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Some estimates on the spin − submanifolds

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Abstract: In this paper, an optimal lower bound is given for the Submanifold Dirac operator in terms of the trace of an Energy − Momentum tensor, scalar curvature and mean curvature. In the equality case, it is proven that the submanifold is Einstein if the normal bundle is flat.

Key words: Spin geometry, eigenvalues, Dirac operator

1. Introduction

On a compact Riemannian manifold endowed with Spin − structure, one can construct a spinor bundle. On this spinor bundle by using the Levi − Civita connection coming from the structure of the Riemannian manifold, the spinorial Levi − Civita connection can be built. The Dirac operator can be defined with the help of this spinorial Levi − Civita connection [3, 12, 13]. Since the Dirac operator and the spinorial Levi − Civita connection are carried subtle information about the geometry and topology of the manifold, many mathematicians work on them [1, 4, 6, 7, 9–11, 14, 16]. One of these studies is done by the A. Lichnerowicz [15] to bring the lower bounds on a compact $n-$dimensonal Riemannian Spin − manifold :

$$\lambda^2 \geq \frac{1}{4} \inf_M R, \tag{1.1}$$

where R is denoted by the scalar curvature of M. The proof is based on the well − known Schrödinger − Lichnerowicz formula [15]. Subsequently, (1.1) is improved in terms of the Yamabe number and Energy − Momentum tensor [6, 8].

After that, on the compact $n+m-$dimensional Riemannian Spin − manifold N and its $n \geq 2-$dimensional Spin − submanifold M whose normal bundle is also Riemannian Spin similar studies have been done. One of these is done in [10] by O. Hijazi and X. Zhang. They obtain an estimate for the eigenvalue λ_H of the submanifold Dirac operator D_H in terms of the mean curvature as follows:

$$\lambda_H^2 \geq \sup_\beta \inf_{M_\Phi} \Big(\frac{R + R_{\perp,\Phi}}{n\beta^2 - 2\beta + 1} - \frac{(n-1)}{(1-n\beta)^2} \|H\|^2 \Big), \tag{1.2}$$

for some β real function, $\beta \neq \frac{1}{n}$ if $H \neq 0$ for some A. Here $\|H\| = \sqrt{\sum_A H_A^2}$ is the norm of the mean curvature denoted

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by H_A and for any spinor field $\Phi \in \Gamma(\mathbb{S})$, $R_{\perp,\Phi} = -\frac{1}{2}\Big(\sum\limits_{i,j,A,B} R_{ijAB} e^i \cdot e^j \cdot e^A \cdot e^B \cdot \Phi, \frac{\Phi}{|\Phi|^2}\Big)$ is defined on the complement set of zeros M_Φ. In the limiting case, if the normal bundle is flat, they showed that the manifold $(M, g_N\big|_M)$ is an Einstein. Also by using some tricks they showed that λ_H^2 satisfies

$$\lambda_H^2 \geq \inf_{M_\Phi}\Big(\sqrt{\frac{n}{n-1}(R_{\perp,\Phi} + R)} - \|H\|\Big)^2. \tag{1.3}$$

Moreover, with respect to the conformal metric $\overline{g}_N = e^{2u} g_N$, they improved (1.2) and (1.3) as

$$\lambda_H^2 \geq \begin{cases} \frac{1}{4}\inf\limits_{M_\Phi}\Big(\sqrt{\frac{n}{n-1}(\mu_1 + R_{\perp,\Phi})} - \|H\|\Big)^2, & \text{if } n \geq 3 \text{ and } n(R_{\perp,\Phi} + \mu_1) \geq (n-1)\|H\|^2, \\ \frac{1}{4}\inf\limits_{M_\Phi}\Big(\sqrt{\frac{16\pi}{Area(M)} + 2R_{\perp,\Phi}} - \|H\|\Big)^2, & \text{if } n = 2 \text{ and, } \frac{16\pi}{Area(M)} + 2R_{\perp,\Phi} > \|H\|^2, \end{cases} \tag{1.4}$$

where μ_1 is the first eigenvalue of the Yamabe number and $Area(M)$ denotes the area of M. If the limiting cases are achieved and the normal bundle is flat, the norm of the mean curvature is constant and the manifold $(M, g_N\big|_M)$ is Einstein.

Also, by considering the Energy−Momentum tensor defined on the complement of the set zeros of a spinorfield Φ and modified scalar curvature given by

$$Q_\Phi(X,Y) = \frac{1}{2}Re\Big(X \cdot \nabla_Y \Phi + Y \cdot \nabla_X \Phi, \frac{\Phi}{|\Phi|^2}\Big), \tag{1.5}$$

$$R_{\tau,u,\Phi} = R + R_{\perp,\Phi} - 4\tau\Delta u + 4\nabla\tau\nabla u - 4\Big(1 - \frac{1}{n}\Big)\tau^2|du|^2, \tag{1.6}$$

respectively, the following estimate is obtained in [10] as

$$\lambda_H^2 \geq \begin{cases} \frac{1}{4}\sup\limits_{\tau,\beta,u}\inf\limits_{M_\Phi}\Big(\frac{R_{\tau,u,\Phi} + 4|Q_\Phi|^2}{1+n\beta^2 - 2^*\beta} - \Upsilon_{n,\beta,\Phi}\frac{(n-1)}{(n\beta-1)^2}\|H\|^2\Big), & n \geq 2, \\ \frac{1}{4}\sup\limits_{\Theta^Q_{\tau,u,\Phi}}\inf\limits_{M_\Phi}\Big(\sqrt{\frac{n}{n-1}}(R_{\tau,u,\Phi} + 4|Q_\Phi|^2) - \|H\|\Big)^2, & \text{if } n \geq 2 \text{ and } n \text{ is odd} \end{cases} \tag{1.7}$$

where $\Upsilon_{n,\beta,\Phi} = \frac{n\beta^2 - 2\beta + 1}{1+n\beta^2 - 2^*\beta}$, $2^* = 2\left(1 - \left(\frac{Re(\epsilon\omega\cdot\Phi,\Phi)}{|\Phi|^2}\right)\right)$, $\epsilon = (-1)^{m-1}$ and

$$\Theta^Q_{\tau,u,\Phi} = \{(\tau,u) : n(R_{\tau,u,\Phi} + 4|Q_\Phi|^2) > (n-1)\|H\|^2\}. \tag{1.8}$$

Here τ, β and u are real functions. With respect to the conformal metric $\overline{g}_N = e^{2u} g_N$, they improved (1.7) as

$$\lambda_H^2 \geq \begin{cases} \frac{1}{4}\sup\limits_{\tau,\beta,u}\inf\limits_{M_\Phi}\Big(\frac{\hat{R}_{\tau,u,\Phi} + 4|Q_\Phi|^2}{1+n\beta^2 - 2^*\beta} - \Upsilon_{n,\beta,\Phi}\frac{(n-1)}{(n\beta-1)^2}\|H\|^2\Big), & n \geq 2, \\ \frac{1}{4}\sup\limits_{\widetilde{\Omega}^Q_{\tau,u,\Phi}}\inf\limits_{M_\Phi}\Big(\sqrt{\frac{n}{n-1}}(\hat{R}_{\tau,u,\Phi} + 4|Q_\Phi|^2) - \|H\|\Big)^2, & \text{if } n \geq 2 \text{ and } n \text{ is odd,} \end{cases} \tag{1.9}$$

$$\widetilde{\Omega}^Q_{\tau,u,\Phi} = \{(\tau,u) : n(\hat{R}_{\tau,u,\Phi} + 4|Q_\Phi|^2) > (n-1)\|H\|^2\}, \tag{1.10}$$

and

$$\hat{R}_{\tau,u,\Phi} = R + R_{\perp,\Phi} + 4\Big(\frac{n-1}{2} - \tau\Big)\Delta u + 4\nabla\tau\nabla u$$
$$-\Big((n-1)(n-2) + 4(2-n)\tau + 4\big(1-\frac{1}{n}\big)\tau^2\Big)|du|^2. \tag{1.11}$$

In order to obtain geometric information related to the manifold we will extend the modified spinorial Levi − Civita connection given in [1, 10], depending on the trace of the Energy − Momentum tensor.

In the following section, we give some fundamental facts concerning the submanifold Dirac operator [10]. Then we obtain new lower bounds. For this bound, we consider the equality case and we show that the submanifold is an Einstein manifold with constant mean curvature and scalar curvature. In the last section by considering modified scalar curvature given in (1.6) we obtain some optimal bounds in terms of the mean curvature, scalar curvature, and Yamabe number.

2. Submanifold Dirac operator

Consider $(n+m)-$dimensional compact Riemannian *Spin*− manifold N with its $n-$dimensional submanifold M equipped with Spin − structure and reduced Riemanian metric as described in [10]. Accordingly, since the spinor bundle constructed on N and denoted by $\mathbb{S}$ is globally defined on M, we denoted the spinor bundle defined on M by $\mathbb{S}$. On the Riemannian Spin − manifold N and on its submanifold M endowed with induced Riemannian metric, one can construct two Levi − Civita connection denoted by $\widetilde{\nabla}$ and ∇. The corresponding spinorial Levi − Civita connections are constructed by lifting $\widetilde{\nabla}$ and ∇ to the spinorial bundle $\mathbb{S}$ and denoted by the same symbol, respectively. Also, on the spinor bundle $\mathbb{S}$, the following well − known positive definite hermitian metric can be defined as

$$\big(v\cdot\Phi, v\cdot\Psi\big) = |v|^2\big(\Phi,\Psi\big), \tag{2.1}$$

where $'\cdot'$ denotes the Clifford multiplication and $v\in\Gamma(T^*N)$, and $\Phi,\Psi\in\Gamma(\mathbb{S})$ [12]. According to the local coordinates, the Dirac operators are defined on the manifold N and its submanifold M as $\widetilde{D}\Phi = e_i\cdot\widetilde{\nabla}_i\Phi$ and $D\Phi = e_i\cdot\nabla_i\Phi$, respectively. With respect to the metric $(\ ,\ )$, $\widetilde{\nabla},\nabla$ are compatible and D is formally self − adjoint. Moreover, this metric is globally defined along M [10].

The identities used in this paper are given as follows without any need for proof since they are identities mentioned in [10]. Let us have a point $x\in M$ and an orthonormal basis $\{e_\tau\}$ of T_xN with $\{e_A\}$ normal and $\{e_i\}$ tangent to M such that $(\nabla_i e_j) = 0$. It will be used in the upcoming proofs throughout in the whole paper indices are ranged as follows:

$$1\le\tau,\beta,\gamma,\eta\le n+m; \qquad 1\le i,j,k,l\le n,$$
$$n+1\le A,B\le n+m. \tag{2.2}$$

Let e^τ be the coframe at point x. Then the relation between $\widetilde{\nabla}$ and ∇ is given by

$$\widetilde{\nabla}_i = \nabla_i + \frac{1}{2}h_{Aij}e^A\cdot e^j\cdot, \tag{2.3}$$

where $h_{Aij} = h_{Aji} = \big(\widetilde{\nabla}_i e_A, e_j\big)$ is the second fundemantal form. Let $\widetilde{R}_{\tau\beta\gamma\eta}$ and R_{ijkl} and R_{ijAB} be the curvature tensor of N, M and the normal bundle of M, respectively. Recall that $R_{ijAB}=0$ the normal bundle of M is flat. With respect to (2.3), $\widetilde{D}$ is given as

$$\widetilde{D} = D + \frac{1}{2}H_A e^A\cdot, \tag{2.4}$$

where $H_A = \sum_{i=1}^{n} h_{Aii}$ is denoted the component of the mean curvature of M. For any spinor field $\Phi \in \Gamma(\mathbb{S})\big|_M$, the Schrödinger – Lichnerowicz – type formula is described as follows

$$\int_M |D\Phi|^2 \quad = \quad \int_M \Big(|\nabla\Phi|^2 + \frac{1}{4}(R + R_{\perp,\Phi}\Big)|\Phi|^2. \tag{2.5}$$

Considering the operator $\omega_\perp$ defined on $\mathbb{S}$ by,

$$\omega_\perp := (-1)^{\left(m(m-1)/4\right)} e^{A_1} \cdot e^{A_2} ... e^{A_m} \cdot, \tag{2.6}$$

where $\{e^{A_i}\}$ is an orthonormal coframe and D_H satisfies the following relation

$$D_H = \omega_\perp \cdot \widetilde{D}. \tag{2.7}$$

Recall that D_H is formally self – adjoint with respect to the metric $(\ ,\ )$ and satisfies $\widetilde{D}^*\widetilde{D} = D_H^2$, where $\widetilde{D}^* = D - \frac{1}{2}H_A e^A \cdot$ [10].

In the following some optimal eigenvalue estimates are obtained for appropriately constructed modified spinorial Levi – Civita connections.

Theorem 2.1 *On a compact Riemannian Spin – submanifold $M \subset N$ of dimension $n \geq 2$ whose normal bundle is also Spin, any eigenvalue λ_H of the Dirac operator D_H to which attached an eigenspinor Φ satisfies*

$$\lambda_H^2 \geq \quad \frac{1}{4} \sup_{\beta,\kappa_{n,\beta,\Phi}} \ \inf_M \left(\frac{R + R_{\perp,\Phi} + 4\kappa_{n,\beta,\Phi}|trQ_\Phi|^2}{n\beta^2 - 2\beta + 1} - \frac{(n-1)\|H\|^2}{(1-n\beta)^2} \right) \tag{2.8}$$

where β real function, $\beta = \frac{1}{n}$ if $H_A \neq 0$ for some A and $\kappa_{n,\beta,\Phi} = \frac{(n\beta-1)^2}{n}$.

Proof For any real functions $\tau, \beta \neq \frac{1}{n}, \gamma$ and spinor fields $\Phi \in \Gamma(\mathbb{S})$, consider the following modified spinorial Levi – Civita connection

$$\nabla_i^{\tau,\beta,\gamma}\Phi = \nabla_i\Phi + \tau H_A e^A \cdot e^i \cdot \Phi + \beta e^i \cdot \widetilde{D}\Phi + \gamma trQ_\Phi e^i \cdot \Phi. \tag{2.9}$$

For any $1 \leq i \leq n$, the norm square of (2.9) is

$$\begin{aligned} |\nabla_i^{\tau,\beta,\gamma}\Phi|^2 \quad = \quad & |\nabla_i\Phi|^2 + 2\tau H_A Re\big(e^i \cdot \nabla_i\Phi, e^A \cdot \Phi\big) - 2\beta Re\big(e^i \cdot \nabla_i\Phi, \widetilde{D}\Phi\big) \\ & -2trQ_\Phi \gamma Re\big(e^i \cdot \nabla_i\Phi, \Phi\big) + \tau^2\|H\|^2|\Phi|^2 - 2\tau\beta H_A Re\big(e^A \cdot \Phi, \widetilde{D}\Phi\big) \\ & -2\tau\gamma H_A trQ_\Phi Re\big(e^A \cdot \Phi, \Phi\big) + \beta^2|\widetilde{D}\Phi|^2 + 2\beta\gamma trQ_\Phi Re\big(\widetilde{D}\Phi, \Phi\big) \\ & +\gamma^2|trQ_\Phi|^2|\Phi|^2. \end{aligned} \tag{2.10}$$

Summing over i and using the definition $\widetilde{D}$, we get

$$\begin{aligned} |\nabla^{\tau,\beta,\gamma}\Phi|^2 \quad = \quad & |\nabla\Phi|^2 + \big(2\tau + \beta - 2n\tau\beta\big)H_A Re\big(e^A \cdot \Phi, D\Phi\big) + \big(\beta^2 n - 2\beta\big)|\widetilde{D}\Phi|^2 \\ & +\Big(\frac{\beta}{2} + \tau^2 n - \tau\beta n\Big)\|H\|^2|\Phi|^2 + \Big(\gamma^2 n + 2\gamma\beta n - 2\gamma\Big)|trQ_\Phi|^2 \\ & +\big(\beta\gamma n - 2\tau\gamma n\big)H_A trQ_\Phi Re\big(e^A \cdot \Phi, \Phi\big). \end{aligned} \tag{2.11}$$

Integrating the above equation over M and using (2.5), we have

$$\int_M |\widetilde{D}\Phi|^2 v_g = \int_M \Big(|\nabla^{\tau,\beta,\gamma}\Phi|^2 + \frac{1}{4}\Big(R + R_{\perp,\Phi}\Big)|\Phi|^2 + \big(2n\tau\beta - 2\tau - \beta + 1\big)H_A Re\big(D\Phi, e^A \cdot \Phi\big)$$
$$-\big(n\beta^2 - 2\beta\big)|\widetilde{D}\Phi|^2 + \Big(n\tau\beta - \tau^2 n - \frac{\beta}{n} + \frac{1}{4}\Big)\|H\|^2|\Phi|^2 + \big(2\gamma - \gamma^2 n - 2\gamma\beta n\big)|trQ_\Phi|^2|\Phi|^2$$
$$+(2\tau\gamma n - \beta\gamma n)H_A trQ_\Phi Re\big(e^A \cdot \Phi, \Phi\big)\Big) v_g. \tag{2.12}$$

Taking $\tau = \frac{1-\beta}{2(1-n\beta)}$, $\gamma = \frac{1-n\beta}{n}$ and using the fact that $Re\big(e^A \cdot \Phi, \Phi\big) = 0$, we obtain

$$\int_M (1 + n\beta^2 - 2\beta)|\widetilde{D}\Phi|^2 v_g = \int_M \Big(|\nabla^{\tau,\beta,\gamma}\Phi|^2 + \frac{1}{4}\Big(R + R_{\perp,\Phi}\Big)|\Phi|^2$$
$$-\Big(\frac{(1 + n\beta^2 - 2\beta)(1-n)}{4(1-n\beta)^2}\Big)\|H\|^2|\Phi|^2 + \frac{(1-n\beta)^2}{n}|trQ_\Phi|^2|\Phi|^2\Big) v_g \tag{2.13}$$

Since $\int_M \Big|\nabla^{\tau,\beta,\gamma}\Phi|^2 \geq 0$, one can obtain the desired result given in (2.8). □

If λ_H^2 takes its minimum value, then the following theorem is obtained.

Theorem 2.2 *On a compact Riemannian Spin− submanifold $M \subset N$ of dimension $n \geq 2$ whose normal bundle is also Spin, if λ_H^2 takes its minimum value and the normal bundle is flat, then the manifold $(M, g_N\big|_M)$ is an Einstein. Accordingly, the following holds:*

$$R = n\big(n-1\big)\|\widetilde{H}\|^2 + 4n(n-1)|tr\widetilde{Q}_\Phi|^2,$$
$$\lambda_H^2 = \frac{(n-1)^2}{4(1-n\beta_0)^4}|H|^2 + \frac{n^3\beta_0^2 - 2\beta_0 n^2 + 2n - 1}{\beta_0^2 n^3 - 2\beta_0 n^2 + n^2}|trQ_\Phi|^2,$$
$$R_{ij} = (n-1)\Big(\frac{(n\beta_0^2 - 2\beta_0 + 1)^2}{(1-n\beta_0)^4}\|H\|^2 + \frac{4}{n^2}|trQ_\Psi|^2\Big)\delta_{ij}, \tag{2.14}$$

where β_0 is chosen such that the right hand side of (2.8) takes its maximum.

Proof In the equality case of (2.8), $\nabla_i^{\tau,\beta,\gamma}\Phi \equiv 0$. This means

$$\nabla_i\Phi = -\frac{\widetilde{H}_A}{2}e^A \cdot e^i \cdot \Phi - tr\widetilde{Q}_\Phi e^i \cdot \Phi, \qquad D\Phi = -\frac{n}{2}\widetilde{H}_A e^A \cdot \Phi + ntr\widetilde{Q}_\Phi\Phi \tag{2.15}$$

where $\widetilde{H}_A = \frac{n\beta_0^2-2\beta_0+1}{(1-n\beta_0)^2} H_A$ and $tr\widetilde{Q}_\Phi = \frac{1}{n} trQ_\Phi$. Accordingly $d|\Phi|^2 = 0$. Also, if the normal bundle is flat, then

$$\begin{aligned}\sum_{k,l}\frac{1}{4}R_{ijkl}e^k\cdot e^l\cdot\Phi &= \Big(\nabla_j\nabla_i-\nabla_i\nabla_j\Big)\Phi\\ &= \nabla_j\big(-\frac{1}{2}\widetilde{H}_Ae^A\cdot e^i - tr\widetilde{Q}_\Phi e^i\big)\cdot\Phi-\nabla_i\big(-\frac{1}{2}\widetilde{H}_Ae^A\cdot e^j-tr\widetilde{Q}_\Phi e^j\big)\cdot\Phi\\ &= \frac{1}{2}\Big(\nabla_i(\widetilde{H}_Ae^A)\cdot e^j-\nabla_j(\widetilde{H}_Ae^A)\cdot e^i\Big)\cdot\Phi+\frac{1}{2}\widetilde{H}_Ae^A\cdot\Big(e^j\cdot\nabla_i\Phi-e^i\cdot\nabla_j\Phi\Big)\\ &\quad+\Big(\nabla_i(tr\widetilde{Q}_\Phi)e^j-\nabla_j(tr\widetilde{Q}_\Phi)e^i\Big)\cdot\Phi+tr\widetilde{Q}_\Phi\Big(e^j\cdot\nabla_i\Phi-e^i\cdot\nabla_j\Phi\Big)\end{aligned} \tag{2.16}$$

Considering Clifford multiplication of the above equality with e^j, yields

$$\begin{aligned}\sum_k\frac{1}{2}R_{ik}e^k\cdot\Phi &= \sum_{j,k,l}\frac{1}{4}R_{ijkl}e^j\cdot e^k\cdot e^l\cdot\Phi\\ &= (\frac{n-2}{2})\nabla_i(\widetilde{H}_Ae^A)\cdot\Phi+\frac{1}{2}e^i\cdot D(\widetilde{H}_Ae^A)\cdot\Phi+(\frac{n-2}{2})\widetilde{H}_Ae^A\cdot\nabla_i\Phi\\ &\quad+\frac{n}{4}\|\widetilde{H}\|^2e^i\cdot\Phi+ntr\widetilde{Q}_\Phi^2e^i\cdot\Phi+(2-n)tr\widetilde{Q}_\Phi\nabla_i\Phi+(2-n)\nabla_i(tr\widetilde{Q}_\Phi)\Phi\\ &\quad+ei\cdot D(tr\widetilde{Q}_\Phi)\cdot\Phi.\end{aligned} \tag{2.17}$$

Again, Clifford multiplication of (2.17) with e^i, one has

$$\begin{aligned}-\frac{1}{2}R\Phi &= \sum_{i,k}\frac{1}{2}R_{ik}e^i\cdot e^k\cdot\Phi\\ &= -(n-1)D\Big(\widetilde{H}_Ae^A\Big)\cdot\Phi-n\Big(\frac{n-1}{2}\Big)\|\widetilde{H}\|^2\Psi-2(n-1)D\Big(tr\widetilde{Q}_\Phi\Big)\cdot\Phi-2n(n-1)|tr\widetilde{Q}_\Phi|^2\Phi.\end{aligned} \tag{2.18}$$

Take inner product of the equality (2.18) with Φ and compare its real and imaginary parts to obtain

$$R \;=\; n(n-1)\|\widetilde{H}\|^2+4n(n-1)|tr\widetilde{Q}_\Phi|^2 \text{ and } D\Big(\widetilde{H}_Ae^A\Big)+2D\Big(tr\widetilde{Q}_\Phi\Big)=0. \tag{2.19}$$

Since $\nabla^{\tau,\beta,\gamma}\equiv 0$, one gets

$$\lambda_H^2=\frac{1}{4}\underset{\beta}{sup}\;\underset{M}{inf}\Bigg(\frac{R_{\perp,\Phi}+R}{n\beta_0^2-2\beta_0+1}-\frac{(n-1)\|H\|^2}{(1-n\beta_0)^2}+\frac{(n\beta_0-1)^2}{(n^2\beta_0-2n\beta_0+n)}|trQ_\Phi|^2\Bigg). \tag{2.20}$$

This and (2.19) imply that, $\widetilde{H}$ and $tr\widetilde{Q}_\Phi$ are constant. Therefore

$$\begin{aligned}R_{ij} &= (n-1)\Big(|\widetilde{H}|^2+4|tr\widetilde{Q}_\Psi|^2\Big)\delta_{ij}\\ &= (n-1)\Big(\frac{(n\beta_0^2-2\beta_0+1)^2)}{(1-n\beta_0)^4}\|H\|+\frac{4}{n^2}|trQ_\Psi|^2\Big)\delta_{ij}\end{aligned} \tag{2.21}$$

and

$$\lambda_H^2 = \frac{(n-1)^2}{4(1-n\beta_0)^4}\|H\|^2 + \frac{n^3\beta_0^2 - 2\beta_0 n^2 + 2n - 1}{\beta_0^2 n^3 - 2\beta_0 n^2 + n^2}|trQ_\Phi|^2. \tag{2.22}$$

□

In (2.24), by taking

$$(1-n\beta)^2 = \frac{(n-1)\|H\|^2}{\sqrt{\frac{n}{n-1}(R_{\perp,\Phi}+R)} - \|H\|} \tag{2.23}$$

and $\kappa_{n,\beta,\Phi} = \frac{(1-n\beta)^2}{n}$ in (2.8) as in [10], we get the following estimates under the condition $n(R+R_{\perp,\Phi}+4\kappa_{n,\beta,\Phi}|trQ_\Phi|^2) > (n-1)\|H\|^2$,

$$\lambda_H^2 \geq \frac{1}{4}\underset{\kappa_{n,\beta,\Phi}}{sup}\ \underset{M}{inf}\left(\sqrt{\frac{n}{n-1}(R_{\perp,\Phi}+R+4\kappa_{n,\beta,\Phi}|trQ_\Phi|^2)} - \|H\|\right)^2. \tag{2.24}$$

3. Conformal lower bounds of eigenvalues

Let (N, g_N) be an $(n+m)-$dimensional compact Riemannian Spin$-$manifold. Then, the isometry G_u between $SO_{g_N}N$ and $SO_{\overline{g}_N}N$ can be given by the conformal change of metric $\overline{g}_N = e^{2u}g_N$ for any real function u on N. Accordingly, G_u induces an isometry between the $Spin_n$ principal bundles, $Spin_{g_N}$ and $Spin_{\overline{g}_N}$. Also, it induces an isometry between the corresponding spinor bundles $\mathbb{S}$ and $\overline{\mathbb{S}}$ of N. Moreover, the natural Hermitian metrics $(\ ,\ )_{g_N}$ and $(\ ,\ )_{\overline{g}_N}$ defined on $\mathbb{S}$ and $\overline{\mathbb{S}}$, respectively, satisfies

$$(\Psi, \Phi)_{g_N} = (\overline{\Psi}, \overline{\Phi})_{\overline{g}_N}, \tag{3.1}$$

where $\Psi, \Phi \in \Gamma(\mathbb{S})$ and $\overline{\Psi} = G_u\Psi, \overline{\Phi} = G_u\Phi \in \Gamma(\overline{\mathbb{S}})$. As well as, the Clifford multiplication on $\Gamma(\overline{\mathbb{S}})$ denoted by $\bar{\cdot}$ and given as follows

$$\overline{e^i}\,\bar{\cdot}\,\overline{\Psi} = \overline{e^i \cdot \Psi}. \tag{3.2}$$

In the following, we give some relations with respect to the conformal change of the metric by using the same arguments given in [11]. Let Ω the regular class of N, given as

$$\Omega = \{u \in C^\infty(N, \mathbb{R}),\ du(e_A)\big|_M = 0, \text{for all A}\}. \tag{3.3}$$

Now, we give some identities which are obtained in [11] to extend our estimates in terms of the Yamabe number and area of M.

Let $\overline{g} = e^{2u}\big|_M g$ be a regular conformal metric. Then the following identities are held [10]:

$$\overline{D}_H(e^{-\left((n-1)/2\right)u}\overline{\Phi}) = e^{-\left((n+1)/2\right)u}\overline{D_H\Phi} \tag{3.4}$$

$$\overline{H}_{ijA} = e^{-u}\big(H_{ijA} + du(e_A)\big). \tag{3.5}$$

Also, with respect to $\overline{g}_N = e^{2u} g_N$, one has $\overline{R}_{ijAB} = e^{-2u} R_{ijAB}$ and $\overline{R}_{\perp,\Psi} = e^{-2u} R_{\perp,\Phi}$ for $\Psi = e^{-((n-1)/2)u}\Phi$. With respect to $\overline{g}_N = e^{2u} g_N$, applying $\overline{\Phi}$ to the equation (2.13), we get

$$\begin{aligned}\int_M (1+n\beta^2-2\beta)|\overline{D}_H\ \overline{\Phi}|^2 v_g &= \int_M \Big(|\overline{\nabla}^{\tau,\beta,\gamma}\ \overline{\Phi}|^2 + \frac{1}{4}(R_{\perp,\overline{\Phi}+\overline{R}})|\overline{\Phi}|^2 \\ &\quad -(\frac{(1+n\beta^2-2\beta)(1-n)}{4(1-n\beta)^2})\|\overline{H}\|^2|\Phi|^2 + \frac{(1-n\beta)^2}{n}|trQ_{\overline{\Phi}}|^2|\overline{\Phi}|^2\Big)v_{\overline{g}}.\end{aligned} \tag{3.6}$$

Accordingly, the following corollaries are obtained.

Corollary 3.1 *Under the same conditions as in Theorem 2.1, for some real functions β, u on N any eigenvalue λ_H of D_H satisfies*

$$\lambda_H^2 \geq \ \tfrac{1}{4} \sup_{u,\beta,\kappa_{n,\beta,\Phi}} \inf_{M_\Phi}\left(\tfrac{(\overline{R}_{\perp,\Phi}+\overline{R})e^{2u}+4\kappa_{n,\beta,\Phi}|trQ_\Phi|^2}{n\beta^2-2\beta+1} - \tfrac{(n-1)\|H\|^2}{(1-n\beta)^2}\right), \tag{3.7}$$

where $\overline{R}$ is denoted the scalar curvature of M associated to a regular conformal metric $\overline{g} = e^{2u}\big|_M g$.

Corollary 3.2 *Under the same conditions as in Theorem 2.1, if $n \geq 2$ and the scalar curvature $n\big((\overline{R}_{\perp,\Phi}+\overline{R})e^{2u} + 4\kappa_{n,\beta,\Phi}|trQ_\Phi|^2\big) - (n-1)\|H\|^2 > 0$ for some regular conformal metric $\overline{g} = e^{2u} g$ and real function τ on N. Let λ_H be any eigenvalue of the Dirac operator D_H to which is attached an eigenspinor Ψ. Then for $\Phi = e^{-\left(\frac{(n-1)}{2}u\right)}\Psi$, one has*

$$\lambda_H^2 \geq \ \tfrac{1}{4} \sup_{u,\beta,\kappa_{n,\beta,\Phi}} \inf_{M_\Phi}\left(\sqrt{\tfrac{n}{n-1}\big((\overline{R}_{\perp,\Phi}+\overline{R})e^{2u}+4\kappa_{n,\beta,\Phi}|trQ_\Phi|^2\big)} - \|H\|\right)^2 \tag{3.8}$$

where $\overline{R}$ is the scalar curvature of M associated to a regular conformal metric $\overline{g} = e^{2u}\big|_M g$, for some real functions a, u on N.

Using the fact that $\mu_1 = \sup_u \inf_M \big(\overline{R}e^{2u}\big)$ we obtain the following corollary.

Corollary 3.3 *Under the same conditions as in Theorem 2.1, if $n \geq 3$ and $n\big(\mu_1 + R_{\perp,\Phi}) + 4\kappa_{n,\beta,\Phi}|trQ_\Psi|^2\big) - (n-1)\|H\|^2 > 0$, then*

$$\lambda_H^2 \geq \ \tfrac{1}{4} \sup_{\kappa_{n,\beta,\Phi}} \inf_{M}\left(\sqrt{\tfrac{n}{n-1}\big(\mu_1 + R_{\perp,\Phi}+4\kappa_{n,\beta,\Phi}|trQ_\Phi|^2\big)} - \|H\|\right)^2 \tag{3.9}$$

where μ_1 is the first eigenvalue of the Yamabe operator.

As we know the relation between the Yamabe number denoted by μ and μ_1 is $\mu_1 \geq \frac{\mu}{Vol(M,g)^{2/n}}$. Using this relation we obtain the following inequality.

Corollary 3.4 *Under the same conditions as in Theorem 2.1, if $n \geq 3$ and $n\big(\mu + R_{\perp,\Phi} + 4\kappa_{n,\beta,\Phi}|trQ_\Phi|^2\big) - (n-1)Vol(M^n,g)^{2/n}\|H\|^2 > 0$, then any eigenvalue λ_H of D_H satisfies*

$$\lambda_H^2 \quad \geq \quad \frac{1}{4} \sup_{\kappa_{n,\beta,\Phi}} \inf_{M_\Phi}\left(\sqrt{\frac{n}{(n-1)Vol(M^n,g)^{2/n}}\big(\mu_1 + R_{\perp,\Phi}+4\kappa_{n,\beta,\Phi}|trQ_\Phi|^2\big)} - \|H\|\right)^2 \tag{3.10}$$

where μ is the Yamabe number.

Again by considering the fact that $\underset{u}{sup}\ \underset{M}{inf}\left(\overline{R}e^{2u}\right)=\frac{8\Pi}{Area(M)}$, we obtain the following corollary.

Corollary 3.5 *Under the same conditions as in Theorem 2.1, if M is compact surface of genus zero and*

$$\frac{16\pi}{Area(M)}+2R_{\perp,\Phi}+8\kappa_{n,\beta,\Phi}|trQ_\Phi|^2-\|H\|^2>0, \tag{3.11}$$

then any eigenvalue λ_H of D_H satisfies

$$\lambda_H^2 \geq \frac{1}{4}\underset{\kappa_{n,\beta,\Phi}}{sup}\ \underset{M_\Phi}{inf}\Big(\sqrt{\frac{16\pi}{Area(M)}+2R_{\perp,\Phi}+8\kappa_{n,\beta,\Phi}|trQ_\Phi|^2}-\|H\|\Big)^2, \tag{3.12}$$

where $Area(M)$ is denoted the area of M.

4. Generalized conformal lower bounds

In this section, we optimized the eigenvalue estimation obtained in [10] by modifying the spinorial Levi−Civita connection in terms of the Energy−Momentum tensor Q_Φ and its trace trQ_Φ. Using the Dirac operator defined on M [10] as follows

$$D\Phi=-\lambda_H\epsilon\omega_\perp\cdot\Phi-\frac{H_A}{2}e^A\cdot\Phi, \tag{4.1}$$

where $\epsilon=(-1)^{m-1}$, one gets

$$|D\Phi|^2 = \lambda_H^2|\Phi|^2+\frac{\|H\|^2}{4}|\Phi|^2+\lambda_H H_A Re(\epsilon\omega_\perp\cdot\Phi,e^A\cdot\Phi). \tag{4.2}$$

Integrating over M, then combining (2.5) with (4.2), one has

$$\int_M|\nabla\Phi|^2v_g = \int_M\Big(\lambda_H^2|\Phi|^2+\frac{\|H\|^2}{4}|\Phi|^2+\lambda_H H_A Re(\epsilon\omega_\perp\cdot\Phi,e^A\cdot\Phi) -\Big(\frac{R+R_{\perp,\Phi}}{4}\Big)|\Phi|^2\Big)v_g. \tag{4.3}$$

The following estimates are the extension of the results obtained in [1], in terms of $R_{\tau,u,\Phi}$.

Theorem 4.1 *Let $M\subset N$ be an $n\geq 2$ dimensional compact Riemannian Spin− submanifold with its Spin normal bundle. Then any eigenvalue λ_H of D_H to which attached an eigenspinor Φ satisfies*

$$\lambda_H^2\geq \frac{1}{4}\underset{\tau,q,u,\kappa_{n,q,\Phi}}{sup}\ \underset{M_\Phi}{inf}\left(\frac{R_{\tau,u,\Phi}+4\kappa_{n,q,\Phi}|trQ_\Phi|^2}{nq^2-2q+1}-\frac{(n-1)\|H\|^2}{(1-nq)^2}\right) \tag{4.4}$$

where $q,\kappa_{n,q,\Phi}=\frac{(1-nq)^2}{n}$ are real functions.

Proof For real functions τ,β,γ,q and p, we define the following modified spinorial Levi−Civita connection

$$\nabla_i^M\Phi = \nabla_i\Phi+\tau\nabla_i u\Phi+\beta\nabla_j ue^i\cdot e^j\cdot\Phi-\frac{\gamma}{2}H_Ae^i\cdot e^A\cdot\Phi-q\lambda_H\epsilon\omega_\perp\cdot\Phi +ptrQ_\Phi e^i\cdot\Phi. \tag{4.5}$$

One can easily compute

$$\begin{aligned}
|\nabla_i^M \Phi|^2 \quad = \quad & |\nabla_i \Phi|^2 + 2\tau Re\big(\nabla_i \Phi, \nabla_i u \Phi\big) + 2\beta Re\big(\nabla_i \Phi, \nabla_i u e^i \cdot e^j \cdot \Phi\big) \\
& -\gamma H_A Re\big(\nabla_i \Phi, e^i \cdot e^A \cdot \Phi\big) - 2q\lambda_H Re\big(\nabla_i \Phi, \epsilon e^i \cdot \omega \cdot \Phi\big) \\
& +2ptrQ_\Phi Re\big(\nabla_i \Phi, e^i \cdot \Phi\big) + \tau^2 |\nabla_i u|^2 |\Phi|^2 + \beta^2 |du|^2 |\Phi|^2 \\
& +2\tau\beta Re\big(\nabla_i u\Phi, \nabla_j u e^i \cdot e^j \cdot \Phi\big) - 2\tau\gamma Re\big(\nabla_i u\Phi, e^i \cdot e^A \cdot \Phi\big) \\
& -2\tau q\lambda_H Re\big(\nabla_i u\Phi, \epsilon e^i \cdot \omega_\perp \cdot \Phi\big) + 2\tau ptrQ_\Phi Re\big(\nabla_i u\Phi, e^i \cdot \Phi\big) \\
& -\beta\gamma H_A Re\big(\nabla_j u e^j \cdot \Phi, e^A \cdot \Phi\big) - 2\beta q\lambda_H Re\big(\nabla_j u e^j \cdot \Phi, \epsilon\omega_\perp \cdot \Phi\big) \\
& +2\beta ptrQ_\Phi Re\big(\nabla_j u e^j \cdot \Phi, \Phi\big) + \frac{\gamma^2}{4}\|H\|^2 |\Phi|^2 + \gamma H_A q\lambda_H Re(e^A \cdot \Phi, \epsilon\omega_\perp \cdot \Phi\big) \\
& -\gamma H_A ptrQ_\Phi Re(e^A \cdot \Phi, \Phi\big) + q^2\lambda_H^2 |\Phi|^2 + p^2 |trQ_\Phi|^2 |\Phi|^2.
\end{aligned} \tag{4.6}$$

Summing over i, yields

$$\begin{aligned}
|\nabla^M \Phi|^2 \quad = \quad & |\nabla \Phi|^2 + 2\tau \nabla_i u Re\big(\nabla_i \Phi, \Phi\big) + \big(\gamma qn - q - \gamma\big)\lambda_H H_A Re\big(e^A \Phi, \epsilon\omega_\perp \cdot \Phi\big) \\
& +\big(\frac{n\gamma^2 - 2\gamma}{4}\big)\|H\|^2 |\Phi|^2 + \big(nq^2 - 2q\big)\lambda_H^2 |\Phi|^2 + \big(np^2 - 2p)|trQ_\Phi|^2 |\Phi|^2 \\
& -2qpn\lambda_H trQ_\Phi Re\big(\epsilon\omega_\perp \cdot \Phi, \Phi\big) + \big(\tau^2 - 2\tau\beta + n\beta^2\big)|du|^2 |\Phi|^2 \\
& +\big(\tau\gamma - \beta\gamma n\big) H_A Re\big(du \cdot \Phi, e^A \cdot \Phi\big) + \big(2\tau q - 2\beta qn\big) Re\big(du \cdot \Phi, \epsilon\omega_\perp \cdot \Phi\big).
\end{aligned} \tag{4.7}$$

Taking $\beta = \frac{\tau}{n}$, one gets

$$\begin{aligned}
|\nabla^M \Phi|^2 \quad = \quad & |\nabla \Phi|^2 + 2\tau \nabla_i u Re\big(\nabla_i \Phi, \Phi\big) + \big(\gamma qn - q - \gamma\big)\lambda_H H_A Re\big(e^A \Phi, \epsilon\omega_\perp \cdot \Phi\big) \\
& +\big(\frac{n\gamma^2 - 2\gamma}{4}\big)\|H\|^2 |\Phi|^2 + \big(nq^2 - 2q\big)\lambda_H^2 |\Phi|^2 + \big(np^2 - 2p)|trQ_\Phi|^2 |\Phi|^2 \\
& -2qpn\lambda_H trQ_\Phi Re\big(\epsilon\omega_\perp \cdot \Phi, \Phi\big) + \big(\tau^2 - \frac{2\tau^2}{n} + \frac{n\tau^2}{n}\big)|du|^2 |\Phi|^2.
\end{aligned} \tag{4.8}$$

Integrating over M and plugging (4.3) into (4.8), we obtain

$$\begin{aligned}
\int_M |\nabla^M \Phi|^2 \quad = \quad & \int_M \Big(\lambda_H^2 |\Phi|^2 + \frac{\|H\|^2}{4}|\Phi|^2 + \lambda_H H_A Re\big(\epsilon\omega_\perp \cdot \Phi, e^A \cdot \Phi\big) - \big(\frac{R + R_\perp}{4}\big)|\Phi|^2 \\
& +\tau\Delta u|\Phi|^2 - \nabla a \nabla u|\Phi|^2 + \big(1 - \frac{1}{n}\big)\tau^2 |du|^2 |\Phi|^2 + \big(nq^2 - 2q\big)\lambda_H^2 |\Phi|^2 \\
& +\big(\gamma qn - q - \gamma\big)\lambda_H H_A Re\big(e^A \Phi, \epsilon\omega_\perp \cdot \Phi\big) + \big(\frac{n\gamma^2 - 2\gamma}{4}\big)\|H\|^2 |\Phi|^2 \\
& +\big(np^2 - 2p)|trQ_\Phi|^2 |\Phi|^2 - 2qpn\lambda_H trQ_\Phi Re\big(\epsilon\omega_\perp \cdot \Phi, \Phi\big)\Big).
\end{aligned}$$

Considering the modified scalar curvature (1.6) and using the fact that $trQ_\Psi = -\lambda_H Re\big(\epsilon\omega_\perp \cdot \Phi, \frac{\Phi}{|\Phi|^2}\big)$ with $\gamma = \frac{q-1}{nq-1}$, one gets

$$\int_M \lambda_H^2|\Phi|^2 \quad \geq \quad \int_M \left(\frac{nR_{\tau,u,\Phi}+4(1-nq)^2|trQ_\Psi|^2}{n\big(nq^2-2q+1\big)} - \frac{(n-1)|H|^2}{(1-nq)^2}\right). \tag{4.9}$$

□

In the following corollaries, we obtain an estimate with respect to a regular conformal metric $\overline{g}_N = e^{2u}g_N$. Under the conformal metric, the modified spinorial Levi − Civita connection is written as:

$$\begin{aligned}\overline{\nabla}^M_{\overline{e_i}}\ \overline{\Phi} \quad &= \quad \overline{\nabla}_{\overline{e_i}}\ \overline{\Phi} + \tau e^{-u}\nabla_i u\overline{\Phi} + \beta e^{-u}\nabla_j u\overline{e^i}\ \overline{\cdot}\ \overline{e^j}\ \overline{\cdot}\ \overline{\Phi} - \frac{\gamma}{2}\overline{H}_A\overline{e^i}\ \overline{\cdot}\ \overline{e^A}\ \overline{\cdot}\ \overline{\Phi} \\ &-q\epsilon\ \overline{\lambda}_H\overline{e^i}\ \overline{\cdot}\ \overline{\omega}_\perp\ \overline{\cdot}\overline{\Phi} + ptrQ_{\overline{\Phi}}\ \overline{e^i}\ \overline{\cdot}\ \overline{\Phi}.\end{aligned} \tag{4.10}$$

Here $\overline{H}_A = e^{-u}H_A$. Also, the scalar curvature defined on the manifold N with respect to the conformal change of the metric is expressed as follows:

$$\begin{aligned}\hat{R}_{\tau,u,\Phi} \quad &= \quad R + R_{\perp,\Phi} + 4(\frac{n-1}{2}-\tau)\Delta u + 4\nabla\tau\nabla u - \Big((n-1)(n-2)+4(2-n)\tau \\ &+4\big(1-\frac{1}{n}\big)\tau^2\Big)|du|^2.\end{aligned} \tag{4.11}$$

Using the spinorial modified Levi − Civita connection $\overline{\nabla}^M_{\overline{e_i}}\ \overline{\Phi}$ and the similar procedure as in Theorem 4.1, we obtain the following inequality

$$\int_M \lambda_H^2|\Phi|^2 \quad \geq \quad \int_M \left(\frac{n\hat{R}_{\tau,u,\Phi}+4(1-nq)^2|trQ_\Psi|^2}{n\big(nq^2-2q+1\big)} - \frac{(n-1)|H|^2}{(1-nq)^2}\right). \tag{4.12}$$

As a result, the following corollary is obtained.

Corollary 4.2 *Under the same conditions as in Theorem 4.1, if $n \geq 2$, then any eigenvalue of D_H satisfies*

$$\lambda_H^2 \geq \quad \tfrac{1}{4}\sup_{\tau,q,u,\kappa_{n,q,\Phi}}\ \inf_{M_\Phi}\left(\tfrac{\hat{R}_{\tau,u,\Phi}+4\kappa_{n,q,\Phi}|trQ_\Phi|^2}{nq^2-2q+1} - \tfrac{(n-1)\|H\|^2}{(1-nq)^2}\right) \tag{4.13}$$

where $q, \kappa_{n,q,\Phi} := \frac{(1-nq)^2}{n}$.

Now we consider the following modified spinorial Levi − Civita connection constructed by Energy − Momentum tensor Q_Φ and by its trace trQ_Φ,

$$\begin{aligned}\nabla^L_i\Phi \quad &= \quad \nabla_i\Phi + \tau\nabla_i u\Phi + \beta\nabla_j ue^i\cdot e^j\cdot\Phi - \frac{\gamma}{2}H_Ae^i\cdot e^A\cdot\Phi - q\lambda_H\epsilon\cdot\omega_\perp\cdot\Phi \\ &+ptrQ_\Phi e^i\cdot\Phi + Q_{ij,\Phi}e^j\cdot\Phi.\end{aligned} \tag{4.14}$$

where τ, β, γ, q and p real functions.

Theorem 4.3 *Under the same conditions as in Theorem 4.1 if $n \geq 2$, then any eigenvalue λ_H of D_H satisfies*

$$\lambda_H^2 \geq \quad \tfrac{1}{4}\sup_{\tau,q,u,\kappa_{n,q,\Phi}}\ \inf_{M_\Phi}\left(\tfrac{R_{\tau,u,\Phi}+4(\kappa_{n,q,\Phi}|trQ_\Phi|^2+|Q_\Phi|^2)}{nq^2-2q+1} - \tfrac{(n-1)\|H\|^2}{(1-nq)^2}\right) \tag{4.15}$$

where $q, \kappa_{n,q,\Phi} = \frac{(1-nq)^2}{n}$ *are real functions.*

Proof The norm square of (4.14) is

$$\begin{aligned}
|\nabla_i^L\Phi|^2 &= |\nabla_i\Phi|^2 + 2\tau Re\big(\nabla_i\Phi, \nabla_i u\Phi\big) + 2\beta Re\big(\nabla_i\Phi, \nabla_i u e^i\cdot e^j\cdot\Phi\big)\\
&= -\gamma H_A Re\big(\nabla_i\Phi, e^i\cdot e^A\cdot\Phi\big) - 2q\lambda_H Re\big(\nabla_i\Phi, \epsilon e^i\cdot\omega\cdot\Phi\big)\\
&\quad +2ptrQ_\Phi Re\big(\nabla_i\Phi, e^i\cdot\Phi\big) + 2Re\big(\nabla_i\Phi, Q_{ij,\Phi}e^j\cdot\Phi\big) + \tau^2|\nabla_i u|^2|\Phi|^2\\
&\quad +2\tau\beta Re\big(\nabla_i u\Phi, \nabla_j u e^i\cdot e^j\cdot\Phi\big) - 2\tau\gamma Re\big(\nabla_i u\Phi, e^i\cdot e^A\cdot\Phi\big)\\
&\quad -2\tau q\lambda_H Re\big(\nabla_i u\Phi, \epsilon e^i\cdot\omega_\perp\cdot\Phi\big) + 2\tau ptrQ_\Phi Re\big(\nabla_i u\Phi, e^i\cdot\Phi\big)\\
&\quad +2\tau Re\big(\nabla_i u\Phi, Q_{ij,\Phi}e^j\cdot\Phi\big) + \beta^2|du|^2|\Phi|^2\\
&\quad -\beta\gamma H_A Re\big(\nabla_j u e^j\cdot\Phi, e^A\cdot\Phi\big) - 2\beta q\lambda_H Re\big(\nabla_j u e^j\cdot\Phi, \epsilon\omega_\perp\cdot\Phi\big)\\
&\quad +2\beta ptrQ_\Phi Re\big(\nabla_j u e^j\cdot\Phi, \Phi\big) + 2\beta Re\big(\nabla_j u e^i\cdot e^j\cdot\Phi, Q_{ij,\Phi}e^j\cdot\Phi\big)\\
&\quad +\frac{\gamma^2}{4}\|H\|^2|\Phi|^2 + \gamma H_A q\lambda_H Re(e^A\cdot\Phi, \epsilon\omega_\perp\cdot\Phi\big) - \gamma H_A ptrQ_\Phi Re(e^A\cdot\Phi, \Phi\big)\\
&\quad -\gamma H_A Re\big(e^i\cdot e^A\cdot\Phi, Q_{ij,\Phi}e^j\cdot\Phi\big) + q^2\lambda_H^2|\Phi|^2 + p^2|trQ_\Phi|^2|\Phi|^2\\
&\quad +2ptrQ_\Phi Re\big(e^i\cdot\Phi, Q_{ij,\Phi}e^j\cdot\Phi\big) + |Q_{ij,\Phi}|^2|\Phi|^2. \qquad (4.16)
\end{aligned}$$

Summing over i, yields

$$\begin{aligned}
|\nabla^L\Phi|^2 &= |\nabla\Phi|^2 + 2\tau\nabla_i u Re\big(\nabla_i\Phi, \Phi\big) + \big(\gamma qn - q - \gamma\big)\lambda_H H_A Re\big(e^A\Phi, \epsilon\omega_\perp\cdot\Phi\big)\\
&\quad +\big(\frac{n\gamma^2 - 2\gamma}{4}\big)\|H\|^2|\Phi|^2 + \big(nq^2 - 2q\big)\lambda_H^2|\Phi|^2 + np^2|trQ_\Phi|^2|\Phi|^2\\
&\quad -\big(2q + 2qpn\big)\lambda_H trQ_\Phi Re\big(\epsilon\omega_\perp\cdot\Phi, \Phi\big) - |Q|^2|\Phi|^2 + \big(\tau^2 - 2\tau\beta + n\beta^2\big)|du|^2|\Phi|^2\\
&\quad +\big(\tau\gamma - \beta\gamma n\big)H_A Re\big(du\cdot\Phi, e^A\cdot\Phi\big) + \big(2\tau q - 2\beta qn\big)Re\big(du\cdot\Phi, \epsilon\omega_\perp\cdot\Phi\big). \qquad (4.17)
\end{aligned}$$

Taking $\beta = \frac{\tau}{n}$, one gets

$$\begin{aligned}
|\nabla^L\Phi|^2 &= |\nabla\Phi|^2 + 2\tau\nabla_i u Re\big(\nabla_i\Phi, \Phi\big)\\
&\quad +\big(\gamma qn - q - \gamma\big)\lambda_H H_A Re\big(e^A\Phi, \epsilon\omega_\perp\cdot\Phi\big) + \big(\frac{n\gamma^2 - 2\gamma}{4}\big)\|H\|^2|\Phi|^2\\
&\quad +\big(nq^2 - 2q\big)\lambda_H^2|\Phi|^2 + np^2|trQ_\Phi|^2|\Phi|^2 - \big(2q + 2qpn\big)\lambda_H trQ_\Phi Re\big(\epsilon\omega_\perp\cdot\Phi, \Phi\big)\\
&\quad -|Q|^2|\Phi|^2 + \big(\tau^2 - \frac{2\tau^2}{n} + \frac{n\tau^2}{n}\big)|du|^2|\Phi|^2
\end{aligned} \qquad (4.18)$$

Integrating over M and plugging (4.2), one has

$$\begin{aligned}\int_M |\nabla^L\Phi|^2 &= \int_M \Bigg(\lambda_H^2|\Phi|^2 + \frac{\|H\|^2}{4}|\Phi|^2 + \lambda_H H_A Re\big(\epsilon\omega_\perp\cdot\Phi, e^A\cdot\Phi\big) \\ &-(\frac{R+R_\perp}{4})|\Phi|^2 + \tau\Delta u|\Phi|^2 - \nabla a\nabla u|\Phi|^2 \\ &+\big(\gamma qn - q - \gamma\big)\lambda_H H_A Re\big(e^A\Phi, \epsilon\omega_\perp\cdot\Phi\big) + (\frac{n\gamma^2-2\gamma}{4})\|H\|^2|\Phi|^2 \\ &+\big(nq^2-2q\big)\lambda_H^2|\Phi|^2 + np^2|trQ_\Phi|^2|\Phi|^2 - \big(2q+2qpn\big)\lambda_H trQ_\Phi Re\big(\epsilon\omega_\perp\cdot\Phi,\Phi\big) \\ &-|Q|^2|\Phi|^2 + \big(1-\frac{1}{n}\big)\tau^2|du|^2|\Phi|^2\Bigg).\end{aligned} \tag{4.19}$$

Using the fact that $trQ_\Phi = -\lambda_H Re\big(\epsilon\omega_\perp\cdot\Phi, \frac{\Phi}{|\Phi|^2}\big)$ with $\gamma = \frac{q-1}{nq-1}$, one gets

$$\int_M \lambda_H^2|\Phi|^2 \geq \int_M \Bigg(\frac{nR_{\tau,u,\Phi} + 4n|Q_\Phi|^2 + 4(1-nq)^2|trQ_\Phi|^2}{\big(nq^2-2q+1\big)} - \frac{(n-1)\|H\|^2}{(1-nq)^2}\Bigg). \tag{4.20}$$

□

As in the Corollary 4.2, considering the following modified spinorial Levi−Civita connection expressed with respect to the regular metric $\overline{g}_N = e^{2u}g_N$,

$$\begin{aligned}\overline{\nabla}^L_{\overline{e_i}}\overline{\Phi} &= \overline{\nabla}_{\overline{e_i}}\overline{\Phi} + \tau e^{-u}\nabla_i u\overline{\Phi} + \beta e^{-u}\nabla_j\overline{e^i}\,\overline{\cdot}\,\overline{e^j}\,\overline{\cdot}\,\overline{\Phi} - \frac{\gamma}{2}\overline{H}_A\overline{e^i}\,\overline{\cdot}\,\overline{e^A}\,\overline{\cdot}\,\overline{\Phi} \\ &-q\epsilon\,\overline{\lambda}_H\overline{e^i}\,\overline{\cdot}\,\overline{\omega}_\perp\,\overline{\cdot}\overline{\Phi} + ptrQ_{\overline{\Phi}}\,\overline{e^i}\,\overline{\cdot}\,\overline{\Phi} + Q_{\overline{e^i},\,\overline{e^j},\,\overline{\Phi}}\,\overline{e^j}\,\overline{\cdot}\,\overline{\Phi},\end{aligned} \tag{4.21}$$

and following similar procedure used in the proof of Theorem 4.3, we obtain the following inequality

$$\int_M \lambda_H^2|\Phi|^2 \geq \int_M \Bigg(\frac{n\hat{R}_{\tau,u,\Phi} + 4(1-nq)^2|trQ_\Phi|^2 + 4n|Q_\Phi|^2}{n\big(nq^2-2q+1\big)} - \frac{(n-1)\|H\|^2}{(1-nq)^2}\Bigg). \tag{4.22}$$

Then we obtain the following corollary.

Corollary 4.4 *Under the same conditions as in Theorem 4.3, if $n \geq 2$, then any eigenvalue λ_H of the Dirac operator D_H to which is attached an eigenspinor Ψ satisfies*

$$\lambda_H^2 \geq \frac{1}{4}\sup_{\tau,q,u,\kappa_{n,q,\Phi}}\inf_{M_\Phi}\Bigg(\frac{\hat{R}_{\tau,u,\Phi}+4(\kappa_{n,q,\Phi}|trQ_\Phi|^2+|Q_\Phi|^2)}{nq^2-2q+1} - \frac{(n-1)\|H\|^2}{(1-nq)^2}\Bigg). \tag{4.23}$$

The authors confirm that the present manuscript does not report on or involve the use of any animal or human data or tissue (i.e. it is “Not applicable” in this study).

so it is “Not applicable”.

study are available within the article in the Figures and Tables presented in the manuscript.

of interest. Also, they confirm that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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