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Existence results for impulsive dynamic singular nonlinear Sturm–Liouville equations on infinite intervals

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Abstract: The purpose of this study is to investigate an impulsive dynamic singular nonlinear Sturm–Liouville problem on infinite intervals. The existence and uniqueness of the solutions of such problem will be investigated by considering Weyl's limit-circle case.

Key words: Dynamic equations on time scales, singular nonlinear boundary value problems, boundary value problems with impulses

1. Introduction

The Sturm–Liouville problems that arise while trying to solve the partial differential equations encountered in various branches of science with the method of separating the variables have been attracting the attention of researchers for a long time. Such equations are extensively investigated under many states and various boundary conditions. Impulsive boundary conditions are one of the mentioned boundary conditions. The emergence of equations containing such boundary conditions in some physical problems has made the subject interesting (see [14–16]). In [2, 3, 17, 18], Sturm–Liouville problems are studied under impulsive boundary conditions. In [10–12], impulsive q-Sturm–Liouville problems were investigated.

In the 1990s, Hilger introduced the concept of time scale to the literature. With the help of this concept, it is aimed to examine differential equations and difference equations under a single structure. Thus, it has become necessary to investigate all the cases considered in the classical derivative case on the time scale (see [4]). Equations with impulsive boundary conditions have started to be investigated on the time scale (see [5, 8, 9, 19–22]). Recently, in [1], the authors investigated some spectral properties of impulsive dynamic Sturm–Liouville problems.

In this study, the impulsive singular nonlinear Sturm–Liouville problem is studied on a time scale. Some existence and uniqueness theorems are proved by considering such an equation in Weyl's limit-circle case over

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the interval $(-\infty, \infty)$. Some existence theorems for second-order impulsive time scale boundary value problems are given in [8, 9, 19–22]. However, none of these studies have addressed the singular nonlinear Sturm–Liouville problem in Weyl's limit-circle case. As it is known, for the Sturm-Liouville problems in the singular case, Weyl's limit circle/limit point situations occur. Investigating these situations is crucial for the spectral theory of Sturm-Liouville problems. In this study, the problems investigated for classical Sturm-Liouville problems are investigated on the time scale. Thus, under impulsive conditions, the Sturm-Lioville equation and the Sturm-Liouville difference equation will be examined under a single roof.

2. The fundamental problem

We shall consider the dynamic equation:

$$Lu := -\left[p(\eta)\, u^{\Delta}(\eta)\right]^{\nabla} + q(\eta)\, u(\eta) = \Gamma(\eta, u), \ \eta \in J \subset \mathbb{T}, \tag{2.1}$$

where $J := J_1 \cup J_2,\ J_1 := (-\infty, d),\ J_2 := (d, \infty),\ d > 0,\ \mathbb{T}$ is a time scale and $u = u(\eta)$ is a sought solution.

We work in the Hilbert space $H = L^2_{\nabla}(J_1) \dot{+} L^2_{\nabla}(J_2)$ (of the real-valued functions) endowed with the inner product

$$\langle \Upsilon, \Sigma \rangle_H := \int_{-\infty}^{d} \Upsilon^{(1)} \Sigma^{(1)} \nabla\eta + \alpha \int_{d}^{\infty} \Upsilon^{(2)} \Sigma^{(2)} \nabla\eta$$

and norm

$$\|\Upsilon\| := \left(\int_{-\infty}^{d} (\Upsilon^{(1)}(\eta))^2 \nabla\eta + \alpha \int_{d}^{\infty} (\Upsilon^{(2)}(\eta))^2 \nabla\eta \right)^{\frac{1}{2}},$$

where

$$\Upsilon(\eta) = \begin{cases} \Upsilon^{(1)}(\eta), & \eta \in J_1 \\ \Upsilon^{(2)}(\eta), & \eta \in J_2, \end{cases} \quad \Sigma(\eta) = \begin{cases} \Sigma^{(1)}(\eta), & \eta \in J_1 \\ \Sigma^{(2)}(\eta), & \eta \in J_2. \end{cases}$$

Let

$$\mathcal{D}_{\max} = \left\{ u \in H : \begin{array}{c} u \text{ is } \Delta\text{-locally absolutely continuous and } pu^{\Delta} \\ \text{is } \nabla\text{-locally absolutely continuous functions} \\ \text{on } J, \text{ one-sided limits } u(d\pm) \text{ and } \left(pu^{\Delta}\right)(d\pm) \\ \text{exist and finite, } U(d+) = \Pi U(d-) \text{ and } Lu \in H \end{array} \right\},$$

where

$$U(\eta) := \begin{pmatrix} u(\eta) \\ p(\eta)\, u^{\Delta}(\eta) \end{pmatrix},$$

Π is the 2×2 real matrix with $\det \Pi = 1/\alpha > 0$. Then the *maximal operator* $L_{\max}$ on $\mathcal{D}_{\max}$ is defined by $L_{\max} u = Lu$. For $u_1, u_2 \in \mathcal{D}_{\max}$, we have

$$\int_{-\infty}^{\infty} \left[(Lu_1)(\eta) u_2(\eta) - u_1(\eta)(Lu_2)(\eta)\right] \nabla\eta$$

$$= [u_1, u_2](\infty) - [u_1, u_2](d+) + [u_1, u_2](d-) - [u_1, u_2](-\infty), \tag{2.2}$$

where

$$[u_1, u_2](\eta) := W_{\Delta}(u_1, u_2)(\eta) = p(\eta)\left\{u_1(\eta) u_2^{\Delta}(\eta) - u_1^{\Delta}(\eta) u_2(\eta)\right\}.$$

and the limits $[u_1,u_2](\pm\infty)=\lim_{\eta\to\pm\infty}[u_1,u_2](\eta)$ exist and are finite.

We shall use the following assumptions.

(H1) p is a nabla differentiable function on J, q is a real-valued continuous function on J, p^{∇} is continuous on J and $p(t)\neq 0$ for all $t\in J$. $d\in\mathbb{T}$ is a regular point for L and one-sided limits $q(d\pm)$, $p^{\nabla}(d\pm)$ exist. Furthermore, Weyl's limit-circle case holds for L.

(H2) $\Gamma(\eta,u)$ is real-valued and continuous in $(\eta,\zeta)\in J\times\mathbb{R}$, and

$$|\Gamma(\eta,u)|\leq\Sigma(\eta)+\kappa|\zeta| \tag{2.3}$$

for all (η,ζ) in $J\times\mathbb{R}$, where $\Sigma(\eta)\geq 0$, $\Sigma\in H$, and κ is a positive constant.

Denote by

$$\rho(\eta)=\begin{cases}\rho^{(1)}(\eta), & \eta\in J_1\\ \rho^{(2)}(\eta), & \eta\in J_2\end{cases}\quad\text{and }\sigma(\eta)=\begin{cases}\sigma^{(1)}(\eta), & \eta\in J_1\\ \sigma^{(2)}(\eta), & \eta\in J_2\end{cases}$$

the solutions of the equation $Lu=0$ satisfying

$$\rho^{(1)}(0)=0,\ p(0)\rho^{(1)^{\Delta}}(0)=1,\ \sigma^{(1)}(0)=-1,\ p(0)\sigma^{(1)^{\Delta}}(0)=0, \tag{2.4}$$

and impulsive conditions

$$\begin{aligned}V_1(d+)&=\Pi V_1(d-),\\ V_2(d+)&=\Pi V_2(d-),\end{aligned} \tag{2.5}$$

where

$$V_1(\eta):=\begin{pmatrix}\rho(\eta)\\ p(\eta)\rho^{\Delta}(\eta)\end{pmatrix}\text{ and }V_2(\eta):=\begin{pmatrix}\sigma(\eta)\\ p(\eta)\sigma^{\Delta}(\eta)\end{pmatrix}.$$

If we set $W_{\Delta}^{(i)}:=W_{\Delta}\left(\rho^{(i)},\sigma^{(i)}\right)$ $(\eta\in J_i,\ i=1,2)$, then we have $W_{\Delta}^{(1)}=(1/\alpha)W_{\Delta}^{(2)}$. Let $W_{\Delta}:=W_{\Delta}^{(1)}=(1/\alpha)W_{\Delta}^{(2)}$. Since $W_{\Delta}\left(\rho^{(1)},\sigma^{(1)}\right)=1$, ρ and σ form a fundamental system of solutions of Eq. (2.1). From (**H1**), we see that $\rho,\ \sigma\in H$ and $\rho,\ \sigma\in\mathcal{D}_{\max}$. Thus, for every $u\in\mathcal{D}_{\max}$, $[u,\rho]_{\pm\infty}$ and $[u,\sigma]_{\pm\infty}$ exist and are finite.

From conditions (2.4)-(2.5), we get the following relations

$$\begin{aligned}
[u,\rho]_{-\infty}&=u(0)-\int_{-\infty}^{0}\rho(\eta)(Lu)(\eta)\nabla\eta,\\
[u,\sigma]_{-\infty}&=p(0)u^{\Delta}(0)-\int_{-\infty}^{0}\sigma(\eta)(Lu)(\eta)\nabla\eta,\\
[u,\rho]_{\infty}&=u(0)+\int_{0}^{\infty}\rho(\eta)(Lu)(\eta)\nabla\eta,\\
[u,\sigma]_{\infty}&=p(0)u^{\Delta}(0)+\int_{0}^{\infty}\sigma(\eta)(Lu)(\eta)\nabla\eta.
\end{aligned} \tag{2.6}$$

Now, we will consider the following problem

$$Lu = \Gamma\left(\eta, u\right), \ \eta \in J, \tag{2.7}$$

$$\begin{aligned} &[u,\rho]_{-\infty}\cos\gamma + [u,\sigma]_{-\infty}\sin\gamma = \varsigma_1, \\ &[u,\rho]_{\infty}\cos\beta + [u,\sigma]_{\infty}\sin\beta = \varsigma_2, \end{aligned} \tag{2.8}$$

$$U\left(d+\right) = \Pi U\left(d-\right), \tag{2.9}$$

where $\varsigma_1, \varsigma_2, \gamma, \beta \in \mathbb{R}$, $U = \begin{pmatrix} u \\ pu^{\Delta} \end{pmatrix}$, $\det \Pi = 1/\alpha > 0$, and

(H3) $\qquad \omega := \cos\gamma\sin\beta - \cos\beta\sin\gamma \neq 0.$

For $\mathbb{T} = \mathbb{R}$ case, the problem (2.7) - (2.9) has been addressed by Guseinov and Yaslan without impulsive boundary conditions [6].

3. The corresponding Green function

In this section, the Green function corresponding to the problem will be set up.

Consider the problem

$$-\left[p\left(\eta\right)u^{\Delta}\left(\eta\right)\right]^{\nabla} + q\left(\eta\right)u(\eta) = h\left(\eta\right), \quad h \in H, \tag{3.1}$$

$$\left.\begin{aligned} &[u,\rho]_{-\infty}\cos\gamma + [u,\sigma]_{-\infty}\sin\gamma = 0, \\ &U\left(d+\right) = \Pi U\left(d-\right), \\ &[u,\rho]_{\infty}\cos\beta + [u,\sigma]_{\infty}\sin\beta = 0 \end{aligned}\right\}, \tag{3.2}$$

where $\gamma, \beta \in \mathbb{R}$, and $\eta \in J$.

Let us define

$$\Theta\left(\eta\right) = \rho\left(\eta\right)\cos\gamma + \sigma\left(\eta\right)\sin\gamma, \ \Xi\left(\eta\right) = \rho\left(\eta\right)\cos\beta + \sigma\left(\eta\right)\sin\beta, \tag{3.3}$$

where $W_{\Delta}\left(\Theta, \Xi\right) = \omega$. It is obvious that Θ and Ξ are solutions of the equation $Lu = 0$ and Θ, $\Xi \in H$. Moreover, we infer that

$$[\Theta,\rho]_{\eta} = \Theta\left(0\right) = -\sin\gamma, \ [\Theta,\sigma]_{\eta} = p\left(0\right)\Theta^{\Delta}\left(0\right) = \cos\gamma, \ \eta \in J_1, \tag{3.4}$$

$$[\Xi,\rho]_{\eta} = \Xi\left(0\right) = -\sin\beta, \ [\Xi,\sigma]_{\eta} = p\left(0\right)\Xi^{\Delta}\left(0\right) = \cos\beta, \ \eta \in J_1, \tag{3.5}$$

$$[\Theta,\rho]_{-\infty} = -\sin\gamma, \ [\Theta,\sigma]_{-\infty} = \cos\gamma,$$

$$[\Xi,\rho]_{\infty} = -(1/\alpha)\sin\beta, \ [\Xi,\sigma]_{\infty} = (1/\alpha)\cos\beta, \tag{3.6}$$

$$\Phi\left(d+\right) = \Pi\Phi\left(d-\right), \ \Phi(\eta) := \begin{pmatrix} \Theta\left(\eta\right) \\ p\left(\eta\right)\Theta^{\Delta}\left(\eta\right) \end{pmatrix}, \tag{3.7}$$

$$\Psi\left(d+\right) = \Pi\Psi\left(d-\right), \ \Psi(\eta) := \begin{pmatrix} \Xi\left(\eta\right) \\ p\left(\eta\right)\Xi^{\Delta}\left(\eta\right) \end{pmatrix}. \tag{3.8}$$

Green's function of (3.1)-(3.2) is given by the formula

$$G(\eta,\xi)=\begin{cases}\frac{\Theta(\eta)\Xi(\xi)}{\omega}, & \text{if } -\infty<\eta\leq\xi<\infty,\ \eta\neq d,\ \xi\neq d,\\ \frac{\Theta(\xi)\Xi(\eta)}{\omega}, & \text{if } -\infty<\xi\leq\eta<\infty,\ \eta\neq d,\ \xi\neq d.\end{cases} \tag{3.9}$$

$G(\eta,\xi)$ is a Hilbert–Schmidt kernel, i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|G(\eta,\xi)|^2\nabla\eta\nabla\xi<\infty, \tag{3.10}$$

since $\Theta,\ \Xi\in H$.

Theorem 3.1 *The unique solution of (3.1)-(3.2) is given by*

$$u(\eta)=\langle G(\eta,.),h(.)\rangle, \tag{3.11}$$

where $\eta\in J$.

Proof By the method of variation of constants, we see that

$$\begin{aligned}u(\eta)&=k_1\Theta^{(1)}(\eta)+k_2\Xi^{(1)}(\eta)\\&+\frac{\Xi^{(1)}(\eta)}{\omega}\int_{-\infty}^{\eta}\Theta^{(1)}(\xi)h(\xi)\nabla\xi\\&+\frac{\Theta^{(1)}(\eta)}{\omega}\int_{\eta}^{d}\Xi^{(1)}(\xi)h(\xi)\nabla\xi,\ \eta\in J_1\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}u(\eta)&=k_3\Theta^{(2)}(\eta)+k_4\Xi^{(2)}(\eta)\\&+\frac{\alpha}{\omega}\Xi^{(2)}(\eta)\int_{d}^{\eta}\Theta^{(2)}(\xi)h(\xi)\nabla\xi\\&+\frac{\alpha}{\omega}\Theta^{(2)}(\eta)\int_{\eta}^{\infty}\Xi^{(2)}(\xi)h(\xi)\nabla\xi,\ \eta\in J_2.\end{aligned} \tag{3.13}$$

where $k_i\,(i=1,2,3,4)$ is arbitrary.

From (3.12) and (3.13), it may be concluded that

$$\begin{aligned}p(\eta)u^{\Delta}(\eta)&=k_1p(\eta)\Theta^{(1)^{\Delta}}(\eta)+k_2p(\eta)\Xi^{(1)^{\Delta}}(\eta)\\&+\frac{p(\eta)\Xi^{(1)^{\Delta}}(\eta)}{\omega}\int_{-\infty}^{\eta}\Theta^{(1)}(\xi)h(\xi)\nabla\xi\\&+\frac{p(\eta)\Theta^{(1)^{\Delta}}(\eta)}{\omega}\int_{\eta}^{d}\Xi^{(1)}(\xi)h(\xi)\nabla\xi,\ \eta\in J_1,\end{aligned}$$

and

$$p(\eta)u^{\Delta}(\eta)=k_3p(\eta)\Theta^{(2)^{\Delta}}(\eta)+k_4p(\eta)\Xi^{(2)^{\Delta}}(\eta)$$

$$+\frac{\alpha}{\omega}p\left(\eta\right)\Xi^{(2)^{\Delta}}\left(\eta\right)\int_{d}^{\eta}\Theta^{(2)}\left(\xi\right)h\left(\xi\right)\nabla\xi$$

$$+\frac{\alpha}{\omega}p\left(\eta\right)\Theta^{(2)^{\Delta}}\left(\eta\right)\int_{\eta}^{\infty}\Xi^{(2)}\left(\xi\right)h\left(\xi\right)\nabla\xi,\ \ \eta\in J_{2}.$$

Moreover, we have

$$[u,\rho]_{\eta}=p(\eta)\left\{u(\eta)\rho^{\Delta}(\eta)-u^{\Delta}(\eta)\rho(\eta)\right\}$$

$$=k_{1}[\Theta^{(1)},\rho]_{\eta}+k_{2}\left[\Xi^{(1)}\left(\eta\right),\rho\right]_{\eta}$$

$$\frac{1}{\omega}\left[\Xi^{(1)}\left(\eta\right),\rho\right]_{\eta}\int_{-\infty}^{\eta}\Theta^{(1)}\left(\xi\right)h\left(\xi\right)\nabla\xi$$

$$+\frac{1}{\omega}\left[\Theta^{(1)}\left(\eta\right),\rho\right]_{\eta}\int_{\eta}^{d}\Xi^{(1)}\left(\xi\right)h\left(\xi\right)\nabla\xi,\ \ \eta\in J_{1},$$

$$[u,\rho]_{\eta}=k_{3}[\Theta^{(2)},\rho]_{\eta}+k_{4}[\Xi^{(2)},\rho]_{\eta}$$

$$+\frac{\alpha}{\omega}[\Xi^{(2)},\rho]_{\eta}\int_{d}^{\eta}\Theta^{(2)}\left(\xi\right)h\left(\xi\right)\nabla\xi$$

$$+\frac{\alpha}{\omega}[\Theta^{(2)},\rho]_{\eta}\int_{\eta}^{\infty}\Xi^{(2)}\left(\xi\right)h\left(\xi\right)\nabla\xi,\ \ \eta\in J_{2},$$

and

$$[u,\sigma]_{\eta}=p(\eta)\left\{u(\eta)\sigma^{\Delta}(\eta)-u^{\Delta}(\eta)\sigma(\eta)\right\}$$

$$=k_{1}[\Theta^{(1)},\sigma]_{\eta}+k_{2}\left[\Xi^{(1)}\left(\eta\right),\sigma\right]_{\eta}$$

$$\frac{1}{\omega}\left[\Xi^{(1)}\left(\eta\right),\sigma\right]_{\eta}\int_{-\infty}^{\eta}\Theta^{(1)}\left(\xi\right)h\left(\xi\right)\nabla\xi$$

$$+\frac{1}{\omega}\left[\Theta^{(1)}\left(\eta\right),\sigma\right]_{\eta}\int_{\eta}^{d}\Xi^{(1)}\left(\xi\right)h\left(\xi\right)\nabla\xi,\ \ \eta\in J_{1},$$

$$[u,\sigma]_{\eta}=k_{3}[\Theta^{(2)},\sigma]_{\eta}+k_{4}[\Xi^{(2)},\sigma]_{\eta}$$

$$+\frac{\alpha}{\omega}[\Xi^{(2)},\sigma]_{\eta}\int_{d}^{\eta}\Theta^{(2)}\left(\xi\right)h\left(\xi\right)\nabla\xi$$

$$+\frac{\alpha}{\omega}[\Theta^{(2)},\sigma]_{\eta}\int_{\eta}^{\infty}\Xi^{(2)}\left(\xi\right)h\left(\xi\right)\nabla\xi,\ \ \eta\in J_{2}.$$

Thus, we obtain

$$[u,\rho]_{-\infty} = k_1[\Theta^{(1)},\rho]_{-\infty} + k_2\left[\Xi^{(1)}(\eta),\rho\right]_{-\infty}$$

$$+\frac{1}{\omega}\left[\Theta^{(1)},\rho\right]_{-\infty}\int_{-\infty}^{d}\Xi^{(1)}(\xi)\,h(\xi)\,\nabla\xi$$

$$= -k_1\sin\gamma - k_2\sin\beta - \frac{1}{\omega}\sin\gamma\int_{-\infty}^{d}\Xi^{(1)}(\xi)\,h(\xi)\,\nabla\xi \tag{3.14}$$

and

$$[u,\sigma]_{-\infty} = k_1[\Theta^{(1)},\sigma]_{-\infty} + k_2\left[\Xi^{(1)}(\eta),\sigma\right]_{-\infty}$$

$$+\frac{1}{\omega}\left[\Theta^{(1)}(\eta),\sigma\right]_{-\infty}\int_{-\infty}^{d}\Xi^{(1)}(\xi)\,h(\xi)\,\nabla\xi$$

$$= k_1\cos\gamma + k_2\cos\beta + \frac{1}{\omega}\cos\gamma\int_{-\infty}^{d}\Xi^{(1)}(\xi)\,h(\xi)\,\nabla\xi. \tag{3.15}$$

From (3.14), (3.15), and (3.2), we have $k_2 = 0$.

Likewise, we find

$$[u,\rho]_{\infty} = k_3[\Theta^{(2)},\rho]_{\infty} + k_4[\Xi^{(2)},\rho]_{\infty}$$

$$+\frac{\alpha}{\omega}[\Xi^{(2)},\rho]_{\infty}\int_{d}^{\infty}\Theta^{(2)}(\xi)\,h(\xi)\,\nabla\xi$$

$$= -\frac{k_3}{\alpha}\sin\gamma - \frac{k_4}{\alpha}\sin\beta + \frac{\alpha}{\omega\alpha}\sin\beta\int_{-\infty}^{d}\Xi^{(1)}(\xi)\,h(\xi)\,\nabla\xi \tag{3.16}$$

and

$$[u,\sigma]_{\infty} = k_3[\Theta^{(2)},\sigma]_{\infty} + k_4[\Xi^{(2)},\sigma]_{\infty}$$

$$+\frac{\alpha}{\omega}[\Xi^{(2)},\sigma]_{\infty}\int_{d}^{\infty}\Theta^{(2)}(\xi)\,h(\xi)\,\nabla\xi$$

$$= \frac{k_3}{\alpha}\cos\gamma + \frac{k_2}{\alpha}\cos\beta + \frac{\alpha}{\omega\alpha}\cos\beta\int_{-\infty}^{d}\Xi^{(1)}(\xi)\,h(\xi)\,\nabla\xi. \tag{3.17}$$

From (3.2), we obtain $k_3 = 0$.

Similarly, we find

$$U(d+) = \begin{pmatrix} u(d+) \\ p(d+)\,u^{\Delta}(d+) \end{pmatrix} = \begin{pmatrix} k_4\Xi^{(2)}(d+) \\ k_4 p(d+)\,\Xi^{(2)^{\Delta}}(d+) \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{\alpha}{\omega}\Theta^{(2)}(d+)\int_d^{\infty}\Xi^{(2)}(\xi)\,h(\xi)\,\nabla\xi \\ \frac{\alpha}{\omega}p(d+)\,\Theta^{(2)^{\Delta}}(d+)\int_d^{\infty}\Xi^{(2)}(\xi)\,h(\xi)\,\nabla\xi \end{pmatrix}$$

$$= k_4\begin{pmatrix} \Xi^{(2)}(d+) \\ p(d+)\,\Xi^{(2)^{\Delta}}(d+) \end{pmatrix}$$

$$+\frac{\alpha}{\omega}\int_d^{\infty}\Xi^{(2)}(\xi)\,h(\xi)\,\nabla\xi\begin{pmatrix} \Theta^{(2)}(d+) \\ p(d+)\,\Theta^{(2)^{\Delta}}(d+) \end{pmatrix}$$

$$= k_4\Psi(d+) + \left\{\frac{\alpha}{\omega}\int_d^{\infty}\Xi^{(2)}(\xi)\,h(\xi)\,d_q\xi\right\}\Phi(d+)$$

and

$$U(d-) = \begin{pmatrix} u(d-) \\ p(d-)\,u^{\Delta}(d-) \end{pmatrix} = \begin{pmatrix} k_1\Theta^{(1)}(d-) \\ k_1 p(d-)\,\Theta^{(1)^{\Delta}}(d-) \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{\Xi^{(1)}(d-)}{\omega}\int_{-\infty}^{d}\Theta^{(1)}(\xi)\,h(\xi)\,\nabla\xi \\ \frac{p(d-)\Xi^{(1)^{\Delta}}(d-)}{\omega}\int_{-\infty}^{d}\Theta^{(1)}(\xi)\,h(\xi)\,\nabla\xi \end{pmatrix}$$

$$= k_1\begin{pmatrix} \Theta^{(1)}(d-) \\ p(d-)\,\Theta^{(1)^{\Delta}}(d-) \end{pmatrix}$$

$$+\frac{1}{\omega}\int_{-\infty}^{d}\Theta^{(1)}(\xi)\,h(\xi)\,\nabla\xi\begin{pmatrix} \Xi^{(1)}(d-) \\ p(d-)\,\Xi^{(1)^{\Delta}}(d-) \end{pmatrix}$$

$$= k_1\Phi(d-) + \left\{\frac{1}{\omega}\int_{-\infty}^{d}\Theta^{(1)}(\xi)\,h(\xi)\,d_q\xi\right\}\Psi(d-).$$

It follows from (3.2) that

$$k_4\Psi(d+) + \left\{\frac{\alpha}{\omega}\int_d^{\infty}\Xi^{(2)}(\xi)\,h(\xi)\,\nabla\xi\right\}\Phi(d+)$$

$$= \Pi\left\{k_1\Phi(d-) + \left\{\frac{1}{\omega}\int_{-\infty}^{d}\Theta^{(1)}(\xi)\,h(\xi)\,\nabla\xi\right\}\Psi(d-)\right\}.$$

Using (3.6) and (3.8), we obtain

$$\Phi(d-)\left\{\frac{\alpha}{\omega}\int_d^\infty \Xi^{(2)}(\xi)\,h(\xi)\,\nabla\xi - k_1\right\}$$

$$= \Psi(d-)\left\{\frac{1}{\omega}\int_{-\infty}^d \Theta^{(1)}(\xi)\,h(\xi)\,\nabla\xi - k_4\right\}$$

$$\begin{pmatrix} \Theta^{(1)}(d-) \\ p(d-)\,\Theta^{(1)^\Delta}(d-) \end{pmatrix}\left\{\frac{\alpha}{\omega}\int_d^\infty \Xi^{(2)}(\xi)\,h(\xi)\,\nabla\xi - k_1\right\}$$

$$= \begin{pmatrix} \Xi^{(1)}(d-) \\ p(d-)\,\Xi^{(1)^\Delta}(d-) \end{pmatrix}\left\{\frac{1}{\omega}\int_{-\infty}^d \Theta^{(1)}(\xi)\,h(\xi)\,\nabla\xi - k_4\right\}.$$

Therefore, we find

$$k_4\Xi^{(1)}(d-) - k_1\Theta^{(1)}(d-)$$

$$= \left\{\frac{1}{\omega}\int_{-\infty}^d \Theta^{(1)}(\xi)\,h(\xi)\,\nabla\xi\right\}\Xi^{(1)}(d-)$$

$$-\left\{\frac{\alpha}{\omega}\int_d^\infty \Xi^{(2)}(\xi)\,h(\xi)\,\nabla\xi\right\}\Theta^{(1)}(d-),$$

$$k_4 p(d-)\,\Xi^{(1)^\Delta}(d-) - k_1 p(d-)\,\Theta^{(1)^\Delta}(d-)$$

$$= \left\{\frac{1}{\omega}\int_{-\infty}^d \Theta^{(1)}(\xi)\,h(\xi)\,\nabla\xi\right\}p(d-)\,\Xi^{(1)^\Delta}(d-)$$

$$\left\{\frac{\alpha}{\omega}\int_d^\infty \Xi^{(2)}(\xi)\,h(\xi)\,\nabla\xi\right\}p(d-)\,\Theta^{(1)^\Delta}(d-),$$

i.e.,

$$\begin{pmatrix} \Xi^{(1)}(d-) & \Theta^{(1)}(d-) \\ p(d-)\,\Xi^{(1)^\Delta}(d-) & p(d-)\,\Theta^{(1)^\Delta}(d-) \end{pmatrix}\begin{pmatrix} k_4 \\ -k_1 \end{pmatrix}$$

$$= \begin{pmatrix} \Xi^{(1)}(d-) & \Theta^{(1)}(d-) \\ p(d-)\,\Xi^{(1)^\Delta}(d-) & p(d-)\,\Theta^{(1)^\Delta}(d-) \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\omega}\int_{-\infty}^d \Theta^{(1)}(\xi)\,h(\xi)\,\nabla\xi \\ -\frac{\alpha}{\omega}\int_d^\infty \Xi^{(2)}(\xi)\,h(\xi)\,\nabla\xi \end{pmatrix}.$$

Then we conclude that

$$k_1 = \frac{\alpha}{\omega}\int_d^\infty \Xi^{(2)}(\xi)\,h(\xi)\,\nabla\xi,\ \ k_4 = \frac{1}{\omega}\int_{-\infty}^d \Theta^{(1)}(\xi)\,h(\xi)\,\nabla\xi,$$

due to

$$\begin{vmatrix} \Xi^{(1)}(d-) & \Theta^{(1)}(d-) \\ p(d-)\,\Xi^{(1)^{\Delta}}(d-) & p(d-)\,\Theta^{(1)^{\Delta}}(d-) \end{vmatrix} = -\omega \neq 0.$$

Hence

$$u(\eta) = \Theta^{(1)}(\eta)\,\frac{\alpha}{\omega}\int_{d}^{\infty}\Xi^{(2)}(\xi)\,h(\xi)\,\nabla\xi$$

$$+\frac{\Xi^{(1)}(\eta)}{\omega}\int_{-\infty}^{\eta}\Theta^{(1)}(\xi)\,h(\xi)\,\nabla\xi$$

$$+\frac{\Theta^{(1)}(\eta)}{\omega}\int_{\eta}^{d}\Xi^{(1)}(\xi)\,h(\xi)\,\nabla\xi,\ \ \eta\in J_1,$$

and

$$u(\eta) = \Xi^{(2)}(\eta)\,\frac{1}{\omega}\int_{-\infty}^{d}\Theta^{(1)}(\xi)\,h(\xi)\,\nabla\xi$$

$$+\frac{\alpha}{\omega}\Xi^{(2)}(\eta)\int_{d}^{\eta}\Theta^{(2)}(\xi)\,h(\xi)\,\nabla\xi$$

$$+\frac{\alpha}{\omega}\Theta^{(2)}(\eta)\int_{\eta}^{\infty}\Xi^{(2)}(\xi)\,h(\xi)\,\nabla\xi,\ \ \eta\in J_2.$$

□

Theorem 3.2 *The unique solution of (3.1), (2.8), (2.9) is given by*

$$u(\eta) = w(\eta) + \langle G(\eta,.)\,,h(.)\rangle,$$

where

$$w(\eta) = \frac{\varsigma_1}{\omega}\Theta(\eta) - \frac{\varsigma_2}{\omega}\Xi(\eta)\,.$$

Proof From (3.4)-(3.8), we deduce that $w(\eta)$ is a unique solution of equation $Lu = 0$ satisfying (2.8)-(2.9). From Theorem 3.1 that $\langle G(\eta,.)\,,h(.)\rangle$ is a unique solution of Eq. (3.1) satisfying (3.2). □

It follows from Theorem 3.2 that the problem (2.1), (2.8), (2.9) in H is equivalent to the following equation

$$u(\eta) = w(\eta) + \langle G(\eta,.)\,,\Gamma(.,u(.))\rangle, \tag{3.18}$$

where $\eta \in J$.

Now, we shall consider Eq. (3.18).

Let $T : H \to H$ be an operator defined as

$$(Tu)(\eta) = w(\eta) + \langle G(\eta,.)\,,\Gamma(.,u(.))\rangle, \tag{3.19}$$

where $\eta \in J$, and $u, w \in H$. By (3.18), we find $u = Tu$.

4. The existence theorem

In this section, the existence and uniqueness of the solutions will be proved.

Theorem 4.1 *Assume the hypotheses (**H1**)-(**H3**) hold. In addition, let there exist a number $K>0$ such that*

$$\int_{-\infty}^{d}\left|\Gamma^{(1)}\left(\eta,u_1^{(1)}(\eta)\right)-\Gamma^{(1)}\left(\eta,u_2^{(1)}(\eta)\right)\right|^2\nabla\eta$$

$$+\alpha\int_{d}^{\infty}\left|\Gamma^{(2)}\left(\eta,u_1^{(2)}(\eta)\right)-\Gamma^{(2)}\left(\eta,u_2^{(2)}(\eta)\right)\right|^2\nabla\eta$$

$$\leq K^2\left(\int_{-\infty}^{d}\left|u_1^{(1)}(\eta)-u_2^{(1)}(\eta)\right|^2\nabla\eta+\alpha\int_{d}^{\infty}\left|u_1^{(2)}(\eta)-u_2^{(2)}(\eta)\right|^2\nabla\eta\right)$$

$$=K^2\left\|u_1-u_2\right\|^2 \tag{4.1}$$

for all $u_1,u_2\in H$. If

$$K\left(\int_{-\infty}^{d}\int_{-\infty}^{d}\left|G(\eta,\xi)\right|^2\nabla\eta\nabla\xi+\alpha\int_{d}^{\infty}\int_{d}^{\infty}\left|G(\eta,\xi)\right|^2\nabla\eta\nabla\xi\right)^{1/2}<1, \tag{4.2}$$

then the boundary-value problem (2.1), (2.8), (2.9) has a unique solution in H.

Proof Let $u_1,u_2\in H$. Then we have

$$\left|(Tu_1)(\eta)-(Tu_2)(\eta)\right|^2$$

$$=\left|\left\langle G(\eta,.),\left[\Gamma(.,u_1(.))-\Gamma(.,u_2(.))\right]\right\rangle\right|^2$$

$$\leq\left\|G(\eta,.)\right\|^2\left\|\Gamma(.,u_1(.))-\Gamma(.,u_2(.))\right\|^2$$

$$\leq K^2\left\|G(\eta,.)\right\|^2\left\|u_1-u_2\right\|^2,\ \eta\in J.$$

Thus we find

$$\left\|Tu_1-Tu_2\right\|\leq\theta\left\|u_1-u_2\right\|,$$

where

$$\theta=K\left(\int_{-\infty}^{d}\int_{-\infty}^{d}\left|G(\eta,\xi)\right|^2\nabla\eta\nabla\xi+\alpha\int_{d}^{\infty}\int_{d}^{\infty}\left|G(\eta,\xi)\right|^2\nabla\eta\nabla\xi\right)^{1/2}.$$

Since $\theta<1$, we conclude that T is a contraction operator. □

Theorem 4.2 *Assume the hypotheses (**H1**)-(**H3**) hold. In addition, let there exist numbers* $M,\ K>0$ *such that*

$$\int_{-\infty}^{d}\left|\Gamma^{(1)}\left(\eta,u_1^{(1)}(\eta)\right)-\Gamma^{(1)}\left(\eta,u_2^{(1)}(\eta)\right)\right|^2\nabla\eta$$

$$+\alpha\int_{d}^{\infty}\left|\Gamma^{(2)}\left(\eta,u_1^{(2)}(\eta)\right)-\Gamma^{(2)}\left(\eta,u_2^{(2)}(\eta)\right)\right|^2\nabla\eta$$

$$\leq K^2\left(\int_{-\infty}^{d}\left|u_1^{(1)}(\eta)-u_2^{(1)}(\eta)\right|^2\nabla\eta+\alpha\int_{d}^{\infty}\left|u_1^{(2)}(\eta)-u_2^{(2)}(\eta)\right|^2\nabla\eta\right)$$

$$=K^2\left\|u_1-u_2\right\|^2, \tag{4.3}$$

where $u_1,u_2\ \in\ S_M=\{u\in H:\|u\|\leq M\}$ *and* K *may depend on* M*. If*

$$\left\{\int_{-\infty}^{d}\left|w^{(1)}(\eta)\right|^2\nabla\eta+\alpha\int_{d}^{\infty}\left|w^{(2)}(\eta)\right|^2\nabla\eta\right\}^{1/2}$$

$$+\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|G(\eta,\xi)\right|^2\nabla\eta\nabla\xi\right)^{1/2}$$

$$\times\sup_{u\in S_M}\left\{\begin{array}{c}\int_{-\infty}^{d}\left|\Gamma^{(1)}\left(\xi,u_1^{(1)}(\xi)\right)-\Gamma^{(1)}\left(\xi,u_2^{(1)}(\xi)\right)\right|^2\nabla\xi\\ +\alpha\int_{d}^{\infty}\left|\Gamma^{(2)}\left(\xi,u_1^{(2)}(\xi)\right)-\Gamma^{(2)}\left(\xi,u_2^{(2)}(\xi)\right)\right|^2\nabla\xi\end{array}\right\}^{1/2}$$

$$\leq M \tag{4.4}$$

and

$$K\left(\int_{-\infty}^{d}\int_{-\infty}^{d}\left|G(\eta,\xi)\right|^2\nabla\eta\nabla\xi+\alpha\int_{d}^{\infty}\int_{d}^{\infty}\left|G(\eta,\xi)\right|^2\nabla\eta\nabla\xi\right)^{1/2}<1, \tag{4.5}$$

then the boundary-value problem (2.1), (2.8), (2.9) has a unique solution with

$$\int_{-\infty}^{d}\left|u^{(1)}(\eta)\right|^2\nabla\eta+\alpha\int_{d}^{\infty}\left|u^{(2)}(\eta)\right|^2\nabla\eta\leq M^2.$$

Proof Let $u\in S_M$. Then we see that

$$\|Tu\|=\|w+\langle G(\eta,.),\Gamma(.,u(.))\rangle\|\leq\|w\|+\|\langle G(\eta,.),\Gamma(.,u(.))\rangle\|$$

$$\leq\|w\|+\left(\int_{-\infty}^{d}\int_{-\infty}^{d}\left|G(\eta,\xi)\right|^2\nabla\eta\nabla\xi+\alpha\int_{d}^{\infty}\int_{d}^{\infty}\left|G(\eta,\xi)\right|^2\nabla\eta\nabla\xi\right)^{1/2}$$

$$\times \sup_{u \in S_M} \left\{ \begin{array}{c} \int_{-\infty}^{d} \left| \Gamma^{(1)}\left(\xi, u_1^{(1)}(\xi)\right) - \Gamma^{(1)}\left(\xi, u_2^{(1)}(\xi)\right) \right|^2 \nabla\xi \\ +\alpha \int_{d}^{\infty} \left| \Gamma^{(2)}\left(\xi, u_1^{(2)}(\xi)\right) - \Gamma^{(2)}\left(\xi, u_2^{(2)}(\xi)\right) \right|^2 \nabla\xi \end{array} \right\}^{1/2} \leq M.$$

Therefore, we deduce that $T : S_M \to S_M$.

As in the proof of Theorem 4.1, we infer that

$$\|Tu_1 - Tu_2\| \leq \theta \|u_1 - u_2\|, \; u_1, u_2 \in S_M.$$

By the Banach fixed point theorem, we get the desired result. □

5. The existence theorem without the uniqueness

In this section, the existence theorem of solutions without the uniqueness condition will be proved.

Theorem 5.1 *Under conditions (**H1**)-(**H3**), T is a completely continuous operator.*

Proof Let $u_0 \in H$. Then, we obtain

$$|(Tu)(\eta) - (Tu_0)(\eta)|^2$$

$$= |\langle G(\eta, .), [\Gamma(., u(.)) - \Gamma(., u_0(.))]\rangle|^2 \leq \|G(\eta, .)\|^2$$

$$\times \left\{ \begin{array}{c} \int_{-\infty}^{d} \left| \Gamma^{(1)}\left(\xi, u^{(1)}(\xi)\right) - \Gamma^{(1)}\left(\xi, u_0^{(1)}(\xi)\right) \right|^2 \nabla\xi \\ +\alpha \int_{d}^{\infty} \left| \Gamma^{(2)}\left(\xi, u^{(2)}(\xi)\right) - \Gamma^{(2)}\left(\xi, u_0^{(2)}(\xi)\right) \right|^2 \nabla\xi \end{array} \right\}.$$

which implies that

$$\|Tu - Tu_0\|^2$$

$$\leq K \left\{ \begin{array}{c} \int_{-\infty}^{d} \left| \Gamma^{(1)}\left(\xi, u^{(1)}(\xi)\right) - \Gamma^{(1)}\left(\xi, u_0^{(1)}(\xi)\right) \right|^2 \nabla\xi \\ +\alpha \int_{d}^{\infty} \left| \Gamma^{(2)}\left(\xi, u^{(2)}(\xi)\right) - \Gamma^{(2)}\left(\xi, u_0^{(2)}(\xi)\right) \right|^2 \nabla\xi \end{array} \right\}, \tag{5.1}$$

where

$$K = \int_{-\infty}^{d} \int_{-\infty}^{d} |G(\eta, \xi)|^2 \nabla\eta\nabla\xi + \alpha \int_{d}^{\infty} \int_{d}^{\infty} |G(\eta, \xi)|^2 \nabla\eta\nabla\xi.$$

Let F be an operator defined as $Fu(\eta) = \Gamma(\eta, u(\eta))$. By (**H2**), this operator is continuous in H ([13]). Then, for any $\epsilon > 0$, we can find a $\delta > 0$ such that

$$\left\{ \begin{array}{c} \int_{-\infty}^{d} \left| \Gamma^{(1)}\left(\xi, u^{(1)}(\xi)\right) - \Gamma^{(1)}\left(\xi, u_0^{(1)}(\xi)\right) \right|^2 \nabla\xi \\ +\alpha \int_{d}^{\infty} \left| \Gamma^{(2)}\left(\xi, u^{(2)}(\xi)\right) - \Gamma^{(2)}\left(\xi, u_0^{(2)}(\xi)\right) \right|^2 \nabla\xi \end{array} \right\} < \frac{\epsilon^2}{K^2}$$

when $\|u-u_0\|<\delta$. Therefore, by (5.1),

$$\|Tu-Tu_0\|<\epsilon,$$

which implies that T is continuous.

Let $U=\{u\in H:\|u\|\leq\varkappa\}$. By (4.4), we find

$$\|Tu\|\leq\|w\|+\left\{\begin{array}{c} K\int_{-\infty}^{d}\left|\Gamma^{(1)}\left(\xi,u^{(1)}\left(\xi\right)\right)\right|^{2}\nabla\xi \\ +\alpha K\int_{d}^{\infty}\left|\Gamma^{(2)}\left(\xi,u^{(2)}\left(\xi\right)\right)\right|^{2}\nabla\xi \end{array}\right\}^{1/2},$$

for all $u\in U$. Moreover, by (2.3), we conclude that

$$\int_{-\infty}^{d}\left|\Gamma^{(1)}\left(\xi,u^{(1)}\left(\xi\right)\right)\right|^{2}\nabla\xi+\alpha\int_{d}^{\infty}\left|\Gamma^{(2)}\left(\xi,u^{(2)}\left(\xi\right)\right)\right|^{2}\nabla\xi$$

$$\leq\int_{-\infty}^{d}\left[\Sigma^{(1)}\left(\xi\right)+\kappa\left|u^{(1)}\left(\xi\right)\right|\right]^{2}\nabla\xi$$

$$+\alpha\int_{d}^{\infty}\left[\Sigma^{(2)}\left(\xi\right)+\kappa\left|u^{(2)}\left(\xi\right)\right|\right]^{2}\nabla\xi$$

$$\leq 2\int_{-\infty}^{d}\left[\left(\Sigma^{(1)}\right)^{2}\left(\xi\right)+\kappa^{2}\left|u^{(1)}\left(\xi\right)\right|^{2}\right]\nabla\xi$$

$$+2\alpha\int_{d}^{\infty}\left[\left(\Sigma^{(2)}\right)^{2}\left(\xi\right)+\kappa^{2}\left|u^{(2)}\left(\xi\right)\right|^{2}\right]\nabla\xi$$

$$=2\left(\|\Sigma\|^{2}+\kappa^{2}\|u\|^{2}\right)\leq 2\left(\|\Sigma\|^{2}+\kappa^{2}\varkappa^{2}\right).$$

Consequently, we have, for every $u\in U$,

$$\|Tu\|\leq\|w\|+\left[2K\left(\|\Sigma\|^{2}+\kappa^{2}\varkappa^{2}\right)\right]^{1/2}.$$

Furthermore, for all $u\in U,$ we see that

$$\int_{-\infty}^{d}\left|\left(Tu^{(1)}\right)\left(\eta+h\right)-\left(Tu^{(1)}\right)\left(\eta\right)\right|^{2}\nabla\eta$$

$$+\alpha\int_{d}^{\infty}\left|\left(Tu^{(2)}\right)\left(\eta+h\right)-\left(Tu^{(2)}\right)\left(\eta\right)\right|^{2}\nabla\eta$$

$$=\left\|\left\langle\left[G\left(\eta+h,.\right)-G\left(\eta,.\right)\right],\Gamma\left(.,u\left(.\right)\right)\right\rangle\right\|^{2}$$

$$\leq \left\{ \begin{array}{c} \int_{-\infty}^{d} \int_{-\infty}^{d} \left| G(\eta+h,\xi) - G(\eta,\xi) \right|^2 \nabla\eta\nabla\xi \\ +\alpha \int_{d}^{\infty} \int_{d}^{\infty} \left| G(\eta+h,\xi) - G(\eta,\xi) \right|^2 \nabla\eta\nabla\xi \end{array} \right\}$$

$$\times \left\{ \begin{array}{c} \int_{-\infty}^{d} \left| \Gamma^{(1)}\left(\xi, u^{(1)}(\xi)\right) \right|^2 \nabla\xi \\ +\alpha \int_{d}^{\infty} \left| \Gamma^{(2)}\left(\xi, u^{(2)}(\xi)\right) \right|^2 \nabla\xi \end{array} \right\}$$

$$\leq 2\left(\|g\|^2 + \kappa^2 \varkappa^2 \right) \left\{ \begin{array}{c} \int_{-\infty}^{d} \int_{-\infty}^{d} \left| G(\eta+h,\xi) - G(\eta,\xi) \right|^2 \nabla\eta\nabla\xi \\ +\alpha \int_{d}^{\infty} \int_{d}^{\infty} \left| G(\eta+h,\xi) - G(\eta,\xi) \right|^2 \nabla\eta\nabla\xi \end{array} \right\}.$$

By (3.10), for any $\epsilon > 0$ and every $u \in U,$ we can find a $\delta > 0$ such that

$$\int_{-\infty}^{d} \left| Tu^{(1)}(\eta+h) - Tu^{(1)}(\eta) \right|^2 \nabla\eta$$

$$+\alpha \int_{d}^{\infty} \left| Tu^{(2)}(\eta+h) - Tu^{(2)}(\eta) \right|^2 \nabla\eta < \epsilon^2,$$

where $h < \delta.$

Moreover, for every $u \in U,$ we deduce that

$$\int_{-\infty}^{-N} \left| Tu^{(1)}(\eta) \right|^2 \nabla\eta + \alpha \int_{N}^{\infty} \left| Tu^{(2)}(\eta) \right|^2 \nabla\eta$$

$$\leq \int_{-\infty}^{-N} \left| w^{(1)}(\eta) \right|^2 \nabla\eta + \alpha \int_{N}^{\infty} \left| w^{(2)}(\eta) \right|^2 \nabla\eta$$

$$+2(\|\Sigma\|^2 + \kappa^2\varkappa^2) \left(\int_{-\infty}^{-N} \|G(\eta,.)\|^2 \nabla\eta + \alpha \int_{N}^{\infty} \|G(\eta,.)\|^2 \nabla\eta \right).$$

It follows from (3.10) that for given $\epsilon > 0$ there exists a $N > 0$, depending only on ϵ such that

$$\int_{-\infty}^{-N} \left| Tu^{(2)}(\eta) \right|^2 \nabla\eta + \alpha \int_{N}^{\infty} \left| Tu^{(2)}(\eta) \right|^2 \nabla\eta < \epsilon^2,$$

for all $u \in U.$

Thus, T is a completely continuous operator. □

Theorem 5.2 *Assume that the hypotheses (**H1**)-(**H3**) hold. Further, let there exist a number $M > 0$ such that*

$$\left\{ \int_{-\infty}^{d} \left| w^{(1)}(\eta) \right|^2 \nabla\eta + \alpha \int_{d}^{\infty} \left| w^{(2)}(\eta) \right|^2 \nabla\eta \right\}^{1/2}$$

$$+ \left(\int_{-\infty}^{d} \int_{-\infty}^{d} |G(\eta,\xi)|^2 \nabla\eta\nabla\xi + \alpha \int_{d}^{\infty} \int_{d}^{\infty} |G(\eta,\xi)|^2 \nabla\eta\nabla\xi \right)^{1/2}$$

$$\times \sup_{u\in S_M}\left\{\begin{array}{c}\int_{-\infty}^{d}\left|\Gamma^{(1)}\left(\xi,u_1^{(1)}(\xi)\right)-\Gamma^{(1)}\left(\xi,u_2^{(1)}(\xi)\right)\right|^2\nabla\xi\\+\alpha\int_{d}^{\infty}\left|\Gamma^{(2)}\left(\xi,u_1^{(2)}(\xi)\right)-\Gamma^{(2)}\left(\xi,u_2^{(2)}(\xi)\right)\right|^2\nabla\xi\end{array}\right\}^{1/2}$$

$$\leq M,$$

where $S_M=\{u\in H:\|u\|\leq M\}$. *Then the boundary-value problem (2.1), (2.8), (2.9) has a unique solution with*

$$\int_{-\infty}^{d}\left|u^{(1)}(\eta)\right|^2\nabla\eta+\alpha\int_{d}^{\infty}\left|u^{(2)}(\eta)\right|^2\nabla\eta\leq M^2.$$

Proof Let us consider the operator T defined as (3.19). By Theorems 4.2 and 5.1, we conclude that $T: S_M\to S_M$. Using Schauder's fixed point theorem, the theorem follows because S_M is bounded, convex, and closed. □

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