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## Modification of the sector theorem of Kondo-Tanaka

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**Abstract:** Kondo-Tanaka proved that if a rotationally symmetric plane  $M_m$  is von Mangoldt or Cartan-Hadamard outside a compact set and has finite total curvature, then it has a sector with no pair of cut points. We show that the condition of finite total curvature can be removed.

**Key words:** Radial curvature, critical point, surface of revolution, finite topological type, finite total curvature, cut point, conjugate point

### 1. Introduction

Let  $(M, p)$  denote a complete, noncompact Riemannian manifold  $M$  with arbitrarily chosen basepoint  $p$ . Let  $(M_m, o)$  denote a rotationally symmetric plane  $M_m$  together with its origin  $o$ , where  $M_m$  equals  $\mathbb{R}^2$  equipped with a smooth, complete Riemannian metric  $g_m := dr^2 + m^2(r)d\theta^2$  with  $m(0) = 0$  and  $m'(0) = 1$ . Given  $M_m$ , define a sector of angular measure  $\delta$ ,  $V(\delta)$ , as  $V(\delta) := \{q \in M_m \mid 0 < \theta(q) < \delta\}$ . It is assumed that always  $0 < \delta \leq \pi$ .

Global Riemannian geometry seeks to relate geometric data to topological data. It is often of particular interest if we can show that a certain set of traits imply that a noncompact manifold  $M$  is topologically finite, i.e. that it is homeomorphic to the interior of a compact set with boundary. The Toponogov comparison theorem was extended in [3] to open complete manifolds with radial sectional curvature bounded below by the curvature of a rotationally symmetric plane,  $M_m$ , with finite total curvature and a sector with no pair of cut points. Kondo-Tanaka used this extended version to prove the following important result:

**Theorem 1.1** (Main Theorem of [3]) *Let  $(M, p)$  be a complete open Riemannian  $n$ -manifold whose radial curvature at basepoint  $p$  is bounded below by that of a noncompact rotationally symmetric plane  $M_m$  with finite total curvature and a sector with no pair of cut points. Then  $M$  is of finite topological type.*

It is natural to wonder what types of rotationally symmetric planes satisfy the condition of Theorem 1.1 requiring a sector with no pair of cut points. A von Mangoldt or Cartan-Hadamard plane contains  $V(\pi)$  free of any pair of cut points; recall that in a von Mangoldt plane,  $G_m$  (the sectional curvature function) is nonincreasing in  $r$ ; in a Cartan-Hadamard plane,  $G_m \leq 0$  for all  $r$ . In [4], Kondo-Tanaka show that the cut-point-free sector requirement for  $M_m$  can be dropped if the total curvature is strictly less than  $2\pi$ . But the requirement stands when the total curvature is equal to  $2\pi$ .

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Apart from applications such as in Theorem 1.1, the behavior of geodesics and their cut points in rotationally symmetric planes is a substantive area of study in its own right; see for example [5], chapter 7. All things considered, the *Sector Theorem*, a result in [3], is of interest:

**Theorem 1.2** (Sector Theorem) *Let  $M_m$  be a noncompact rotationally symmetric plane that is von Mangoldt or Cartan-Hadamard outside a compact set. If  $M_m$  admits finite total curvature, then there exists a sector that has no pair of cut points.*

In the above theorem,  $M_m$  being von Mangoldt or Cartan-Hadamard outside a compact set means respectively that  $G_m$  is nonincreasing or  $G_m \leq 0$  on  $[R, \infty)$  for some  $R \in (0, \infty)$ . Equivalently, we can say that  $M_m$  is von Mangoldt or Cartan-Hadamard outside  $\overline{B_R(o)}$  to denote the above. In the **main result of this paper**, we remove the requirement of finite total curvature in Theorem 1.2:

**Theorem 1.3** (Main result) *Let  $M_m$  be a noncompact rotationally symmetric plane that is von Mangoldt or Cartan-Hadamard outside a compact set. Then there exists a sector that has no pair of cut points.*

## 2. Notations, conventions, and definitions

Theorem 1.1 uses the notion of the radial sectional curvature of  $M$  being bounded below by that of  $M_m$ . We here define this notion:

**Definition 2.1** *Let  $G$  be the sectional curvature function for  $M$ , and for any meridian  $\mu(t)$  emanating from  $o = \mu(0)$ , define the **radial sectional curvature function** of  $M_m$  as  $G_m \circ \mu : [0, \infty) \rightarrow \mathbb{R}$ . We say that  $(M, p)$  has **radial sectional curvature bounded below** by that of  $(M_m, o)$  if, along every unit-speed minimal geodesic  $\gamma : [0, a) \rightarrow M$  emanating from  $p = \gamma(0)$ , we have  $G(\sigma_t) \geq G_m(\mu(t))$  for all  $t \in [0, a)$  and all 2-dimensional subspaces  $\sigma_t$  spanned by  $\gamma'(t)$  and an element of  $T_{\gamma(t)}M$ .*

Two theorems foundational to Theorem 1.1 are the Isotopy Lemma and the extended Toponogov comparison theorem. The Isotopy Lemma, given as Theorem 2.1 below, is a part of the critical point theory of distance functions by Grove-Shiohama [2]; recall that given  $(M, p)$ , a point  $q \in M$  is a *critical point of  $d(\cdot, p)$*  (the distance function to  $p$ ) if, given any  $v \in T_qM$ , there exists a minimal geodesic  $\gamma$  emanating from  $q$  to  $p$  such that  $\angle(\dot{\gamma}(0), v) \leq \frac{\pi}{2}$ :

**Theorem 2.1** (Isotopy Lemma) *Given  $(M, p)$ , suppose that for  $R_1, R_2$  with  $0 < R_1 < R_2 \leq \infty$ ,  $\overline{B_{R_2}(p)} \setminus B_{R_1}(p)$  has no critical point of  $d(\cdot, p)$ . Then  $\overline{B_{R_2}(p)} \setminus B_{R_1}(p)$  is homeomorphic to  $\partial B_{R_1}(p) \times [R_1, R_2]$ .*

The extended Toponogov comparison theorem is developed in [3]. It enables us to compare triangles in  $M$  and  $M_m$  and draw conclusions on corresponding angle inequalities. Its application requires the existence of a cut-point-free sector in  $M_m$ , and hence the requirement in Theorem 1.1. Theorem 1.1 works by showing that if its conditions are satisfied, then the critical points of  $d(\cdot, p)$  are confined to  $B_R(p)$ ,  $R < \infty$ .

All geodesics are parametrized by arclength. The term *segments* refers to minimizing geodesics. Let  $\partial_r, \partial_\theta$  denote the vector fields dual to  $dr, d\theta$  on  $\mathbb{R}^2$ . Given  $q \neq o$ , denote its polar coordinates by  $\theta_q, r_q$ . Let  $\mu_q, \eta_q$  denote the geodesics defined on  $[0, \infty)$  that start at  $q$  in the direction of  $\partial_r, -\partial_r$ , respectively. Set

$\kappa_{\gamma(t)} := \angle(\dot{\gamma}(t), \partial_r)$ . For all geodesic segments  $\gamma : [0, \ell] \rightarrow M_m$ , assume  $r_{\gamma(\ell)} \geq r_{\gamma(0)}$ . We write  $\dot{r}$ ,  $\dot{\theta}$ , and  $\dot{\gamma}$  for the derivatives of  $r_{\gamma(t)}$ ,  $\theta_{\gamma(t)}$ , and  $\gamma(t)$  by  $t$ , while  $m'$  denotes  $\frac{dm}{dr}$ , and proceed similarly for higher derivatives. The *total curvature* of  $M_m$ ,  $c(M_m)$ , takes on the usual definition:

$$c(M_m) := \int_{M_m} G_m dM \tag{2.2}$$

We state below an important result (by Alexis Clairaut), which gives rise to the terms Clairaut constant and Clairaut relation, used in this paper:

**Theorem 2.3** *Let  $\gamma$  be a geodesic in a rotationally symmetric plane  $M_m$  such that  $\gamma$  does not intersect the origin. Then there exists a constant  $c$  such that  $m(r) \sin \kappa_{\gamma(s)} = c$  for all  $s$ .*

A useful consequence of the above result is that since  $0 \leq \sin \kappa_{\gamma(s)} \leq 1$  for all  $s$ ,  $0 \leq c \leq m(r_{\gamma(s)})$ , where  $c = m(r_{\gamma(s)})$  only at points where  $\gamma$  is tangent to a parallel and  $c = 0$  when  $\gamma$  is tangent to a meridian.

**Definition 2.2** *In the setting of Theorem 2.3, the constant  $c$  is called the **Clairaut constant** of geodesic  $\gamma$ ; the equation  $m(r) \sin \kappa_{\gamma(s)} = c$  is called the **Clairaut relation**.*

**3. Proof of Theorem 1.3**

**Remark 3.1** We acknowledge that our proof is a modification of the proof in [3]; much of the relevant original content in [3] remains the same, so we hereby give due credit and respect to Kondo-Tanaka.

Broadly speaking, Lemmas 3.1, 3.2, 3.4, 3.5, and 3.6 are technical lemmas used to prove Lemma 3.8. Lemmas 3.7, 3.8, and 3.9 are used to prove Lemma 3.10. The culminating proof at the end of this section uses Lemmas 3.10, 3.11, and 3.12.

Lemma 3.1 below states a useful relationship between Clairaut constants and their corresponding geodesic segments in an important limiting process that will be used later.

**Lemma 3.1** (Lemma 3.1, [3]) *Given  $M_m$ , let  $V_i := V(\frac{1}{i})$  for each  $i = 1, 2, \dots$ . Assume that there exist a constant  $r_0 > 0$  and a sequence  $\{\sigma_i : [0, \ell_i] \rightarrow V_i\}$  of geodesic segments such that  $\sigma_i([0, \ell_i]) \cap \overline{B_{r_0}(o)} \neq \emptyset$  for each  $i$  and that  $\liminf_{i \rightarrow \infty} r(\sigma_i(\ell_i)) > r_0$ . Then,  $\lim_{i \rightarrow \infty} c_i = 0$  holds, where  $c_i$  denotes the Clairaut constant of  $\sigma_i$ .*

Lemmas 3.2 and 3.4 give an important condition needed for a geodesic in  $M_m$  to have a conjugate point.

**Lemma 3.2** (Propositions 7.2.1, 7.2.2, [5]) *Given  $q \in M_m$ , let  $\gamma : [0, s] \rightarrow M_m$ ,  $\gamma(0) = q$  be a geodesic not tangent to the parallel or meridian through  $q$ . If  $\dot{r}_\gamma$  is nonzero on  $[0, s)$ , then there exists a Jacobi field  $X(t)$  along  $\gamma$  that can be expressed as*

$$X(t) = \text{sign} \left( \frac{\pi}{2} - \kappa_\gamma \right) \dot{r}(t) \int_{d(o,q)}^{r(t)} \frac{m(r)}{\sqrt{m^2(r) - c^2}} dr \left\{ -c \frac{\partial}{\partial r_{\gamma(t)}} + \dot{r}(t) \frac{\partial}{\partial \theta_{\gamma(t)}} \right\} \tag{3.3}$$

on  $[0, s)$ , where  $c$  is the Clairaut constant of  $\gamma$ .

**Lemma 3.4** Given  $q \in M_m$ , let  $\gamma : [0, s] \rightarrow M_m$ ,  $\gamma(0) = q$  be a geodesic that is not tangent to the parallel or meridian through  $q$ . If  $\dot{r}_\gamma$  is nonzero on  $[0, s]$ , then there exists no conjugate point of  $q$  along  $\gamma|_{[0, s]}$ .

**Proof** Each additive term in the expression for  $X(t)$  in Lemma 3.2 carries  $\dot{r}(t)$ . Hence,  $\dot{r}(t)$  nonzero on  $[0, s]$  implies that the Jacobi field  $X(t)$  is nonzero on  $[0, s]$ .  $\square$

Lemma 3.5 states conditions in which a geodesic segment cannot be tangent to a parallel in  $M_m$  from below.

**Lemma 3.5** Let  $M_m$  be such that  $\liminf_{r \rightarrow \infty} m(r) > 0$ . Let  $\{\sigma_i : [0, \ell_i] \rightarrow M_m\}$  be a sequence of minimal geodesics such that  $\ell_i \rightarrow \infty$ ,  $c_i \neq 0$ , and  $c_i \rightarrow 0$ . Then there exists  $L > 0$  such that for all  $i \geq L$ , there does not exist any value  $t$  at which both  $\dot{r}_{\sigma_i(t)} = 0$  and  $\ddot{r}_{\sigma_i(t)} < 0$  hold.

**Proof** By contradiction; suppose that for any  $L > 0$ , there exists  $i \geq L$  such that  $\dot{r}_{\sigma_i(t_i)} = 0$  and  $\ddot{r}_{\sigma_i(t_i)} < 0$  for some  $t_i$ . Choose such a subsequence and denote it  $\{\sigma_i\}$ . By reflectional symmetry and uniqueness of geodesics,  $r_{\sigma_i}$  attains its absolute maximum at  $t_i$ . Since  $c_i = m(r_{\sigma_i(t_i)})$ ,  $c_i \rightarrow 0$ , we have  $m(r_{\sigma_i(t_i)}) \rightarrow 0$ . Since  $\liminf_{r \rightarrow \infty} m(r) > 0$ ,  $m(r_{\sigma_i(t_i)}) \rightarrow 0$  implies  $r_{\sigma_i(t_i)} \rightarrow 0$ . But this is impossible, since  $\ell_i \rightarrow \infty$  and  $\sigma_i$  is a minimal geodesic.  $\square$

**Remark 3.2** Given any  $q \in M$ ,  $M$  a complete Riemannian manifold, we define the *segment domain* of  $q$  as

$$\{v \in T_q M \mid \exp_q tv : [0, 1] \rightarrow M \text{ is a minimal geodesic}\}$$

It is well known that the segment domain of any  $q \in M$  is star-shaped and closed. The *interior* of the segment domain of  $q$ , denoted  $I(q)$ , is likewise defined as

$$\{v \in T_q M \mid \exp_q tv : [0, 1) \rightarrow M \text{ is a minimal geodesic}\}$$

Note that  $\exp_q$  is one-to-one on  $I(q)$ , so if  $x$  is in the image of  $I(q)$ , denoted  $I(q)^*$ , there exists a unique minimizing geodesic  $\gamma$  connecting  $q$  to  $x$ , and there exists  $\epsilon > 0$  such that  $\gamma$  minimizes on  $(0, d(q, x) + \epsilon)$ . Hence, if  $x$  is conjugate to  $q$ ,  $x$  cannot be in  $I(q)^*$ .

**Lemma 3.6** Let  $\{\sigma_i : [0, \ell_i] \rightarrow M_m\}$  be a sequence of minimal geodesics converging to  $\sigma : [0, \ell] \rightarrow M_m$ , where  $\sigma$  is a subarc of a meridian. For all  $i$  large enough,  $\sigma_i(\ell_i)$  is in  $I(\sigma_i(0))^*$  and  $\sigma_i(0)$  is in  $I(\sigma_i(\ell_i))^*$ .

**Proof** Since any subarc of a meridian is distance-minimizing,  $\sigma(\ell)$  is in  $I(\sigma(0))^*$ . Hence for  $i$  large enough,  $\sigma_i(\ell_i)$  is also in  $I(\sigma(0))^*$ . It follows that  $\sigma(0)$  is in  $I(\sigma_i(\ell_i))^*$ , since the above implies that  $\sigma(0)$  is joined to  $\sigma_i(\ell_i)$  by a unique minimal geodesic and  $\sigma(0)$  cannot be conjugate to  $\sigma_i(\ell_i)$ . So for  $i$  large enough,  $\sigma_i(0)$  is in  $I(\sigma_i(\ell_i))^*$ . It must also follow that  $\sigma_i(\ell_i)$  is in  $I(\sigma_i(0))^*$ .  $\square$

Below we give the original version of Lemma 3.3 in [3], followed by our modified version. Both versions are key to our proof of Theorem 1.3.

**Lemma 3.7** (Lemma 3.3, [3]) Let  $M_m$  have finite total curvature. For each  $r > 0$ , there exists a number  $\delta(r) \in (0, \pi)$  such that  $\sigma([0, \ell]) \cap \overline{B_r(o)} = \emptyset$  holds for any minimal geodesic segment  $\sigma : [0, \ell] \rightarrow V(\delta(r)) \subset M$ , along which  $\sigma(0)$  is conjugate to  $\sigma(\ell)$ .

**Lemma 3.8** *Let  $M_m$  be such that  $\liminf_{r \rightarrow \infty} m(r) > 0$ . For each  $r > 0$ , there exists a number  $\delta(r) \in (0, \pi)$  such that  $\sigma([0, \ell]) \cap \overline{B_r(o)} = \emptyset$  holds for any minimal geodesic segment  $\sigma : [0, \ell] \rightarrow V(\delta(r)) \subset M$ , along which  $\sigma(0)$  is conjugate to  $\sigma(\ell)$ .*

**Proof** By contradiction. To establish the existence of  $\delta(r) \in (0, \pi)$ , all we need to do is show that there exists  $\delta(r) > 0$ , since we have  $|\theta(\sigma(0)) - \theta(\sigma(\ell))| < \pi$  for any minimal geodesic segment  $\sigma : [0, \ell] \rightarrow M \setminus \{o\}$ . Put  $V_i := V(\frac{1}{i})$  for each  $i$ . Assume that there exists a constant  $r_0 > 0$  and a sequence of minimal geodesic segments  $\{\sigma_i : [0, \ell_i] \rightarrow V_i\}$ , with  $\sigma_i(0)$  conjugate to  $\sigma_i(\ell_i)$  along  $\sigma_i$ , such that  $\sigma_i([0, \ell_i]) \cap \overline{B_{r_0}(o)} \neq \emptyset$  for each  $i$ .

We want to establish that the sequence of Clairaut constants,  $\{c_i\}$ , converges to 0 as  $i \rightarrow \infty$ . We do this by showing that  $\lim_{i \rightarrow \infty} \ell_i = \infty$ ; indeed, this implies  $\liminf_{i \rightarrow \infty} r_{\sigma_i(\ell_i)} > r_0$ , whereupon by Lemma 3.1  $\{c_i\} \rightarrow 0$ .

Suppose  $\lim_{i \rightarrow \infty} \ell_i < \infty$  or does not exist. Then there exists  $M < \infty$  such that given any  $N$ , there exists  $i \geq N$  such that  $\ell_i \leq M$ . Then we have a subsequence of  $\{\sigma_i\}$  such that the endpoints  $\{\sigma_i(0)\}$ ,  $\{\sigma_i(\ell_i)\}$  are confined to a compact set. Let  $\{\sigma_i\}$  denote this subsequence. Since each  $\sigma_i$  is a minimal geodesic,  $\{\sigma_i\}$  must lie in a bounded set. By the Arzela-Ascoli theorem, there exists a geodesic  $\sigma$  to which some subsequence  $\{\sigma_{i_j}\}$  converges, and by construction  $\sigma$  must be a subarc of a meridian. Let  $\sigma(0)$  be the point to which  $\{\sigma_{i_j}(0)\}$  converges and let  $\sigma(\ell)$  be the point to which  $\{\sigma_{i_j}(\ell_{i_j})\}$  converges. For  $j$  large enough,  $\sigma_{i_j}(0)$  is in  $I(\sigma_{i_j}(\ell_{i_j}))^*$  and  $\sigma_{i_j}(\ell_{i_j})$  is in  $I(\sigma_{i_j}(0))^*$  by Lemma 3.6. Remark 3.2 implies that  $\sigma_{i_j}(0)$  cannot be conjugate to  $\sigma_{i_j}(\ell_{i_j})$ , a contradiction. Hence we establish that  $\liminf_{i \rightarrow \infty} r_{\sigma_i(\ell_i)} > r_0$ .

Since  $\sigma_i(0)$  and  $\sigma_i(\ell_i)$  are conjugate, there exists a positive parameter value  $a_i$  at which  $\dot{r}_{\sigma_i} = 0$  by Lemma 3.4. From our work above, we have  $c_i \rightarrow 0$  and  $\ell_i \rightarrow \infty$ , and by assumption  $\liminf_{r \rightarrow \infty} m(r) > 0$ , so by Lemma 3.5, there exists  $J$  such that for all  $i > J$ , we cannot have  $\ddot{r}_{\sigma_i}(a_i) < 0$ . From this point on, assume  $i > J$  always. Since  $\sigma_i$  is tangent to a parallel from above,  $r_{\sigma_i(a_i)}$  is the absolute minimum of  $r_{\sigma_i}$ , implying  $r_{\sigma_i(a_i)} \in B_{r_0}(o)$ .

Let  $u_i \in [a_i, \ell_i]$  be a parameter value of  $\sigma_i$  such that  $r_{\sigma_i(u_i)} = r_0$ . Set  $\Delta_i :=$  the triangle  $o\sigma_i(a_i)\sigma_i(u_i)$ . This triangle lies in  $\overline{B_{r_0}(o)} \cap V_i$ . The angle at  $\sigma_i(a_i)$  equals  $\frac{\pi}{2}$  by construction. The angle at  $o < \frac{1}{i}$ , so it tends to 0 as  $i \rightarrow \infty$ . This implies that the area of  $\Delta_i$  tends to 0 as  $i \rightarrow \infty$ .

Now consider the angle at  $\sigma(u_i)$ . On the one hand, since  $c_i \rightarrow 0$ , the angle at  $\sigma(u_i)$  must go to 0. On the other hand, the curvature function  $G_m(r)$  attains its maximum and minimum on  $[0, r_0]$ , so  $\int_{\Delta_i} G_m \rightarrow 0$  as  $i \rightarrow \infty$ . The Gauss-Bonnet theorem gives  $\{\text{sum of the interior angles}\} = \pi + \int_{\Delta_i} G_m$ , so we have  $\{\text{sum of the interior angles}\} \rightarrow \pi$  as  $i \rightarrow \infty$ . This means that the angle at  $\sigma_i(u_i)$  must approach  $\frac{\pi}{2}$  as  $i \rightarrow \infty$ , a contradiction. □

**Lemma 3.9** *Suppose  $M_m$  is von Mangoldt or Cartan-Hadamard outside a compact set. If  $\liminf_{r \rightarrow \infty} m(r) = 0$ , then  $M_m$  has finite total curvature.*

**Proof** We prove our claim by showing that  $\lim_{r \rightarrow \infty} m'(r)$  exists and is finite. Let  $R > 0$  be such that  $M_m$  is von Mangoldt or Cartan-Hadamard on  $M_m \setminus \overline{B_R(o)}$ . There exists  $r_0 > R$  at which  $m' < 0$ , for if  $m'(r) \geq 0$  for all  $r > R$ , then  $\liminf_{r \rightarrow \infty} m(r) > 0$ . Because  $m(r) > 0$  on  $r > 0$ , we cannot have  $m'(r) \leq m'(r_0)$  on

$[r_0, \infty)$ . Hence there exists  $r_1 > r_0$  such that  $m'(r_1) < 0$  and  $m''(r_1) > 0$ . Also  $G_m(r_1) < 0$ . Since  $M_m$  is von Mangoldt or Cartan-Hadamard on  $(R, \infty)$ ,  $G_m(r) \leq 0$  on  $[r_1, \infty)$ , implying  $m''(r) \geq 0$  on  $[r_1, \infty)$ . We claim  $m' < 0$  on  $[r_1, \infty)$ . Indeed, if for some  $r \geq r_1$   $m' \geq 0$ , then  $m''[r_1, \infty) \geq 0$  implies  $m' \geq 0$  for all  $r \geq r_1$ , implying  $\liminf_{r \rightarrow \infty} m(r) > 0$ . Since  $m'$  is an increasing function on  $[r_1, \infty)$  that is bounded above by 0, it must converge to a finite number. □

**Lemma 3.10** *Let  $M_m$  be von Mangoldt or Cartan-Hadamard outside a compact set. Then for each  $r > 0$ , there exists a constant number  $\delta(r) \in (0, \pi)$  such that  $\sigma([0, \ell]) \cap \overline{B_r(o)} = \emptyset$  holds for any minimal geodesic segment  $\sigma : [0, \ell] \rightarrow V(\delta(r)) \subset M$ , along which  $\sigma(0)$  is conjugate to  $\sigma(\ell)$ .*

**Proof** Either  $\liminf_{r \rightarrow \infty} m(r) > 0$  or  $\liminf_{r \rightarrow \infty} m(r) = 0$ . If  $\liminf_{r \rightarrow \infty} m(r) > 0$ , then the claim holds by Lemma 3.8. If  $\liminf_{r \rightarrow \infty} m(r) = 0$ , then Lemma 3.9 applies, so  $M_m$  has finite total curvature. Lemma 3.7 then implies the claim. □

**Lemma 3.11** *If  $q \in V(\delta)$  has a cut point in  $V(\delta)$ , then  $q$  must also have a conjugate point in  $V(\delta)$ .*

**Proof** Suppose  $q \in V(\delta)$  has a cut point  $x \in V(\delta)$ . If  $x$  is conjugate to  $q$ , we are done, so suppose not. Then let  $\alpha, \beta$  be minimal geodesics connecting  $q$  to  $x$  and bounding a region  $D$ ; since  $\alpha, \beta$  cannot be tangent to any meridian,  $D \subset V(\delta)$ . The boundary of  $D$  only meets  $C_q$ , the set of cut points of  $q$ , at  $x$  because  $\alpha, \beta$  are minimal. Since  $C_q$  is a tree by Theorem 4.2.1 in [5], the interior of  $D$  contains an endpoint of  $C_q$  that is conjugate to  $q$ . □

**Lemma 3.12** *Let  $M_m$  be von Mangoldt or Cartan-Hadamard outside  $\overline{B_{R_1}(o)}$ . Then for any  $\delta \leq \pi$ , there does not exist a geodesic  $\sigma : [0, \ell] \rightarrow V(\delta) \setminus \overline{B_{R_1}(o)}$  containing conjugate points.*

**Proof** (This proof is modeled on the proofs of Theorem 3.4 in [3] and Theorem 7.3.1 in [5]). Let  $\gamma_x$  be the minimal geodesic joining  $q$  to  $x$ . Suppose  $M_m \setminus \overline{B_{R_1}(o)}$  is Cartan-Hadamard. Then  $G_m \leq 0$  along  $\gamma_x$ , implying that  $q$  cannot be conjugate to  $x$  along  $\gamma_x$ .

Suppose  $M_m \setminus \overline{B_{R_1}(o)}$  is von Mangoldt. By Corollary 4.2.1 in [5] and Lemmas 2.5.19 and 2.5.22 in [1], we can find a normal cut point  $y$  in  $C_q$  (see Definition 2.5.12 in [1]) arbitrarily close to  $x$  such that  $d(q, x) < d(q, y)$  and  $\theta_x < \theta_y < \pi$ . By Remark 2.5.21 in [1], there exists a minimal geodesic  $\beta_y$  connecting  $q$  to  $y$  such that

$$\angle(\dot{\beta}_y(0), \dot{\eta}_q(0)) < \angle(\dot{\gamma}_x(0), \dot{\eta}_q(0)), \tag{3.13}$$

and since  $y$  can be made arbitrarily close to  $x$ , we can ensure that  $\beta_y$  does not intersect  $\overline{B_{R_1}(o)}$ .

We now show that

$$\ell(\gamma_x) < \ell(\beta_y) \quad \text{and} \quad r_{\gamma_x(s)} > r_{\beta_y(s)} \tag{3.14}$$

for all  $s \in (0, \ell(\gamma_x))$ . For each  $s \in (0, \ell(\gamma_x))$ , since  $\theta_y > \theta_x$ , there exists a unique value  $t(s)$  of  $\beta_y$  giving us

$$\theta_{\gamma(s)} = \theta_{\beta_y(t(s))}. \tag{3.15}$$

Since  $\gamma_x, \beta_y$  cannot intersect in their interiors we have  $r_{\beta_y(t(s))} < r_{\gamma_x(s)}$ . Hence for any given  $s$ , the set

$$S_s := \{t \in (0, \ell(\beta_y)) \mid r_{\beta_y(t)} < r_{\gamma_x(s)}\} \tag{3.16}$$

is nonempty. Now fix  $s_0 \in (0, \ell(\gamma_x))$ . Let  $(a, b)$  be the connected component of  $S_{s_0}$  containing  $t(s_0)$ . If we show that  $s_0 \in (a, b)$ , then we will have  $r_{\gamma_x(s_0)} > r_{\beta_y(s_0)}$ . If  $(0, \ell(\gamma_x)) \subseteq (a, b)$  then  $s_0 \in (a, b)$  and there is nothing to prove, so we can assume  $a > 0$  or  $b < \ell(\gamma_x)$ . We have

$$r_{\gamma_x(s_0)} = r_{\beta_y(a)} = r_{\beta_y(b)}, \quad 0 \leq \theta_{\beta_y(a)} < \theta_{\gamma_x(s_0)} = \theta_{\beta_y(t(s_0))} < \theta_{\beta_y(b)} < \pi \tag{3.17}$$

so the conditions for Lemma 7.3.2 in [5] are satisfied. It follows that

$$a = d(q, \beta_y(a)) < s_0 = d(q, \gamma_x(s_0)) < d(q, \beta_y(b)) = b, \tag{3.18}$$

implying  $s_0 \in (a, b)$  and therefore  $r_{\beta_y(s_0)} < r_{\gamma_x(s_0)}$ . Since  $s_0$  was arbitrary and  $M_m \setminus \overline{B_{R_1}(o)}$  is von Mangoldt, we have  $G_m(r_{\gamma_x(s)}) \leq G_m(r_{\beta_y(s)})$  for all  $s \in [0, \ell(\gamma_x)]$ . Recalling that  $q$  is conjugate to  $x$  along  $\gamma_x$  and applying the Sturm Comparison Theorem, we have that  $q$  is conjugate to  $\beta_y(t)$  along  $\beta_y$  for some  $t \in (0, \ell(\gamma_x)]$ . But this is impossible, since  $\beta_y$  minimizes the distance from  $q$  to  $y$  and  $\ell(\beta_y) > \ell(\gamma_x)$ . Hence  $q$  cannot be conjugate to  $x$  along  $\gamma_x$ , and this completes our proof. □

**Proof** [Proof of Theorem 1.3] Let  $M_m$  be von Mangoldt or Cartan-Hadamard outside  $\overline{B_{R_0}(o)}$  for some  $R_0 > 0$ . Fix any  $R_1 > R_0$ , and in the setting of Lemma 3.10, let  $\delta(R_1) \in (0, \pi)$  be the number such that if  $\sigma : [0, \ell] \rightarrow V(\delta(R_1))$  is a minimal geodesic along which  $\sigma(0)$  is conjugate to  $\sigma(\ell)$ , then

$$\sigma[0, \ell] \cap \overline{B_{R_1}(o)} = \emptyset. \tag{3.19}$$

Proceeding by contradiction, if  $q \in V(\delta(R_1))$  has a cut point in  $V(\delta(R_1))$ , then by Lemma 3.11 there exists a point  $x \in V(\delta(R_1))$  and a geodesic  $\gamma_x$  such that  $q$  is conjugate to  $x$  along  $\gamma_x$ . By Lemma 3.10,  $\gamma_x$  does not intersect  $\overline{B_{R_1}(o)}$ . But if  $\gamma_x$  lies entirely in  $M_m \setminus \overline{B_{R_1}(o)}$ , then by Lemma 3.12,  $q$  cannot be conjugate to any point along  $\gamma_x$ . □

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