Turkish Journal of Mathematics

Volume 47 | Number 7 Article 10

11-9-2023

Proximality and transitivity in relation to points that are asymptotic to themselves

KAROL GRYSZKA

Follow this and additional works at: https://journals.tubitak.gov.tr/math

Part of the Mathematics Commons

Recommended Citation

GRYSZKA, KAROL (2023) "Proximality and transitivity in relation to points that are asymptotic to themselves," *Turkish Journal of Mathematics*: Vol. 47: No. 7, Article 10. https://doi.org/10.55730/1300-0098.3476
Available at: https://journals.tubitak.gov.tr/math/vol47/iss7/10

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact academic.publications@tubitak.gov.tr.

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 1974 – 1990
© TÜBİTAK
doi:10.55730/1300-0098.3476

Proximality and transitivity in relation to points that are asymptotic to themselves

Karol GRYSZKA*
Institute of Mathematics, Pedagogical University of Kraków, Podchorażych 2, Kraków, Poland

Received: 14.08.2023 • **Accepted/Published Online:** 21.09.2023 • **Final Version:** 09.11.2023

Abstract: We discuss dynamical systems that exhibit at least one weakly asymptotically periodic point. In the general case we prove that the system becomes trivial (it is either a periodic point or a fixed point) provided it is equicontinuous and transitive. This result can be used to provide a simple characterization of periodic points in transitive systems. We also discuss systems whose orbits are both proximal and weakly asymptotically periodic. As a result, we obtain a more general tool to detect mutual dynamics between two close orbits which need not be bounded (or have the empty limit set).

Key words: Weak asymptotic periodicity, periodicity, ω-limit set, equicontinuity, transitive system, proximal relation, regionally proximal relation

1. Introduction

In [10] the concept of weak asymptotic periodicity (i.e. x is such if $(x, F^T(x))$ is an asymptotic pair for some $T > 0$) was introduced to study dynamical systems on general metric spaces (that need not be compact) whose orbits show asymptotic recurrence–like behaviour. The general approach there provided basic information about the structure of the limit set or the characterization of such orbits via the limit set behaviour. The study of asymptotic pairs itself is important. It appears both in relation to the entropy of the system or to chaos. In particular, Downarowicz and Lacroix show that a zero entropy system has an extension with no asymptotic pairs [6], while Blanchard et al. show that in positive entropy systems there are ”lots of” asymptotic pairs [5]. Moreover, a pair of points which is proximal and not asymptotic is a Li-Yorke pair: in 1975 Li and Yorke introduced such pairs in their definition of chaos [14] (see also the book [2] and references therein). Note that other forms of periodicity, including those for nonautonomous dynamical systems (see for instance [15]) are also considered in chaotic (in particular: transitive) systems.

In this article, we would like to briefly describe the dynamical structure of the systems that are both transitive and have at least one weakly asymptotically periodic orbit. Since weak asymptotic periodicity is a more general approach than classic asymptotic periodicity, we thus extend the properties of the latter to ones of the former.

*Correspondence: karol.gryszka@up.krakow.pl

2010 *AMS Mathematics Subject Classification:* 37B20, 37B25, 37B05

This work is licensed under a Creative Commons Attribution 4.0 International License.

Furthermore, Theorem 6 in the article [10] shows that if F is equicontinuous at a periodic point y and $y \in \omega(x)$ for some $x \in X$, then x is weakly asymptotically periodic. Here we also consider a different case where the limit set might be empty. In particular, we prove the following results.

Main Theorem 1 *Let (X, F) be a dynamical system and $x \in X$ be a transitive point. The following conditions are equivalent: (1) x is periodic, (2) x is asymptotically periodic, (3) x is weakly asymptotically periodic.*

Main Theorem 2 *Let X be a compact metric space and (X, F) be a transitive dynamical system with a discrete spectrum. If $x \in X$ is weakly asymptotically periodic, then (X, F) is a periodic orbit.*

Main Theorem 3 *Let (X, F) be a dynamical system that is uniformly equicontinuous at x and y. Assume that x and y are regionally proximal of order $d \geq 1$ along arithmetic progression. Then the following conditions are equivalent: (1) x is weakly asymptotically T-periodic, (2) y is weakly asymptotically T-periodic.*

2. Preliminaries

Following [1], [2] we introduce basic notation and definitions. Throughout this paper (X, d) is a metric space with metric d and without isolated points (unless stated otherwise). The metric space itself need not be compact.

2.1. Dynamical systems

A *dynamical system* is a continuous map $F\colon X \to X$. We denote if briefly by (X, F). By F^n we denote the n-th iteration of F. The set $\mathcal{O}(x) = \{F^n(x) \mid n \in \mathbb{N}\}$ is called the *orbit* of x. The ω*-limit set* $\omega(x)$ consists of all points $y \in X$ such that there exists a strictly increasing sequence of natural numbers $(a_n)_{n \in \mathbb{N}}$ such that $F^{a_n}(x) \to y$ as $n \to +\infty$.

We call x a *fixed point* if $F^n(x) = x$ for every $n \in \mathbb{N}$ and a *T-periodic point* ($T > 0$) if $F^T(x) = x$ (note that we do not emphasise that T is the smallest number with such property). We say that x is asymptotic to y if $\lim_{n \to +\infty} d(F^n(x), F^n(y)) = 0$. We call x *asymptotically periodic* if there exists a T-periodic point y such that x is asymptotic to y. We call x *weakly asymptotically T-periodic point* if it is asymptotic to $F^T(x)$. We say that x is *weakly asymptotically periodic* if it is weakly asymptotically T-periodic for some $T > 0$.

The set $\mathcal{E}$ of *equicontinuous points* consists of all points x such that for every $\varepsilon > 0$ there is $\delta > 0$ such that if $d(z, x) < \delta$, then for each $n \in \mathbb{N}$ we have $d(F^n(z), F^n(x)) < \varepsilon$. We call $x \in \mathcal{E}$ a *point of equicontinuity* of F (we also say that F is *equicontinuous* at x). If $\mathcal{E} = X$, then we say that (X, F) is *equicontinuous.* A point x is *uniformly equicontinuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $d(z, F^r(x)) < \delta$ for some $z \in X$ and $r \geq 0$, then for each $n \in \mathbb{N}$, $d(F^n(z), F^{n+r}(x)) < \varepsilon$.

We call (Y, G) a *subsystem* of (X, F) if $X \subset Y$ is a closed and F-invariant set (that is $F(Y) \subset Y$ and $F|_Y = G$). In the paper we abuse the notation and denote the map of the subsystem by the same letter F.

The *recurrence relation* $\mathcal{R}$ is defined as the set of all pairs (x, y), such that for every $\varepsilon > 0$ there is $n > 0$ with the property $d(y, F^n(x)) < \varepsilon$. The *nonwandering relation* $\mathcal{N}$ is defined as the set of all pairs (x, y), such that for all ε and $\delta > 0$ there is $n > 0$ and $z \in X$ such that $d(z, x) < \delta$ and $d(y, F^n(z)) < \varepsilon$.

2.2. Transitive systems

Following [1], [8] we recall the concept of transitive systems. We call a point transitive if $\omega(x) = X$ and a system (X, F) transitive if it has a transitive point. If for every $x \in X$ the set $\mathcal{O}(x)$ is dense in X, then we call (X, F) minimal (or equivalently, (X, F) is minimal if every point is transitive).

We present the following examples. They will be used in the next sections of the paper.

Example 2.1 *Consider the map $G : \mathbb{S}^1 \to \mathbb{S}^1$ which is an irrational rotation. Then every point in this system has a dense orbit (the system is minimal) and all points have the exact same dynamics, which in this case is that there is no point that weakly asymptotically periodic for any value of T (see also Example 2.1 in [10]).*

Example 2.2 *Let $2^{\mathbb{N}}$ be the set of all one-sided infinite sequences whose terms are 0s and 1s. Define the mapping*

$$R_1(u)_n = \begin{cases} mod_2(u_n + 1), & \text{if } u_0 = \ldots = u_{n-1} = 1, \\ u_n, & \text{otherwise.} \end{cases}$$

This map can informally be described as "add 1 and carry", where the addition is at the leftmost term and the carry proceeds to the right. Then $(2^{\mathbb{N}}, R_1)$ is the dyadic adding machine. Dyadic adding machines are special cases of adding machines, described for instance by the periodic structure. In particular, adding machines are minimal, equicontinuous, and have no periodic points (see [13, Chapter 4] for other kinds of adding machines and their properties).

2.3. Proximal pairs

We recall various concepts of proximality. We only consider proper pairs, that is $x \neq y$. Let X be a set consisting of at least two different elements. A pair $(x, y) \in X \times X$ is *proximal* if

$$\liminf_{n \to +\infty} d(F^n(x), F^n(y)) = 0.$$

A pair $(x, y) \in X \times X$ is *weakly proximal* if there exist sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \to +\infty} d(F^{a_n}(x), F^{b_n}(y)) = 0.$$

Note that in the definition of weakly proximal pair the sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ can be arbitrary, not necessarily increasing. A dynamical system (X, F) is (*weakly*) *proximal* if all pairs $(x, y) \in X \times X$ are (weakly) proximal. A pair $(x, y) \in X \times X$ is said to be *regionally proximal of order d* if for any $\varepsilon > 0$, there exists $x', y' \in X$ and a vector $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that $d(x, x') < \varepsilon$, $d(y, y') < \varepsilon$ and

$$d(F^{\underline{n} \cdot \epsilon}(x'), F^{\underline{n} \cdot \epsilon}(y')) < \varepsilon \text{ for any } \epsilon \in \{0, 1\}^d, \epsilon \neq \underline{0}.$$

Here $\underline{n} \cdot \epsilon = \sum_{i=1}^{d} n_i \epsilon_i$ and $\underline{0} = (0, \ldots, 0) \in \mathbb{Z}^d$. The case $d = 1$ reduces to the usual definition of a regionally proximal pair.

2.4. Auxiliarry lemmas

We conclude this section by recalling the following properties of weakly asymptotically periodic points (see [10]).

Lemma 2.3 *Let (X, F) be a dynamical system. If x is weakly asymptotically T-periodic and $\omega(x) \neq \emptyset$, then every point $y \in \omega(x)$ is either T-periodic or fixed.*

Lemma 2.4 *If (X, F) and (Y, G) are two conjugated dynamical systems on compact metric spaces with conjugating mapping φ and $x \in X$ is weakly asymptotically T-periodic, then $\varphi(x)$ is weakly asymptotically T-periodic.*

3. Transitivity

The goal of this section is to describe points that are both weakly asymptotically periodic and transitive. The motivation for our research comes from the following Theorem 3.1 and Corollary 3.2. Note that later we will prove a stronger result (Theorem 3.5). Moreover, we formulate all results for periodic points, but with proper adjustments, they can also be formulated for fixed points.

Theorem 3.1 *Let (X, F) be an equicontinuous dynamical system. If $M \subset X$ is a minimal set that contains at least one weakly asymptotically periodic orbit, then every orbit in M is weakly asymptotically periodic.*

Proof Let $x \in M$ be weakly asymptotically T-periodic and $y \neq x$. By the minimality, $y \in \overline{\mathcal{O}(x)}$. Fix $\varepsilon > 0$. By the equicontinuity, there is $\delta > 0$ such that if for some $z \in X$ we have $d(z, y) < \delta$, then for every $n \geq 0$,

$$d(F^n(z), F^n(y)) < \frac{\varepsilon}{3}.$$

Since y is in the closure of the orbit of x, there is $c \geq 0$ such that

$$d(F^c(x), y) < \delta.$$

Applying the equicontinuity at y with $z = F^c(x)$ we conclude that for every $n \geq 0$:

$$d(F^{c+n}(x), F^n(y)) < \frac{\varepsilon}{3}.$$

A point x is weakly asymptotically T-periodic, so there is N such that for every $n \geq N$ we have

$$d(F^{n+T}(x), F^n(x)) < \frac{\varepsilon}{3}.$$

Finally, for every $n \geq N$,

$$d(F^{T+n}(y), F^n(y)) \leq d(F^{n+T}(y), F^{n+T+c}(x)) + d(F^{n+T+c}(x), F^{n+c}(x)) + d(F^{n+c}(x), F^n(y)) < \varepsilon.$$

This proves that y is weakly asymptotically T-periodic. □

Corollary 3.2 *Let (X, F) be a dynamical system on a compact metric space. Assume that F is transitive and equicontinuous and there exists $x \in X$ which is weakly asymptotically periodic. Then every point in X is weakly asymptotically periodic.*

Proof This follows from the well-known fact that transitive and equicontinuous systems on a compact topological space are minimal. We can now apply Theorem 3.1. □

The existence of weakly asymptotically periodic points is necessary even in the compact minimal sets, since in those systems there can be no weakly asymptotically periodic points.

We now present a different approach to the above results which will in turn allow us to generalize the thesis or Theorem 3.1. We actually show that the structure of such a system can be completely described. The basis is the following theorem (see Theorem 2.42 in [13]).

Theorem 3.3 *Let X be a compact metric space and (X,F) be a minimal equicontinuous system. Then (X,F) is conjugated to an isometry and there exists an Abelian group structure on X with a continuous group operator and a point $x \in X$ such that $F(x) = x + a$.*

Lemma 3.4 *If (X,F) is an isometry and $x \in X$ is weakly asymptotically T-periodic, then x is periodic with period T.*

Proof Let $\varepsilon > 0$ and $d(F^{n_0}(x), F^{n_0+T}(x)) < \varepsilon$ for some n_0 sufficiently large. Suppose that $d(F^{n_0}(x), F^{n_0+T}(x)) = \varepsilon' > 0$ for some $0 < \varepsilon' < \varepsilon$. Since F is an isometry, $d(F^n(x), F^{n+T}(x)) = \varepsilon'$ for all $n \geq n_0$. That contradicts with the definition of weak asymptotic periodicity, since $\lim_{n\to+\infty} d(F^n(x), F^{n+T}(x)) = 0$.

From the above we have that $d(F^{n_0}(x), F^{n_0+T}(x)) = 0$ for some $n_0 \geq 0$. Then by the isometry property again, $d(F^{n_0+k}(x), F^{n_0+k+T}(x)) = 0$ for all $k \geq 0$, so $F^{n_0}(x)$ is periodic and thus x is eventually periodic with period T (i.e. $F^m(x)$ is periodic for some $m > 0$). Since F is an isometry, it is injective and x is periodic. □

Theorem 3.5 *Let (X,F) be an equicontinuous dynamical system. If $M \subset X$ is a compact minimal set that contains a weakly asymptotically periodic point, then M is a periodic orbit.*

Proof Since (M,F) is minimal and equicontinuous, by Theorem 3.3 it is conjugated via φ to an isometry, which we denote by (Y,G). By Lemma 2.4 if $x \in M$ is weakly asymptotically T-periodic, then $\varphi(x)$ is such as well.

By Lemma 3.4 the point $\varphi(x)$ is a periodic point under the action of G, therefore x is such under the action of F.

Finally, since M is minimal and $x \in M$, $\mathcal{O}(x) = M$. □

Corollary 3.6 *Let (X,F) be an equicontinuous dynamical system. If $M \subset X$ is a compact invariant set such that (M,F) is a transitive subsystem and M contains a weakly asymptotically periodic point, then M is a periodic orbit.*

Theorem 3.5 allows us to give a characterization of periodic points.

Theorem 3.7 *Let (X,F) be an equicontinuous and transitive system on a compact metric space and $x \in X$. The following conditions are equivalent:*

1. *x is periodic,*
2. *x is asymptotically periodic,*

3. x is weakly asymptotically periodic.

Proof The implication (2) $\Rightarrow$ (3) follows without additional assumptions and is proved in [10, Theorem 1.]. Furthermore, (3) $\Rightarrow$ (1) follows from Theorem 3.5. □

Assume that the system has at least one weakly asymptotically periodic point. One could ask if adding just the transitivity of the system is sufficient to ensure the system is a periodic point. It turns out it is not (obviously). Consider the following example.

Example 3.8 *Take $X = [0,1]$ and consider the logistic map $f(x) = 4x(1-x)$. Then (X, f) is transitive and $x = \frac{5-\sqrt{5}}{8}$ is weakly asymptotically 2-periodic (in fact it is periodic). Since (X, f) is not minimal, it cannot be equicontinuous. Therefore, transitive systems with dense sets of weakly asymptotically periodic points need not be reduced to periodic orbits.*

The following example shows that if the point is weakly asymptotically periodic and its orbit is dense in the system, it does not need to be periodic. This is particularly important when one defines transitive systems to be the ones having at least one dense orbit (the assumption about isolated points becomes crucial).

Example 3.9 *Consider the following system given with the aid of Figure 1. The external orbit is 7-periodic while the orbit of x_0 is asymptotically approaching the external one. Then the orbit of x_0 is weakly asymptotically 7-periodic and it is dense in the system, however, the system is composed of $\mathcal{O}(x)$ and the periodic orbit.*

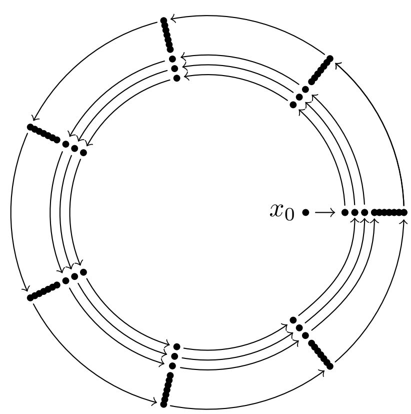

Figure 1. A sketch of the nonminimal system with weakly asymptotically periodic orbit which has a dense orbit.

We now show that if one combines transitivity and weak asymptotic periodicity properly, the system becomes trivial.

Lemma 3.10 *Assume that (X, F) is a dynamical system and $x \in X$ is both recurrent (that is $x \in \omega(x)$) and weakly asymptotically T-periodic. Then x is a T-periodic orbit.*

Proof The proof follows easily from Lemma 2.3. □

Theorem 3.11 *Let (X, F) be a transitive system and suppose that a point x is both weakly asymptotically periodic and transitive. Then (X, F) is a periodic orbit.*

Proof By the assumption, $\omega(x) = X$. In particular, $x \in \omega(x)$, hence by Lemma 3.10, x is a periodic orbit.

Since x is periodic, we have $\mathcal{O}(x) = \omega(x) = X$, which completes the proof. □

The difference between Theorem 3.11 and Theorem 3.5 is that in the former we assume the point with the dense limit set is weakly asymptotically periodic, while in the latter we just assume there is some point with a dense limit set. Thus, one can ask if a similar characterization as the one in Theorem 3.7 can be provided. The answer is positive.

Main Theorem 1 *Let (X, F) be a dynamical system and $x \in X$ be a transitive point. The following conditions are equivalent:*

1. *x is periodic,*
2. *x is asymptotically periodic,*
3. *x is weakly asymptotically periodic.*

Proof The implication $(3) \Rightarrow (1)$ follows from Theorem 3.11. □

Main Theorem 1 has an advantage over Theorem 3.7 in the following way: one does not have to impose compactness of the space.

Corollary 3.12 *Adding machines have no weakly asymptotically periodic points.*

Proof By Example 2.2 adding machines are minimal and cannot have periodic points. By Main Theorem 1 if there was a weakly asymptotically periodic point, it would have to be periodic. □

Corollary 3.13 *If the topological space is infinite and totally disconnected, then any dynamical system of that space that is minimal and equicontiunous cannot have any weakly asymptotically periodic point (see [13, Chapter 4]).*

Remark 3.14 *The classic definition of Devaney's chaos (see [4]) implies the existence of a transitive point with a dense set of periodic orbits. When we consider such systems with weakly asymptotically periodic points it turns out there are some limitations to that system. Chaotic (in particular transitive) systems cannot be equicontinuous (see for instance [1] ,[13]), therefore, we may only presume the orbit which is transitive is also weakly asymptotically periodic, otherwise independent consideration could lead to nothing – see Example 3.8.*

Main Theorem 1 can have a variety of consequences. Let us mention another one in the following theorem.

Theorem 3.15 *Under the assumptions of Main Theorem 1, let x be weakly asymptotically periodic with two coprime periods A and B. Then x is a fixed point.*

Proof By Theorem 3 in [10], x is weakly asymptotically T-periodic for every value of T. Applying Lemma 2.3 and Main Theorem 1 we see that x has to be periodic with two coprime periods, hence it must be a fixed point. □

4. Spectrum

In this section, we discuss weakly asymptotically periodic points in connection with the spectrum of the mapping.

Let (X, F) be a dynamical system. Recall that $f \in \mathcal{C}(X, \mathbb{C})$ is an *eigenfunction* that corresponds to a *eigenvalue* λ if $f \circ F = \lambda \cdot f$. We say that (X, F) has a *discrete spectrum*, if the set of its eigenfunctions is the basis of the vector space $\mathcal{C}(X, \mathbb{C})$, that is the set of linear combinations of eigenfunctions spans a dense subset of $\mathcal{C}(X, \mathbb{C})$ in the uniform convergence topology.

Our present goal is to prove the following theorem.

Main Theorem 2 *Let X be a compact metric space and (X, F) be a transitive dynamical system with a discrete spectrum. If $x \in X$ is weakly asymptotically periodic, then (X, F) is a periodic orbit.*

The theorem follows easily from the following proposition and the results introduced in Section 3. The proposition is a variation of Halmos and von Neumann's Theorem (see [11, 13]). For Reader's convenience we include the sketch proof of the proposition below.

Proposition 4.1 *Let X be a compact metric space and (X, F) be a transitive dynamical system with discrete spectrum. Then (X, F) is equicontinuous.*

Proof Let $(f_n)_{n \in \mathbb{N}}$ be the sequence of eigenfunctions which is dense in $\mathcal{C}(X, \mathbb{C})$. Set

$$\varrho(x, y) := \sum_{i=0}^{+\infty} \frac{|f_i(x) - f_i(y)|}{2^i \|f_i\|}.$$

Then ϱ is a metric on X which is equivalent to d. It can be checked that F is an isometry on (X, ϱ), and hence F is equicontinuous. □

Proof [Proof of Main Theorem 2] Let x be weakly asymptotically periodic.

We know from Proposition 4.1 that (X, F) is equicontinuous. It follows from Theorem 3.7 ((3) $\implies$ (1)) that x is periodic. □

It is interesting to find other conditions on the metric spaces X that mimic the thesis of Theorem 3.11 or of Main Theorem 2. The main assumption still has to be based on the existence of a weakly asymptotically periodic orbit. We therefore finish this section with the following question for further consideration: can one impose different nontrivial condition to the weakly asymptotically periodic orbit or to the space that would make the system trivial?

5. Proximality relation

Theorem 6 in the article [10] shows that if F is equicontinuous at a periodic point y and $y \in \omega(x)$ for some $x \in X$, then x is weakly asymptotically periodic (i.e. $(x, F^T(x))$ is an asymptotic pair for some $T > 0$). Here, we would like to consider a different case where the limit set might be empty. Clearly, that is not the case for dynamical systems on compact spaces, however, in other cases different scenarios can occur.

Assume that we do not have the limit set relation between two points. In general, this can be substituted by various recurrence-like relations. In this article, we suggest proximality as the main replacement. We also do a quick check for recurrence and nonwandering relations.

Note the following important example.

Example 5.1 (see Example 2.1 in [10]) *Let $X = [0, +\infty)$ and define a map*

$$F(x) = \ln(e^x + 1), \qquad x \in [0, +\infty).$$

Then $x = 0$ is weakly asymptotically T-periodic for every value of $T > 0$. However, x has an empty limit set and therefore the theory developed in [10] does not apply. On the other hand, it is easy to check that every point in this system is weakly asymptotically periodic. It is also easy to check that F is equicontinuous and proximal.

Clearly, by using equicontinuity and proximality one should be able to obtain the same result as in [10, Theorem 6], but with the lack of the limit set. In particular, one expects a more general result with proximality relation. This is exactly the main purpose of this section.

Throughout this section we will use the following two conditions:

(C) x is weakly asymptotically T-periodic,

(D) y is weakly asymptotically T-periodic.

By (C) $\Longleftrightarrow$ (D) we mean that these conditions are equivalent.

Theorem 5.2 *Let (X, F) be a dynamical system and $x, y \in X$. Assume that (x, y) is a weakly proximal pair, $T > 0$ and F is uniformly equicontinuous at y. Then (C) $\Longleftrightarrow$ (D).*

Proof Assume that (C) holds and fix $\varepsilon > 0$. Then there is $\delta > 0$ such that if for some $z \in X$ and $r \geq 0$ we have $d(z, F^r(y)) < \delta$, then for every $n \geq 0$,

$$d(F^n(z), F^{n+r}(y)) < \frac{\varepsilon}{3}.$$

A pair (x, y) is weakly proximal, hence there are sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ and a number $K \geq 0$ such that for every $k \geq K$ we have

$$d(F^{a_k}(x), F^{b_k}(y)) < \delta.$$

Let $N_1 := b_K$ and $c := a_K - b_K \in \mathbb{Z}$. Using just introduced notation the latter inequality takes the form

$$d(F^{N_1+c}(x), F^{N_1}(y)) < \delta.$$

Then, by the uniform equicontinuity at y applied to $r = N_1$ and $z = F^{N_1+c}(x)$ we conclude that for every $n \geq 0$:

$$d(F^{N_1+c+n}(x), F^{N_1+n}(y)) < \frac{\varepsilon}{3}.$$

Equivalently, for every $n \geq N_1$,

$$d(F^{n+c}(x), F^n(y)) < \frac{\varepsilon}{3}.$$

A point x is weakly asymptotically T-periodic, so there is N_2 such that for every $n \geq N_2$ we have

$$d(F^{n+T}(x), F^n(x)) < \frac{\varepsilon}{3}.$$

Define $N := \max\{N_1, N_2\}$. Then for every $n \geq N_2$,

$$d(F^{T+n}(y), F^n(y)) \leq d(F^{n+T}(y), F^{n+T+c}(x)) + d(F^{n+T+c}(x), F^{n+c}(x)) + d(F^{n+c}(x), F^n(y)) < \varepsilon.$$

This proves that y is weakly asymptotically T-periodic.

To show that (D) implies (C) we mimic the above with appropriate adjustments in the constant N. We put $N := \max\{N_1, N_2\} + c$ and then we take $n \geq N$ and adjust the triangle inequality expansion as follows:

$$d(F^{T+n}(x), F^n(x)) \leq d(F^{n+T}(x), F^{n+T-c}(y)) + d(F^{n+T-c}(y), F^{n-c}(y)) + d(F^{n-c}(y), F^n(x)) < \varepsilon.$$

This completes the proof. □

The same thesis can be obtained by changing the equicontinuity condition to a different point. By applying some straightforward and necessary changes in the proof of Theorem 5.2 we have the following.

Corollary 5.3 *In Theorem 5.2 uniform equicontinuity at y can be replaced by the uniform equicontinuity at x.*

Note a simple application of Theorem 5.2.

Corollary 5.4 *If $y \in \omega(x)$ and y or x is a point of uniform equicontinuity, then (C) $\iff$ (D).*

Proof Indeed, as $y \in \omega(x)$, there is a sequence $(a_n)_{n\in\mathbb{N}}$ satisfying

$$\lim_{n\to+\infty} d(F^{a_n}(x), y) = 0.$$

Take $b_n = 0$ for each $n \in \mathbb{N}$. Then (x, y) is a weakly proximal pair and consequently, assumptions of Theorem 5.2 are satisfied. □

Theorem 5.5 *Let (X, F) be a dynamical system, $(x, y) \in \mathcal{R}$ and $\mathcal{O}(x) \cap \mathcal{O}(y) = \emptyset$.*

(1) If x is weakly asymptotically periodic, then y is periodic or fixed.

(2) If y or x is a point of uniform equicontinuity, then (C) $\iff$ (D).

Proof Since $y \notin \mathcal{O}(x)$, $d(F^n(x), y) > 0$ for all $n \in \mathbb{N}$. In particular, choosing $\varepsilon_k = 2^{-k}$ we obtain an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$d(F^{n_k}(x), y) < 2^{-k}$$

and thus $y \in \omega(x)$. We can use Theorem 2 in [10] to complete the proof of (1). Corollary 5.4 implies (2). □

The nonwandering relation (which is close to the proximality relation) is too weak to obtain the same thesis, even at equicontinuous points. The corresponding example (see Example 6.3) is moved to the next section.

The following theorem is stated in the weaker setting just to emphasize a direct application of the above theory.

Theorem 5.6 *Let (X,F) be (weakly) proximal dynamical system such that F is uniformly equicontinuous at each point $x \in X$. Then the following conditions are equivalent:*

(E) there exists x which is (weakly) asymptotically T-periodic,

(F) every x is (weakly) asymptotically T-periodic.

Proof We only need to show that (E) implies (F). Let x be weakly asymptotically T-periodic and pick any $y \in X \setminus \{x\}$. Then (x,y) is a (weakly) proximal pair. By Theorem 5.2, y is (weakly) asymptotically T-periodic. □

Remark 5.7 *The dynamical system $([0,+\infty),F)$ in Example 5.1 has at least one weakly asymptotically T-periodic point $x=0$ for every value of T. It is easy to verify that the system is proximal and uniformly equicontinuous at each point, thus by Theorem 5.6 all points from the domain are weakly asymptotically T-periodic for every value of T.*

Remark 5.8 *Theorem 5.6 has a rather interesting counterpart which we explain with the aid of the map G from Example 2.1. In that example, orbits are dense in the circle. The point $x=0$ is not weakly asymptotically periodic for any value of T. Since the mapping is an isometry and the system is weakly proximal, by Theorem 5.6 one cannot find any weakly asymptotically periodic point in this system.*

Remark 5.9 *Note that if (X,d) is a compact metric space, then in Theorem 5.6 we can replace "uniformly equicontinuous at each point" with "equicontinuous". This theorem also creates a sort of dichotomy for equicontinuous and proximal dynamical systems. For such systems, either all orbits are weakly asymptotically periodic or none is such. Furthermore, if there exist two weakly asymptotically periodic orbits with coprime periods, then the dynamical system cannot be proximal and equicontinuous at the same time (see Theorem 3 in [10]).*

The equicontinuity is essential for the results provided in this section. For the associated example, see Example 6.1.

Weak proximality in above results cannot be trivially relaxed to regional proximality (of order d). We cannot assume that the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ consist of weakly asymptotically periodic points and expect x and y (the limit points of respective sequences) to be weakly asymptotically periodic. This is true even for periodic points.

Example 5.10 *Consider a full binary shift (Σ_2,σ). Denote by $s_2(n)$ the sum modulo 2 of all digits of the ingeter n in base 2 representation and take $x=(s_2(n) \mod 2)_{n\in\mathbb{N}}$. Then x is the Prouhet–Thue–Morse sequence [3]. It is cube–free and so it is not periodic. The sequence $(y_n)_{n\in\mathbb{N}}$, where $y_n=(x_{[0,2^n)})^\infty$, contains only periodic points and $\lim_{n\to+\infty} y_n = x$.*

The notion of regionally proximal pairs of order d was introduced in [12] to build a relation that would describe maximal equicontinuous factors of minimal systems. Here we observe that a variation of Theorem 5.2 holds for regionally proximal pair of order $d \geq 1$. Note that we assume that the map is uniformly equicontinuous at two points.

Main Theorem 3 *Let (X,F) be a dynamical system that is uniformly equicontinuous at x and y. Assume that x and y are regionally proximal of order $d \geq 1$. Then (C) $\iff$ (D).*

Proof Fix $\varepsilon > 0$. Let $\delta > 0$ be the common constant assigned to $\frac{\varepsilon}{3}$ in the definition of uniform equicontinuity of x and y and assume that $\delta \leq \frac{\varepsilon}{3}$. Thus

$$d(x,x') < \delta \quad \text{and} \quad d(y,y') < \delta$$

imply that for any $n \in \mathbb{N}$,

$$d(F^n(y'),F^n(y)) < \frac{\varepsilon}{3} \quad \text{and} \quad d(F^n(x'),F^n(x)) < \frac{\varepsilon}{3}.$$

Now choose x', y' and $\underline{n}$ from the definition of regional proximality of order d so that

$$d(F^{\underline{n}\cdot\epsilon}(x'),F^{\underline{n}\cdot\epsilon}(y')) < \delta \qquad \text{for any } \epsilon \in \{0,1\}^d, \epsilon \neq \underline{0}.$$

and

$$d(x,x') < \delta \quad \text{and} \qquad d(y,y') < \delta.$$

In particular,

$$[d(F^{\underline{n}\cdot\epsilon}(y'),F^{\underline{n}\cdot\epsilon}(y)) < \frac{\varepsilon}{3} \qquad \text{and} \qquad d(F^{\underline{n}\cdot\epsilon}(x'),F^{\underline{n}\cdot\epsilon}(x)) < \frac{\varepsilon}{3} \qquad \text{for any} \qquad \epsilon \in \{0,1\}^d, \epsilon \neq \underline{0}.]$$

Thus,

$$d(F^{\underline{n}\cdot\epsilon}(x),F^{\underline{n}\cdot\epsilon}(y)) \leq d(F^{\underline{n}\cdot\epsilon}(x),F^{\underline{n}\cdot\epsilon}(x')) + d(F^{\underline{n}\cdot\epsilon}(x'),F^{\underline{n}\cdot\epsilon}(y')) + d(F^{\underline{n}\cdot\epsilon}(y'),F^{\underline{n}\cdot\epsilon}(y)) \leq \varepsilon$$
$$\text{for any } \epsilon \in \{0,1\}^d, \epsilon \neq \underline{0}.$$

Therefore, x and y are weakly proximal and we can apply Theorem 5.2. □

One can prove Main Theorem 3 using the following reduction. Let $\mathbf{RP}^{[d]}$ be the set of all pairs in $X \times X$ that are regionally proximal of order d in a given dynamical system (X,F). It follows from the definition that

$$\ldots \subseteq \mathbf{RP}^{[d+1]} \subseteq \mathbf{RP}^{[d]} \subseteq \ldots \subseteq \mathbf{RP}^{[2]} \subseteq \mathbf{RP}^{[1]}$$

and $\mathbf{RP}^{[1]}$ is the set of regionally proximal pairs. If $(x,y) \in \mathbf{RP}^{[d]}$, then x and y are regionally proximal and it is enough to prove the case $d = 1$.

In [7] the authors introduce the so-called *regionally proximal relation of order d along arithmetic progressions* in the set $X \times X$, denoted by $\mathbf{AP}^{[d]}$. Namely, $(x,y) \in \mathbf{AP}^{[d]}$ if for any $\varepsilon > 0$ there are $x', y' \in X$ and $n \in \mathbb{N}$ such that $d(x,x') < \varepsilon$, $d(y,y') < \varepsilon$ and

$$d(F^{ni}(x'),F^{ni}(y')) < \varepsilon \qquad \text{for any } 1 \leq i \leq d.$$

Note the immediate consequence of Main Theorem 3.

Corollary 5.11 *Let (X,F) be a dynamical system that is uniformly equicontinuous at x and y. Assume that x and y are regionally proximal of order $d \geq 1$ along arithmetic progression. Then (C) $\iff$ (D).*

Proof It follows easily from the fact that for each $d \in \mathbb{N}$, $\mathbf{AP}^{[d]} \subseteq \mathbf{RP}^{[d]}$. Indeed, take $\underline{n} = (n, \ldots, n) \in \mathbb{Z}^d$, then $\underline{n} \cdot \epsilon \in \{n, 2n, \ldots, dn\}$ for any $\varepsilon \in \{0,1\}^d$, $\epsilon \neq \underline{0}$ and the inclusion follows. □

Denote $\mathbf{AP}^{[\infty]} := \bigcap_{d \geq 1} \mathbf{AP}^{[d]}$. The following shows that any weakly asymptotically periodic point gives rise to a regionally proximal pair of arbitrary order along arithmetic progression. Since $\mathbf{AP}^{[d]} \subseteq \mathbf{RP}^{[d]}$, it also gives rise to a regionally proximal pair of arbitrary order.

Theorem 5.12 *Let (X, F) be a dynamical system and $x \in X$ is weakly asymptotically T-periodic. Then $(x, F^T(x)) \in \mathbf{AP}^{[\infty]}$.*

Proof Let $\varepsilon > 0$ and $N \geq 0$ be such that for any $n \geq N$ we have

$$d(F^n(x), F^{n+T}(y)) < \varepsilon.$$

It follows that for any $d \geq 1$ and for any $1 \leq i \leq d$,

$$d(F^{Ni}(x), F^{Ni}(F^T(x))) < \varepsilon,$$

hence for any $d \geq 1$, $(x, F^T(x)) \in \mathbf{AP}^{[d]}$. □

It is essential that in Main Theorem 3 both x and y are points of equicontinuity. The necessity of that assumption is described by Example 6.3.

Note that Main Theorem 3 gives us another proof of the second part of Theorem 5.5 (if $y \in \omega(x)$, then x and y are regionally proximal).

Remark 5.13 *By [5] and [6] if (X, F) is a noninvertible dynamical system on a compact metric space, then a set of points belonging to a proper asymptotic pair has measure 1 for any ergodic measure of positive entropy. Moreover, a positive measure is necessary since if there are no asymptotic pairs (recall that we set pairs to be proper, i.e. $x \neq y$), the entropy is zero. If in such a rich set of asymptotic pairs, we find one of the form $(x, F^T(x))$ with F being uniformly equicontinous at x, then any point y that is proximal (or asymptotic) to x is weakly asymptotically T-periodic. This in particular applies to any $y \in \omega(x)$ (even without equicontinuity, see Theorem 2 in [10]).*

6. Examples

In this section, we provide a series of dynamical systems that highlight the necessity of assumptions used in previous sections.

The following example is a classic symbolic dynamics system. We follow the notation and definitions from [13].

Example 6.1 *Consider a full binary shift (Σ_2, σ) equipped with the metric*

$$d(x, y) = \begin{cases} 0, & x = y \\ 2^{-k}, & k = \min\{n \in \mathbb{N} \mid x_n \neq y_n\}. \end{cases}$$

Here σ denotes the usual shift map $\sigma((x_0, x_1, x_2, \ldots)) = (x_1, x_2, x_3, \ldots)$. Then (Σ_2, d) is a compact metric space. We use the usual notation $x = x_0 x_1 x_2 \ldots$ for elements in Σ_2.

Let $a = 01$ and $b = 00$ and set $x \in \Sigma_2$ as $x = a^\infty$. Consider a sequence $(w_k)_{k\in\mathbb{N}}$ of finite-length words defined inductively:

$$w_0 = 0, \qquad w_1 = ab, \qquad w_k = w_{k-1}a^k b^k, \qquad k \geq 2.$$

Set $y_k = w_k 0^\infty$. Then each $y_k \in \Sigma_2$ and the sequence $(y_k)_{k\in\mathbb{N}}$ is convergent; let y denote its limit.

We claim that the following conditions are satisfied:

(1) (Σ_2, σ) has no point of equicontinuity,

(2) (x, y) a is proximal pair,

(3) x is weakly asymptotically 2-periodic,

(4) y is not weakly asymptotically periodic for any value of T.

Take $\varepsilon = \frac{1}{2}$ from the definition of equicontinuity and any $\delta > 0$. Let u be any point in Σ_2 and $v \neq u$ be such that $d(u, v) < \delta$. Let $k = \min\{n \in \mathbb{N} \mid u_n \neq v_n\}$. Then $d(\sigma^{k+1}(u), \sigma^{k+1}(v)) = 1$, but that contradicts with the choice of ε. Condition (1) is therefore satisfied.

For each $n \in \mathbb{N}$ define a number $a_n = 2n(n+1)$. Such numbers form a sequence satisfying the conditions

$$\sigma^{a_n}(x) = x, \qquad \sigma^{a_n}(y)_{[0,4n+4)} = a^{n+1}b^{n+1}.$$

Note that the latter equality completely describes y. In particular, $\sigma^{a_n}(x)_{[0,2n+3)} = \sigma^{a_n}(y)_{[0,2n+3)} = (01)^{n+1}0$ but at the same time, $1 = (\sigma^{a_n}(x))_{2n+3} \neq (\sigma^{a_n}(y))_{2n+3} = 0$. Consequently

$$d(\sigma^{a_n}(x), \sigma^{a_n}(y)) = 2^{-(2n+3)}$$

and (x, y) is a proximal pair. Therefore, condition (2) is satisfied.

The sequence $x = 01010101\ldots$ is 2-periodic, hence condition (3) is satisfied.

To show condition (4) it is sufficient to check even values of T. Indeed, if T were odd, then by Corollary 2.1 in [10] it would satisfy the property with an even multiplicity of T which is an even number again.

Let $T = 2\ell$ and $\varepsilon = 2^{-k}$ for some $\ell > 0$ and $k > 1$. Assume $N > 0$ is such that if $n \geq N$, then $d(\sigma^{n+2\ell}(y), \sigma^n(y)) < 2^{-k}$. By the construction of y we can find $n_0 \geq N$ satisfying

$$\sigma^{n_0}(y)_{[0,2\ell)} = 0^{2\ell} = b^\ell,$$
$$\sigma^{n_0+2\ell}(y)_{[0,2\ell)} = a^\ell.$$

Moreover, there are infinitely many pairwise different numbers with the last two properties. They form a sequence $(n_i)_{i\in\mathbb{N}}$. For each such n_i we have

$$d(\sigma^{n_i}(y), \sigma^{n_i+2\ell}(y)) = \frac{1}{2} > 2^{-k}$$

for every $k > 1$. Thus y cannot be weakly asymptotically 2ℓ-periodic.

Conditions (1)–(4) are therefore met and they show that equicontinuity is essential in Theorem 5.2 and also in Theorem 5.6. Proximality however cannot be skipped by default, as without this condition we have no control over the mutual relation between x and y.

Remark 6.2 *Example 6.1 can be further generalized to transitive systems that are not equicontinuous. In such systems, a pair consisting of a transitive point and an arbitrary one is (at least) a weakly proximal pair and also the system is, in some way, not so trivial. The actual structure of such a system is a subject for different research.*

Example 6.3 *Consider a dynamical system (X, F), where $X \subset \mathbb{C}$ is defined as follows. Let $n > 0$ and denote*

$$U_n = \{z \in \mathbb{C} \mid z^n = 1\}.$$

For simplicity, let us also denote $H = \left\{\frac{1}{n} \mid n > 4\right\}$. Put $x_0 := \frac{3}{2}$. Take any two positive natural numbers a and b and define

$$X := U_a \cdot (1 + H) \cup U_b \cdot (2 - H) \cup \{x_0\}.$$

We now define a homeomorphism F on X with the aid of Figure 2.

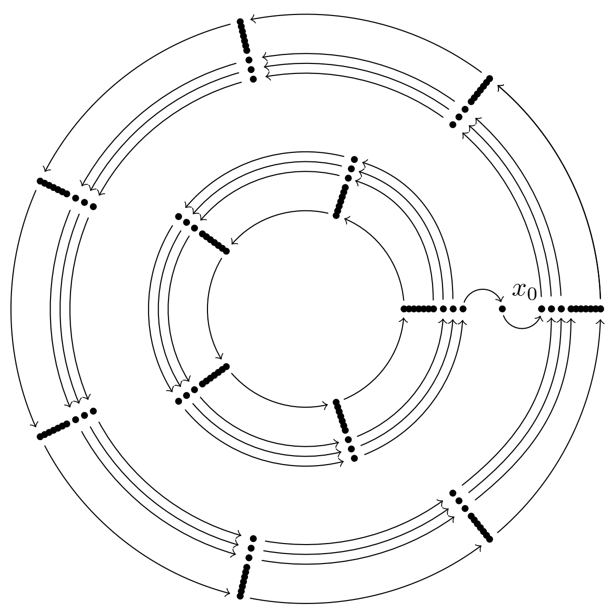

Figure 2. A sketch of the dynamical system (X, F) for the case $a = 5$ and $b = 7$. The scale and relative proportions are not preserved.

It can be explicitly defined using the following formulas. Let us first define F on sets U_a and U_b:

$$F(x) = \begin{cases} x \exp\left(\frac{2\pi i}{a}\right), & x \in U_a, \\ x \exp\left(\frac{2\pi i}{b}\right), & x \in U_b. \end{cases}$$

On the remaining portion of the space we define it by saying what is the n-th positive and n-th negative iteration *of* x_0*:*

$$F^n(x_0) = \left(2 - \frac{1}{5 + E\left(\frac{n-1}{b}\right)}\right) \exp\left(2\pi i \left(\frac{n-1}{b} \mod 1\right)\right), \qquad n > 0,$$

$$F^{-n}(x_0) = \left(1 + \frac{1}{5 + E\left(\frac{n-1}{a}\right)}\right) \exp\left(2\pi i \left(\frac{a+1-n}{a} \mod 1\right)\right), \qquad n > 0.$$

The function $E(x)$ *used above is the integer part of* x*. See Figure* **??** *for the sketch of* (X, F)*.*

One can verify that if $x \in U_a$ *and* $y \in U_b$*, then* $(x, y) \in \mathcal{N}$*. Moreover,* x *is* a*-periodic,* y *is* b*-periodic and* $y \in \mathcal{E}$*. This is however not enough to ensure that* a *and* b *have the exact same periodicity.*

By reversing the direction in which points are iterated under F *and changing the number on the subscript of the sets* U_n *to different ones (particularly to obtain points with coprime periods) we see that no set of assumptions such as proximity or (uniform) equicontinuity can ensure any desired relation between* x *and* y*. In particular, equicontinuity posed in Theorem 5.5 is essential and equicontinuity of both points in Main Theorem 3 is also essential.*

Remark 6.4 *Example 6.1 also shows that equicontinuity is essential for theorems in Section 3 - there is a weakly asymptotically periodic point which is not equicontinuous, but selected orbits that are proximal are not weakly asymptotically periodic*

References

[1] Akin E, Auslander J, Berg K. In: Bergelson V, Martch P, Rosenblatt J (editors). Conference in Ergodic Theory and Probability. Berlin, New York: De Gruyter, 2011; pp. 25-40. https://doi.org/10.1515/9783110889383.25

[2] Alligood KT, Sauer TD, Yorke JA. Chaos: An Introduction to Dynamical Systems. Springer - Verlang, New York, 1996. https://doi.org/10.1007/b97589

[3] Allouche JP, Shallit J. The Ubiquitous Prouhet–Thue–Morse Sequence. In: Ding C, Helleseth T, Niederreiter H (editorss). Sequences and Their Applications. Discrete Mathematics and Theoretical Computer Science. Springer, London, 1999; pp. 1-16. https://doi.org/10.1007/978-1-4471-0551-0_1

[4] Banks J, Brooks J, Cairns G, Davis G, Stacey P. On Devaney's definition of chaos. The American Mathematical Monthly 1992; 99 (4): 332-334. https://doi.org/10.2307/2324899

[5] Blanchard F, Host B, Ruette S. Asymptotic pairs in positive-entropy systems. Ergodic Theory and Dynamical Systems 2002; 22 (3): 671-686. https://doi.org/10.1017/S0143385702000342

[6] Downarowicz T, Lacroix Y. Topological entropy zero and asymptotic pairs. Israel Journal of Mathematics 2012; 189: 323-336. https://doi.org/10.1007/s11856-011-0174-6

[7] Glasner E, Huang W, Shao S, Xiangdong Y. Regionally proximal relation of order d along arithmetic progressions and nilsystems. Science China Mathematics 2020; 63: 1757-1776. https://doi.org/10.1007/s11425-019-1607-5

[8] Glasner E, Weiss B. Sensitive dependence on initial conditions. Nonlinearity 1993; 6: 1067-1075. https://doi.org/10.1088/0951-7715/6/6/014

[9] Gottschalk WH, Hedlund GA. Topological Dynamics. American Mathematical Society Colloquium Publications, vol. 36. Providence, American Mathematical Society, 1955. https://doi.org/doi:10.2307/3609454

[10] Gryszka K. On weak asymptotic periodicity. International Journal of Bifurcation and Chaos 2020; 30 (2): 2050030. https://doi.org/10.1142/S0218127420500303

[11] Halmos PR, von Neumann J. Operator methods in classical mechanics II. Annals of Mathematics 1942; 43: 332-350. https://doi.org/10.2307/1968872

[12] Host B, Kra B, Maass A. Nilsequences and a structure theory for topological dynamical systems. Advances in Mathematics 2010; 224: 103-129. https://doi.org/10.1016/j.aim.2009.11.009

[13] Kůrka P. Topological and Symbolic Dynamics. Paris: Société mathématique de France, 2003.

[14] Li TY, Yorke JA. Period three implies chaos. The American Mathematical Monthly 1975; 82 (10): 985-992. https://doi.org/10.2307/2318254

[15] Pravec P. Remarks on definitions of periodic points for nonautonomous dynamical system. Journal of Difference Equations and Applications 2019; 25: 1372-1381. https://doi.org/10.1080/10236198.2019.1641496