4888. https://doi.org/10.1002/mma.5700

[22] Xiao-Chuang J, Jun-Guo L, Qing-Hao Z. Delay-dependent and order-dependent asymptotic stability conditions for Riemann–Liouville fractional-order systems with time delays. Computational and Applied Mathematics 2023; 42: 116. https://doi.org/10.1007/s40314-023-02257-2

[23] Xu S, Lam J. Robust control and filtering of singular systems.Lecture Notes in Control and Information Sciences, 332. Springer-Verlag, Berlin, 2006.

[24] Yang C, Zhang Q, Zhou L. Stability analysis and design for nonlinear singular systems. Lecture Notes in Control and Information Sciences, 435. Springer, Heidelberg, 2013.

[25] Yiğit A, Tunç C. Asymptotical stability of nonlinear fractional neutral systems with unbounded delay. Applied Analysis and Optimization 2023; 7 (1): 63-77.

[26] Yiğit A, Sivasundaram S, Tunç C. Stability for fractional order delay singular systems. Nonlinear studies (NS) 2022; 29 (3): 865-879.

[27] Zhang H, Ye R, Cao J, Ahmed A, Li X, Ying W. Lyapunov functional approach to stability analysis of Riemann–Liouville fractional neural networks with time-varying delays. Asian Journal of Control 2018; 20 (5): 1938-1951. https://doi.org/10.1002/asjc.1675

[28] Zhai C, Ren J. The unique solution for a fractional q-difference equation with three-point boundary conditions. Indagationes Mathematicae 2018; 29 (3): 948-961. https://doi.org/10.1016/j.indag.2018.02.002