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On strong solvability of one nonlocal boundary value problem for Laplace equation in rectangle

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Abstract: One nonlocal boundary value problem for the Laplace equation in a bounded domain is considered in this work. The concept of a strong solution to this problem is introduced. The correct solvability of this problem in the Sobolev spaces generated by the weighted mixed norm is proved by the Fourier method. In a classic statement, this problem has been earlier considered by E.I.Moiseev [34]. A similar problem has been treated by M.E.Lerner and O.A.Repin [30].

Key words: Laplace equation, nonlocal problem, weighted Sobolev space, strong solution

1. Introduction

Consider the following (formal for now) nonlocal boundary value problem for the Laplace equation:

$$u_{xx} + u_{yy} = 0\,, \quad 0 < x < 2\pi\,, \quad 0 < y < h\,, \tag{1.1}$$

$$u\left(x,0\right) = \varphi\left(x\right)\,, \quad u\left(x,h\right) = \psi\left(x\right), \quad 0 < x < 2\pi, \tag{1.2}$$

$$u_x\left(0,y\right) = 0, \quad u\left(0,y\right) = u\left(2\pi,y\right), \quad 0 < y < h \tag{1.3}$$

Such problems have specific peculiarities compared to the ones with local conditions. Earlier, F.I.Frankl [21]; [22, p.453-456] considered the problem with nonlocal boundary conditions for a mixed-type equation. Bitsadze-Samarski problem [13] for elliptic equations is also nonlocal with supports on the part of the boundary of the domain, and these supports are free of other boundary conditions. In [28], N.I. Ionkin and E.I. Moiseev solved the boundary value problem for multidimensional parabolic equations with nonlocal conditions, whose supports are the characteristic and the improper parts of the boundary of the domain. In a classic statement, the problem (1.1)-(1.3) has been considered in [34] and [30].

In this work, we consider the problem (1.1)-(1.3) in a weighted Sobolev space with the weight belonging to the Muckenhoupt class. We define a concept of a strong solution to this problem. And, using the Fourier method, we prove the correct solvability of this problem.

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Note that the study of solvability of elliptic equations regarding so-called nonstandard function spaces and also in the weighted Sobolev spaces faces some difficulties compared to the weightless case. That is why the number of research works dedicated to this field has been growing in recent years (see, e.g., [2, 3, 7–10, 12, 14–17, 19, 31, 35]), and the elaboration of a corresponding theory is far from complete. Note also that the same problem was considered in the work [31] regarding unbounded rectangular.

2. Auxiliary concepts and facts

We will use standard notations. N will be the set of positive integers, while $\alpha = (\alpha_1; \alpha_2) \in Z^+ \times Z^+$ will denote a multiindex, where $Z^+ = N \bigcup \{0\}$. Denote $\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$, where $|\alpha| = \alpha_1 + \alpha_2$. By $|M|$ we will denote the Lebesgue measure of the set M; $\bar{M}$ will be the closure of M. $C^\infty \left(\bar{M}\right)$ will stand for the infinitely differentiable functions on $\bar{M}$, and $C_0^\infty (M)$ will denote the infinitely differentiable and finite functions on M. Throughout this paper we will assume that p' is a conjugate number of p, $1 < p < +\infty$: $\frac{1}{p'} + \frac{1}{p} = 1$. $d\sigma$ is an area element.

Let us define our weighted Sobolev space. Let $\nu : [0, 2\pi] \to (0, +\infty)$ be some weight function, $\Pi = (0, 2\pi) \times (0, h)$. Denote by $L_{p;\nu} (\Pi)$ a Banach space of functions on Π with the mixed norm

$$\|f\|_{L_{p,\nu}(\Pi)} = \int_0^h \left(\int_0^{2\pi} |f(x;y)|^p \nu(x)\, dx \right)^{\frac{1}{p}} dy\ , \quad 1 < p < +\infty.$$

Denote by $W_{p,\nu}^2 (\Pi)$ a Sobolev space with the norm

$$\|u\|_{W_{p;\nu}^2} = \sum_{|\alpha| \leq 2} \|\partial^\alpha u\|_{L_{p,\nu}(\Pi)}\,.$$

Now denote by $L_{p,\nu} (I)$, where $I = (0, 2\pi)$, a weighted Lebesgue space with the norm

$$\|f\|_{L_{p,\nu}(I)} = \left(\int_I |f(x)|^p \nu(x)\, dx \right)^{\frac{1}{p}}.$$

We will also consider the weighted Sobolev space $W_{p,\nu}^2 (I)$, with the norm

$$\|f\|_{W_{p,\nu}^2(I)} = \|f\|_{L_{p,\nu}(I)} + \|f'\|_{L_{p,\nu}(I)} + \|f''\|_{L_{p,\nu}(I)}\,.$$

We will need the class of Muckenhoupt weights $A_p (I)$. This is a class of $2\pi-$ periodic functions (i.e. the class of functions ν periodically extended to the real axis with period 2π), satisfying the condition

$$\sup_{J \subset I} \left(\frac{1}{|J|} \int_J \nu(t)\, dt \right) \left(\frac{1}{|J|} \int_I |\nu(t)|^{-\frac{1}{p-1}} dt \right)^{p-1} < +\infty,$$

where *sup* is taken over all intervals $J \subset I$, and $|J|$ is a length of the interval J.

We will also need some concepts and facts from the theory of bases in a Banach space. Related to these facts and concepts one can see [4] and references therein. Let X be a Banach space.

Definition 2.1 *A system* $\{u_n\}_{n\in N} \subset X$ *is called a basis if any element* $f \in X$ *is uniquely represented as a series*

$$f = \sum_{n=1}^{\infty} c_n u_n,$$

convergent in the norm X.

Definition 2.2 *A system* $\{u_n\}_{n\in N} \subset X$ *is called complete in* X *if* $\overline{Sp}\{u_n\} = X$ *and minimal in* X *if* $u_n \notin \overline{Sp}\{u_k\}_{k\neq n}$.

It is known that each basis of the space X is a complete and minimal system in X, the converse is not true in general.

Minimum criterion. The system $\{u_n\}_{n\in N}$ is minimal in X if and only if there exists a biorthogonal system, i.e. there exists a system $\{v_n\}_{n\in N} \subset X^*$ such that $\langle u_n, v_k\rangle = v_k(u_n) = \delta_{nk}$, where δ_{nk} is the Kronecker symbol.

Basis criterion. The system $\{u_n\}_{n\in N} \subset X$ is a basis of the space X if and only if the following conditions are satisfied:

1) $\{u_n\}_{n\in N}$ is complete and minimal in X;

2) uniformly bounded projectors

$$P_n f = \sum_{k=1}^{n} \langle f, v_k\rangle u_k,$$

where $\{v_k\}_{k\in N}$ is a biorthogonal system.

Definition 2.3 *A system* $\{u_n\}_{n\in N} \subset X$ *is called a basis with brackets in* X *if there exists a sequence of integers* $0 = n_0 < n_1 < n_2 < \dots$ *such that each element of* $f \in X$ *is uniquely represented as a series*

$$f = \sum_{k=0}^{\infty} \sum_{i=n_k+1}^{n_{k+1}} c_i u_i,$$

convergent in the norm X.

To obtain our main results, we will use the basicity of the classical trigonometric system in the weighted Lebesgue spaces. From the results obtained by R.A.Hunt, W.S.Young in [27], the statement below follows immediately:

Proposition 2.4 *Trigonometric system* $\{1; \cos nx; \sin nx\}_{n\in N}$ *forms a basis for the weighted Lebesgue space* $L_{p,\nu}(I)$, $1 < p < +\infty \Leftrightarrow \nu \in A_p(I)$.

Note that the basicity problems of perturbed trigonometric systems have been also studied in [5, 6, 25]. The same problems have been studied in [4, 11, 24, 26, 36] for the eigenfunction systems of some differential operators.

It is known that if $\nu \in A_p(I)$, $1 < p < +\infty$, then $C_0^\infty(I)$ is dense in $L_{p,\nu}(I)$. In fact,

$$\nu \in A_p(I) \;\Rightarrow\; \exists\, p_0 \in (1,+\infty) \;\Rightarrow\; \nu \in L_{p_0}(I)$$

(see, e.g., [18, 20, 23]). We have

$$\int_I |f|^p \nu\, dx \le \left(\int_I |f|^{pp_0'} dx\right)^{\frac{1}{p_0'}} \left(\int_I \nu^{p_o} dx\right)^{\frac{1}{p_0}}.$$

Hence it follows that

$$\|f\|_{L_{p,\nu}(I)} \le C\,\|f\|_{L_{p_1}(I)}, \tag{2.1}$$

where $p_1 = pp_0'$ and $C > 0$ is a constant independent of f, i.e. the continuous embedding $L_{p_1}(I) \subset L_{p,\nu}(I)$ is true. As $C_0^\infty(I)$ is dense in $L_{p_1}(I)$, from the inequality (2.1) it follows that it is also dense in $L_{p;\nu}(I)$. Consequently, $L_{p_1}(I)$ is densely embedded into $L_{p,\nu}(I)$.

To obtain our main results we will extensively use the following Minkowski inequality (see, e.g., [29, p.24]) for integrals.

Proposition 2.5 *[29] Let $(M_k;\, \sigma_{M_k};\, \mu_k)$, $k = \overline{1,2}$, be measurable spaces with σ-finite measures μ_k and $F(x;y)$ be a $\mu_1 \times \mu_2$-measurable function. Then*

$$\left\|\int_{M_1} F(x;y)\, d\mu_1(x)\right\|_{L_p(\mu_2)} \le \int_{M_1} \|F(x;\cdot)\|_{L_p(\mu_2)}\, d\mu_1(x),$$

where

$$\|f\|_{L_p(\mu_2)} = \left(\int_{M_2} |f|^p\, d\mu_2\right)^{\frac{1}{p}}.$$

From $\nu \in A_p(I)$, $1 < p < +\infty$, it is obvious that $L_{p,\nu}(I) \subset L_1(I)$. Then, the continuous embedding $W_{p,\nu}^2(\Pi) \subset W_1^2(\Pi)$ is also true. Consequently, every function $u \in W_{p,\nu}^2(\Pi)$ has traces $u|_{\partial\Pi}$ and $u_x|_{\partial\Pi}$ as functions of space $L_1(\partial\Omega; d\sigma)$ on the boundary $\partial\Pi$ (correctly defined with respect to the Lebesgue measure on $\partial\Pi$).

Let $u \in W_{p,\nu}^2(\Pi)$, and $\xi \in [0,h]$ be an arbitrary number. Denote $I_\xi = \{(x,\xi) : x \in I\}$. Obviously, $u \in W_1^2(\Pi)$. Denote the trace of the function $u(x,y)$ on I_ξ, $\xi \in (0,h)$, by $F_\xi(x) : F_\xi(x) = u(x,\xi)$, $0 < x < 2\pi$. Let us show that $F_\xi \in L_{p,\nu}(I)$. Assume $\Pi_\xi = \{(x,y) : x \in I,\ y \in (0,\xi)\}$. It is absolutely clear that $u \in W_{p,\nu}^2(\Pi_\xi)$. From the condition $v \in A_p(I)$ it follows that $v \in A_p(\Pi_\xi)$ (can be verified directly), and so $C^\infty\left(\bar{\Pi}_\xi\right)$ is dense in $W_{p,\nu}^2(\Pi_\xi)$. Let us first assume that $u \in C^\infty\left(\bar{\Pi}_\xi\right)$. Without loss of generality, we suppose that $u(x;0) = 0$, $\forall\, x \in I$. Then, we have

$$F_\xi(x) = u(x;\xi) = \int_0^\xi \frac{\partial u(x,y)}{\partial y} dy, \quad \text{a.e. } x \in I.$$

Applying Minkowski's inequality (Proposition 2.5), we obtain

$$\|F_\xi\|_{L_{p,\nu}(I)} \le \left\|\frac{\partial u}{\partial y}\right\|_{L_{p,\nu}(\Pi_\xi)} \le \|u\|_{W_{p,\nu}^2(\Pi_\xi)}.$$

Using this estimate and the fact that $C^{\infty}\left(\bar{\Pi}_{\xi}\right)$ is dense in $W_{p,\nu}^{2}\left(\Pi_{\xi}\right)$, absolutely similar to the weightless case we can prove that the trace of $\forall\, u\in W_{p,\nu}^{2}\left(\Pi\right)$ on I_{ξ} satisfies the estimate

$$\left\| F_{\xi}\right\|_{L_{p,\nu}(I)}\leq\left\|u\right\|_{W_{p,\nu}^{2}(\Pi)}.$$

If u satisfies (1.1), then it is known that $u\in C^{\infty}\left(\Pi\right)\ \Rightarrow\ F_{y}\left(x\right)=u\left(x,y\right)\ ,\ \forall\, x\in I$.

So let us introduce the following

Definition 2.6 *A function $u\in W_{p,\nu}^{2}\left(\Pi\right)$ is called a strong solution of the problem (1.1)-(1.3) if the equality (1.1) is satisfied for a.e. $(x;y)\in\Pi$ and its trace $u|_{\partial\Pi}$ satisfies the relations (1.2), (1.3).*

Introduce the systems of functions $\left\{u_{n}\left(x\right)\right\}_{n\in Z^{+}}$ and $\left\{\vartheta_{n}\left(x\right)\right\}_{n\in Z^{+}}$, where

$$u_{2n}\left(x\right)=\cos nx,\ \ n\in Z^{+},\quad u_{2n-1}\left(x\right)=x\sin nx,\ \ n\in N, \tag{2.2}$$

$$\vartheta_{0}\left(x\right)=\frac{1}{2\pi^{2}}\left(2\pi-x\right),\vartheta_{2n}\left(x\right)=\frac{1}{\pi^{2}}\left(2\pi-x\right)\cos nx,\vartheta_{2n-1}\left(x\right)=\frac{1}{\pi^{2}}\sin nx\,,\ \ n\in N. \tag{2.3}$$

Note that these systems are biorthogonal, which can be verified directly. To obtain our main result, we will significantly use the following theorem.

Theorem 2.7 *Let $\nu\in A_{p}\left(I\right),\ 1<p<+\infty$. Then the system (2.2) forms a basis for $L_{p,\nu}\left(I\right)$.*

Proof Conjugate space of $L_{p;\nu}\left(I\right)$ is $L_{p',\nu}\left(I\right)$. It is absolutely clear that the system (2.3) belongs to $L_{p',\nu}\left(I\right)$ and is biorthonormalized to the system (2.2) (see [34]). It follows that (2.2) is minimal in $L_{p,\nu}\left(I\right)$. On the other hand, from [1] it follows that the system (2.2) forms a basis with brackets for $L_{p}\left(I\right)$ for every $p\in\left(1,+\infty\right)$, and, consequently, it is complete in $L_{p_{1}}\left(I\right)$, where the number p_{1} is the same as in inequality (2.1). Then from the embedding $L_{p_{1}}\left(I\right)\subset L_{p,\nu}\left(I\right)$ it follows that (2.2) is complete and, consequently, complete and minimal in $L_{p,\nu}\left(I\right)$.

Let us prove the basicity of the system (2.2) for $L_{p,\nu}\left(I\right)$. Consider the projectors

$$P_{n}\left(f\right)=\sum_{k=0}^{n}\left\langle f,\vartheta_{k}\right\rangle u_{k},\forall\ n\in Z^{+},\forall f\in L_{p,\nu}\left(I\right),$$

where

$$\left\langle f,g\right\rangle=\int_{0}^{2\pi}f\left(x\right)g\left(x\right)dx.$$

From the basicity with brackets of the system (2.2) for $L_{p,\nu}\left(I\right)$ it follows that

$$\exists C>0:\left\|P_{2n}\left(f\right)\right\|_{L_{p,\nu}(I)}\leq C\left\|f\right\|_{L_{p,\nu}(I)},\forall\ n\in N. \tag{2.4}$$

On the other hand, from (2.2), (2.3) we have

$$\exists M>0:\left\|u_{n}\right\|_{L_{p,\nu}(I)}\leq M,\quad\left\|\vartheta_{n}\right\|_{L_{p',\nu}(I)}\leq M,\quad\forall\ n\in N. \tag{2.5}$$

Considering the relations (2.4), (2.5), we obtain

$$\|P_{2n+1}(f)\|_{L_{p,\nu}(I)} = \|P_{2n}(f) + \langle f, \vartheta_{2n+1}\rangle u_{2n+1}\|_{L_{p,\nu}(I)} \le \|P_{2n}(f)\|_{L_{p,\nu}(I)} +$$

$$+ \|\langle f, \vartheta_{2n+1}\rangle u_{2n+1}\|_{L_{p,\nu}(I)} \le C\|f\|_{L_{p,\nu}(I)} + \|f\|_{L_{p,\nu}(I)} \|u_{2n+1}\|_{L_{p,\nu}(I)} \|\vartheta_{2n+1}\|_{L_{p',\nu}(I)} \le$$

$$\le \left(C + M^2\right) \|f\|_{L_{p,\nu}(I)} . \tag{2.6}$$

From (2.4), (2.6) it follows that the projectors $\{P_n\}_{n\in Z^+}$ are uniformly bounded, and, according to the criterion for basicity, this means that the system (2.2) forms a basis for $L_{p,\nu}(I)$. The theorem is proved. □

3. Main results

In this section, we will study the existence and uniqueness of strong solution of the problem (1.1)-(1.3) in the sense of Definition 2.6. First, denote $\Gamma_0 = \{(0; y) : 0 < y < h\}$ and $\Gamma_{2\pi} = \{(2\pi; y), 0 < y < h\}$. Consider the following nonlocal problem

$$\Delta u = 0 , \ (x; y) \in \Pi, \tag{3.1}$$

$$u|_{I_0} = \varphi , \ u|_{I_h} = \psi, \ u|_{\Gamma_0} = u|_{\Gamma_{2\pi}} , \ u_x|_{\Gamma_0} = 0. \tag{3.2}$$

By the solution of this problem, we mean a function $u \in W^2_{p;\nu}(\Pi)$, which satisfies the equality (3.1) a.e. in Π and whose traces satisfy the relations (3.2) on the boundary $\partial\Pi = I_0 \bigcup I_h \bigcup \Gamma_0 \bigcup \Gamma_{2\pi}$. Let us first prove the uniqueness of the solution. The following theorem is true:

Theorem 3.1 *Let $\nu \in A_p(I)$, $1 < p < +\infty$, and the functions φ, $\psi \in W^2_{p,\nu}(I)$ satisfy the conditions $\varphi(2\pi) - \varphi(0) = \varphi'(0) = 0$, $\psi(2\pi) - \psi(0) = \psi'(0) = 0$. If the problem (3.1),(3.2) has a solution in $W^2_{p,\nu}(\Pi)$, then it is unique.*

Proof Suppose $u(x, y) \in W^2_{p;\nu}(\Pi)$ is a solution of the problem (3.1), (3.2). Consider $U_n(y) = \langle u(\cdot, y), \vartheta_n(\cdot)\rangle$, i.e.

$$\left.\begin{array}{r} U_0(y) = \frac{1}{2\pi^2}\int_0^{2\pi} u(x,y)\,(2\pi - x)\,dx \ , \\ U_{2n}(y) = \frac{1}{\pi^2}\int_0^{2\pi} u(x,y)\,(2\pi - x)\cos nx\,dx \ , \\ U_{2n-1}(y) = \frac{1}{\pi^2}\int_0^{2\pi} u(x,y)\,\sin nx\,dx \ , \ \ n \in N \ . \end{array}\right\} \tag{3.3}$$

From Theorems 1.1.1-1.1.3 of [32, pp. 13-15] it follows that the functions $U_n(y)$ are twice differentiable and they can be differentiated under the integral sign. Since the function $u(x, y)$ satisfies the equation (3.1), multiplying it by $\sin nx$ (by $(2\pi - x)\cos nx$) and integrating over I, we obtain the following relations for $U_{2n-1}(y)$ (respectively, for $U_{2n}(y)$):

$$U''_{2n-1}(y) - n^2 U_{2n-1}(y) = 0 \ , \ y \in (0.h) , \tag{3.4}$$

$$U''_{2n}(y) - n^2 U_{2n}(y) = -2nU_{2n-1}(y) \ , \ y \in (0.h) . \tag{3.5}$$

By the Newton-Leibniz formula, we have

$$u(x,\xi) = u(x,0) + \int_0^\xi \frac{\partial u(x,y)}{\partial y}\, dy = \varphi(x) + \int_0^\xi \frac{\partial u(x,y)}{\partial \xi}\, d\xi, \quad \text{a.e. } x \in I.$$

Consequently

$$|u(x,\xi) - \varphi(x)| \le \int_0^\xi \left| \frac{\partial u(x,y)}{\partial y} \right| dy, \quad \text{a.e. } x \in I.$$

Hence it immediately follows that

$$\int_I |u(x,\xi) - \varphi(x)|\, dx \le \int_I \int_0^\xi \left| \frac{\partial u(x,y)}{\partial y} \right| dy\, dx. \tag{3.6}$$

We have $|\Pi_\xi| \to 0$ as $\xi \to +0$. Then from (3.6) it follows that

$$u_\xi(\cdot) \to \varphi(\cdot)\ ,\ \xi \to +0, \tag{3.7}$$

in the norm of the space $L_1(I)$.

Similarly, we have

$$u(x,\xi) = u(x,h) - \int_\xi^h \frac{\partial u(x,y)}{\partial y}\, dy = \psi(x) - \int_\xi^h \frac{\partial u(x,y)}{\partial y}\, dy, \text{ a.e. } x \in I.$$

Hence

$$\int_I |u(x,\xi) - \psi(x)|\, dx \le \int_I \int_\xi^h \left| \frac{\partial u(x,y)}{\partial y} \right| dy\, dx. \tag{3.8}$$

As $|\Pi \backslash \Pi_\xi| \to 0$ when $\xi \to h - 0$, from (3.8) it follows that

$$u_\xi(\cdot) \to \psi(\cdot)\ ,\ \xi \to h - 0, \tag{3.9}$$

in the norm of the space $L_1(I)$.

On the other hand, it is clear that $U_n(y) \in W_1^2(0,h)$. Hence it immediately follows that there exist limits

$$\lim_{y \to +0} U_n(y) = U_n(0)\ ,\ \lim_{y \to h-0} U_n(y) = U_n(h)\,,\ \ \forall n \in Z^+.$$

By (3.7) and (3.9), from the last two relations it immediately follows that

$$U_n(0) = \varphi_n, \quad U_n(h) = \psi_n, \quad \forall\, n \in Z^+, \tag{3.10}$$

where

$$\begin{cases} \varphi_0 = \frac{1}{2\pi^2} \int_0^{2\pi} \varphi(x)(2\pi - x)\, dx, \\ \varphi_{2n-1} = \frac{1}{\pi^2} \int_0^{2\pi} \varphi(x) \sin nx\, dx, \\ \varphi_{2n} = \frac{1}{\pi^2} \int_0^{2\pi} \varphi(x)(2\pi - x) \cos nx\, dx, \quad n \in N; \end{cases} \tag{3.11}$$

$$\begin{cases} \psi_0 = \frac{1}{2\pi^2}\int_0^{2\pi}\psi(x)(2\pi - x)\,dx, \\ \psi_{2n-1} = \frac{1}{\pi^2}\int_0^{2\pi}\psi(x)\sin nx\,dx, \\ \psi_{2n} = \frac{1}{\pi^2}\int_0^{2\pi}\psi(x)(2\pi - x)\cos nx\,dx, \quad n \in N. \end{cases} \tag{3.12}$$

The solution of the problem (3.4), (3.10) has the following expression

$$U_{2n-1}(y) = \psi_{2n-1}\frac{\sinh ny}{\sinh nh} + \varphi_{2n-1}\frac{\sinh n(h-y)}{\sinh nh}, \forall\, n \in N, \tag{3.13}$$

and the solution of the problem (3.5), (3.10) is

$$U_0(y) = \frac{\psi_0 - \varphi_0}{h}y + \varphi_0, \tag{3.14}$$

$$+\varphi_{2n}\frac{\sinh n(h-y)}{\sinh nh} - y\left(\psi_{2n-1}\frac{\cosh ny}{\sinh nh} - \varphi_{2n-1}\frac{\cosh n(h-y)}{\sinh nh}\right), \forall\, n \in N. \tag{3.15}$$

Now we can proceed to the proof of the uniqueness of the solution. For this, it suffices to prove that the corresponding homogeneous problem has only a trivial solution. In fact, if $\varphi(x) = \psi(x) \equiv 0$, then $\varphi_n = \psi_n = 0$, $\forall n \in Z^+$, and from the formulas (3.13)-(3.15) it follows that $U_n(y) = 0$, $\forall\, y \in (0,h)$, $\forall\, n \in Z^+$. As $u_y \in L_{p,\nu}(I)$, $\forall y \in (0,h)$, the basicity of the system (2.2) for $L_{p,\nu}(I)$ implies $u_y(x) = 0$ a.e. $x \in I$ and $\forall\, y \in (0,h)$. Hence it follows that $u(x;y) = 0$ a.e. $(x;y) \in \Pi$. Consequently, the homogeneous problem has only a trivial solution, and this completes the proof of uniqueness. □

Now let us prove the existence of the solution. The following theorem is true.

Theorem 3.2 *Let the weight function $\nu(x)$ belong to the class $A_p(I)$, $1 < p < +\infty$, and the boundary functions $\varphi(x)$ and $\psi(x)$ belong to the space $W^2_{p;\nu}(I)$ and satisfy the conditions*

$$\varphi(0) - \varphi(2\pi) = \varphi'(0) = 0, \;\; \psi(0) - \psi(2\pi) = \psi'(0) = 0.$$

Then the problem (1.1)-(1.3) has a (unique) solution in $W^2_{p;\nu}(\Pi)$.

Proof Consider the function

$$u(x,y) = U_0(y) + \sum_{n=1}^{\infty} U_n(y)\, u_n(x) = U_0(y) +$$

$$+\sum_{k=1}^{\infty}\left(U_{2k}(y)\cos kx + U_{2k-1}(y)\, x\sin kx\right), \;\; (x,y) \in \Pi, \tag{3.16}$$

where the coefficients $U_0(y)$, $U_{2k}(y(U_{2k}(y)))$, $U_{2k-1}(y(U_{2k-1}(y)))$, $k \in N$, are defined by (3.13)-(3.15). Let us show that the function $u(x,y)$ belongs to $W^2_{p;\nu}(\Pi)$. Denote by $u_{\alpha_1,\alpha_2}(x,y)$ the sum of the series obtained by the formal differentiation of the series (3.16), i.e.

$$u_{\alpha_1,\alpha_2}(x,y) = U_0^{(\alpha_2)}(y) + \sum_{n=1}^{\infty} U_n^{(\alpha_2)}(y)\, u_n^{(\alpha_1)}(x), \tag{3.17}$$

where $\alpha_1, \alpha_2 \in Z^+, \alpha_1 + \alpha_2 = 0, 1, 2;$ $u_{0,0}(x, y) = u(x, y)$ and $U_n^{(\alpha_2)}(y) = \frac{d^{\alpha_2} U_n}{dy^{\alpha_2}}$; $u_n^{(\alpha_1)}(x) = \frac{d^{\alpha_1} u_n}{dx^{\alpha_1}}$.
Let us first consider the following member of series (3.16)

$$u_1(x, y) = \sum_{k=1}^{\infty} U_{2k-1}(y)\ x \sin kx.$$

So, differentiating this series formally term-by-term, we have

$$\frac{\partial^2 u_1}{\partial y^2} = \sum_{k=1}^{\infty} U''_{2k-1}(y)\ x \sin kx = \sum_{k=1}^{\infty} k^2 U_{2k-1}(y)\ x \sin kx, \tag{3.18}$$

$$\frac{\partial u_1}{\partial x} = \sum_{k=1}^{\infty} U_{2k-1}(y) \sin kx + \sum_{k=1}^{\infty} k U_{2k-1}(y)\ x \cos kx, \tag{3.19}$$

$$\frac{\partial^2 u_1}{\partial x^2} = 2\sum_{k=1}^{\infty} k U_{2k-1}(y) \cos kx - \sum_{k=1}^{\infty} k^2 U_{2k-1}(y)\ x \sin kx. \tag{3.20}$$

Denote

$$w(x, y) = \sum_{k=1}^{\infty} k^2 U_{2k-1}(y)\ x \sin kx.$$

Let us show that the function $w(x, y)$ belongs to $L_{p;\nu}(\Pi)$. Let

$$\varphi''_{2k-1} = \frac{1}{\pi^2} \int_0^{2\pi} \varphi''(x) \sin kx\, dx, \psi''_{2k-1} = \frac{1}{\pi^2} \int_0^{2\pi} \psi''(x) \sin kx\, dx.$$

From (3.11), integrating by parts, we obtain

$$\varphi_{2k-1} = -\frac{1}{\pi^2 k} \int_0^{2\pi} \varphi(x)\ d\cos kx = -\frac{1}{\pi^2 k} \left(\varphi(2\pi) - \varphi(0) - \int_0^{2\pi} \varphi'(x) \cos kx\, dx \right) =$$

$$= \frac{1}{\pi^2 k} \int_0^{2\pi} \varphi'(x) \cos kx\, dx = -\frac{1}{\pi^2 k^2} \int_0^{2\pi} \varphi''(x) \sin kx\, dx = -\frac{1}{k^2} \varphi''_{2k-1}.$$

Similarly, from (3.12) we have

$$\psi_{2k-1} = -\frac{1}{k^2} \psi''_{2k-1}.$$

Thus

$$w(x, y) = -\sum_{k=1}^{\infty} \left(\psi''_{2k-1} \frac{\sinh ky}{\sinh kh} + \varphi''_{2k-1} \frac{\sinh k(h-y)}{\sinh kh} \right) x \sin kx\, .$$

It is known that if $\nu \in A_p(I)$, then $\exists\ \alpha > 1 : \nu \in L_\alpha(I)$ (see, e.g., [23, p. 395]). Let $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. Applying Holder's inequality, we obtain

$$\left(\int_0^{2\pi} |w(x, y)|^p \nu(x)\, dx \right)^{\frac{1}{p}} \leq \left(\int_0^{2\pi} v^\alpha(x)\, dx \right)^{\frac{1}{\alpha}} \left(\int_0^{2\pi} |w(x, y)|^{p\alpha'} dx \right)^{\frac{1}{p\alpha'}} = c \left(\int_0^{2\pi} |w(x, y)|^{p_1} dx \right)^{\frac{1}{p_1}},$$

where $c=\left(\int_0^{2\pi} v^{\alpha}(x)\,dx\right)^{\frac{1}{\alpha}}$ (consequently, the constant c does not depend on $w(x,y)$) and $p_1=p\alpha'$. Let us consider the cases $p\geq 2$ and $1<p<2$. Consider the following separate cases regarding p.

I. $p\geq 2$. Then $p_1=p\alpha'>2$. From the previous inequality, we have

$$\left(\int_0^{2\pi} |w(x,y)|^p \nu(x)\,dx\right)^{\frac{1}{p}} \leq c\sum_{k=1}^{\infty}|U_{2k-1}(y)|\left(\int_0^{2\pi}|u_{2k-1}(x)|^{p_1}dx\right)^{\frac{1}{p_1}} \leq$$

$$\leq c_1\sum_{k=1}^{\infty}|U_{2k-1}(y)| \leq c_1\sum_{k=1}^{\infty}\left|\psi''_{2k-1}\frac{\sinh ky}{\sinh kh}+\varphi''_{2k-1}\frac{\sinh k(h-y)}{\sinh kh}\right| \leq$$

$$\leq c_1\sum_{k=1}^{\infty}\left(\left|\psi''_{2k-1}\right|\frac{\sinh ky}{\sinh kh}+\left|\varphi''_{2k-1}\right|\frac{\sinh k(h-y)}{\sinh kh}\right).$$

Hence, first integrating with respect to $y\in(0,h)$ and then applying Holder's inequality for any $\beta\in(1,\infty)$, we obtain

$$\|w\|_{L_{p,\nu}(\Pi)} \leq c_1\sum_{k=1}^{\infty}\left(\frac{\left|\psi''_{2k-1}\right|}{\sinh kh}\int_0^h \sinh ky dy+\frac{\left|\varphi''_{2k-1}\right|}{\sinh kh}\int_0^h \sinh k(h-y)\,dy\right) \leq$$

$$\leq c_1\sum_{k=1}^{\infty}\frac{\left|\psi''_{2k-1}\right|+\left|\varphi''_{2k-1}\right|}{\sinh kh}\int_0^h \sinh ky dy \leq$$

$$\leq c_1\sum_{k=1}^{\infty}\frac{\cosh kh-1}{k\sinh kh}\left(\left|\psi''_{2k-1}\right|+\left|\varphi''_{2k-1}\right|\right) \leq$$

$$\leq c_2\sum_{k=1}^{\infty}\frac{1}{k}\left(\left|\psi''_{2k-1}\right|+\left|\varphi''_{2k-1}\right|\right) \leq c_2\left(\sum_{k=1}^{\infty}\frac{1}{k^{\beta'}}\right)^{\frac{1}{\beta'}}\left(\left(\sum_{n=1}^{\infty}\left|\varphi''_n\right|^{\beta}\right)^{\frac{1}{\beta}}+\left(\sum_{n=1}^{\infty}\left|\psi''_n\right|^{\beta}\right)^{\frac{1}{\beta}}\right).$$

Now, assuming $\beta\geq 2$ and applying classical Hausdorff-Young inequality (see, e.g. [37, p.154]), we have

$$\|w\|_{L_{p,\nu}(\Pi)} \leq c_3\left(\|\psi''\|_{L_{\beta'}(I)}+\|\varphi''\|_{L_{\beta'}(I)}\right). \tag{3.21}$$

It is known that if $\nu\in A_p(I)$, $1<p<+\infty$, then $\exists\, q:\ 1<q<p \ \Rightarrow \nu\in A_q(I)$. Let $r=\frac{p}{q}$ and $g\in L_{p,\nu}(I)$. Then $1<r<p$ and we have

$$\left(\int_I |g|^r dx\right)^{\frac{1}{r}} = \left(\int_I |g|^{\frac{p}{q}}\nu^{\frac{1}{q}}\nu^{-\frac{1}{q}}dx\right)^{\frac{1}{r}} \leq \left(\int_I |g|^p\nu\,dx\right)^{\frac{1}{qr}}\left(\int_I \nu^{-\frac{q'}{q}}dx\right)^{\frac{1}{q'r}} =$$

$$=\left(\int_I |g|^p\nu\,dx\right)^{\frac{1}{p}}\left(\int_I \nu^{-\frac{1}{q-1}}dx\right)^{\frac{q-1}{p}}.$$

As $-\frac{q'}{q}=\frac{1}{1-q}$, from $\nu\in A_q(I)$ it follows that $\nu^{-\frac{1}{q-1}}\in L_1(I)$. Then, the last inequality means $g\in L_r(I)$ and

$$\|g\|_{L_r(I)} \leq c\,\|g\|_{L_{p;\nu}(I)},$$

where $c>0$ is a constant independent of g. Also note that the continuous embedding $L_{p,\nu}(I)\subset L_\alpha(I)$ is true for every $\alpha\in(1,r)$. Let us choose β big enough to satisfy the condition $1<\beta'<r$. Then from (3.21) we obtain

$$\|w\|_{L_{p;\nu}(\Pi)} \le c\left(\|\psi''\|_{L_{p;\nu}(I)}+\|\varphi''\|_{L_{p;\nu}(I)}\right).$$

II. $p\in(1,2)$. As in the previous case, note that there exists a number $\alpha>1$ such that $\nu\in L_\alpha(I)$. But then $\nu\in L_s(I)$ for every $s\in(1,\alpha)$. Therefore, choosing $\alpha>1$ close enough to 1, we can provide that $p_1=p\,\alpha'>2$ (this is possible, because $\alpha'\to+\infty$ as $\alpha\to 1+0$). With this, further considerations are carried out similar to the previous case.

Other series from (3.18)-(3.20), and, consequently, all series from (3.17) are estimated in a similar way. So, as a result, we obtain

$$\|u\|_{W^2_{p;\nu}(\Pi)} \le c\left(\|\varphi\|_{W^2_{p;\nu}(I)}+\|\psi\|_{W^2_{p;\nu}(I)}\right),$$

where $c>0$ is a constant independent of φ and ψ. The fulfillment of equation (3.1) by $u(\cdot;\cdot)$ can be verified directly. Let us verify the fulfillment of boundary conditions. Denote the trace operators on Γ_0, $\Gamma_{2\pi}$, I_0 and I_h by θ_0, $\theta_{2\pi}$, T_0 and T_h, respectively. Let us show that $T_0u=\varphi$. From the boundedness of the trace operator $T_0\in\left[W^2_{p;\nu}(\Pi)\,;\,L_{p;\nu}(I)\right]$ it follows that if $u_m\to u$ in $W^2_{p;\nu}(\Pi)$, then $u_m/_{I_0}\to u/_{I_0}$ in $L_{p;\nu}(I)$.

Now, let us consider the following functions

$$u_m(x,y)=U_0(y)+\sum_{n=1}^{m}\left(U_{2n}(y)\cos nx+U_{2n-1}(y)\,x\sin nx\right),\quad (x;y)\in\Pi, m\in N.$$

We have

$$T_0u_m=u_m(x,0)=U_0(0)+\sum_{n=1}^{m}\left(U_{2n}(0)\cos nx+U_{2n-1}(0)\,x\sin nx\right)=$$

$$=\frac{1}{2\pi^2}\int_0^{2\pi}\varphi(\tau)(2\pi-\tau)\,d\tau+\sum_{n=1}^{m}\left(\frac{1}{\pi^2}\int_0^{2\pi}\varphi(\tau)(2\pi-\tau)\cos n\tau d\tau\,\cos nx+\frac{1}{\pi^2}\int_0^{2\pi}\varphi(\tau)\sin n\tau d\tau\;x\sin nx\right).$$

The basicity of the system (2.2) for $L_{p;\nu}(I)$ implies $T_0u_m\to\varphi$, $m\to\infty$, in $L_{p;\nu}(I)$. Consequently, $T_0u=\varphi$.

Absolutely similar we can show that $T_hu_m\to\psi$, $m\to\infty$, in $L_{p;\nu}(I)$. Consequently, $T_hu=\psi$.

Consider the operators θ_0 and $\theta_{2\pi}$. It is clear that $\theta_0u_m=\theta_{2\pi}u_m$, $\forall\, m\in N$. Obviously, $\theta_0u_m\to\theta_0u$ and $\theta_{2\pi}u_m\to\theta_{2\pi}u\;\Rightarrow\;\theta_0u=\theta_{2\pi}u$. Thus, the boundary conditions (3.2) are fulfilled. Other trace relations can be proved in a similar way.

The theorem is proved. □

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