Turkish Journal of Mathematics

Volume 47 | Number 7 Article 6

11-9-2023

# Fibonomial matrix and its domain in the spaces $\ell_p$ and $\ell_{\infty}$

MUHAMMET CİHAT DAĞLI

TAJA YAYING

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 1915 – 1931
© TÜBİTAK
doi:10.55730/1300-0098.3472

# Fibonomial matrix and its domain in the spaces $\ell_p$ and $\ell_\infty$

**Muhammet Cihat DAĞLI**[1,*], **Taja YAYING**[2]
[1]Department of Mathematics, Akdeniz University, 07058-Antalya, Turkiye
[2]Department of Mathematics, Dera Natung Government College, Itanagar, India

**Received:** 04.02.2022 • **Accepted/Published Online:** 01.09.2023 • **Final Version:** 09.11.2023

**Abstract:** In this paper, we introduce the fibonomial sequence spaces $b_p^{r,s,F}$ and $b_\infty^{r,s,F}$, and show that these are BK-spaces. Also, we prove that these new spaces are linearly isomorphic to $\ell_p$ and $\ell_\infty$. Moreover, we determine the $\alpha$-, $\beta$-, $\gamma$-duals for these new spaces and characterize some matrix classes. The final section is devoted to the investigation of some geometric properties of the newly defined space $b_p^{r,s,F}$.

**Key words:** Fibonomial sequence spaces, Schauder basis, $\alpha$-, $\beta$-, $\gamma$-duals, matrix transformations, geometric properties

## 1. Introduction

The theory of sequence spaces can be regarded as a fundamental subject in summability that has many important applications, mainly in functional analysis. The classical summability theory deals with the generalization of the convergence of sequences or series of real or complex numbers. The idea is to assign a limit of some sort to divergent sequences or series by considering a transform of a sequence or series rather than the original sequence or series. One can ask why we employ the special transformations represented by infinite matrices instead of general linear operators? The answer to this question is, in many cases, the general linear operators between two sequence spaces is given by an infinite matrix. So, the theory of matrix transformations has always been of great interest in the study of sequence spaces. The study of the general theory of matrix transformations was motivated by special results in summability theory [6].

Indeed, the theory of matrix transformations deals with establishing the necessary and sufficient conditions on the entries of a matrix to map a sequence space $X$ into a sequence space $Y$. This is a natural generalization of the problem to characterize all summability methods given by infinite matrices that preserve convergence.

In this study, $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{R}$ denotes the set of all real numbers. For simplicity in notation, here and in what follows, the summation without limits runs from $0$ to $\infty$, if not stated.

A sequence space is a linear subspace of the set of all real-valued sequences $\omega$. As well known examples, we give $\ell_\infty, c, c_0$ and $\ell_p$ as the set of all bounded sequences, the set of all convergent sequences, the set of all convergent to zero sequences and the set of all sequences constituting $p-$absolutely convergent series,

*Correspondence: mcihatdagli@akdeniz.edu.tr
2010 *AMS Mathematics Subject Classification:* 46A45, 46B45, 40H05, 40C05.

respectively. These are Banach spaces with the following norms

$$\|x\|_{\ell_\infty} = \|x\|_c = \|x\|_{c_0} = \sup_{k\in\mathbb{N}} |x_k| \quad \text{and} \quad \|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p},$$

respectively. A Banach space with all coordinate functionals $p_k$ denoted by $p_k(x) = x_k$ are continuous is referred as a BK-space.

Let $T = (t_{nk})$ be an infinite matrix with real entries $t_{nk}$ and $T_n$ be the sequence in the $n$th row of the matrix $T$ for each $n \in \mathbb{N}$. The $T-$transform of a sequence $x = (x_k) \in \omega$ is the sequence $Tx$ obtained by the usual matrix product and its entries are stated as

$$(Tx)_n = \sum_k t_{nk} x_k$$

provided that the series is convergent for each $n \in \mathbb{N}$. The matrix $T$ is called as a matrix mapping from a sequence space $\lambda$ to a sequence space $\mu$ if the sequence $Tx$ exists and $Tx \in \mu$ for all $x \in \lambda$. The collection of all infinite matrices from $\lambda$ to $\mu$ is denoted by $(\lambda, \mu)$.

Recall that the set

$$\lambda_T = \{x \in \omega : Tx \in \lambda\}$$

is called the domain of the infinite matrix $T$ in the space $\lambda$. For the last two decades, the concept of domains of special triangular matrices has attacted many scholars. One may refer to these nice articles [2–5, 7, 9–12, 14, 16, 31, 32] and the textbook [6] for relevant studies.

The Euler sequence spaces $e_p^r = (\ell_p)_{E^r}$, $e_0^r = (c_0)_{E^r}$, $e_c^r = c_{E^r}$,and $e_\infty^r = (\ell_\infty)_{E^r}$ are introduced by Altay et al. [1, 2], where $E^r = (e_{nk}^r)$ denotes the Euler means of order $r$ defined by

$$e_{nk}^r = \begin{cases} \binom{n}{k} (1-r)^{n-k} r^k, & \text{if } 0 \leq k \leq n; \\ 0, & \text{if } k > n; \end{cases}$$

for all $k, n \in \mathbb{N}$. Apart from the papers [1, 2], the studies concerning the construction of new sequence spaces by employing Euler matrix via the matrix domain of a particular limitation method have been considered by Altay and Polat [3], Kara and Başarır [14], Karakaya and Polat [15], Polat and Başar [26], and Mursaleen et al. [23].

Bişgin [9, 10] gave further extension of Euler sequence spaces by introducing binomial sequence spaces $b_p^{r,s} = (\ell_p)_{B^{r,s}}$, $b_0^{r,s} = (c_0)_{B^{r,s}}$, $b_c^{r,s} = c_{B^{r,s}}$ and $b_\infty^{r,s} = (\ell_\infty)_{B^{r,s}}$ by means of the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$

$$b_{nk}^{r,s} = \begin{cases} \dfrac{1}{(s+r)^n} \binom{n}{k} r^k s^{n-k}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{if } k > n. \end{cases}$$

The author discussed various topological and geometric properties in [9]. It was shown that the spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ are BK-spaces and linearly isomorphic to $\ell_p$ and $\ell_\infty$, respectively. Also, the Schauder basis and $\alpha$-, $\beta$- and $\gamma$-dual of these spaces were determined.

For more details about binomial sequence spaces and their generalizations, the readers can consult the studies [13, 20–22, 29, 30].

Let $(F_n)$ denote the sequence of Fibonacci numbers defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. Thus $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$ are the first few Fibonacci numbers. In recent years, few authors discussed Fibonacci (or $F$-) or Golden calculus in their papers [18, 19, 24, 25].

The Fibonomial coefficient (cf. [19]) is defined by

$$\binom{n}{k}_F = \frac{F_n!}{F_k! F_{n-k}!},$$

where $0 \leq k \leq n$ and

$$F_n! = F_n F_{n-1} \ldots F_1, \quad F_0! = 1$$

denotes the $F$-factorial. Note that $\binom{n}{0}_F = 1$ and $\binom{n}{k}_F = 0$ for $n < k$.

The followings are some properties (cf. [19]) sufficed by fibonomial coefficients:

$$\binom{n}{k}_F = \binom{n}{n-k}_F,$$

$$\binom{n}{k}_F \binom{k}{i}_F = \binom{n}{i}_F \binom{n-i}{k-i}_F,$$

$$(x+y)_F^n = \sum_{k=0}^{n} \binom{n}{k}_F x^k y^{n-k}. \text{ (Fibonomial Theorem)}$$

In this paper, motivated by [10], we introduce the Fibonomial sequence spaces $b_p^{r,s,F}$ and $b_\infty^{r,s,F}$ which include the spaces $\ell_p$ and $\ell_\infty$, respectively. Also, we demonstrate that the sequence spaces $b_p^{r,s,F}$ and $b_\infty^{r,s,F}$ are BK-spaces and linearly isomorphic to the spaces $\ell_p$ and $\ell_\infty$, respectively. Furthermore, we offer $\alpha-$dual, $\beta-$dual and $\gamma-$dual for these spaces and characterize some matrix classes. In the final section, we exhibit some geometric properties of the space $b_p^{r,s,F}$.

## 2. Fibonomial sequence spaces

Let $s, r$ be nonzero real numbers such that $s + r \neq 0$. Then, we introduce fibonomial matrix $B^{r,s,F} = \left(b_{nk}^{r,s,F}\right)$ defined by

$$b_{nk}^{r,s,F} = \begin{cases} \frac{1}{(s+r)_F^n} \binom{n}{k}_F r^k s^{n-k}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{if } k > n. \end{cases}$$

One can easily verify that the following properties are valid for $rs > 0$ :

*i)* $\left\| B^{r,s,F} \right\| < \infty$,

*ii)* $\lim_{n\to\infty} b_{nk}^{r,s,F} = 0$ for each $k$,

*iii)* $\lim_{n\to\infty} \sum_k b_{nk}^{r,s,F} = 1$.

This leads us to the fact that the fibonomial matrix is regular for $rs > 0$. Here and henceforth, we assume that $rs > 0$ unless otherwise stated.

By taking into consideration the fibonomial matrix, let us introduce the fibonomial sequence spaces $b_p^{r,s,F}$ and $b_\infty^{r,s,F}$ as follows:

$$\begin{aligned} b_p^{r,s,F} &= \left\{ x=(x_k)\in\omega : \sum_n \left| \frac{1}{(s+r)_F^n} \sum_{k=0}^{n} \binom{n}{k}_F r^k s^{n-k} x_k \right|^p < \infty \right\}, \\ b_\infty^{r,s,F} &= \left\{ x=(x_k)\in\omega : \sup_n \left| \frac{1}{(s+r)_F^n} \sum_{k=0}^{n} \binom{n}{k}_F r^k s^{n-k} x_k \right| < \infty \right\}. \end{aligned}$$

That is to say that

$$b_p^{r,s,F} = (\ell_p)_{B^{r,s,F}} \quad \text{and} \quad b_\infty^{r,s,F} = (\ell_\infty)_{B^{r,s,F}}.$$

Define a sequence $y=(y_k)$ by

$$y_k = \left(B^{r,s,F}x\right)_k = \frac{1}{(s+r)_F^k} \sum_{i=0}^{k} \binom{k}{i}_F r^i s^{k-i} x_i, \tag{2.1}$$

which shall be known as $B^{r,s,F}$-transform of the sequence $x=(x_k)$ in the rest of the paper.

It is worth notable that with the help of Theorem 4.3.2 of Wilansky [28], the fibonomial sequence spaces $b_p^{r,s,F}$ and $b_\infty^{r,s,F}$ are BK-spaces depending on the norms $\|x\|_{b_p^{r,s,F}} = \left\|B^{r,s,F}x\right\|_{\ell_p} = \left(\sum_n \left|\left(B^{r,s,F}x\right)_n\right|^p\right)^{1/p}$ and $\|x\|_{b_\infty^{r,s,F}} = \left\|B^{r,s,F}x\right\|_{\ell_\infty} = \sup_{n\in\mathbb{N}} \left|\left(B^{r,s,F}x\right)_n\right|$, respectively, where $1\le p<\infty$.

**Theorem 2.1** *The fibonomial sequence spaces $b_p^{r,s,F}$ and $b_\infty^{r,s,F}$ are linearly isomorphic to $\ell_p$ and $\ell_\infty$, respectively.*

**Proof** We present that $b_p^{r,s,F}$ linearly isomorphic to $\ell_p$ for $1\le p<\infty$. To do this, we must provide that there exists a linear transformation between these spaces which is injective, surjective and norm-preserving. For any $x\in b_p^{r,s,F}$, let $L: b_p^{r,s,F}\to\ell_p$ be a transformation such that $L(x)=B^{r,s,F}x$. The linearity of $L$ is clear by using the fact that any matrix transformation is linear. Also, the transformation $L$ is injective by employing that if $L(x)=(0,0,...,0,...)$, then $x=(0,0,...,0,...)$. For any sequence $y=(y_k)\in\ell_p$, if the sequence $x=(x_k)$ is denoted for $n\in\mathbb{N}$ by

$$x_k = \frac{1}{r^k}\sum_{i=0}^{k}\binom{k}{i}_F (-s)^{k-i}(s+r)_F^i\, y_i, \tag{2.2}$$

then, we have

$$\begin{aligned} \|x\|_{b_p^{r,s,F}} = \left\|B^{r,s,F}x\right\|_{\ell_p} &= \left(\sum_{n=1}^{\infty}\left|\left(B^{r,s,F}x\right)_n\right|^p\right)^{1/p} \\ &= \left(\sum_{n=1}^{\infty}\left|\frac{1}{(s+r)_F^n}\sum_{k=0}^{n}\binom{n}{k}_F r^k s^{n-k}x_k\right|^p\right)^{1/p} \\ &= \left(\sum_{n=1}^{\infty}\left|\frac{1}{(s+r)_F^n}\sum_{k=0}^{n}\binom{n}{k}_F s^{n-k}\sum_{i=0}^{k}\binom{k}{i}_F (-s)^{k-i}(s+r)_F^i\, y_i\right|^p\right)^{1/p} \end{aligned}$$

$$= \left( \sum_{n=1}^{\infty} |y_n|^p \right)^{1/p} = \|y\|_{\ell_p} = \|L(x)\|_{\ell_p} < \infty.$$

Thus, $L$ is norm-preserving and $x \in b_p^{r,s,F}$, consequently, $L$ is surjective. The other case of the theorem can be verified analogously. Hence, the proof is completed. □

Before determining the Schauder basis for the matrix domain of our special triangular matrix, we recall the definition of Schauder basis. The sequence $(\delta_n)$ is called a Schauder basis for the space $\lambda$ if given any $x \in \lambda$, there exists a unique sequence of scalars $\tau_n$ such that

$$\left\| x - \sum_{k=0}^{n} \tau_k \delta_k \right\| \to 0, \quad \text{as } n \to \infty$$

for a normed space $(\lambda, \|.\|)$ and a sequence $\delta_n$ in $\lambda$. Then, we write

$$x = \sum_{k=0}^{\infty} \tau_k \delta_k.$$

**Theorem 2.2** *Let* $\mu_k = \left\{ B^{r,s,F} x \right\}_k$ *be given for all* $k \in \mathbb{N}$. *Let the sequence* $s^{(k)}(r,s,F) = \left\{ s_n^{(k)}(r,s,F) \right\}_{n \in \mathbb{N}}$ *be denoted as the elements of the binomial sequence space* $b_p^{r,s,F}$ *by*

$$s_n^{(k)}(r,s,F) = \begin{cases} \frac{1}{r^n} \binom{n}{k}_F (-s)^{n-k} (s+r)_F^k, & \text{if } 0 \le k \le n; \\ 0, & \text{if } k > n. \end{cases}$$

*Then, the sequence* $\left\{ s^{(0)}(r,s,F), s^{(1)}(r,s,F), \ldots \right\}$ *is a basis for the space* $b_p^{r,s,F}$ *and any* $x$ *in* $b_p^{r,s,F}$ *is uniquely determined as* $x = \sum_k \mu_k s^{(k)}(r,s,F)$, *where* $1 \le p < \infty$.

**Proof** Given any $x = (x_k) \in b_p^{r,s,F}$ for $1 \le p < \infty$. Then, we consider for all non negative integer $m$ that

$$x^{[m]} = \sum_{k=0}^{m} \mu_k s^{(k)}(r,s,F).$$

Now, applying the fibonomial matrix $B^{r,s,F} = \left( b_{nk}^{r,s,F} \right)$ to $x^{[m]}$ leads to

$$B^{r,s,F} x^{[m]} = \sum_{k=0}^{m} \mu_k B^{r,s,F} s^{(k)}(r,s,F) = \sum_{k=0}^{m} \left( B^{r,s,F} x \right)_k e^{(k)}$$

and

$$\left\{ B^{r,s,F} \left( x - x^{[m]} \right) \right\}_n = \begin{cases} \left( B^{r,s,F} x \right)_n, & \text{if } n > m; \\ 0, & \text{if } 0 \le n \le m; \end{cases}$$

for all $n, m \in \mathbb{N}$. For any given $\varepsilon > 0$, there exists a non-negative integer $m_0$ such that

$$\sum_{n=m_0+1}^{\infty} \left| \left( B^{r,s,F} x \right)_n \right|^p \le \left( \frac{\varepsilon}{2} \right)^p$$

for all $m \geq m_0$. So, we have

$$\begin{aligned}\left\|x - x^{[m]}\right\|_{b_p^{r,s,F}} &= \left(\sum_{n=m+1}^{\infty} \left|\left(B^{r,s,F}x\right)_n\right|^p\right)^{1/p} \\ &\leq \left(\sum_{n=m_0+1}^{\infty} \left|\left(B^{r,s,F}x\right)_n\right|^p\right)^{1/p} \\ &\leq \frac{\varepsilon}{2} < \varepsilon, \quad \text{for all } m \geq m_0,\end{aligned}$$

which concludes that

$$x = \sum_k \mu_k s^{(k)}(r,s,F).$$

To prove the uniqueness of this representation, let

$$x = \sum_k \mu'_k s^{(k)}(r,s,F)$$

be another representation of $x$. Then, it is readily seen for every $n \in \mathbb{N}$ that

$$\begin{aligned}\left(B^{r,s,F}x\right)_n &= \sum_k \mu'_k \left\{B^{r,s,F} s^{(k)}(r,s,F)\right\}_n \\ &= \sum_k \mu'_k e_n^{(k)} \\ &= \mu'_n,\end{aligned}$$

which contradicts the representation $\left(B^{r,s,F}x\right)_n = \mu_n$ for every $n \in \mathbb{N}$. So, the proof is completed. □

Combining the fact that $b_p^{r,s,F}$ is Banach space for $1 \leq p < \infty$ together with Theorem 2.2, we get the following corollary.

**Corollary 2.3** *The fibonomial sequence space* $b_p^{r,s,F}$ *is separable for* $1 \leq p < \infty$.

## 3. $\alpha$-, $\beta$-, $\gamma$-duals

This section is devoted to present $\alpha$-dual, $\beta$-dual and $\gamma$-dual of fibonomial sequence spaces $b_p^{r,s,F}$ and $b_\infty^{r,s,F}$.

The multiplier space of $\lambda$ and $\mu$ is the set $S(\lambda,\mu)$ defined by

$$S(\lambda,\mu) = \{u \in \omega : zu \in \mu \text{ for all } z \in \lambda\}.$$

By adopting this notation, the $\alpha$-dual, $\beta$-dual and $\gamma$-dual of a sequence space $\lambda$ are defined by

$$\lambda^\alpha = S(\lambda,\ell_1), \quad \lambda^\beta = S(\lambda,cs) \text{ and } \lambda^\gamma = S(\lambda,bs).$$

Here, $cs$ and $bs$ denote the spaces of all convergent and bounded series, respectively.

We begin with the following lemma, which is an effective tool to discover the $\alpha$-dual, $\beta$-dual and $\gamma$-dual of the fibonomial sequence spaces $b_p^{r,s,F}$ and $b_\infty^{r,s,F}$. Note that $\mathcal{F}$ denotes the family of all finite subsets of $\mathbb{N}$ and $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p \leq \infty$.

**Lemma 3.1** *([27]) $T=(t_{nk})\in(\ell_1,\ell_1)$ if and only if*

$$\sup_{k\in\mathbb{N}}\sum_n |t_{nk}|<\infty.$$

*$T=(t_{nk})\in(\ell_1,\ell_\infty)$ if and only if*

$$\sup_{n,k\in\mathbb{N}} |t_{nk}|<\infty. \tag{3.1}$$

*$T=(t_{nk})\in(\ell_1,c)$ if and only if (3.1) holds and*

$$\lim_{n\to\infty} t_{nk} \text{ exists} \tag{3.2}$$

*for each $k\in\mathbb{N}$. $T=(t_{nk})\in(\ell_p,\ell_\infty)$ if and only if*

$$\sup_n \sum_k |t_{nk}|^q<\infty, \tag{3.3}$$

*where $1<p<\infty$. $T=(t_{nk})\in(\ell_p,c)$ if and only if (3.2) and (3.3) hold for $1<p<\infty$. $T=(t_{nk})\in(\ell_p,\ell_1)$ if and only if*

$$\sup_{K\in F}\sum_k\left|\sum_{n\in K}^{\infty} t_{nk}\right|^q<\infty \quad \text{for } 1<p<\infty.$$

*$T=(t_{nk})\in(\ell_\infty,\ell_\infty)=(c,\ell_\infty)$ if and only if (3.3) holds for $q=1$. $T=(t_{nk})\in(\ell_\infty,c)$ if and only if (3.2) holds and*

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}|t_{nk}|=\sum_{k=0}^{\infty}\left|\lim_{n\to\infty}t_{nk}\right|. \tag{3.4}$$

**Theorem 3.2** *Define the sets*

$$\sigma_1^{r,s,F} = \left\{b=(b_k)\in\omega:\sup_{K\in F}\sum_k\left|\sum_{n\in K}\binom{n}{k}_F(-s)^{n-k}r^{-n}(s+r)_F^k\,b_n\right|^q<\infty\right\},$$

$$\sigma_2^{r,s,F} = \left\{b=(b_k)\in\omega:\sup_{k\in\mathbb{N}}\sum_n\left|\binom{n}{k}_F(-s)^{n-k}r^{-n}(s+r)_F^k\,b_n\right|<\infty\right\}.$$

*Then, $\left\{b_1^{r,s,F}\right\}^\alpha=\sigma_2^{r,s,F}$ and $\left\{b_p^{r,s,F}\right\}^\alpha=\sigma_1^{r,s,F}$, where $1<p\le\infty$.*

**Proof** For any $b=(b_n)\in\omega$, one can write from (2.2) that

$$b_nx_n=\sum_{k=0}^{n}\binom{n}{k}_F(-s)^{n-k}r^{-n}(s+r)_F^k\,b_ny_k=\left(G^{r,s,F}y\right)_n$$

for all $n\in\mathbb{N}$. So, we have $bx=(b_nx_n)\in\ell_1$ whenever $x=(x_k)\in b_1^{r,s,F}$ or $x=(x_k)\in b_p^{r,s,F}$ if and only if $G^{r,s,F}y\in\ell_1$ whenever $y=(y_k)\in\ell_1$ or $y=(y_k)\in\ell_p$, respectively, for $1<p\le\infty$. This tells us

$b=(b_n)\in\left\{b_1^{r,s,F}\right\}^{\alpha}$ or $b=(b_n)\in\left\{b_p^{r,s,F}\right\}^{\alpha}$ if and only if $G^{r,s,F}\in(\ell_1,\ell_1)$ or $G^{r,s,F}\in(\ell_p,\ell_1)$, respectively, for $1<p\leq\infty$. Combine the relevant part of Lemma 3.1 and these facts to get

$$b=(b_n)\in\left\{b_1^{r,s,F}\right\}^{\alpha}$$

if and only if

$$\sup_{k\in\mathbb{N}}\sum_{n}\left|\binom{n}{k}_F(-s)^{n-k}r^{-n}(s+r)_F^k b_n\right|<\infty$$

or

$$b=(b_n)\in\left\{b_p^{r,s,F}\right\}^{\alpha}$$

if and only if

$$\sup_{K\in\mathcal{F}}\sum_{k}\left|\sum_{n\in K}\binom{n}{k}_F(-s)^{n-k}r^{-n}(s+r)_F^k b_n\right|^q<\infty,$$

respectively, where $1<p\leq\infty$. So, the proof is completed. □

**Theorem 3.3** *Define the sets*

$$\sigma_3^{r,s,F}=\left\{b=(b_k)\in\omega:\sum_{i=k}^{\infty}\binom{i}{k}_F(-s)^{i-k}r^{-i}(s+r)_F^k b_i \text{ exists for each } k\in\mathbb{N}\right\},$$

$$\sigma_4^{r,s,F}=\left\{b=(b_k)\in\omega:\sup_{n,k}\left|\sum_{i=k}^{n}\binom{i}{k}_F(-s)^{i-k}r^{-i}(s+r)_F^k b_i\right|<\infty\right\},$$

$$\sigma_5^{r,s,F}=\left\{b=(b_k)\in\omega:\lim_{n\to\infty}\sum_{k}\left|\sum_{i=k}^{n}\binom{i}{k}_F(-s)^{i-k}r^{-i}(s+r)_F^k b_i\right|=\sum_{k}\left|\sum_{i=k}^{\infty}\binom{i}{k}_F(-s)^{i-k}r^{-i}(s+r)_F^k b_i\right|\right\},$$

$$\sigma_6^{r,s,F}=\left\{b=(b_k)\in\omega:\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\sum_{i=k}^{n}\binom{i}{k}_F(-s)^{i-k}r^{-i}(s+r)_F^k b_i\right|^q<\infty\right\},\ (1<q<\infty)$$

$$\sigma_7^{r,s,F}=\left\{b=(b_k)\in\omega:\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\sum_{i=k}^{n}\binom{i}{k}_F(-s)^{i-k}r^{-i}(s+r)_F^k b_i\right|<\infty\right\}.$$

*Then,*

1. $\left\{b_1^{r,s,F}\right\}^{\beta}=\sigma_3^{r,s,F}\cap\sigma_4^{r,s,F}$, $\left\{b_1^{r,s,F}\right\}^{\gamma}=\sigma_4^{r,s,F}$.

2. $\left\{b_p^{r,s,F}\right\}^{\beta}=\sigma_3^{r,s,F}\cap\sigma_6^{r,s,F}$, $\left\{b_p^{r,s,F}\right\}^{\gamma}=\sigma_6^{r,s,F}$, $1<p<\infty$.

3. $\left\{b_\infty^{r,s,F}\right\}^{\gamma}=\sigma_7^{r,s,F}$.

**Proof** We only present the proof of $\left\{b_p^{r,s,F}\right\}^\beta = \sigma_3^{r,s,F} \cap \sigma_6^{r,s,F}$ for $1 < p < \infty$ to avoid unnecessary repetitions of similar statements. For any $b = (b_n) \in \omega$, it follows from (2.2) that

$$\begin{aligned}\sum_{k=0}^{n} b_k x_k &= \sum_{k=0}^{n} \left( \frac{1}{r^k} \sum_{i=0}^{k} \binom{k}{i}_F (-s)^{k-i} (s+r)_F^i \, y_i \right) b_k \\ &= \sum_{k=0}^{n} \left( \sum_{i=k}^{n} \binom{i}{k}_F (-s)^{i-k} r^{-i} (s+r)_F^k \, b_i \right) y_k \\ &= \left(M^{r,s,F} y\right)_n ,\end{aligned}$$

for all $n \in \mathbb{N}$. Here, $M^{r,s,F} = \left(m_{nk}^{r,s,F}\right)$ denotes the matrix, defined by

$$m_{nk}^{r,s,F} = \begin{cases} \sum_{i=k}^{n} \binom{i}{k}_F (-s)^{i-k} r^{-i} (s+r)_F^k \, b_i, & \text{if } 0 \le k \le n; \\ 0, & \text{if } k > n; \end{cases}$$

for all $n, k \in \mathbb{N}$. So, $bx = (b_n x_n) \in cs$ whenever $x = (x_k) \in b_p^{r,s,F}$ if and only if $M^{r,s,F} y \in c$ whenever $y = (y_k) \in \ell_p$ for $1 < p < \infty$, from which one concludes that $b = (b_k) \in \left\{b_p^{r,s,F}\right\}^\beta$ if and only if $M^{r,s,F} \in (\ell_p, c)$, where $1 < p < \infty$. Considering these facts and the related part of Lemma 3.1 yields that

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{i=k}^{n} \binom{i}{k}_F (-s)^{i-k} r^{-i} (s+r)_F^k \, b_i \right|^q < \infty$$

and

$$\sum_{i=k}^{\infty} \binom{i}{k}_F (-s)^{i-k} r^{-i} (s+r)_F^k \, b_i \text{ exists for each } k \in \mathbb{N}$$

for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. As a result, we reach that $\left\{b_p^{r,s,F}\right\}^\beta = \sigma_3^{r,s,F} \cap \sigma_6^{r,s,F}$ for $1 < p < \infty$, as desired. ☐

## 4. Matrix mappings

In the current section, we characterize some class of matrix mappings from the spaces $b_p^{r,s,F}$ and $b_\infty^{r,s,F}$ to the space $\mu \in \{\ell_\infty, c, c_0, \ell_1\}$. We begin with the following theorem which is fundamental in our investigation.

**Theorem 4.1** *Let $1 \le p \le \infty$ and $\lambda \subset \omega$. Then $\Phi = (\varphi_{nk}) \in (b_p^{r,s,F}, \lambda)$ if and only if $\Theta^{(n)} = (\theta_{jk}^{(n)}) \in (\ell_p, c)$ for each $n \in \mathbb{N}$, and $\Theta = (\theta_{nk}) \in (\ell_p, \lambda)$, where*

$$\theta_{jk}^{(n)} = \begin{cases} 0 & (k > j), \\ \sum_{l=k}^{j} \binom{l}{k}_F (-s)^{l-k} r^{-l} (s+r)_F^k \, \varphi_{nl} & (0 \le k \le j), \end{cases}$$

*and*

$$\theta_{nk} = \sum_{l=k}^{\infty} \binom{l}{k}_F (-s)^{l-k} r^{-l} (s+r)_F^k \, \varphi_{nl}. \tag{4.1}$$

*for all $n, k \in \mathbb{N}$.*

**Proof** The proof is similar to the proof of Theorem 4.1 of [16]. Hence details are omitted. □

Recalling the well-known results of Stielglitz and Tietz [27] and taking in account Theorem 4.1, we obtain the following results:

**Corollary 4.2** *The following statements hold:*

1. $\Phi \in (b_1^{r,s,F}, \ell_\infty)$ *if and only if*

$$\lim_{j\to\infty} \theta_{jk}^{(n)} \text{ exists for all } n,k \in \mathbb{N}, \tag{4.2}$$

$$\sup_{n,k\in\mathbb{N}} \left|\theta_{jk}^{(n)}\right| < \infty, \tag{4.3}$$

$$\sup_{n,k\in\mathbb{N}} |\theta_{nk}| < \infty, \tag{4.4}$$

2. $\Phi \in (b_1^{r,s,F}, c)$ *if and only if (4.2) and (4.3) hold, and (4.4) and*

$$\lim_{n\to\infty} \theta_{nk} \text{ exists for all } k \in \mathbb{N}, \tag{4.5}$$

*also hold.*

3. $\Phi \in (b_1^{r,s,F}, c_0)$ *if and only if (4.2) and (4.3) hold, and (4.4) and*

$$\lim_{n\to\infty} \theta_{nk} = 0 \text{ for all } k \in \mathbb{N}, \tag{4.6}$$

*also hold.*

4. $\Phi \in (b_1^{r,s,F}, \ell_1)$ *if and only if (4.2) and (4.3) hold, and*

$$\sup_{k\in\mathbb{N}} \sum_{n=0}^{\infty} |\theta_{nk}| < \infty, \tag{4.7}$$

*also holds.*

**Corollary 4.3** *The following statements hold:*

1. $\Phi \in (b_p^{r,s,F}, \ell_\infty)$ *if and only if (4.2) holds, and*

$$\sup_{j\in\mathbb{N}} \sum_{k=0}^{j} \left|\theta_{jk}^{(n)}\right|^k < \infty, \tag{4.8}$$

$$\sup_{n\in\mathbb{N}} \sum_{k=0}^{n} |\theta_{nk}|^k < \infty, \tag{4.9}$$

*also hold.*

2. $\Phi \in (b_p^{r,s,F}, c)$ *if and only if (4.2) and (4.8) hold, and (4.5) and (4.9) also hold.*

3. $\Phi \in (b_p^{r,s,F}, c_0)$ *if and only if (4.2) and (4.8) hold, and (4.6) and (4.9) also hold.*

4. $\Phi \in (b_p^{r,s,F}, \ell_1)$ *if and only if (4.2) and (4.8) hold, and*

$$\sup_{N\in\mathcal{N}} \sum_{k=0}^{\infty} \left| \sum_{n\in N} \theta_{nk} \right|^p < \infty, \tag{4.10}$$

*also holds.*

**Corollary 4.4** *The following statements hold:*

1. $\Phi \in (b_\infty^{r,s,F}, \ell_\infty)$ *if and only if (4.2) and*

$$\lim_{j\to\infty} \sum_{k=0}^{j} \left|\theta_{jk}^{(n)}\right| = \sum_{k=0}^{j} \left| \lim_{j\to\infty} \theta_{jk}^{(n)} \right| \text{ for each } n \in \mathbb{N} \tag{4.11}$$

*hold, and (4.9) also holds with* $p = 1$.

2. $\Phi \in (b_\infty^{r,s,F}, c)$ *if and only if (4.2) and (4.11) hold, and (4.5) and*

$$\lim_{n\to\infty} \sum_{k=0}^{n} |\theta_{nk}| = \sum_{k=0}^{n} \left| \lim_{n\to\infty} \theta_{nk} \right|, \tag{4.12}$$

*also hold.*

3. $\Phi \in (b_\infty^{r,s,F}, c_0)$ *if and only if (4.2) and (4.11) hold, and*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \theta_{nk} = 0, \tag{4.13}$$

*also holds.*

4. $\Phi \in (b_\infty^{r,s,F}, \ell_1)$ *if and only if (4.2) and (4.11) hold, and (4.10) also holds with* $p = 1$.

Recently Başar and Altay [7] developed a lemma which is extensively used in characterizing matrix mappings between two sequence spaces.

**Lemma 4.5** *[7] Let $\lambda$ and $\mu$ be any two sequence spaces, $\Phi$ be an infinite matrix and $\Theta$ be a triangle. Then, $\Phi \in (\lambda, \mu_\Theta)$ if and only if $\Theta\Phi \in (\lambda, \mu)$.*

Using Lemma 4.5 together with Corollaries 4.2, 4.3 and 4.4, we derive following classes of matrix mappings:

**Corollary 4.6** *Let $\Phi = (\varphi_{nk})$ be an infinite matrix and define the matrix $\tilde{C} = (\tilde{C}_{nk})$ by*

$$\tilde{C}_{nk} = \sum_{v=0}^{n} q^v \frac{C_v(q)C_{n-v}(q)}{C_{n+1}(q)} \varphi_{vk}, \ \ (n, k \in \mathbb{N})$$

*where $(C_n(q))$ is sequence of $q$-Catalan numbers, $0 < q < 1$. Then, the necessary and sufficient conditions that $\Phi$ belongs to any one of the classes $(b_1^{r,s,F}, c_0(\tilde{C}))$, $(b_1^{r,s,F}, c(\tilde{C}))$, $(b_p^{r,s,F}, c_0(\tilde{C}))$, $(b_p^{r,s,F}, c(\tilde{C}))$, $(b_\infty^{r,s,F}, c_0(\tilde{C}))$ and $(b_\infty^{r,s,F}, c(\tilde{C}))$ can be determined from the respective ones in Corollaries 4.2, 4.3 and 4.4, by replacing the entries of the matrix $\Phi$ by those of matrix $\tilde{C}$, where $c(\tilde{C})$ and $c_0(\tilde{C})$ are q-Catalan sequence spaces defined by Yaying et al. [32].*

**Corollary 4.7** *Let $\Phi = (\varphi_{nk})$ be an infinite matrix and define the matrix $C^{(q)} = (c_{nk}^{(q)})$ by*

$$c_{nk}^{(q)} = \sum_{v=0}^{n} \frac{q^v}{[n+1]_q} \varphi_{vk}, \quad (n, k \in \mathbb{N})$$

*where $[n]_q$ is the $q$-analog of $n \in \mathbb{N}$ and $0 < q < 1$. Then, the necessary and sufficient conditions that $\Phi$ belongs to any one of the classes $(b_1^{r,s,F}, X_p^q)$, $(b_1^{r,s,F}, X_\infty^q)$, $(b_p^{r,s,F}, X_p^q)$, $(b_p^{r,s,F}, X_\infty^q)$, $(b_\infty^{r,s,F}, X_p^q)$, and $(b_\infty^{r,s,F}, X_\infty^q)$ can be determined from the respective ones in Corollaries 4.2, 4.3 and 4.4, by replacing the entries of the matrix $\Phi$ by those of matrix $C^{(q)}$, where $X_p^q$ and $X_\infty^q$ are $q$-Cesàro sequence spaces defined by Yaying et al. [31].*

## 5. Geometric properties

In this final section, we investigate certain geometric properties of the space $b_p^{r,s,F}$. Before proceeding, we define certain geometric notions which are the basis for our examination. We use the notation $\mathsf{B}(\lambda)$ to represent the unit ball in $\lambda$.

**Definition 5.1** *[8] A Banach space $\lambda$ has the weak Banach-Saks property if every weakly null sequence $(t_n)$ in $\lambda$ has a subsequence $(t_{n_k})$ whose Cesàro means sequence is norm convergent to zero, that is,*

$$\lim_{n\to\infty} \left\| \frac{1}{n+1} \sum_{k=0}^{n} t_{n_k} \right\| = 0.$$

*Moreover, $\lambda$ possesses Banach-saks property if every bounded sequence in $\lambda$ has a subsequence whose Cesàro means sequence is norm convergent.*

**Definition 5.2** *[17] A Banach space $\lambda$ has the Banach-Saks type $p$, if every weakly null sequence $(t_n)$ has a subsequence $(w_n)$ such that, for some $J > 0$,*

$$\left\| \sum_{k=0}^{n} t_{n_k} \right\| \leq J(n+1)^{1/p},$$

*for all $n \in \mathbb{N}$.*

**Theorem 5.3** *The sequence space $b_p^{r,s,F}$ is of Banach-Saks type $p$.*

**Proof** We take sequence of positive numbers $(\tau_k)$ satisfying $\sum_{k=0}^{\infty} \tau_k \leq \frac{1}{2}$. Assume $(h_k)$ to be a weakly null sequence in $\mathsf{B}(b_p^{r,s,F})$. Fix $t_0 = c_0 = 0$ and $t_1 = c_{r_1} = c_1$. Then there exists $j_1 \in \mathbb{N}$ such that

$$\left\| \sum_{k=j_1+1}^{\infty} t_1(k)\varepsilon^{(k)} \right\|_{b_p^{r,s,F}} < \tau_1.$$

By hypothesis, $(h_n)$ is a weakly null sequence, which in turn implies that $h_n \to 0$ coordinatewise. Thus, there exists $n_2 \in \mathbb{N}$ satisfying

$$\left\| \sum_{k=0}^{j_1} h_n(k)\varepsilon^{(k)} \right\|_{b_p^{r,s,F}} < \tau_1,$$

whenever $n \geq n_2$. Again fix $t_2 = h_{n_2}$. Then there exists $j_2 > j_1$ with

$$\left\| \sum_{k=j_2+1}^{\infty} t_2(k)\varepsilon^{(k)} \right\|_{b_p^{r,s,F}} < \tau_2.$$

Again there exists $n_3 > n_2$ (since $h_r \to 0$ coordinatewise) such that

$$\left\| \sum_{k=0}^{j_2} h_n(k)\varepsilon^{(k)} \right\|_{b_p^{r,s,F}} < \tau_2,$$

whenever $n \geq n_3$.

Continuing this process, we get two increasing sequences $(n_k)$ and $(j_k)$ such that

$$\left\| \sum_{k=0}^{j_v} h_n(k)\varepsilon^{(k)} \right\|_{b_p^{r,s,F}} < \tau_v,$$

for all $n \geq n_{k+1}$ and

$$\left\| \sum_{k=j_v+1}^{\infty} t_v(k)\varepsilon^{(k)} \right\|_{b_p^{r,s,F}} < \tau_v,$$

where $t_v = h_{n_v}$. Thus

$$\begin{aligned}\left\| \sum_{v=0}^{n} t_v \right\|_{b_p^{r,s,F}} &= \left\| \sum_{v=0}^{n} \left( \sum_{k=0}^{j_{v-1}} t_v(k)\varepsilon^{(k)} + \sum_{k=j_{v-1}+1}^{j_v} t_v(k)\varepsilon^{(k)} + \sum_{k=j_v}^{\infty} t_v(k)\varepsilon^{(k)} \right) \right\|_{b_p^{r,s,F}} \\ &\leq \left\| \sum_{v=0}^{n} \left( \sum_{k=j_{v-1}+1}^{j_v} t_v(k)\varepsilon^{(k)} \right) \right\|_{b_p^{r,s,F}} + 2\sum_{v=0}^{n} \tau_v.\end{aligned}$$

Since $h_n \in \mathsf{B}(b_p^{r,s,F})$ and $\|h\|_{b_p^{r,s,F}} = \sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} b_{nk}^{r,s,F} h_k\right|$, we realise that $\|h\|_{b_p^{r,s,F}} \le 1$. This leads us to the fact that

$$\left\|\sum_{v=0}^{n}\left(\sum_{k=j_{v-1}+1}^{j_v} t_v(k)\varepsilon^{(k)}\right)\right\|_{b_p^{r,s,F}}^{p} = \sum_{v=0}^{n}\sum_{k=j_{v-1}+1}^{j_v}\left|\sum_{m=0}^{k} b_{km}^{r,s,F} t_v(m)\right|^p$$

$$\le \sum_{v=0}^{n}\sum_{k=0}^{\infty}\left|\sum_{m=0}^{k} b_{km}^{r,s,F} t_v(m)\right|^p \le n+1.$$

Since $1 \le (n+1)^{1/p}$ for all $n \in \mathbb{N}$ and $1 \le p < \infty$, we immediately deduce that

$$\left\|\sum_{v=0}^{n} t_v\right\|_{b_p^{r,s,F}} \le (n+1)^{1/p} + 1 \le 2(n+1)^{1/p}.$$

Thus, we conclude that $b_p^{r,s,F}$ possesses Banach-Saks type $p$. □

Now, we give an estimation for Gurarii's modulus of convexity for the space $b_p^{r,s,F}$.

**Definition 5.4** *The Gurarii's modulus of convexity of a normed linear space $\lambda$ is defined by*

$$\beta_\lambda(\tau) = \inf\left\{1 - \inf_{0\le\varsigma\le1}\|\varsigma t + (1-\varsigma)w\| : t, w \in \mathsf{B}(\lambda), \|t-w\| = \tau\right\} \text{ where } 0 \le \tau \le 2.$$

**Theorem 5.5** *The Gurarii's modulus of convexity of the normed space $b_p^{r,s,F}$ is*

$$\beta_{b_p^{r,s,F}}(\tau) \le 1 - \left(1 - \left(\frac{\tau}{2}\right)^p\right)^{1/p}, \text{ where } 0 \le \tau \le 2.$$

**Proof** Let $t$ be a sequence in $b_p^{r,s,F}$. Then

$$\|t\|_{b_p^{r,s,F}} = \left\|B^{r,s,F}t\right\|_{\ell_p} = \left(\sum_{n=0}^{\infty}\left|\left(B^{r,s,F}t\right)_n\right|^p\right)^{1/p}.$$

Let $0 \le \tau \le 2$ and $D^{r,s,F}$ be the inverse of $B^{r,s,F}$. Consider the following two sequences:

$$t = \left(\left(D^{r,s,F}\left(1 - \left(\frac{\tau}{2}\right)^p\right)\right)^{1/p}, D^{r,s,F}\left(\frac{\tau}{2}\right), 0, 0, \ldots\right),$$

$$w = \left(\left(D^{r,s,F}\left(1 - \left(\frac{\tau}{2}\right)^p\right)\right)^{1/p}, D^{r,s,F}\left(\frac{-\tau}{2}\right), 0, 0, \ldots\right).$$

Then we notice that

$$\|t\|_{b_p^{r,s,F}} = \left\|B^{r,s,F}t\right\|_{\ell_p} = \left|\left(1 - \left(\frac{\tau}{2}\right)^p\right)^{1/p}\right|^p + \left|\frac{\tau}{2}\right|^p$$

$$= 1 - \left(\frac{\tau}{2}\right)^p + \left(\frac{\tau}{2}\right)^p = 1,$$

$$\|w\|_{b_p^{r,s,F}} = \left\|B^{r,s,F}w\right\|_{\ell_p} = \left|\left(1-\left(\frac{\tau}{2}\right)^p\right)^{1/p}\right|^p + \left|\frac{-\tau}{2}\right|^p$$
$$= 1-\left(\frac{\tau}{2}\right)^p + \left(\frac{\tau}{2}\right)^p = 1.$$

$$\|t-w\|_{b_p^{r,s,F}} = \left\|B^{r,s,F}t - B^{r,s,F}w\right\|_{\ell_p}$$
$$= \left(\left|\left(1-\left(\frac{\tau}{2}\right)^p\right)^{1/p} - \left(1-\left(\frac{\tau}{2}\right)^p\right)^{1/p}\right|^p + \left|\frac{\tau}{2} - \left(\frac{-\tau}{2}\right)\right|^p\right)^{1/p}$$
$$= \tau.$$

Eventually, for $0 \leq \varsigma \leq 1$, we have

$$\inf_{0\leq\varsigma\leq 1} \|\varsigma t + (1-\varsigma)w\|_{b_p^{r,s,F}}$$
$$= \inf_{0\leq\varsigma\leq 1} \left\|\varsigma B^{r,s,F}t + (1-\varsigma)B^{r,s,F}w\right\|_{\ell_p}$$
$$= \inf_{0\leq\varsigma\leq 1} \left\{\left|\varsigma\left(1-\left(\frac{\tau}{2}\right)^p\right)^{1/p} + (1-\varsigma)\left(1-\left(\frac{\tau}{2}\right)^p\right)^{1/p}\right|^p + \left|\varsigma\left(\frac{\tau}{2}\right) + (1-\varsigma)\left(\frac{-\tau}{2}\right)\right|\right\}^{1/p}.$$
$$= \inf_{0\leq\varsigma\leq 1} \left\{1-\left(\frac{\tau}{2}\right)^p + |2\varsigma-1|^p\left(\frac{\tau}{2}\right)^p\right\}^{1/p}$$
$$= \left(1-\left(\frac{\tau}{2}\right)^p\right)^{1/p}.$$

Consequently, $\beta_{b_p^{r,s,F}}(\tau) \leq 1 - \left(1-\left(\frac{\tau}{2}\right)^p\right)^{1/p}$. This completes the proof. □

**Corollary 5.6** *The following results hold:*

*(a) If* $\tau = 2$, *then* $\beta_{b_p^{r,s,F}}(\tau) \leq 1$. *Hence* $b_p^{r,s,F}$ *is strictly convex.*

*(b) If* $0 < \tau < 2$, *then* $0 < \beta_{b_p^{r,s,F}}(\tau) < 1$. *Hence* $b_p^{r,s,F}$ *is uniformly convex.*

## 6. Conclusions

The construction of a new sequence space using the domain of a special infinite matrix has attracted by many scholars. In the present study, inspired by Fibonacci calculus, which has become increasingly popular in recent years, and motivated by the significant papers concerning binomial sequence spaces, we introduced the fibonomial sequence spaces, which are Banach spaces, and investigated some properties such as Schauder basis, special duals and some matrix classes. We concluded the study by examining some geometric properties of the resulting space $b_p^{r,s,F}$. We expect that our results presented here might be a reference for further studies in this field.

## References


[1] Altay B, Başar F. On some Euler sequence spaces of non-absolute type. Ukrainian Mathematical Journal 2005; 54: 1-17. https://doi.org/10.1007/s11253-005-0168-9

[2] Altay B, Başar F, Mursaleen M. On the Euler sequence spaces which include the spaces $\ell_p$ and $\ell_\infty$ I. Information Sciences 2006; 176 (10): 1450-1462. https://doi.org/10.1016/j.ins.2005.05.008

[3] Altay B, Polat H. On some new Euler difference sequence spaces. Southeast Asian Bulletin of Mathematics 2006; 30: 209-220.

[4] Aydın C, Başar F. On the new sequence spaces which include the spaces $c_0$ and $c$. Hokkaido Mathematical Journal 2004; 33 (2): 383-398. https://doi.org/10.14492/hokmj/1285766172

[5] Aydın C, Başar F. Some new difference sequence spaces. Applied Mathematics and Computation 2004; 157 (3): 677-693. https://doi.org/10.1016/j.amc.2003.08.055

[6] Başar F. Summability Theory and its Applications, 2nd edn. CRC Press/Taylor and Francis Group, Boca Raton, London, New York: 2022.

[7] Başar F, Altay B. On the space of sequences of $p$-bounded variation and related matrix mappings. Ukrainian Mathematical Journal 2003; 55: 136-147. https://doi.org/10.1023/A:1025080820961

[8] Beauzamy B. Banach-Saks properties and spreading models. Mathematica Scandinavica 1997; 44: 357-384. https://doi.org/10.7146/math.scand.a-11818

[9] Bişgin MC. The binomial sequence spaces of nonabsolute type. Journal of Inequalities and Applications 2016; 2016: 309. https://doi.org/10.1186/s13660-016-1256-0

[10] Bişgin MC. The binomial sequence spaces which include the spaces $\ell_p$ and $\ell_\infty$ and geometric properties. Journal of Inequalities and Applications 2016; 2016: 304. https://doi.org/10.1186/s13660-016-1252-4

[11] Candan M. A new approach on the spaces of generalized Fibonacci difference null and convergent sequences. Mathematica Aeterna 2015; 5 (1): 191-210.

[12] Candan M. Some Characteristics of Matrix Operators on Generalized Fibonacci Weighted Difference Sequence Space. Symmetry 2022; 14: 1283. https://doi.org/10.3390/sym14071283

[13] Choudhary A, Raj K, Mursaleen M. Compact operators on spaces of binomial fractional difference sequences. Mathematical Sciences 2022; 16: 79-85. https://doi.org/10.1007/s40096-021-00396-3

[14] Kara EE, Başarır M. On compact operators and some Euler $B^{(m)}$ difference sequence spaces. Journal of Mathematical Analysis and Applications 2011; 379: 499-511. https://doi.org/10.1016/j.jmaa.2011.01.028

[15] Karakaya V, Polat H. Some new paranormed sequence spaces defined by Euler and difference operators. Acta Scientiarum Mathematicarum 2010; 76: 87-100. https://doi.org/10.1007/BF03549822

[16] Kirişçi M, Başar F. Some new sequence spaces derived by the domain of generalized difference matrix. Computers & Mathematics with Applications 2010; 60: 1299-1309. https://doi.org/10.1016/j.camwa.2010.06.010

[17] Knaust H. Orlicz sequence spaces of Banach-Saks type. Archiv der Mathematik 1992; 59: 562-565. https://doi.org/10.1007/BF01194848

[18] E. Krot, An introduction to finite fibonomial calculus, (2005) arXiv:math/0503210.

[19] Kuş S, Tuğlu N, Kim T. Bernouli $F$-polynomials and Fibo-Bernoulli matrices. Advances in Difference Equations 2019; 2019: 145. https://doi.org/10.1186/s13662-019-2084-6

[20] Meng J, Song M. Binomial difference sequence space of order $m$. Advances in Difference Equations 2017; 2017: 241. https://doi.org/10.1186/s13662-017-1291-2

[21] Meng J, Song M. On some binomial difference sequence spaces. Kyungpook Mathematical Journal 2017; 57: 631-640. https://doi.org/10.5666/KMJ.2017.57.4.631

[22] Meng J, Song M. On some binomial $B^{(m)}$ -difference sequence spaces. Journal of Inequalities and Applications 2017; 2017: 194. https://doi.org/10.1186/s13660-017-1470-4

[23] Mursaleen M, Başar F, Altay B. On the Euler sequence spaces which include the spaces $\ell_p$ and $\ell_\infty$ II. Nonlinear Analysis 2006; 65 (3): 707-717. https://doi.org/10.1016/j.na.2005.09.038

[24] Ozvatan M. Generalized golden-Fibonacci calculus and applications. Ph.D. thesis, Izmir Institute of Technology, 2018.

[25] Pashaev OK, S. Nalci. Golden quantum oscillator and Binet-Fibonacci Calculus. Journal of Physics A: Mathematical and Theoretical 2012; 45: 015303. https://doi.org/10.1088/1751-8113/45/1/015303

[26] Polat H, Başar F. Some Euler spaces of difference sequences of order $m$. Acta Mathematica Scientia 2007; 27: 254-266. https://doi.org/10.1016/S0252-9602(07)60024-1

[27] Stieglitz M, Tietz H. Matrix transformationen von folgenraumen eine ergebnisbersicht. Mathematische Zeitschrift 1977; 154: 1-16. https://doi.org/10.1007/BF01215107

[28] Wilansky A. Summability through Functional Analysis, North-Holland Mathematics Studies 85. Amsterdam-New York-Oxford; 1984.

[29] Yaying T, Das A, Hazarika B, Baliarsingh P. Compactness of binomial difference sequence spaces of fractional order and sequence spaces. Rendiconti del Circolo Matematico di Palermo Series 2 2019; 68: 459-476. https://doi.org/10.1007/s12215-018-0372-8

[30] Yaying T, Hazarika B. On sequence spaces generated by binomial difference operator of fractional order. Mathematica Slovaca 2019; 69 (4): 901-918. https://doi.org/10.1515/ms-2017-0276

[31] Yaying T, Hazarika B, Mursaleen M. On sequence space derived by the domain of $q$-Cesàro matrix in $\ell_p$ space and the associated operator ideal. Journal of Mathematical Analysis and Applications 2021; 493 (1): 124453. https://doi.org/10.1016/j.jmaa.2020.124453

[32] Yaying T, Kara Mİ, Hazarika B, Kara EE. A study on $q$-analogue of Catalan sequence spaces. Filomat 2023; 37 (3): 839-850. https://doi.org/10.2298/FIL2303839Y