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Inequalities involving general fractional integrals of p -convex functions

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Abstract: The Hermite-Hadamard type inequalities involving fractional integral operations for p -convex functions with respect to another function are studied. Then, the inequalities via Riemann-Liouville and Hadamard fractional integrals are presented specially. Using the obtained results, some inequality relations among special functions including beta and incomplete beta functions, gamma and incomplete gamma functions, and hypergeometric functions are presented.

Key words: p -Convexity, Hermite-Hadamard inequality, generalized fractional integral

1. Introduction

Convex functions are mostly characterized by inequalities, some of which involve integral inequalities attributed to Hermite-Hadamard, Ostrowski, Fejer inequalities. In the last five decades, the characterization of convex functions has been improved by using more generalized integrals, namely, the fractional integrals, which have been used in the vast area of science from engineering to medicine [31]. The leading fractional integrals may be accepted as the Riemann-Liouville and Hadamard fractional operators among them. Also, there are many like Caputo, Katugampola, conformable and general fractional operations with respect to the other functions [19, 22, 25]. Recently, the Hermite-Hadamard inequalities and different inequalities for convex functions are generalized via fractional integrals including Riemann-Liouville, Hadamard, and other types of fractional integrals [6, 7, 12, 16, 23, 26, 27, 33].

During that time, on the other side, a lot of extensions and generalizations of convex functions are introduced. B -convex, B^{-1} -convex, s -convex in different senses, and p -convex functions can be counted among the most novel ones [2–5, 10, 21, 30]. The basic properties and characterizations via inequalities are studied by different authors [11, 20, 21, 32]. In this study, we cope with p -convex functions. Different convexity types called p -convexity are seen in the literature. The notion used in this paper is given in [15] with the definition of p -convex sets. Then, the p -convex functions are defined in [30]. It has applications to the fixed point theory, optimization, etc [13–15].

The characterization of different types of convex functions via Hermite-Hadamard inequality are studied by many researchers [1, 8, 9, 28, 29, 33]. Based on these results, more generalized versions of some of these inequalities via fractionals are given [16–18, 26, 33–36]. In this study, using some results based on

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the Hermite-Hadamard inequality for p -convex functions, we set the Hermite-Hadamard type inequalities via general fractional operations p -convex functions with respect to another function. Since the used fractional integral operators are generalizations of the Riemann-Liouville and Hadamard fractional, the results including these main fractionals are also given with theorems. Furthermore, as applications, some inequality relations between special functions including beta, incomplete beta and gamma, incomplete gamma, and hypergeometric functions are presented by using the main results.

2. Preliminaries

2.1. Definitions and Notations of p -convexity

Before recalling definitions, the following notations which will be used throughout the paper should be given.

\mathbb{R}^n denotes the n -dimensional vector space, i.e. $\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i \in \{1, 2, \dots, n\}\}$.

Definition 2.1 [30] Let $A \subset \mathbb{R}^n$ and $0 < p \leq 1$. The set A is called p -convex set if $\lambda x + \mu y \in A$, for all $x, y \in A$ and $\lambda, \mu \in [0, 1]$ such that $\lambda^p + \mu^p = 1$.

All real intervals including zero or accepting it as a boundary point are p -convex.

Definition 2.2 [30] Let $A \subset \mathbb{R}^n$ be a p -convex set and $f : A \rightarrow \mathbb{R}$. If for all $x, y \in A$ and $\lambda, \mu \in [0, 1]$ such that $\lambda^p + \mu^p = 1$;

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y) \tag{2.1}$$

then f is called p -convex function.

Example 2.3 [28] Let n is a positive integer. Then $f(x) = x^{2^n}$ is a p -convex function in $(0, +\infty)$.

With some different notations, (2.1) can be expressed in the following forms. From the equation $\lambda^p + \mu^p = 1$, one has $\mu = (1 - \lambda^p)^{\frac{1}{p}}$; so (2.1) turns to

$$f(\lambda x + (1 - \lambda^p)^{\frac{1}{p}} y) \leq \lambda f(x) + (1 - \lambda^p)^{\frac{1}{p}} f(y) .$$

Also, instead of λ , one can be written that $t^{\frac{1}{p}}$, $t \in [0, 1]$, thus (2.1) converts into the below inequality

$$f(t^{\frac{1}{p}} x + (1 - t)^{\frac{1}{p}} y) \leq t^{\frac{1}{p}} f(x) + (1 - t)^{\frac{1}{p}} f(y) .$$

In [9] where the Hermite-Hadamard inequality for p -convex is obtained, the following relations are also included.

Lemma 2.4 Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a non-decreasing p -convex function. If $a, b \in [0, \infty)$ with $a < b$, then

$$f\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right) \leq f\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)\right) \tag{2.2}$$

and

$$f\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)\right) \leq \frac{1}{2^{\frac{1}{p}}}\left[(f(a) + f(b))\left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)\right]. \tag{2.3}$$

2.2. Fractional integral types

In this subsection, the definition of the fractional integrals which are used for the results of the article are given. Before the definitions, it is useful to remind the gamma and beta functions which are used further of the article, respectively:

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx \quad \text{for } \alpha > 0,$$

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \quad \text{for } \alpha, \beta > 0 .$$

Definition 2.5 [25] Let $f : [a, b] \rightarrow \mathbb{R}$ where $a, b \in \mathbb{R}$ such that $a < b$ and $f \in L_1[a, b]$. The left-sided Riemann-Liouville integral $J_{a+}^\alpha f$ and the right-sided Riemann-Liouville integral $J_{b-}^\alpha f$ of order $\alpha > 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively.

Definition 2.6 [25] Let $f : [a, b] \rightarrow \mathbb{R}$ where $a, b \in \mathbb{R}$ such that $a < b$ and $f \in L_1[a, b]$. The left-sided Hadamard fractional integral J_{a+}^α of order $\alpha > 0$ of f is defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt, \quad x > a$$

provided that the integral exists. The right-sided Hadamard fractional integral J_{b-}^α of order $\alpha > 0$ of f is defined by

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt, \quad x < b$$

provided that the integral exists.

Definition 2.7 [25] Let $f : [a, b] \rightarrow \mathbb{R}$ where $a, b \in \mathbb{R}$ such that $a < b$ and $f \in L_1[a, b]$. Let $g : [a, b] \rightarrow \mathbb{R}$ be an increasing and positive valued function on $(a, b]$, having a continuous derivative $g'(x)$ on (a, b) . The left-sided fractional integral of f with respect to the function g on $[a, b]$ of order $\alpha > 0$ is defined by

$$I_{a+;g}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t)}{[g(x) - g(t)]^{1-\alpha}} dt, \quad x > a$$

provided that the integral exists. The right-sided fractional integral of f with respect to the function g on $[a, b]$ of order $\alpha > 0$ is defined by

$$I_{b-;g}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t)}{[g(t) - g(x)]^{1-\alpha}} dt, \quad x < b$$

provided that the integral exists.

3. Main results

The new types of Hermite-Hadamard inequalities via generalized fractional integrals will be proved in this section. Additionally, because these general fractional integrals can be turned to the Riemann-Liouville and the Hadamard fractional integrals that are the most famous fractional integrals, some new inequalities will be obtained by this relation of the integrals.

Theorem 3.1 *Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be non-decreasing p -convex function where $a, b \geq 0$ such that $a < b$, $f \in L_1[0, +\infty)$ and $\alpha > 0$. Let $g : [a, b] \rightarrow \mathbb{R}$ be an increasing and positive valued function on (a, b) , having a continuous derivative $g'(x)$ on (a, b) . The following left-sided fractional inequality holds:*

$$\frac{f\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} \leq I^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right);g} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1)2^{\frac{1}{p}-1}} \left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)\right]^{\alpha} + \frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)} \int_{\frac{a+b}{2^{\frac{1}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g(u)\right]^{\alpha} du . \tag{3.1}$$

Proof Let $t \in [\frac{1}{2}, 1]$ to get positive values of the following coefficient. To prove (3.1), both side of (2.2) is multiplied by the coefficient

$$\frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)\right]^{1-\alpha}} \left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1}-(1-t)^{\frac{1}{p}-1}\right)\right)$$

then integrate over t , one has

$$\int_{\frac{1}{2}}^1 f\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right) \frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)\right]^{1-\alpha}} \left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1}-(1-t)^{\frac{1}{p}-1}\right)\right) dt$$

$$\leq \int_{\frac{1}{2}}^1 f\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right) \frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right)\right)\right]^{1-\alpha}} \left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1}-(1-t)^{\frac{1}{p}-1}\right)\right) dt .$$

After the substitution by $\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right) = u$ and division by $\Gamma(\alpha)$ to the both side of the inequality, one

has

$$\begin{aligned}
 f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) \int_{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}}^{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}} \frac{g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g(u)\right]^{1-\alpha}} du &\leq \int_{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}}^{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}} \frac{f(u)g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g(u)\right]^{1-\alpha}} du \\
 f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) \frac{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}}{\alpha} &\leq \int_{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}}^{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}} \frac{f(u)g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g(u)\right]^{1-\alpha}} du \\
 \frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) \left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} &\leq I^{\alpha}_{\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)^+;g} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right).
 \end{aligned}$$

To prove the other side of (3.1), using the same coefficient and the procedure as the first part of the proof to the inequality (2.3) the following is obtained

$$\begin{aligned}
 &\int_{\frac{1}{2}}^1 f\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)\right) \frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)\right)\right]^{1-\alpha}} \left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1} - (1-t)^{\frac{1}{p}-1}\right)\right) dt \\
 &\leq \int_{\frac{1}{2}}^1 \frac{1}{2^{\frac{1}{p}}} \left[(f(a) + f(b))\left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)\right] \frac{g'\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)\right)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g\left(\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)\right)\right]^{1-\alpha}} \left(\frac{a+b}{2^{\frac{1}{p}}p}\left(t^{\frac{1}{p}-1} - (1-t)^{\frac{1}{p}-1}\right)\right) dt
 \end{aligned}$$

with the same substitution $\frac{a+b}{2^{\frac{1}{p}}}\left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right) = u$ and use the integration by parts,

$$\begin{aligned}
 \int_{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}}^{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}} \frac{f(u)g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g(u)\right]^{1-\alpha}} du &\leq \frac{(f(a) + f(b))}{a+b} \int_{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}}^{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}} \frac{ug'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g(u)\right]^{1-\alpha}} du \\
 \int_{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}}^{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}} \frac{f(u)g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g(u)\right]^{1-\alpha}} du &\leq \frac{(f(a) + f(b))}{a+b} \left[\frac{a+b}{\alpha 2^{\frac{2}{p}-1}} \left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha} \right. \\
 &\quad \left. + \int_{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}}^{\frac{\frac{a+b}{2^{\frac{2}{p}-1}}}{2^{\frac{1}{p}}}} \frac{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g(u)\right]^{\alpha}}{\alpha} du \right]
 \end{aligned}$$

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_{\frac{\frac{a+b}{2^{\frac{1}{p}}}}{2^{\frac{2}{p}-1}}^{\frac{a+b}{2^{\frac{1}{p}}}}} \frac{f(u)g'(u)}{\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g(u)\right]^{1-\alpha}} du &\leq \frac{(f(a) + f(b))}{\alpha\Gamma(\alpha)2^{\frac{2}{p}-1}} \left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^\alpha \\ &+ \frac{(f(a) + f(b))}{\alpha\Gamma(\alpha)(a+b)} \int_{\frac{\frac{a+b}{2^{\frac{1}{p}}}}{2^{\frac{2}{p}-1}}^{\frac{a+b}{2^{\frac{1}{p}}}}} \left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g(u)\right]^\alpha du \\ I^\alpha \left(\frac{\frac{a+b}{2^{\frac{1}{p}}}}{2^{\frac{2}{p}-1}}\right)^+ ;g \left(\frac{a+b}{2^{\frac{1}{p}}}\right) &\leq \frac{f(a) + f(b)}{\Gamma(\alpha + 1)2^{\frac{2}{p}-1}} \left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^\alpha \\ &+ \frac{f(a) + f(b)}{\Gamma(\alpha + 1)(a+b)} \int_{\frac{\frac{a+b}{2^{\frac{1}{p}}}}{2^{\frac{2}{p}-1}}^{\frac{a+b}{2^{\frac{1}{p}}}}} \left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g(u)\right]^\alpha du . \end{aligned}$$

Combining the both of the obtained sides, we have the deduced inequality. □

Using the left part of (3.1), we have the following:

Corollary 3.2 *Let f and g be given in Theorem 3.1. For $x > 0$ and $0 < p < 1$,*

$$\frac{f\left(\frac{x}{2^{\frac{1}{p}-1}}\right) \left[g(x) - g\left(\frac{x}{2^{\frac{1}{p}-1}}\right)\right]^\alpha}{\Gamma(\alpha + 1)} \leq I^\alpha \left(\frac{x}{2^{\frac{1}{p}-1}}\right)^+ ;g f(x) . \tag{3.2}$$

To prove (3.2), $x = \frac{a+b}{2^{\frac{1}{p}}}$ is taken in (3.1).

Theorem 3.3 *Let $a, b \in \mathbb{R}_+$, $a < b$ and $g : [a, b] \rightarrow \mathbb{R}$ be an increasing and positive valued function on (a, b) , having a continuous derivative $g'(x)$ on (a, b) . Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be non-decreasing p -convex function, $f \in L_1[0, +\infty)$ and $\alpha > 0$. The following right sided general fractional inequality holds:*

$$\begin{aligned} \frac{f\left(\frac{\frac{a+b}{2^{\frac{1}{p}}}}{2^{\frac{2}{p}-1}}\right) \left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g\left(\frac{\frac{a+b}{2^{\frac{1}{p}}}}{2^{\frac{2}{p}-1}}\right)\right]^\alpha}{\Gamma(\alpha+1)} &\leq I^\alpha \left(\frac{\frac{a+b}{2^{\frac{1}{p}}}}{2^{\frac{2}{p}-1}}\right)^- ;g f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1)2^{\frac{1}{p}}} \left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^\alpha \\ &- \frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)} \int_{\frac{\frac{a+b}{2^{\frac{1}{p}}}}{2^{\frac{2}{p}-1}}^{\frac{a+b}{2^{\frac{1}{p}}}}} \left[g(u) - g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)\right]^\alpha du . \end{aligned} \tag{3.3}$$

Proof Let $t \in [\frac{1}{2}, 1]$. The p -convexity of the function f yields (2.2), if this inequality is multiplied by

$\frac{g' \left(\frac{a+b}{2^{\frac{1}{p}}} \left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}} \right) \right)}{\left[g \left(\frac{a+b}{2^{\frac{1}{p}}} \left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}} \right) \right) - g \left(\frac{a+b}{2^{\frac{2}{p}-1}} \right) \right]^{1-\alpha}} \left(\frac{a+b}{2^{\frac{1}{p}p}} \left(t^{\frac{1}{p}-1} - (1-t)^{\frac{1}{p}-1} \right) \right)$ and integrated over $t \in [\frac{1}{2}, 1]$, we have the

left side of (3.3) following the same way as in the first part of proof of the Theorem 3.1. To prove the right side

of (3.3), the coefficient $\frac{g' \left(\frac{a+b}{2^{\frac{1}{p}}} \left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}} \right) \right)}{\left[g \left(\frac{a+b}{2^{\frac{1}{p}}} \left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}} \right) \right) - g \left(\frac{a+b}{2^{\frac{2}{p}-1}} \right) \right]^{1-\alpha}} \left(\frac{a+b}{2^{\frac{1}{p}p}} \left(t^{\frac{1}{p}-1} - (1-t)^{\frac{1}{p}-1} \right) \right)$ is multiplied by (2.3)

and integration of both sides over $t \in [\frac{1}{2}, 1]$ gives

$$\int_{\frac{1}{2}}^1 f \left(\frac{a+b}{2^{\frac{1}{p}}} \left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}} \right) \right) \frac{g' \left(\frac{a+b}{2^{\frac{1}{p}}} \left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}} \right) \right)}{\left[g \left(\frac{a+b}{2^{\frac{1}{p}}} \left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}} \right) \right) - g \left(\frac{a+b}{2^{\frac{2}{p}-1}} \right) \right]^{1-\alpha}} \left(\frac{a+b}{2^{\frac{1}{p}p}} \left(t^{\frac{1}{p}-1} - (1-t)^{\frac{1}{p}-1} \right) \right) dt$$

$$\leq \int_{\frac{1}{2}}^1 \frac{1}{2^{\frac{1}{p}}} \left[(f(a) + f(b)) \left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}} \right) \right] \frac{g' \left(\frac{a+b}{2^{\frac{1}{p}}} \left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}} \right) \right)}{\left[g \left(\frac{a+b}{2^{\frac{1}{p}}} \left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}} \right) \right) - g \left(\frac{a+b}{2^{\frac{2}{p}-1}} \right) \right]^{1-\alpha}} \left(\frac{a+b}{2^{\frac{1}{p}p}} \left(t^{\frac{1}{p}-1} - (1-t)^{\frac{1}{p}-1} \right) \right) dt$$

with the changing variable with $\frac{a+b}{2^{\frac{1}{p}}} \left(t^{\frac{1}{p}} + (1-t)^{\frac{1}{p}} \right) = u$, we deduce that

$$\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \frac{f(u) g'(u)}{\left[g(u) - g \left(\frac{a+b}{2^{\frac{2}{p}-1}} \right) \right]^{1-\alpha}} du \leq \frac{(f(a) + f(b))}{a+b} \int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \frac{u g'(u)}{\left[g(u) - g \left(\frac{a+b}{2^{\frac{2}{p}-1}} \right) \right]^{1-\alpha}} du$$

$$\int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \frac{f(u) g'(u)}{\left[g(u) - g \left(\frac{a+b}{2^{\frac{2}{p}-1}} \right) \right]^{1-\alpha}} du \leq \frac{(f(a) + f(b))}{a+b} \left[\frac{a+b}{\alpha 2^{\frac{1}{p}}} \left[g \left(\frac{a+b}{2^{\frac{1}{p}}} \right) - g \left(\frac{a+b}{2^{\frac{2}{p}-1}} \right) \right]^{\alpha} \right.$$

$$\left. - \frac{1}{\alpha} \int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \frac{\left[g(u) - g \left(\frac{a+b}{2^{\frac{2}{p}-1}} \right) \right]^{\alpha}}{\alpha} du \right]$$

$$\frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \frac{f(u) g'(u)}{\left[g(u) - g \left(\frac{a+b}{2^{\frac{2}{p}-1}} \right) \right]^{1-\alpha}} du \leq \frac{(f(a) + f(b))}{\alpha \Gamma(\alpha) 2^{\frac{1}{p}}} \left[g \left(\frac{a+b}{2^{\frac{1}{p}}} \right) - g \left(\frac{a+b}{2^{\frac{2}{p}-1}} \right) \right]^{\alpha}$$

$$- \frac{(f(a) + f(b))}{\alpha \Gamma(\alpha) (a+b)} \int_{\frac{a+b}{2^{\frac{2}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \left[g(u) - g \left(\frac{a+b}{2^{\frac{2}{p}-1}} \right) \right]^{\alpha} du$$

$$I_{\left(\frac{a+b}{2^{\frac{1}{p}}}\right)^-;g}^{\alpha} f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1)2^{\frac{1}{p}}}\left[g\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha} - \frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}^{\frac{a+b}{2^{\frac{1}{p}}}}\left[g(u)-g\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\right]^{\alpha} du.$$

Finally, with the consideration of the inequalities together, (3.3) is proved. □

The fractional integrals with respect to the function g are given the Riemann-Liouville fractional. Inequalities involving the Riemann-Liouville fractional integrals of p -convex functions are given below.

Theorem 3.4 *Let $a, b \geq 0$ such that $a < b$, $f : [0, +\infty) \rightarrow \mathbb{R}$ be non-decreasing p -convex function, $f \in L_1[0, +\infty)$ and $\alpha > 0$. The following inequality holds:*

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)(a+b)^{\alpha}\left(2^{\frac{1}{p}}-2\right)^{\alpha}}{\Gamma(\alpha+1)2^{\frac{2\alpha}{p}}} \leq J_{\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)^+}^{\alpha} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \leq \frac{(f(a)+f(b))(a+b)^{\alpha}\left(2^{\frac{1}{p}}-2\right)^{\alpha}\left(2\alpha+2^{\frac{1}{p}}\right)}{\Gamma(\alpha+2)2^{\frac{2\alpha+2}{p}}}. \tag{3.4}$$

Proof To prove (3.4), we use Theorem 3.1. In (3.1), if the function $g(x) = x$ is taken, then (3.4) is obtained. □

Theorem 3.5 *Let $a, b \geq 0$ such that $a < b$, Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be non-decreasing p -convex function, $f \in L_1[0, +\infty)$ and $\alpha > 0$. The following inequality holds*

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)(a+b)^{\alpha}\left(2^{\frac{1}{p}}-2\right)^{\alpha}}{\Gamma(\alpha+1)2^{\frac{2\alpha}{p}}} \leq J_{\left(\frac{a+b}{2^{\frac{1}{p}}}\right)^-}^{\alpha} f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) \leq \frac{(f(a)+f(b))(a+b)^{\alpha}\left(2^{\frac{1}{p}}-2\right)^{\alpha}\left(2+\alpha 2^{\frac{1}{p}}\right)}{\Gamma(\alpha+2)2^{\frac{2\alpha+2}{p}}}. \tag{3.5}$$

Proof Taking $g(x) = x$ in Theorem 3.3 yields the desired result. □

In the case of $\alpha = 1$, (3.4) and (3.5) give the following inequality

$$\frac{f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\left(2^{\frac{1}{p}}-2\right)}{2^{\frac{2}{p}}} \leq \frac{1}{a+b}\int_{\frac{a+b}{2^{\frac{2}{p}-1}}^{\frac{a+b}{2^{\frac{1}{p}}}}f(t) dt \leq \frac{(f(a)+f(b))\left(2^{\frac{2}{p}}-4\right)}{2^{\frac{4}{p}+1}}. \tag{3.6}$$

The used general fractional integral operator in the article gives special essential fractional integrals, one of these is Hadamard fractional integrals. Thus, inequalities for p -convex functions including Hadamard fractional integral operators are proven in the following theorems.

Theorem 3.6 *Let $a, b \geq 0$ such that $a < b$, $f : [0, +\infty) \rightarrow \mathbb{R}$ be non-decreasing p -convex function, $f \in$*

$L_1 [0, +\infty)$ and $\alpha > 0$. The following inequality via Hadamard fractional integrals holds

$$\frac{f\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right) (1-p)^\alpha (\ln 2)^\alpha}{\Gamma(\alpha+1) p^\alpha} \leq \mathbf{J}^\alpha_{\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)^+} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1)} \left[\frac{(\ln 2)^\alpha (1-p)^\alpha}{2^{\frac{2}{p}-1} p^\alpha} + \frac{1}{a+b} \int_{\frac{a+b}{2^{\frac{1}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \left(\ln\left(\frac{a+b}{2^{\frac{1}{p}}u}\right)\right)^\alpha du \right].$$

Proof Taking $g(x) = \ln x$ in (3.1), which is increasing and positive monotone function on $(a, b]$, having a continuous derivative $g'(x) = \frac{1}{x}$ on (a, b) , then one has

$$\begin{aligned} \frac{f\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right) \left[\ln\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - \ln\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)\right]^\alpha}{\Gamma(\alpha+1)} &\leq \mathbf{J}^\alpha_{\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)^+} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1) 2^{\frac{2}{p}-1}} \left[\ln\left(\frac{a+b}{2^{\frac{1}{p}}}\right) - \ln\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)\right]^\alpha \\ &+ \frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)} \int_{\frac{a+b}{2^{\frac{1}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \left[\ln\left(\frac{a+b}{2^{\frac{1}{p}}u}\right) - \ln(u)\right]^\alpha du. \end{aligned}$$

Afterwards,

$$\begin{aligned} \frac{f\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right) \left[\ln\left(2^{\frac{1}{p}-1}\right)\right]^\alpha}{\Gamma(\alpha+1)} &\leq \mathbf{J}^\alpha_{\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)^+} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1) 2^{\frac{2}{p}-1}} \left[\ln\left(2^{\frac{1}{p}-1}\right)\right]^\alpha \\ &+ \frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)} \int_{\frac{a+b}{2^{\frac{1}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \left[\ln\left(\frac{a+b}{2^{\frac{1}{p}}u}\right)\right]^\alpha du. \end{aligned}$$

Thus, the desired inequality is obtained. □

Theorem 3.7 Let $a, b \geq 0$ such that $a < b$, $f : [0, +\infty) \rightarrow \mathbb{R}$ be non-decreasing p -convex function, $f \in L_1 [0, +\infty)$ and $\alpha > 0$. The following inequality for p -convex functions including Hadamard fractional integral operator holds

$$\frac{f\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right) (1-p)^\alpha (\ln 2)^\alpha}{\Gamma(\alpha+1) p^\alpha} \leq \mathbf{J}^\alpha_{\left(\frac{a+b}{2^{\frac{1}{p}}}\right)^-} f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1)} \left[\frac{(1-p)^\alpha (\ln 2)^\alpha}{2^{\frac{1}{p}} p^\alpha} - \frac{1}{a+b} \int_{\frac{a+b}{2^{\frac{1}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \left(\ln\left(\frac{2^{\frac{2}{p}-1}u}{a+b}\right)\right)^\alpha du \right]. \tag{3.7}$$

Proof The general fractional integral operator turns to the Hadamard fractional when $g(x) = \ln x$. By using

this in (3.3), the following is obtained.

$$\frac{f\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)\left[\ln\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-\ln\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} \leq \mathbf{J}^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{p}}}\right)} - f\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1)2^{\frac{1}{p}}}\left[\ln\left(\frac{a+b}{2^{\frac{1}{p}}}\right)-\ln\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)\right]^{\alpha} - \frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)} \int_{\frac{a+b}{2^{\frac{1}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \left[\ln(u)-\ln\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)\right]^{\alpha} du .$$

Then,

$$\frac{f\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)\left[\ln\left(2^{\frac{1}{p}-1}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} \leq \mathbf{J}^{\alpha}_{\left(\frac{a+b}{2^{\frac{1}{p}}}\right)} - f\left(\frac{a+b}{2^{\frac{1}{p}-1}}\right) \leq \frac{f(a)+f(b)}{\Gamma(\alpha+1)2^{\frac{1}{p}}}\left[\ln\left(2^{\frac{1}{p}-1}\right)\right]^{\alpha} - \frac{f(a)+f(b)}{\Gamma(\alpha+1)(a+b)} \int_{\frac{a+b}{2^{\frac{1}{p}-1}}}^{\frac{a+b}{2^{\frac{1}{p}}}} \left[\ln\left(\frac{2^{\frac{1}{p}-1}u}{a+b}\right)\right]^{\alpha} du .$$

Hence, (3.7) is obtained after some editing. □

4. Applications

Some relations involving special functions can be deduced using the results in Section 3. In the following propositions, we exemplified some of them pertaining to the functions which are expressed via integrals. The integral calculations of the propositions have been made by Mathematica 13.1.

In the following proposition, a relation between beta and incomplete beta function is obtained via Theorem 3.4

Proposition 4.1 *Let $n \in \mathbb{N}$, $m = 2^n + 1$, $\alpha > 0$, $0 < x < 1$. Then*

$$\frac{x^{m-1}}{2^{\alpha}\alpha} (x-x^2)^{\alpha} \leq B(m, \alpha) - B_x(m, \alpha) \leq \frac{(x-x^2)^{\alpha}(\alpha x+1)}{(\alpha+1)x^{m-2}2^{2+\alpha-m}} \tag{4.1}$$

where $B_-(\cdot, \cdot)$ is incomplete beta function, namely,

$$B_{\theta}(q, r) = \int_0^{\theta} (1-t)^{r-1}t^{q-1}dt \quad (0 < \theta < 1, 0 < q, r).$$

Proof Let us write $f(x) = x^{m-1}$ in (3.4), which is nondecreasing p -convex function on $[0, \infty)$ by Example 2.3 . Using the fact

$$\int (x-t)^{\alpha-1}t^{m-1}dt = x^{\alpha+m-1}B_{\frac{t}{x}}(m, \alpha), \quad (t < x)$$

we have

$$J^\alpha \left(\frac{a+b}{2^{\frac{1}{p}-1}} \right) + f \left(\frac{a+b}{2^{\frac{1}{p}}} \right) = \frac{(a+b)^{\alpha+m-1}}{2^{\frac{\alpha+m-1}{p}} \Gamma(\alpha)} \left(B(m, \alpha) - B_{2^{1-\frac{1}{p}}}(m, \alpha) \right).$$

To obtain left piece of (4.1), let us use the left part of (3.4). Making some easy simplifications, the following inequality obtained

$$\frac{2^{(m-1)(1-\frac{1}{p})}}{\alpha} \left(\frac{2^{\frac{1}{p}} - 2}{2^{\frac{2}{p}}} \right)^\alpha \leq B(m, \alpha) - B_{2^{1-\frac{1}{p}}}(m, \alpha). \tag{4.2}$$

In a similar way, to show right piece of (4.1), we use right part of (3.4). Making some algebraic manipulations, we have

$$B(m, \alpha) - B_{2^{1-\frac{1}{p}}}(m, \alpha) \leq \frac{(a^{m-1} + b^{m-1}) \left(\frac{2^{\frac{1}{p}} - 2}{2^{\frac{2}{p}}} \right)^\alpha (2\alpha + 2^{\frac{1}{p}})}{(a+b)^{m-1} 2^{\frac{3-m}{p}} (\alpha + 1)}.$$

Using $\frac{a^{m-1} + b^{m-1}}{(a+b)^{m-1}} < 1$, one can write

$$B(m, \alpha) - B_{2^{1-\frac{1}{p}}}(m, \alpha) \leq \frac{\left(\frac{2^{\frac{1}{p}} - 2}{2^{\frac{2}{p}}} \right)^\alpha (2\alpha + 2^{\frac{1}{p}})}{2^{\frac{3-m}{p}} (\alpha + 1)}. \tag{4.3}$$

Letting $x = 2^{1-\frac{1}{p}}$ in (4.2) and (4.3) where $x \in (0, 1)$ and combining them, we have the desired result. □

The following propositions that set some relations between gamma and incomplete gamma (or upper gamma) functions are obtained via Theorem 3.6.

Proposition 4.2 *Let $\alpha > 0$. If $x > 0$, then*

$$2^{-2x} x^\alpha + \alpha \Gamma(\alpha, x) \leq \Gamma(\alpha + 1)$$

where $\Gamma(\cdot, \cdot)$ denotes incomplete gamma functions:

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1} dt.$$

Proof Putting $f(x) = x^2$ at left-hand side piece of (3.6), then using

$$\int \left(\ln \frac{x}{t} \right)^{\alpha-1} t dt = \frac{x^2}{2\alpha} \left(\Gamma \left(\alpha, 2 \ln \frac{x}{t} \right) \right)$$

and making some simplifications, we have

$$\mathbf{J}_{a^+}^\alpha (x^2) = \frac{1}{\Gamma(\alpha)} \frac{x^2}{2\alpha} \left(\Gamma(\alpha) - \Gamma \left(\alpha, 2 \ln \frac{x}{a} \right) \right). \tag{4.4}$$

Making use of (4.4) and $\Gamma(\alpha + 1) = (\alpha)\Gamma(\alpha)$ thus simplifying the expressions in left-hand side piece of (3.6), one has

$$\frac{2^{\alpha-2(\frac{1}{p}-1)} (\ln 2)^\alpha}{\alpha} \left(\frac{1}{p} - 1\right)^\alpha \leq \Gamma(\alpha) - \Gamma\left(\alpha, 2\left(\frac{1}{p} - 1\right) \ln 2\right).$$

Taking $x = 2(\ln 2)(\frac{1}{p} - 1)$ where $x > 0$ for $0 < p < 1$, we have the desired result. □

Proposition 4.3 *Let $\alpha > 0$. If $x > 0$, then*

$$2^{-\frac{x}{\ln 2}-\alpha-1} (\Gamma(\alpha + 1) - \alpha\Gamma(\alpha, 2x)) \leq 2^{-\frac{x}{\ln 2}} x^\alpha + \Gamma(\alpha + 1) - \Gamma(\alpha + 1, x)$$

where gamma and incomplete gamma functions are given as in Proposition 4.2.

Proof Putting $f(x) = x^2$ at right-hand side piece of (3.6) and using

$$\int_{\frac{\frac{a+b}{2^{\frac{1}{p}}-1}}}{\frac{a+b}{2^{\frac{1}{p}}}} \left(\ln\left(\frac{a+b}{2^{\frac{1}{p}}u}\right)\right)^\alpha du = \frac{a+b}{2^{\frac{1}{p}}} \left(\alpha\Gamma(\alpha) - \Gamma\left(\alpha + 1, \left(\frac{1}{p} - 1\right) \ln 2\right)\right)$$

and making some simplifications, we have

$$\frac{1}{2^\alpha 2^{\frac{1}{p}}} \left(\Gamma(\alpha) - \Gamma\left(\alpha, 2\left(\frac{1}{p} - 1\right) \ln 2\right)\right) (a+b)^2 \leq \frac{a^2 + b^2}{\alpha} \left[\frac{(\ln 2)^\alpha (1-p)^\alpha}{2^{\frac{1}{p}-1} p^\alpha} + \left(\alpha\Gamma(\alpha) - \Gamma\left(\alpha + 1, \left(\frac{1}{p} - 1\right) \ln 2\right)\right)\right].$$

Dividing each side by $(a+b)^2$ and using $\frac{a^2+b^2}{(a+b)^2} < 1$, we have

$$\frac{1}{2^\alpha 2^{\frac{1}{p}}} \left(\Gamma(\alpha) - \Gamma\left(\alpha, 2\left(\frac{1}{p} - 1\right) \ln 2\right)\right) \leq \frac{1}{\alpha} \left[\frac{(\ln 2)^\alpha (1-p)^\alpha}{2^{\frac{1}{p}-1} p^\alpha} + \left(\alpha\Gamma(\alpha) - \Gamma\left(\alpha + 1, \left(\frac{1}{p} - 1\right) \ln 2\right)\right)\right].$$

After some algebraic manipulations and simplifications and using $\Gamma(\alpha + 1) = (\alpha)\Gamma(\alpha)$, one has

$$2^{-(\frac{1}{p}-1)-\alpha-1} \left(\Gamma(\alpha + 1) - \alpha\Gamma\left(\alpha, 2\left(\frac{1}{p} - 1\right) \ln 2\right)\right) \leq 2^{-(\frac{1}{p}-1)} \left(\left(\frac{1}{p} - 1\right) \ln 2\right)^\alpha + \Gamma(\alpha + 1) - \Gamma\left(\alpha + 1, \left(\frac{1}{p} - 1\right) \ln 2\right).$$

Taking $x = (\ln 2)(\frac{1}{p} - 1)$ where $x > 0$ for $0 < p < 1$, we have the desired result. □

Using Theorem 3.1, the following result is obtained, which shows a relation between two hypergeometric series with same coefficients.

Proposition 4.4 *Let n, m, r positive integers such that $m = 2^r$ and a, b, p, α positive real numbers with $a < b$ and $0 < p < 1$. Suppose $2(1 + \log_2(\frac{a+b}{b}))^{-1} < p$. Then*

$$(a+b)^{m+n} 2^{-\frac{2m+n}{p}+m} \left(1 - 2^{n(1-\frac{1}{p})}\right)^\alpha \frac{m+n}{n\alpha} \leq {}_2F_1\left(1 - \alpha, \frac{m}{n} + 1; \frac{m}{n} + 2; 1\right) - {}_2F_1\left(1 - a, \frac{m}{n} + 1; \frac{m}{n} + 2; \left(\frac{b}{a+b} 2^{\frac{2}{p}-1}\right)^n\right)$$

where F denotes the hypergeometric function which is defined by

$${}_2F_1(\beta, \gamma; \eta; z) = 1 + \frac{\beta\gamma}{\eta}z + \frac{\beta(\beta+1)\gamma(\gamma+1)}{\eta(\eta+1)}\frac{z^2}{2!} + \dots + \frac{(\beta)_n(\gamma)_n}{(\eta)_n}\frac{z^n}{n!} + \dots$$

on $|z| \leq 1$ where $(\cdot)_n$ is the Pochhammer symbol, i.e.

$$(\mu)_n = \mu(\mu+1)(\mu+2)\dots(\mu+n-1).$$

Proof Let $f(x) = x^m$ and $g(x) = x^n$ in Theorem 3.1, which satisfy the conditions of the theorem. Using

$$\int \frac{nt^{n-1}t^m}{(x^n - t^n)^{1-\alpha}} dt = \frac{n}{m+n}t^{n+m}x^{n(\alpha-1)} \times {}_2F_1\left(1-\alpha, \frac{m}{n}+1; \frac{m}{n}+2; \left(\frac{t}{x}\right)^n\right)$$

we have

$$\begin{aligned} I^\alpha \left(\frac{a+b}{2^{\frac{1}{p}-1}}\right)_{:g} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right) &= \frac{1}{\Gamma(\alpha)} \frac{n}{d+n} \left(\frac{a+b}{2^{\frac{1}{p}}}\right)^{n(\alpha-1)} \left(F\left(1-\alpha, \frac{d}{n}+1; \frac{d}{n}+2; 1\right) \right. \\ &\quad \left. - F\left(1-\alpha, \frac{d}{n}+1; \frac{d}{n}+2; \left(\frac{b}{a+b}2^{\frac{2}{p}-1}\right)^n\right)\right). \end{aligned} \tag{4.5}$$

Note that ${}_2F_1(\beta, \gamma; \eta; z)$ is defined (i.e. series converges) on $-1 \leq z \leq 1$ provided that $\beta + \gamma - \eta < 0$ and η is not a negative integer or zero [24]. Since $2(1 + \log_2(\frac{a+b}{b}))^{-1} < p$, it is clear that $\left(\frac{b}{a+b}2^{\frac{2}{p}-1}\right)^n < 1$. Thus, the hypergeometric functions in (4.5) satisfy the convergency conditions. Making use of the left part of (3.1) and making some algebraic manipulations we have the desired result. \square

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