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On the invariance of hyperstoneanness under lattice isomorphisms

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Abstract: Let X and Y be compact Hausdorff spaces with Y hyperstonean. In this paper, we prove that if $C(X,\mathbb{R})$ and $C(Y,\mathbb{R})$ are lattice isomorphic then these Banach spaces are linearly isometric, and, consequently, X and Y are homeomorphic, which in turn implies that X is also hyperstonean. Actually, we prove more than what is announced in the headline above. This result, in some ways, is a generalization of the well-known Banach-Stone theorem.

Key words: Hyperstonean space, Banach-Stone theorem, lattice isomorphism

1. Introduction and preliminaries

Let X be a locally compact topological space. We denote by $C_0(X)$ the Banach space of all complex-valued continuous functions on X which vanish at infinity, provided with the usual supremum norm. If X is actually compact we use the notation $C(X)$ instead of $C_0(X)$. With the norm $\|f\| = \sup_{x\in X}|f(x)|$, $C(X)$ becomes a Banach space. Then we may introduce an ordering by defining $f \geqq g$ to mean $f(x) \geqq g(x)$ for all x. If f and g are in $C(X)$, so are the infimum function $f \wedge g$ defined by $(f\wedge g)(x) = \min\{f(x), g(x)\}$ and the supremum function $f \vee g$ defined by $(f \vee g)(x) = \max\{f(x), g(x)\}$. These operations make $C(X)$ a lattice.

The well-known Banach-Stone theorem states that if X and Y are locally compact spaces, and if $C_0(X)$ and $C_0(Y)$ are linearly isometric then X and Y are homeomorphic. This theorem has several generalizations, see [1, 3, 5, 17]. The main aim of these generalizations is to obtain a homeomorphism of X onto Y, provided that there exists a linear isomorphism φ of a certain linear subspace of $C_0(X)$ onto a similar linear subspace of $C_0(Y)$ with bound $\|\varphi\|\,\|\varphi^{-1}\| < 2$.

Examples show that 2 is the best number in those results, that is, there are non-homeomorphic locally compact spaces X and Y, and an isomorphism φ of $C_0(X)$ onto $C_0(Y)$ with $\|\varphi\|\,\|\varphi^{-1}\| = 2$, see [4, 7, 9]. So, if $\|\varphi\|\,\|\varphi^{-1}\| \geq 2$, we can not expect X and Y to be homeomorphic except in the case that X and Y are countable, compact metric spaces. In this case, 3 is the best number [12]. However, one might still ask whether some set-theoretic and topological properties are preserved (i.e. a property is possessed by X if and only if it is possessed by Y).

The problem of preservation of set-theoretic and topological properties of X under linear isomorphisms of $C_0(X)$ is very interesting and is, basically, wide open. In [6], Cengiz showed that the cardinality of X is

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preserved. For locally compact metric spaces, separability is also preserved, since separability and metrizability of a locally compact space X is equivalent to separability of $C_0(X)$.

In [8], it is proved that some other important properties are also preserved, and given examples to show that some properties which are equally important are not preserved.

Let us recall that a compact Hausdorff space X is *dispersed* if for some ordinal number α, the αth derived set $X^{(\alpha)}$ is empty, or equivalently, every regular Borel measure μ on X has the form

$$\mu = \sum_{n=1}^{\infty} a_n \delta_{x_n},$$

where $\{a_n\}$ is a sequence of scalars such that $\sum |a_n| < \infty$, $\{x_1, x_2, \ldots\}$ is a countable set in X; and where, for each $x \in X, \delta_x$ denotes the unit point mass at x , see [18].

A locally compact space X is said to be *Borel separable* if the support of each regular Borel measure is separable. Locally compact metric spaces, dispersed spaces, and also Eberlein compacts are Borel separable. (Recall that an Eberlein compact is a compact space which is homeomorphic to a weakly compact subset of a Banach space.)

For a topological space X, the weight $w(X)$ (density $d(X)$) is the smallest among the cardinal numbers of bases for the topology of X (dense subsets of X).

In addition to the above-mentioned invariant properties, under isomorphisms, in [8] is proved the following:

Let X and Y be locally compact Hausdorff spaces such that $C_0(X)$ and $C_0(Y)$ are linearly isomorphic. Then,

(i) X is dispersed if and only if Y is dispersed, in this case $d(X) = d(Y)$.

(ii) $w(X) = w(Y)$.

(iii) If X and Y are Borel separable then X is separable if and only if Y is separable.

In this paper, we prove that one more important property of compact spaces is preserved under isomorphism of spaces of continuous scalar functions.

Generally, it is a rare event in analysis that $C(X)$ is the dual of a Banach space, but, as we shall see soon, it is not so rare at all. Clearly finite spaces fall into this category, and the Stone-Čech compactifications of infinite discrete spaces are trivial examples of infinite spaces of this kind, since, for any discrete space D, $l^\infty(D) \simeq C(\beta D)$ is the dual of $l^1(D)$, where βD denotes the Stone-Čech compactification of D. (For two normed spaces E and F, $E \simeq F$ means that they are linearly isometric.)

Let us recall that a compact Hausdorff space is called *extremally disconnected* if the closure of every open set is open. These spaces are also called *Stonean*. They are precisely the Stone spaces of complete Boolean algebras [14].

Following [2] we call an extended real-valued positive Borel measure on a Stonean space *perfect* if

(i) the measure of every nonempty open set is strictly positive,

(ii) every nonempty open set contains a clopen (closed and open) set with nonzero finite measure,

(iii) the measure of every nowhere dense Borel set is zero (equivalently, the measure of every closed set with an empty interior is zero).

A Stonean space with a perfect measure on it is called *hyperstonean* or *a hyperstonean measure space*.

In [8], Cengiz proved that any arbitrary positive measure is equivalent to a perfect Borel measure on a Stonean space in the sense that for every number $1 \leq p < \infty$, their corresponding L^p spaces are linearly isometric.

Clearly, equivalent measures are equivalent to the same perfect measure. This result shows that the class of all hyperstonean spaces is huge indeed.

It is known that there is essentially one perfect measure on a hyperstonean space meaning that all perfect measures on the same space are equivalent. It is also a fact that in a hyperstonean measure space, any Borel set differs from a clopen set by a null set [8, 17].

Theorem 1.1 *For a Stonean space Ω the following are equivalent:*

i) Ω *is hyperstonean,*

ii) $C(\Omega)$ *is a dual space,*

iii) $C(\Omega)$ *is the dual space of* N*,*

iv) S *is dense in* Ω*.*

Here N is the space of all normal measures, (the regular Borel measures vanishing completely on nowhere dense Borel sets) and S is the union of their supports [6, 17, 18].

2. Main result

Theorem 2.1 *Let X and Ω be compact Hausdorff spaces of which the latter is hyperstonean. If $C(X,\mathbb{R})$ and $C(\Omega,\mathbb{R})$ are lattice isomorphic then they are linearly isometric, or equivalently X and Ω are homeomorphic. Hence, X is hyperstonean as well.*

Throughout the rest of this paper, the notation $C(X)$ will be used for $C(X,\mathbb{R})$, the space of real-valued continuous functions defined on X.

Proof Let T be a lattice isomorphism from $C(X)$ onto $C(\Omega)$ and first assume that there is a finite perfect measure μ on Ω, and let $\nu = \mu \circ T$, that is, $\nu = T^{*}\mu$, where T^{*} denotes the adjoint of T. Thus,

$$\int_X f d\nu = \int_\Omega T f d\mu$$

for all $f \in C(X)$.

For $f = f^{+} - f^{-} \in C(X)$,

$$\|f\|_1 = \int_X |f|\, d\nu = \int_\Omega (Tf^{+} + Tf^{-}) d\mu = \int_\Omega [(Tf)^{+} + (Tf)^{-}] d\mu = \int_\Omega |Tf|\, d\mu = \|Tf\|_1$$

Note that $|Tf| = (Tf)^{+} + (Tf)^{-} = T(f^{+}) + T(f^{-}) = T(|f|)$. Thus, T is a linear isometry from $(C(X), \|\|_1)$ onto $(C(\Omega), \|\|_1)$. Since μ is finite, μ and ν are both regular Borel measures [13], which implies that $C(X)$ and $C(\Omega)$

are dense in $L^1(\nu)$ and $L^1(\mu)$ respectively. Hence, T extends to a linear isometry $T^{'}$ from $L^1(\nu)$ onto $L^1(\mu)$. Therefore, by a slightly modified version of a result by Lamperty [16] and also taking into consideration the fact that in a hyperstonean space, every Borel set is equivalent to a clopen set [2, 14], there exists a measurable function α on Ω and a regular set isomorphism φ from the Borel algebra $\mathcal{B}$ on X onto $\mathcal{K}(\Omega)$ (the algebra of clopen subsets of Ω), defined modulo null sets, such that

$$Tf = \alpha\Phi(f), f \in L^1(\nu),$$

where Φ is the induced mapping from the set of measurable functions on X to the set of measurable functions on Ω characterized by the equation

$$\Phi(\chi_A) = \chi_{\varphi A}, A \in \mathcal{B} \quad ([10]).$$

Since $\Phi(1) = 1$, $\alpha = T(1)$ is continuous.

We claim that $\alpha(\omega) \neq 0$ for all $\omega \in \Omega$. Suppose the contrary and assume $\alpha(\omega_0) = 0$ for some $\omega_0 \in \Omega$.

Let $f \in C(X)$. Then, $|f| \leq A1$ for some constant $A > 0$.

$\Longrightarrow |Tf|(\omega_0) = T(|f|)(\omega_0) \leq AT(1)(\omega_0) = A\alpha(\omega_0) = 0.$

$\Longrightarrow (Tf)(\omega_0) = 0$ for all $f \in C(X)$, which shows that T maps $C(X)$ onto the subspace of $C(\Omega)$ of all functions in $C(\Omega)$ vanishing at ω_0, which is false, of course.

Hence, $\alpha(\omega) \neq 0$ for all $\omega \in \Omega$ as claimed.

So, the function $\frac{1}{\alpha} \in C(\Omega)$; and for each f in $C(X)$, $\Phi(f) = \frac{1}{\alpha}T(f)$ is continuous.

Now let $s = \sum\limits_{i=1}^{n} a_i\chi_{A_i}$ be a measurable simple function on X in its canonical form, i.e. $a_1, a_2, \ldots, a_n$ are its nonzero distinct values and such that $\nu(A_i) > 0$ for all $1 \leq i \leq n$. Then, since each φA_i is clopen,

$$\Phi(s) = \sum_{i=1}^{n} a_i\chi_{\varphi A_i}$$

is a continuous function, and

$$\|\Phi(s)\| = \left\|\sum_{i=1}^{n} a_i\chi_{\varphi A_i}\right\| = \max_{1\leq i\leq n} |a_i| = \|s\|.$$

Now let $f \in C(X)$. Then there exists a sequence of $\mathcal{B}$-simple functions $\{s_n\}$ which converges to f uniformly [15]. Since Φ is norm-preserving on the space of $\mathcal{B}$-measurable simple functions, $\Phi(s_n)$ is a Cauchy sequence in the uniform norm, and therefore it converges uniformly to a function g in $C(\Omega)$ which is obviously $\Phi(f)$, and so

$$\|\Phi(f)\| = \lim_n \|\Phi(s_n)\| = \lim_n \|s_n\| = \|f\|.$$

Since Φ maps the space of $\mathcal{B}$-simple functions onto the space of $\mathcal{K}(\Omega)$-simple functions, and by the Stone-Weierstrass approximation theorem [11], this latter space is uniformly dense in $\mathcal{K}(\Omega)$, we conclude that Φ is surjective. Hence, Φ is a linear isometry from $C(X)$ onto $C(\Omega)$, and, consequently, by the well-known Banach-Stone theorem, X and Ω are homeomorphic which implies that X is hyperstonean as well.

Now the general case: we assume that there is a perfect infinite measure μ on Ω.

Let T be an order-preserving isomorphism from $C(\Omega)$ onto $C(X)$.

Since Ω is extremally disconnected, $C(\Omega)$ is order-complete, therefore, $C(X)$ is order-complete which is equivalent to saying that X is extremally disconnected [14]. So, for any open subset U of X, its closure clU is homeomorphic to its Stone-Čech compactification βU [14].

Let $\mathcal{G} = \{\Omega_i : i \in I\}$ be a maximal family of clopen subsets of Ω with strictly positive finite measure (i.e. $0 < \mu(\Omega_i) < \infty$), and let $1_i = \chi_{\Omega_i}, i \in I$.

Now let, for $i \in I$, $U_i = \{x \in X : (T1_i)(x) > 0\}$.

For each $i \in I$, let $BC(U_i)$ be the space of all bounded continuous functions on U_i.

Note that, for every $i \in I$, we have the norm and lattice equivalence between $BC(U_i)$ and $C(\beta U_i)$. (All functions in this discussion are real-valued.)

Fix $i \in I$. For $f \in BC(U_i)$ let f^β denote its unique continuous extension to $clU_i = \beta U_i$, and furthermore, define $f^\beta(x) = 0$ for all x in $X \setminus U_i$, and finally, let $g = T^{-1}(f^\beta)$.

Then, since $U_i \cap U_j = \emptyset$, $\forall j \neq i$, and since $X \setminus U_j$ are closed, $clU_i \cap U_j = \emptyset$, $\forall j \neq i$.

Since f^β vanishes on U_j, $\forall j \neq i$, g vanishes on Ω_j, $\forall j \neq i$, and so, $g \equiv 0$ on $\Omega_0 = \bigcup_{j \neq i} \Omega_j \subset \Omega \setminus \Omega_i$.

Thus we have $g \equiv 0$ on $cl\Omega_0 \subset \Omega \setminus \Omega_i$. Therefore, $g = g\chi_{\Omega_i}$, i.e. $\{x \in \Omega : g(x) \neq 0\} \subset \Omega_i$.

Noticing that $|g| \leq A1_i$ for some number $A > 0$, we obtain $\{x \in X : (Tg)(x) = f^\beta(x) \neq 0\} \subset U_i$, and so, for each $f \in BC(U_i)$, $f^\beta = f$ which proves that U_i is compact, therefore closed. We also conclude that T maps $C(\Omega_i)$ onto $C(U_i)$, so that by the first part of the proof, each U_i is hyperstonean.

To complete the proof we let $X_0 = \bigcup_i U_i$ and observe that $f \equiv 0$ on X_0 implies $g = T^{-1}(f) \equiv 0$ on $\bigcup_i U_i$, and since this union is dense in Ω it follows that $g \equiv 0$ on Ω which in turn implies that $f \equiv 0$ on X. This discussion shows that X_0 is dense in X. Hence,

$$C(\Omega) \simeq BC(\bigcup_i \Omega_i) \simeq \sum_i \oplus C(\Omega_i) \simeq \sum_i \oplus C(U_i) \simeq BC(\bigcup_i U_i) \simeq C(\beta X_0) \simeq C(X). \qquad \square$$

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