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On the monoid of partial isometries of a cycle graph

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Abstract: In this paper we consider the monoid $\mathcal{DPC}_n$ of all partial isometries of an n-cycle graph C_n. We show that $\mathcal{DPC}_n$ is the submonoid of the monoid of all oriented partial permutations on an n-chain whose elements are precisely all restrictions of a dihedral group of order $2n$. Our main aim is to exhibit a presentation of $\mathcal{DPC}_n$. We also describe Green's relations of $\mathcal{DPC}_n$ and calculate its cardinality and rank.

Key words: Transformations, orientation, partial isometries, cycle graphs, rank, presentations

1. Introduction

Let Ω be a finite set. As usual, let us denote by $\mathcal{PT}(\Omega)$ the monoid (under composition) of all partial transformations on Ω, by $\mathcal{T}(\Omega)$ the submonoid of $\mathcal{PT}(\Omega)$ of all full transformations on Ω, by $\mathcal{I}(\Omega)$ the symmetric inverse monoid on Ω, i.e. the inverse submonoid of $\mathcal{PT}(\Omega)$ of all partial permutations on Ω, and by $\mathcal{S}(\Omega)$ the symmetric group on Ω, i.e. the subgroup of $\mathcal{PT}(\Omega)$ of all permutations on Ω.

Recall that the rank of a (finite) monoid M is the minimum size of all (finite) generating sets of M, i.e. the minimum of the set $\{|X| : X \subseteq M \text{ and } X \text{ generates } M\}$.

Let Ω be a finite set with at least 3 elements. It is well-known that $\mathcal{S}(\Omega)$ has rank 2 (as a semigroup, a monoid, or a group) and $\mathcal{T}(\Omega)$, $\mathcal{I}(\Omega)$, and $\mathcal{PT}(\Omega)$ have ranks 3, 3, and 4, respectively. The survey [13] presents these results and similar ones for other classes of transformation monoids, in particular, for monoids of order-preserving transformations and for some of their extensions. For example, the rank of the extensively studied monoid of all order-preserving transformations of an n-chain is n, which was proved by Gomes and Howie [23] in 1992. More recently, for instance, the papers [5, 16, 17, 19, 21] are dedicated to the computation of the ranks of certain classes of transformation semigroups or monoids.

A monoid presentation is an ordered pair $\langle A \mid R\rangle$, where A is a set, often called an alphabet, and $R \subseteq A^* \times A^*$ is a set of relations of the free monoid A^* generated by A. A monoid M is said to be defined by a presentation $\langle A \mid R\rangle$ if M is isomorphic to A^*/ρ_R, where ρ_R denotes the smallest congruence on A^* containing R.

Given a finite monoid, it is clear that we can always exhibit a presentation for it, at worst by enumerating all elements from its multiplication table, but clearly this is of no interest, in general. So, by determining a

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presentation for a finite monoid, we mean to find in some sense a nice presentation (e.g., with a small number of generators and relations).

A presentation for the symmetric group $\mathcal{S}(\Omega)$ was determined by Moore [29] over a century ago (1897). For the full transformation monoid $\mathcal{T}(\Omega)$, a presentation was given in 1958 by Aĭzenštat [1] in terms of a certain type of two-generator presentation for the symmetric group $\mathcal{S}(\Omega)$, plus an extra generator and seven more relations. Presentations for the partial transformation monoid $\mathcal{PT}(\Omega)$ and for the symmetric inverse monoid $\mathcal{I}(\Omega)$ were found by Popova [31] in 1961. In 1962, Aĭzenštat [2] and Popova [32] exhibited presentations for the monoids of all order-preserving transformations and of all order-preserving partial transformations of a finite chain, respectively, and from the Sixties to the present day, several authors obtained presentations for many classes of monoids. See also [33], the survey [13], and, for example, [8–12, 14, 20, 25].

Now, let $G=(V,E)$ be a finite simple connected graph, where V is the set of vertices and E is the list of edges. The (geodesic) distance between two vertices x and y of G, denoted by $\mathrm{d}_G(x,y)$, is the length of a shortest path between x and y, i.e. the number of edges in a shortest path between x and y.

Let $\alpha\in\mathcal{PT}(V)$. We say that α is a partial isometry or distance preserving partial transformation of G if

$$\mathrm{d}_G(x\alpha,y\alpha)=\mathrm{d}_G(x,y)$$

for all $x,y\in\mathrm{Dom}(\alpha)$. Denote by $\mathcal{DP}(G)$ the subset of $\mathcal{PT}(V)$ of all partial isometries of G. Clearly, $\mathcal{DP}(G)$ is a submonoid of $\mathcal{PT}(V)$. Moreover, as a consequence of the property

$$\mathrm{d}_G(x,y)=0 \quad \text{if and only if} \quad x=y$$

for all $x,y\in V$, it immediately follows that $\mathcal{DP}(G)\subseteq\mathcal{I}(V)$. Furthermore, $\mathcal{DP}(G)$ is an inverse submonoid of $\mathcal{I}(V)$ (see [18]).

Observe that if $G=(V,E)$ is a complete graph, i.e. $E=\{\{x,y\}:x,y\in V, x\neq y\}$, then $\mathcal{DP}(G)=\mathcal{I}(V)$.

On the other hand, for $n\geqslant 2$, consider the undirected path graph P_n with n vertices, i.e.

$$P_n=(\{1,\ldots,n\},\{\{i,i+1\}:i=1,\ldots,n-1\}).$$

Then, obviously, $\mathcal{DP}(P_n)$ coincides with the monoid

$$\mathcal{DP}_n=\{\alpha\in\mathcal{I}(\{1,2,\ldots,n\}):|i\alpha-j\alpha|=|i-j| \text{ for all } i,j\in\mathrm{Dom}(\alpha)\}$$

of all partial isometries on $\{1,2,\ldots,n\}$.

The study of partial isometries on $\{1,2,\ldots,n\}$ was initiated by Al-Kharousi et al. in [3, 4]. The first of these two papers is dedicated to investigating some combinatorial properties of the monoid $\mathcal{DP}_n$ and of its submonoid $\mathcal{ODP}_n$ of all order-preserving (considering the usual order of $\mathbb{N}$) partial isometries, in particular, their cardinalities. The second paper presents the study of some of their algebraic properties, namely Green's structure and ranks. Presentations for both the monoids $\mathcal{DP}_n$ and $\mathcal{ODP}_n$ were given by the first author and Quinteiro in [20]. Moreover, for $2\leqslant r\leqslant n-1$, Bugay et al. in [6] obtained the ranks of the subsemigroups $\mathcal{DP}_{n,r}=\{\alpha\in\mathcal{DP}_n:|\mathrm{Im}(\alpha)|\leqslant r\}$ of $\mathcal{DP}_n$ and $\mathcal{ODP}_{n,r}=\{\alpha\in\mathcal{ODP}_n:|\mathrm{Im}(\alpha)|\leqslant r\}$ of $\mathcal{ODP}_n$.

The monoid $\mathcal{DPS}_n$ of all partial isometries of a star graph with n vertices ($n\geqslant 1$) was considered by the authors in [18]. They determined the rank and size of $\mathcal{DPS}_n$ and described its Green's relations. A presentation for $\mathcal{DPS}_n$ was also exhibited in [18].

Now, for $n \geqslant 3$, consider the cycle graph

$$C_n = (\{1, 2, \ldots, n\}, \{\{i, i+1\} : i = 1, 2, \ldots, n-1\} \cup \{\{1, n\}\})$$

with n vertices. Notice that cycle graphs and cycle subgraphs play a fundamental role in Graph Theory.

This paper is devoted to studying the monoid $\mathcal{DP}(C_n)$ of all partial isometries of C_n, which from now on we denote simply by $\mathcal{DPC}_n$. Observe that $\mathcal{DPC}_n$ is an inverse submonoid of the symmetric inverse monoid $\mathcal{I}_n$.

In Section 2, we start by giving a key characterization of $\mathcal{DPC}_n$, which allows for significantly simpler proofs of various results presented later. Also in this section, a description of the Green's relations of $\mathcal{DPC}_n$ is given and the rank and the cardinality of $\mathcal{DPC}_n$ are calculated. Finally, in Section 3, we determine a presentation for the monoid $\mathcal{DPC}_n$ on $n+2$ generators, from which we deduce another presentation for $\mathcal{DPC}_n$ on 3 generators.

For general background and standard notations, we refer to Howie's book [24] for Semigroup Theory, and [34] for Graph Theory.

We would like to point out that we made use of computational tools, namely GAP*[22].

2. Some properties of $\mathcal{DPC}_n$

We begin this section by introducing some concepts and notations.

For $n \in \mathbb{N}$, let Ω_n be a set with n elements. In general, without loss of generality, Ω_n is considered the chain $\Omega_n = \{1 < 2 < \cdots < n\}$ and $\mathcal{PT}(\Omega_n)$, $\mathcal{I}(\Omega_n)$ and $\mathcal{S}(\Omega_n)$ are denoted simply by $\mathcal{PT}_n$, $\mathcal{I}_n$ and $\mathcal{S}_n$, respectively. For any $\alpha \in \mathcal{PT}_n$, the domain and the image sets of α are denoted by $\operatorname{Dom}(\alpha)$ and $\operatorname{Im}(\alpha)$, respectively. Also, the cardinality of the set $\operatorname{Im}(\alpha)$ is called the rank of α.

A partial transformation $\alpha \in \mathcal{PT}_n$ is called order-preserving [order-reversing] if $x \leqslant y$ implies $x\alpha \leqslant y\alpha$ [$x\alpha \geqslant y\alpha$], for all $x, y \in \operatorname{Dom}(\alpha)$. It is clear that the product of two order-preserving or of two order-reversing transformations is order-preserving and the product of an order-preserving transformation by an order-reversing transformation, or vice-versa, is order-reversing. We denote by $\mathcal{POD}_n$ the submonoid of $\mathcal{PT}_n$ whose elements are all order-preserving or order-reversing transformations.

Let $s = (a_1, a_2, \ldots, a_t)$ be a sequence of t ($t \geqslant 0$) elements from the chain Ω_n. We say that s is cyclic [anticyclic] if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $a_i > a_{i+1}$ [$a_i < a_{i+1}$], where a_{t+1} denotes a_1. Notice that, the sequence s is cyclic [anticyclic] if and only if s is empty or there exists $i \in \{0, 1, \ldots, t-1\}$ such that $a_{i+1} \leqslant a_{i+2} \leqslant \cdots \leqslant a_t \leqslant a_1 \leqslant \cdots \leqslant a_i$ [$a_{i+1} \geqslant a_{i+2} \geqslant \cdots \geqslant a_t \geqslant a_1 \geqslant \cdots \geqslant a_i$] (the index $i \in \{0, 1, \ldots, t-1\}$ is unique unless s is constant and $t \geqslant 2$). We also say that s is oriented if s is cyclic or s is anticyclic (see, for example, [7, 26, 28]). Given a partial transformation $\alpha \in \mathcal{PT}_n$ such that $\operatorname{Dom}(\alpha) = \{a_1 < \cdots < a_t\}$ with $t \geqslant 0$, we say that α is orientation-preserving [orientation-reversing, oriented] if the sequence of its images $(a_1\alpha, \ldots, a_t\alpha)$ is cyclic [anticyclic, oriented]. It is easy to show that the product of two orientation-preserving or of two orientation-reversing transformations is orientation-preserving and the product of an orientation-preserving transformation by an orientation-reversing transformation, or vice-versa, is orientation-reversing. We denote by $\mathcal{POR}_n$ the submonoid of $\mathcal{PT}_n$ of all oriented transformations.

Notice that $\mathcal{POD}_n \cap \mathcal{I}_n$ and $\mathcal{POR}_n \cap \mathcal{I}_n$ are inverse submonoids of $\mathcal{I}_n$.

*https://www.gap-system.org

Let us consider the following permutations of Ω_n (for $n \geqslant 2$) of order n and 2, respectively:

$$g = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}.$$

It is clear that $g, h \in \mathcal{POR}_n \cap \mathcal{I}_n$. Moreover, for $n \geqslant 3$, g together with h generate the well-known dihedral group $\mathcal{D}_{2n}$ of order $2n$ (considered a subgroup of $\mathcal{S}_n$). In fact, for $n \geqslant 3$,

$$\mathcal{D}_{2n} = \langle g, h \mid g^n = 1, h^2 = 1, hg = g^{n-1}h \rangle = \{1, g, g^2, \ldots, g^{n-1}, h, hg, hg^2, \ldots, hg^{n-1}\}$$

and we have

$$g^k = \begin{pmatrix} 1 & 2 & \cdots & n-k & n-k+1 & \cdots & n \\ 1+k & 2+k & \cdots & n & 1 & \cdots & k \end{pmatrix}, \quad \text{i.e.} \quad ig^k = \begin{cases} i+k & 1 \leqslant i \leqslant n-k \\ i+k-n & n-k+1 \leqslant i \leqslant n, \end{cases}$$

and

$$hg^k = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ k & \cdots & 1 & n & \cdots & k+1 \end{pmatrix}, \quad \text{i.e.} \quad ihg^k = \begin{cases} k-i+1 & 1 \leqslant i \leqslant k \\ n+k-i+1 & k+1 \leqslant i \leqslant n, \end{cases}$$

for $0 \leqslant k \leqslant n-1$. Observe that, for $n \in \{1, 2\}$, the dihedral group $\mathcal{D}_{2n} = \langle g, h \mid g^n = 1, h^2 = 1, hg = g^{n-1}h \rangle$ of order $2n$ (also known as the Klein four-group for $n = 2$) cannot be considered a subgroup of $\mathcal{S}_n$. Denote also by $\mathcal{C}_n$ the cyclic group of order n generated by g, i.e. $\mathcal{C}_n = \{1, g, g^2, \ldots, g^{n-1}\}$.

Until the end of this paper, we will consider $n \geqslant 3$. Moreover, for convenience, we will denote $\alpha \in \mathcal{PT}_n$ with $\mathrm{Dom}(\alpha) = \{i_1, \ldots, i_k\}$ $(k \geqslant 1)$ by $\alpha = \begin{pmatrix} i_1 & \cdots & i_k \\ i_1\alpha & \cdots & i_k\alpha \end{pmatrix}$.

Now, notice that,

$$\mathrm{d}_{C_n}(x, y) = \min\{|x-y|, n-|x-y|\} = \begin{cases} |x-y| & \text{if } |x-y| \leqslant \frac{n}{2} \\ n-|x-y| & \text{if } |x-y| > \frac{n}{2}, \end{cases}$$

and so $0 \leqslant \mathrm{d}_{C_n}(x, y) \leqslant \frac{n}{2}$ for all $x, y \in \{1, 2, \ldots, n\}$.

From now on, for any two vertices x and y of C_n, we denote the distance $\mathrm{d}_{C_n}(x, y)$ simply by $\mathrm{d}(x, y)$.

Observe for $x, y \in \Omega_n$ that

$$\mathrm{d}(x, y) = \frac{n}{2} \quad \Leftrightarrow \quad |x-y| = \frac{n}{2} \quad \Leftrightarrow \quad n-|x-y| = \frac{n}{2} \quad \Leftrightarrow \quad |x-y| = n-|x-y|,$$

in which case n is even, and

$$|\{z \in \{1, 2, \ldots, n\} : \mathrm{d}(x, z) = d\}| = \begin{cases} 1 & \text{if } d = \frac{n}{2} \\ 2 & \text{if } d < \frac{n}{2} \end{cases} \tag{2.1}$$

for all $1 \leqslant d \leqslant \frac{n}{2}$. Moreover, it is a routine matter to show that

$$D = \{z \in \{1, 2, \ldots, n\} : \mathrm{d}(x, z) = d\} = \{z \in \{1, 2, \ldots, n\} : \mathrm{d}(y, z) = d'\}$$

implies

$$\mathrm{d}(x, y) = \begin{cases} 0 \text{ (i.e. } x = y) & \text{if } |D| = 1 \\ \frac{n}{2} & \text{if } |D| = 2, \end{cases} \tag{2.2}$$

for all $1 \leqslant d, d' \leqslant \frac{n}{2}$.

Recall that $\mathcal{DP}_n$ is an inverse submonoid of $\mathcal{POD}_n \cap \mathcal{I}_n$. This is an easy fact to prove and was observed by Al-Kharousi et al. in [3, 4]. A similar result is also valid for $\mathcal{DPC}_n$ and $\mathcal{POR}_n \cap \mathcal{I}_n$, as we will deduce below.

First, notice that it is easy to show that both permutations g and h of Ω_n belong to $\mathcal{DPC}_n$ and so the dihedral group $\mathcal{D}_{2n}$ is contained in $\mathcal{DPC}_n$. Furthermore, as we prove next, the elements of $\mathcal{DPC}_n$ are precisely the restrictions of the permutations of the dihedral group $\mathcal{D}_{2n}$. This is a key characterization of $\mathcal{DPC}_n$ that will allow us to prove in a simpler way some of the results that we present later in this paper. Observe that

$$\alpha = \sigma|_{\mathrm{Dom}(\alpha)} \quad \Leftrightarrow \quad \alpha = \mathrm{id}_{\mathrm{Dom}(\alpha)}\sigma \quad \Leftrightarrow \quad \alpha = \sigma \mathrm{id}_{\mathrm{Im}(\alpha)},$$

for any $\alpha \in \mathcal{PT}_n$ and $\sigma \in \mathcal{I}_n$, where $\sigma|_{\mathrm{Dom}(\alpha)}$ denotes the restriction mapping of σ to $\mathrm{Dom}(\alpha)$ and id_U, with $U \subseteq \Omega_n$, denotes the restriction map of the identity mapping id of Ω_n to U.

Lemma 2.1 *For any $\alpha \in \mathcal{PT}_n$, $\alpha \in \mathcal{DPC}_n$ if and only if there exists $\sigma \in \mathcal{D}_{2n}$ such that $\alpha = \sigma|_{\mathrm{Dom}(\alpha)}$. Furthermore, for $\alpha \in \mathcal{DPC}_n$:*

1. *if either $|\mathrm{Dom}(\alpha)| = 1$ or $|\mathrm{Dom}(\alpha)| = 2$ and $\mathrm{d}(\min \mathrm{Dom}(\alpha), \max \mathrm{Dom}(\alpha)) = \frac{n}{2}$ (in which case n is even), then there exist exactly two (distinct) permutations $\sigma, \sigma' \in \mathcal{D}_{2n}$ such that $\alpha = \sigma|_{\mathrm{Dom}(\alpha)} = \sigma'|_{\mathrm{Dom}(\alpha)}$;*

2. *if either $|\mathrm{Dom}(\alpha)| = 2$ and $\mathrm{d}(\min \mathrm{Dom}(\alpha), \max \mathrm{Dom}(\alpha)) \neq \frac{n}{2}$ or $|\mathrm{Dom}(\alpha)| \geqslant 3$, then there exists exactly one permutation $\sigma \in \mathcal{D}_{2n}$ such that $\alpha = \sigma|_{\mathrm{Dom}(\alpha)}$.*

Proof For any $\alpha \in \mathcal{PT}_n$, if $\alpha = \sigma|_{\mathrm{Dom}(\alpha)}$, for some $\sigma \in \mathcal{D}_{2n}$, then $\alpha \in \mathcal{DPC}_n$ since $\mathcal{D}_{2n} \subseteq \mathcal{DPC}_n$ and, clearly, any restriction of an element of $\mathcal{DPC}_n$ also belongs to $\mathcal{DPC}_n$.

Conversely, let us suppose that $\alpha \in \mathcal{DPC}_n$. First, observe that, for each pair $1 \leqslant i, j \leqslant n$, there exists a unique $k \in \{0, 1, \ldots, n-1\}$ such that $ig^k = j$ and there exists a unique $\ell \in \{0, 1, \ldots, n-1\}$ such that $ihg^\ell = j$, where g and h are the permutations defined above. In fact, for $1 \leqslant i, j \leqslant n$ and $k, \ell \in \{0, 1, \ldots, n-1\}$, it is easy to show that

1. if $i \leqslant j$ then $ig^k = j$ if and only if $k = j - i$;

2. if $i > j$ then $ig^k = j$ if and only if $k = n + j - i$;

3. if $i + j \leqslant n$ then $ihg^\ell = j$ if and only if $\ell = i + j - 1$;

4. if $i + j > n$ then $ihg^\ell = j$ if and only if $\ell = i + j - 1 - n$.

Therefore, we may conclude immediately that:

1. any nonempty transformation of $\mathcal{DPC}_n$ has at most two distinct extensions in $\mathcal{D}_{2n}$ and, if there are two distinct, one must be an orientation-preserving transformation and the other an orientation-reversing transformation;

2. any transformation of $\mathcal{DPC}_n$ with rank 1 has two distinct extensions in $\mathcal{D}_{2n}$ (one is an orientation-preserving transformation and the other is an orientation-reversing transformation).

Notice that, as $g^n = g^{-n} = 1$, we also have $ig^{j-i} = j$ and $ihg^{i+j-1} = j$, for all $1 \leqslant i,j \leqslant n$.

Next, suppose that $\mathrm{Dom}(\alpha) = \{i_1 < i_2\}$. Then, there exist $\sigma \in \mathcal{C}_n$ and $\xi \in \mathcal{D}_{2n} \setminus \mathcal{C}_n$ (both unique) such that $i_1\sigma = i_1\alpha = i_1\xi$. Take $D = \{z \in \{1,2,\dots,n\} : \mathrm{d}(i_1\alpha, z) = \mathrm{d}(i_1,i_2)\}$. Then $1 \leqslant |D| \leqslant 2$ and $i_2\alpha, i_2\sigma, i_2\xi \in D$.

Suppose that $i_2\sigma = i_2\xi$ and let $j_1 = i_1\sigma$ and $j_2 = i_2\sigma$. Then $\sigma = g^{j_1-i_1} = g^{j_2-i_2}$ and $\xi = hg^{i_1+j_1-1} = hg^{i_2+j_2-1}$. Hence, we have $j_1 - i_1 = j_2 - i_2$ or $j_1 - i_1 = j_2 - i_2 \pm n$ from the first equality, and $i_1 + j_1 = i_2 + j_2$ or $i_1 + j_1 = i_2 + j_2 \pm n$ from the second. Since $i_1 \neq i_2$ and $i_2 - i_1 \neq n$, it is a routine matter to conclude that the only possibility is to have $i_2 - i_1 = \frac{n}{2}$ (in which case n is even). Thus, $\mathrm{d}(i_1,i_2) = \frac{n}{2}$. By (2.1), it follows that $|D| = 1$ and so $i_2\alpha = i_2\sigma = i_2\xi$, i.e. α is extended by both σ and ξ.

If $i_2\sigma \neq i_2\xi$, then $|D| = 2$ (whence $\mathrm{d}(i_1,i_2) < \frac{n}{2}$), and so either $i_2\alpha = i_2\sigma$ or $i_2\alpha = i_2\xi$. In this case, α is extended by exactly one permutation of $\mathcal{D}_{2n}$.

Now, suppose that $\mathrm{Dom}(\alpha) = \{i_1 < i_2 < \cdots < i_k\}$ for some $3 \leqslant k \leqslant n-1$. Since $\sum_{p=1}^{k-1}(i_{p+1} - i_p) = i_k - i_1 < n$, then there exists at most one index $1 \leqslant p \leqslant k-1$ such that $i_{p+1} - i_p \geqslant \frac{n}{2}$. Therefore, we may take $i,j \in \mathrm{Dom}(\alpha)$ such that $i \neq j$ and $\mathrm{d}(i,j) \neq \frac{n}{2}$ and so, as $\alpha|_{\{i,j\}} \in \mathcal{DPC}_n$, by the above deductions, there exists a unique $\sigma \in \mathcal{D}_{2n}$ such that $\sigma|_{\{i,j\}} = \alpha|_{\{i,j\}}$. Let $\ell \in \mathrm{Dom}(\alpha) \setminus \{i,j\}$. Then

$$\ell\alpha, \ell\sigma \in \{z \in \{1,2,\dots,n\} : \mathrm{d}(i\alpha, z) = \mathrm{d}(i,\ell)\} \cap \{z \in \{1,2,\dots,n\} : \mathrm{d}(j\alpha, z) = \mathrm{d}(j,\ell)\}.$$

In order to obtain a contradiction, suppose that $\ell\alpha \neq \ell\sigma$. Therefore, by (2.1), we have

$$\{z \in \{1,2,\dots,n\} : \mathrm{d}(i\alpha, z) = \mathrm{d}(i,\ell)\} = \{\ell\alpha, \ell\sigma\} = \{z \in \{1,2,\dots,n\} : \mathrm{d}(j\alpha, z) = \mathrm{d}(j,\ell)\}$$

and so, by (2.2), $\mathrm{d}(i,j) = \mathrm{d}(i\alpha, j\alpha) = \frac{n}{2}$, which is a contradiction. Hence, $\ell\alpha = \ell\sigma$. Thus, σ is the unique permutation of $\mathcal{D}_{2n}$ such that $\alpha = \sigma|_{\mathrm{Dom}(\alpha)}$, as required. □

Bearing in mind the previous lemma, it seems appropriate to designate $\mathcal{DPC}_n$ by dihedral inverse monoid on Ω_n.

Since $\mathcal{D}_{2n} \subseteq \mathcal{POR}_n \cap \mathcal{I}_n$, which contains all the restrictions of its elements, we have immediately the following corollary.

Corollary 2.2 *The monoid $\mathcal{DPC}_n$ is contained in $\mathcal{POR}_n \cap \mathcal{I}_n$.* □

Observe that, as $\mathcal{D}_{2n}$ is the group of units of $\mathcal{POR}_n \cap \mathcal{I}_n$ (see [14, 15]), then $\mathcal{D}_{2n}$ also has to be the group of units of $\mathcal{DPC}_n$.

Next, recall that, given an inverse submonoid M of $\mathcal{I}_n$, it is well known that the Green's relations $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{H}$ of M can be described as follows: for $\alpha, \beta \in M$,

- $\alpha \mathscr{L} \beta$ if and only if $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$;
- $\alpha \mathscr{R} \beta$ if and only if $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$;
- $\alpha \mathscr{H} \beta$ if and only if $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$ and $\mathrm{Dom}(\alpha) = \mathrm{Dom}(\beta)$.

In $\mathcal{I}_n$, we also have

- $\alpha \mathscr{J} \beta$ if and only if $|\operatorname{Dom}(\alpha)| = |\operatorname{Dom}(\beta)|$ (if and only if $|\operatorname{Im}(\alpha)| = |\operatorname{Im}(\beta)|$).

Since $\mathcal{DPC}_n$ is an inverse submonoid of $\mathcal{I}_n$, it remains to describe its Green's relation $\mathscr{J}$. In fact, it is a routine matter to prove the following proposition.

Proposition 2.3 *Let $\alpha, \beta \in \mathcal{DPC}_n$. Then $\alpha \mathscr{J} \beta$ if and only if one of the following properties is satisfied:*

1. $|\operatorname{Dom}(\alpha)| = |\operatorname{Dom}(\beta)| \leqslant 1$;
2. $|\operatorname{Dom}(\alpha)| = |\operatorname{Dom}(\beta)| = 2$ *and* $\mathrm{d}(i_1, i_2) = \mathrm{d}(i_1', i_2')$ *where* $\operatorname{Dom}(\alpha) = \{i_1, i_2\}$ *and* $\operatorname{Dom}(\beta) = \{i_1', i_2'\}$;
3. $|\operatorname{Dom}(\alpha)| = |\operatorname{Dom}(\beta)| = k \geqslant 3$ *and there exists* $\sigma \in \mathcal{D}_{2k}$ *such that* $\begin{pmatrix} i_1' & i_2' & \cdots & i_k' \\ i_{1\sigma} & i_{2\sigma} & \cdots & i_{k\sigma} \end{pmatrix} \in \mathcal{DPC}_n$ *where* $\operatorname{Dom}(\alpha) = \{i_1 < i_2 < \cdots < i_k\}$ *and* $\operatorname{Dom}(\beta) = \{i_1' < i_2' < \cdots < i_k'\}$. □

An alternative description of $\mathscr{J}$ can be found in the second author's MSc thesis [30].

Next, we count the number of elements of $\mathcal{DPC}_n$.

Theorem 2.4 *One has* $|\mathcal{DPC}_n| = n2^{n+1} - \frac{(-1)^n+5}{4}n^2 - 2n + 1$.

Proof Let $\mathcal{A}_i = \{\alpha \in \mathcal{DPC}_n : |\operatorname{Dom}(\alpha)| = i\}$ for $i = 0, 1, \ldots, n$. Since the sets $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ are pairwise disjoints, we get $|\mathcal{DPC}_n| = \sum_{i=0}^{n} |\mathcal{A}_i|$.

Clearly, $\mathcal{A}_0 = \{\emptyset\}$, where $\emptyset$ denotes the empty mapping on Ω_n, and $\mathcal{A}_1 = \{\binom{i}{j} : 1 \leqslant i, j \leqslant n\}$, whence $|\mathcal{A}_0| = 1$ and $|\mathcal{A}_1| = n^2$. Moreover, for $i \geqslant 3$, by Lemma 2.1, we have as many elements in $\mathcal{A}_i$ as there are restrictions of rank i of permutations of $\mathcal{D}_{2n}$, i.e. we have $\binom{n}{i}$ distinct elements of $\mathcal{A}_i$ for each permutation of $\mathcal{D}_{2n}$, whence $|\mathcal{A}_i| = 2n\binom{n}{i}$. Similarly, for an odd n, by Lemma 2.1, we have $|\mathcal{A}_2| = 2n\binom{n}{2}$. On the other hand, if n is even, also by Lemma 2.1, we have as many elements in $\mathcal{A}_2$ as there are restrictions of rank 2 of permutations of $\mathcal{D}_{2n}$ minus the number of elements of $\mathcal{A}_2$ that have two distinct extensions in $\mathcal{D}_{2n}$, i.e. $|\mathcal{A}_2| = 2n\binom{n}{2} - |\mathcal{B}_2|$, where

$$\mathcal{B}_2 = \{\alpha \in \mathcal{DPC}_n : |\operatorname{Dom}(\alpha)| = 2 \text{ and } \mathrm{d}(\min \operatorname{Dom}(\alpha), \max \operatorname{Dom}(\alpha)) = \tfrac{n}{2}\}.$$

It is easy to check that

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} i & i+\frac{n}{2} \\ j & j+\frac{n}{2} \end{pmatrix}, \begin{pmatrix} i & i+\frac{n}{2} \\ j+\frac{n}{2} & j \end{pmatrix} : 1 \leqslant i, j \leqslant \frac{n}{2} \right\},$$

whence $|\mathcal{B}_2| = 2(\frac{n}{2})^2 = \frac{1}{2}n^2$. Therefore,

$$|\mathcal{DPC}_n| = \begin{cases} 1 + n^2 + 2n\sum_{i=2}^{n}\binom{n}{i} & \text{if } n \text{ is odd} \\ 1 + n^2 + 2n\sum_{i=2}^{n}\binom{n}{i} - \frac{1}{2}n^2 & \text{if } n \text{ is even} \end{cases} = \begin{cases} n2^{n+1} - n^2 - 2n + 1 & \text{if } n \text{ is odd} \\ n2^{n+1} - \frac{3}{2}n^2 - 2n + 1 & \text{if } n \text{ is even,} \end{cases}$$

as required. □

We finish this section by deducing that $\mathcal{DPC}_n$ has rank 3.

Let

$$e_i = \mathrm{id}_{\Omega_n \setminus \{i\}} = \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix} \in \mathcal{DPC}_n,$$

for $i = 1, 2, \ldots, n$. Clearly, for $1 \leqslant i, j \leqslant n$, we have $e_i^2 = e_i$ and $e_i e_j = \mathrm{id}_{\Omega_n \setminus \{i,j\}} = e_j e_i$. More generally, for any $X \subseteq \Omega_n$, we get $\Pi_{i \in X} e_i = \mathrm{id}_{\Omega_n \setminus X}$.

Now, take $\alpha \in \mathcal{DPC}_n$. Then, by Lemma 2.1, $\alpha = h^i g^j|_{\mathrm{Dom}(\alpha)}$ for some $i \in \{0, 1\}$ and $j \in \{0, \ldots, n-1\}$. Hence, $\alpha = h^i g^j \mathrm{id}_{\mathrm{Im}(\alpha)} = h^i g^j \Pi_{k \in \Omega_n \setminus \mathrm{Im}(\alpha)} e_k$. Therefore, $\{g, h, e_1, e_2, \ldots, e_n\}$ is a generating set of $\mathcal{DPC}_n$. Since $e_i = g^{n-i} e_n g^i$ for all $i \in \{1, 2, \ldots, n\}$, it follows that $\{g, h, e_n\}$ is also a generating set of $\mathcal{DPC}_n$. As $\mathcal{D}_{2n}$ is the group of units of $\mathcal{DPC}_n$, which is a group with rank 2, the monoid $\mathcal{DPC}_n$ cannot be generated by less than three elements. So, we have the following theorem.

Theorem 2.5 *The rank of the monoid $\mathcal{DPC}_n$ is 3.* □

3. Presentations for $\mathcal{DPC}_n$

In this section, we aim to determine a presentation for $\mathcal{DPC}_n$. In fact, we first determine a presentation of $\mathcal{DPC}_n$ on $n+2$ generators and then, by applying Tietze transformations, we deduce a presentation for $\mathcal{DPC}_n$ on 3 generators.

We begin this section by recalling some notions related to the concept of a monoid presentation.

Let A be an alphabet and consider the free monoid A^* generated by A. The elements of A and of A^* are called letters and words, respectively. The empty word is denoted by 1 and we write A^+ to express $A^* \setminus \{1\}$. A pair (u, v) of $A^* \times A^*$ is called a relation of A^* and it is usually represented by $u = v$. To avoid confusion, given $u, v \in A^*$, we will write $u \equiv v$ instead of $u = v$, whenever we want to state precisely that u and v are identical words of A^*. A relation $u = v$ of A^* is said to be a consequence of R if $u \rho_R v$, where $R \subseteq A^* \times A^*$ is a set of relations and recall that ρ_R denotes the smallest congruence on A^* containing R.

Let X be a generating set of a monoid M and let $\phi : A \longrightarrow M$ be an injective mapping such that $A\phi = X$. Let $\varphi : A^* \longrightarrow M$ be the (surjective) homomorphism of monoids that extends ϕ to A^*. We say that X satisfies (via φ) a relation $u = v$ of A^* if $u\varphi = v\varphi$. For more details see, for example, [27, 33].

A direct method to find a presentation for a monoid is described by the following well-known result (see, for example, [33, Proposition 1.2.3]).

Proposition 3.1 *Let M be a monoid generated by a set X, let A be an alphabet and let $\phi : A \longrightarrow M$ be an injective mapping such that $A\phi = X$. Let $\varphi : A^* \longrightarrow M$ be the (surjective) homomorphism that extends ϕ to A^* and let $R \subseteq A^* \times A^*$. Then $\langle A \mid R \rangle$ is a presentation for M if and only if the following two conditions are satisfied:*

1. *The generating set X of M satisfies (via φ) all the relations from R;*
2. *If $u, v \in A^*$ are any two words such that the generating set X of M satisfies (via φ) the relation $u = v$ then $u = v$ is a consequence of R.* □

Given a presentation for a monoid, another method to find a new presentation consists in applying Tietze transformations. For a monoid presentation $\langle A \mid R \rangle$, the four *elementary Tietze transformations* are:

(T1) Adding a new relation $u = v$ to $\langle A \mid R \rangle$, provided that $u = v$ is a consequence of R;

(T2) Deleting a relation $u = v$ from $\langle A \mid R \rangle$, provided that $u = v$ is a consequence of $R \backslash \{u = v\}$;

(T3) Adding a new generating symbol b and a new relation $b = w$, where $w \in A^*$;

(T4) If $\langle A \mid R \rangle$ possesses a relation of the form $b = w$, where $b \in A$, and $w \in (A \backslash \{b\})^*$, then deleting b from the list of generating symbols, deleting the relation $b = w$, and replacing all remaining appearances of b by w.

The next result is well-known (see, for example, [33]):

Proposition 3.2 *Two finite presentations define the same monoid if and only if one can be obtained from the other by a finite number of elementary Tietze transformations* $(T1)$, $(T2)$, $(T3)$, *and* $(T4)$. □

Now, consider the alphabet $A = \{g, h, e_1, e_2, \ldots, e_n\}$ and the set R formed by the following $\frac{n^2+5n+9+(-1)^n}{2}$ monoid relations:

(R_1) $g^n = 1$, $h^2 = 1$ and $hg = g^{n-1}h$;

(R_2) $e_i^2 = e_i$ for $1 \leqslant i \leqslant n$;

(R_3) $e_i e_j = e_j e_i$ for $1 \leqslant i < j \leqslant n$;

(R_4) $ge_1 = e_n g$ and $ge_{i+1} = e_i g$ for $1 \leqslant i \leqslant n-1$;

(R_5) $he_i = e_{n-i+1}h$ for $1 \leqslant i \leqslant n$;

(R_6^{o}) $hge_2e_3 \cdots e_n = e_2e_3 \cdots e_n$ if n is odd;

(R_6^{e}) $hge_2 \cdots e_{\frac{n}{2}} e_{\frac{n}{2}+2} \cdots e_n = e_2 \cdots e_{\frac{n}{2}} e_{\frac{n}{2}+2} \cdots e_n$ and $he_1e_2 \cdots e_n = e_1e_2 \cdots e_n$ if n is even.

We aim to show that the monoid $\mathcal{DPC}_n$ is defined by the presentation $\langle A \mid R \rangle$.

Let $\phi : A \longrightarrow \mathcal{DPC}_n$ be the mapping defined by $g\phi = g$, $h\phi = h$ and $e_i\phi = e_i$, for $1 \leqslant i \leqslant n$, and let $\varphi : A^* \longrightarrow \mathcal{DPC}_n$ be the homomorphism of monoids that extends ϕ to A^*. Notice that we are using the same symbols for the letters of the alphabet A and for the generating set of $\mathcal{DPC}_n$, which simplifies notation and, within the context, will not cause ambiguity.

It is a routine matter to check the following lemma.

Lemma 3.3 *The set of generators* $\{g, h, e_1, e_2, \ldots, e_n\}$ *of* $\mathcal{DPC}_n$ *satisfies (via* φ*) all the relations from* R. □

Observe that this result assures us that, if $u, v \in A^*$ are two words such that the relation $u = v$ is a consequence of R, then $u\varphi = v\varphi$.

Next, in order to prove that any relation satisfied by the generating set of $\mathcal{DPC}_n$ is a consequence of R, we first present a series of three lemmas. In what follows, we denote the congruence ρ_R of A^* simply by ρ.

Lemma 3.4 *If n is even, then the relation*

$$hg^{2j-1}e_1\cdots e_{j-1}e_{j+1}\cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1}\cdots e_n = e_1\cdots e_{j-1}e_{j+1}\cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1}\cdots e_n$$

is a consequence of R for $1\leqslant j\leqslant \frac{n}{2}$.

Proof We proceed by induction on j.

Let $j=1$. Then $hge_2\cdots e_{\frac{n}{2}}e_{\frac{n}{2}+2}\cdots e_n = e_2\cdots e_{\frac{n}{2}}e_{\frac{n}{2}+2}\cdots e_n$ is a relation of R. Next, suppose that $hg^{2j-1}e_1\cdots e_{j-1}e_{j+1}\cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1}\cdots e_n = e_1\cdots e_{j-1}e_{j+1}\cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1}\cdots e_n$ for some $1\leqslant j\leqslant \frac{n}{2}-1$. Then

$$\begin{array}{lll}
 & hg^{2(j+1)-1}e_1\cdots e_je_{j+2}\cdots e_{j+\frac{n}{2}}e_{j+\frac{n}{2}+2}\cdots e_n & \\
\equiv & hg^{2j+1}e_1\cdots e_je_{j+2}\cdots e_{j+\frac{n}{2}}e_{j+\frac{n}{2}+2}\cdots e_n & \\
\rho & hg^{2j}e_nge_2\cdots e_je_{j+2}\cdots e_{j+\frac{n}{2}}e_{j+\frac{n}{2}+2}\cdots e_n & \text{(by } R_4\text{)}\\
\rho & hgg^{2j-1}e_ne_1\cdots e_{j-1}e_{j+1}\cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1}\cdots e_{n-1}g & \text{(by } R_4\text{)}\\
\rho & g^{n-1}hg^{2j-1}e_1\cdots e_{j-1}e_{j+1}\cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1}\cdots e_ng & \text{(by } R_1 \text{ and } R_3\text{)}\\
\rho & g^{n-1}e_1\cdots e_{j-1}e_{j+1}\cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1}\cdots e_ng & \text{(by the induction hyphotesis)}\\
\rho & g^{n-1}e_1\cdots e_{j-1}e_{j+1}\cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1}\cdots e_{n-1}ge_1 & \text{(by } R_4\text{)}\\
\rho & g^{n-1}ge_2\cdots e_je_{j+2}\cdots e_{j+\frac{n}{2}}e_{j+\frac{n}{2}+2}\cdots e_ne_1 & \text{(by } R_4\text{)}\\
\rho & e_1\cdots e_je_{j+2}\cdots e_{j+\frac{n}{2}}e_{j+\frac{n}{2}+2}\cdots e_n & \text{(by } R_1 \text{ and } R_3\text{)},
\end{array}$$

as required. □

Lemma 3.5 *The relation $hg^{2i-1}e_1\cdots e_{i-1}e_{i+1}\cdots e_n = e_1\cdots e_{i-1}e_{i+1}\cdots e_n$ is a consequence of R, for $1\leqslant i\leqslant n$.*

Proof We proceed by induction on i.

Let $i=1$. If n is odd then $hge_2e_3\cdots e_n = e_2e_3\cdots e_n$ is a relation of R. So, suppose that n is even. Then $hge_2\cdots e_{\frac{n}{2}}e_{\frac{n}{2}+2}\cdots e_n = e_2\cdots e_{\frac{n}{2}}e_{\frac{n}{2}+2}\cdots e_n$ is a relation of R, whence

$$hge_2\cdots e_{\frac{n}{2}}e_{\frac{n}{2}+2}\cdots e_ne_{\frac{n}{2}+1}\,\rho\, e_2\cdots e_{\frac{n}{2}}e_{\frac{n}{2}+2}\cdots e_ne_{\frac{n}{2}+1}$$

and so $hge_2e_3\cdots e_n = e_2e_3\cdots e_n$, by R_3.

Now, suppose that $hg^{2i-1}e_1\cdots e_{i-1}e_{i+1}\cdots e_n\,\rho\, e_1\cdots e_{i-1}e_{i+1}\cdots e_n$ for some $1\leqslant i\leqslant n-1$. Then (with steps similar to the previous proof), we have

$$\begin{array}{llll}
hg^{2(i+1)-1}e_1\cdots e_ie_{i+2}\cdots e_n & \equiv & hg^{2i+1}e_1\cdots e_ie_{i+2}\cdots e_n & \\
 & \rho & hg^{2i}e_nge_2\cdots e_ie_{i+2}\cdots e_n & \text{(by } R_4\text{)}\\
 & \rho & hgg^{2i-1}e_ne_1\cdots e_{i-1}e_{i+1}\cdots e_{n-1}g & \text{(by } R_4\text{)}\\
 & \rho & g^{n-1}hg^{2i-1}e_1\cdots e_{i-1}e_{i+1}\cdots e_ng & \text{(by } R_1 \text{ and } R_3\text{)}\\
 & \rho & g^{n-1}e_1\cdots e_{i-1}e_{i+1}\cdots e_ng & \text{(by the induction hyphotesis)}\\
 & \rho & g^{n-1}e_1\cdots e_{i-1}e_{i+1}\cdots e_{n-1}ge_1 & \text{(by } R_4\text{)}\\
 & \rho & g^{n-1}ge_2\cdots e_ie_{i+2}\cdots e_ne_1 & \text{(by } R_4\text{)}\\
 & \rho & e_1\cdots e_ie_{i+2}\cdots e_n & \text{(by } R_1 \text{ and } R_3\text{)},
\end{array}$$

as required. □

Lemma 3.6 *The relation $h^\ell g^m e_1e_2\cdots e_n = e_1e_2\cdots e_n$ is a consequence of R for $\ell, m\geqslant 0$.*

Proof First, we prove that the relation $he_1e_2\cdots e_n = e_1e_2\cdots e_n$ is a consequence of R. Since this relation belongs to R when n is even, it remains to show that $he_1e_2\cdots e_n \,\rho\, e_1e_2\cdots e_n$ when n is odd.

Suppose that n is odd. Hence, by $R_6^{\rm o}$, we have $hge_2e_3\cdots e_ne_1 \,\rho\, e_2e_3\cdots e_ne_1$, so $hge_1e_2\cdots e_n \,\rho\, e_1e_2\cdots e_n$ (by R_3), whence $ge_1e_2\cdots e_n \,\rho\, he_1e_2\cdots e_n$ (by R_1) and then $(ge_1e_2\cdots e_n)^n \,\rho\, (he_1e_2\cdots e_n)^n$. Now, by R_4 and R_3, we have $ge_1e_2\cdots e_n \,\rho\, e_nge_2\cdots e_n \,\rho\, e_ne_1\cdots e_{n-1}g \,\rho\, e_1e_2\cdots e_ng$ and so, by relations R_1, R_3, and R_2, it follows that $(ge_1e_2\cdots e_n)^n \,\rho\, g^n(e_1e_2\cdots e_n)^n \,\rho\, e_1e_2\cdots e_n$. On the other hand, by R_5 and R_3, we have $he_1e_2\cdots e_n \,\rho\, e_ne_{n-1}\cdots e_1h \,\rho\, e_1e_2\cdots e_nh$, whence $(he_1e_2\cdots e_n)^n \,\rho\, h^n(e_1e_2\cdots e_n)^n \,\rho\, he_1e_2\cdots e_n$ by relations R_1, R_3, and R_2, since n is odd. Therefore, $he_1e_2\cdots e_n \,\rho\, e_1e_2\cdots e_n$.

Secondly, we prove that the relation $ge_1e_2\cdots e_n = e_1e_2\cdots e_n$ is a consequence of R. In fact, we have

$$\begin{array}{lll}
ge_1e_2\cdots e_n & \rho & ge_1hge_2\cdots e_n & \text{(by Lemma 3.5)}\\
& \rho & e_nghge_2\cdots e_n & \text{(by } R_4)\\
& \rho & e_ngg^{n-1}he_2\cdots e_n & \text{(by } R_1)\\
& \rho & e_nhe_2\cdots e_n & \text{(by } R_1)\\
& \rho & he_1e_2\cdots e_n & \text{(by } R_5)\\
& \rho & e_1e_2\cdots e_n & \text{(by the first part).}
\end{array}$$

Now, clearly, for $\ell, m \geqslant 0$, $h^\ell g^m e_1e_2\cdots e_n \,\rho\, e_1e_2\cdots e_n$ follows immediately from $ge_1e_2\cdots e_n \,\rho\, e_1e_2\cdots e_n$ and $he_1e_2\cdots e_n\rho e_1e_2\cdots e_n$, which concludes the proof of the lemma. □

We are now in a position to prove the following result.

Theorem 3.7 *The monoid $\mathcal{DPC}_n$ is defined by the presentation $\langle A \mid R\rangle$ on $n+2$ generators.*

Proof In view of Proposition 3.1 and Lemma 3.3, it remains to prove that any relation satisfied by the generating set $\{g, h, e_1, e_2, \ldots, e_n\}$ of $\mathcal{DPC}_n$ is a consequence of R.

Let $u, v \in A^*$ be two words such that $u\varphi = v\varphi$. We aim to show that $u \,\rho\, v$. Take $\alpha = u\varphi$.

It is clear that relations R_1 to R_5 allow us to deduce that $u \,\rho\, h^\ell g^m e_{i_1}\cdots e_{i_k}$ for some $\ell \in \{0,1\}$, $m \in \{0,1,\ldots,n-1\}$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and $0 \leqslant k \leqslant n$. Similarly, we have $v \,\rho\, h^{\ell'} g^{m'} e_{i'_1}\cdots e_{i'_{k'}}$ for some $\ell' \in \{0,1\}$, $m' \in \{0,1,\ldots,n-1\}$, $1 \leqslant i'_1 < \cdots < i'_{k'} \leqslant n$ and $0 \leqslant k' \leqslant n$.

Since $\alpha = h^\ell g^m e_{i_1}\cdots e_{i_k}$, it follows that $\operatorname{Im}(\alpha) = \Omega_n \setminus \{i_1,\ldots,i_k\}$ and $\alpha = h^\ell g^m|_{\operatorname{Dom}(\alpha)}$. Similarly, as also $\alpha = v\varphi$, from $\alpha = h^{\ell'} g^{m'} e_{i'_1}\cdots e_{i'_{k'}}$, we get $\operatorname{Im}(\alpha) = \Omega_n \setminus \{i'_1,\ldots,i'_{k'}\}$ and $\alpha = h^{\ell'} g^{m'}|_{\operatorname{Dom}(\alpha)}$. Hence, $k' = k$ and $\{i'_1,\ldots,i'_k\} = \{i_1,\ldots,i_k\}$. Furthermore, if either $k = n-2$ and $\operatorname{d}(\min\operatorname{Dom}(\alpha), \max\operatorname{Dom}(\alpha)) \neq \frac{n}{2}$ or $k \leqslant n-3$, by Lemma 2.1, we obtain $\ell' = \ell$ and $m' = m$, and so $u \,\rho\, h^\ell g^m e_{i_1}\cdots e_{i_k} \,\rho\, v$.

If $h^{\ell'} g^{m'} = h^\ell g^m$ (including as elements of $\mathcal{D}_{2n}$) then $\ell' = \ell$ and $m' = m$, and so we get again $u \,\rho\, h^\ell g^m e_{i_1}\cdots e_{i_k} \,\rho\, v$.

Therefore, let us suppose that $h^{\ell'} g^{m'} \neq h^\ell g^m$. Hence, by Lemma 2.1, we may conclude that $\alpha = \emptyset$ or $\ell' = \ell - 1$ or $\ell' = \ell + 1$. If $\alpha = \emptyset$, i.e. $k = n$, then $u \,\rho\, h^\ell g^m e_1e_2\cdots e_n \,\rho\, e_1e_2\cdots e_n \,\rho\, h^{\ell'} g^{m'} e_1e_2\cdots e_n \,\rho\, v$ by Lemma 3.6.

Thus, we may suppose that $\alpha \neq \emptyset$ and, without loss of generality, also that $\ell' = \ell + 1$, i.e. $\ell = 0$ and $\ell' = 1$. Let $k = n-2$ and admit that $\operatorname{d}(\min\operatorname{Dom}(\alpha), \max\operatorname{Dom}(\alpha)) = \frac{n}{2}$ (in which case n is even).

Let $\alpha = \begin{pmatrix} i_1 & i_2 \\ j_1 & j_2 \end{pmatrix}$ with $1 \leqslant i_1 < i_2 \leqslant n$. Then $i_2 - i_1 = \frac{n}{2} = \mathrm{d}(i_1, i_2) = \mathrm{d}(j_1, j_2) = |j_2 - j_1|$, and so $j_2 \in \{j_1 - \frac{n}{2}, j_1 + \frac{n}{2}\}$. Let $j = \min\{j_1, j_2\}$ (notice that $1 \leqslant j \leqslant \frac{n}{2}$) and $i = j\alpha^{-1}$. Hence, $\mathrm{Im}(\alpha) = \{j, j + \frac{n}{2}\}$ and $\alpha = g^{n+j-i}|_{\mathrm{Dom}(\alpha)} = hg^{i+j-1-n}|_{\mathrm{Dom}(\alpha)}$ (cf. proof of Lemma 2.1). So, we have

$$u\,\rho\, g^m e_1 \cdots e_{j-1}e_{j+1} \cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1} \cdots e_n \quad \text{and} \quad v\,\rho\, hg^{m'} e_1 \cdots e_{j-1}e_{j+1} \cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1} \cdots e_n$$

and, by Lemma 2.1, $m = rn + j - i$ for some $r \in \{0,1\}$, and $m' = i + j - 1 - r'n$ for some $r' \in \{0,1\}$. Thus, we get

$$\begin{array}{lll} u & \rho & g^m e_1 \cdots e_{j-1}e_{j+1} \cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1} \cdots e_n \\ & \rho & g^m hg^{2j-1} e_1 \cdots e_{j-1}e_{j+1} \cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1} \cdots e_n \quad \text{(by Lemma 3.4)} \\ & \rho & g^m hg^{2j-1+(r-r')n} e_1 \cdots e_{j-1}e_{j+1} \cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1} \cdots e_n \quad \text{(by } R_1\text{)} \\ & \rho & hg^{n-m} g^{m+m'} e_1 \cdots e_{j-1}e_{j+1} \cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1} \cdots e_n \quad \text{(by } R_1\text{)} \\ & \rho & hg^{m'} e_1 \cdots e_{j-1}e_{j+1} \cdots e_{j+\frac{n}{2}-1}e_{j+\frac{n}{2}+1} \cdots e_n \quad \text{(by } R_1\text{)} \\ & \rho & v. \end{array}$$

Finally, consider that $k = n - 1$. Let $i \in \Omega_n$ be such that $\Omega_n \setminus \{i_1, \ldots, i_{n-1}\} = \{i\}$. Then $\mathrm{Im}(\alpha) = \{i\}$ and $\{i_1, \ldots, i_{n-1}\} = \{1, \ldots, i-1, i+1, \ldots, n\}$. Take $a = i\alpha^{-1}$. Then $ag^m = i = ahg^{m'}$. Since $ag^m = a+m-rn$ for some $r \in \{0,1\}$, and $ahg^{m'} = (n-a+1)g^{m'} = r'n - a + 1 + m'$ for some $r' \in \{0,1\}$, in a similar way to what we proved before, we have

$$\begin{array}{lll} u & \rho & g^m e_1 \cdots e_{i-1}e_{i+1} \cdots e_n \\ & \rho & g^m hg^{2i-1} e_1 \cdots e_{i-1}e_{i+1} \cdots e_n \quad \text{(by Lemma 3.5)} \\ & \rho & g^m hg^{2i-1+(r-r')n} e_1 \cdots e_{i-1}e_{i+1} \cdots e_n \quad \text{(by } R_1\text{)} \\ & \rho & hg^{n-m} g^{m+m'} e_1 \cdots e_{i-1}e_{i+1} \cdots e_n \quad \text{(by } R_1\text{)} \\ & \rho & hg^{m'} e_1 \cdots e_{i-1}e_{i+1} \cdots e_n \quad \text{(by } R_1\text{)} \\ & \rho & v, \end{array}$$

as required. □

Notice that, taking into account the relation $h^2 = 1$ of R_1, we could have taken only half of the relations R_5, namely the relations $he_i = e_{n-i+1}h$ with $1 \leqslant i \leqslant \lceil \frac{n}{2} \rceil$, where $\lceil \frac{n}{2} \rceil$ denotes the least integer greater than or equal to $\frac{n}{2}$.

Our next and final goal is, by using Tietze transformations, to deduce a new presentation on 3 generators from the previous presentation for $\mathcal{DPC}_n$.

Since we have $e_i = hg^{i-1}e_nhg^{i-1}$ (as transformations) for all $i \in \{1, 2, \ldots, n\}$, we will proceed as follows: first, by applying T1, we add the relations $e_i = hg^{i-1}e_nhg^{i-1}$ for $1 \leqslant i \leqslant n$; secondly, we apply T4 to each of the relations $e_i = hg^{i-1}e_nhg^{i-1}$ with $i \in \{1, 2, \ldots, n-1\}$ and, in some cases, by convenience, we also replace e_n by $hg^{n-1}e_nhg^{n-1}$; finally, by using the relations R_1, we simplify the new relations obtained, eliminating the trivial ones or those that are deduced from others. Performing this procedure for each of the sets of relations R_1 to $R_6^{\mathrm{o}}/R_6^{\mathrm{e}}$, and renaming e_n by e, we may routinely obtain the following set Q of $\frac{n^2-n+13+(-1)^n}{2}$ many monoid relations on the alphabet $B = \{g, h, e\}$:

(Q_1) $g^n = 1$, $h^2 = 1$ and $hg = g^{n-1}h$;

(Q_2) $e^2 = e$ and $ghegh = e$;

(Q_3) $eg^{j-i}eg^{n-j+i} = g^{j-i}eg^{n-j+i}e$ for $1 \leqslant i < j \leqslant n$;

(Q_4) $hg(eg)^{n-2}e = (eg)^{n-2}e$ if n is odd;

(Q_5) $hg(eg)^{\frac{n}{2}-1}g(eg)^{\frac{n}{2}-2}e = (eg)^{\frac{n}{2}-1}g(eg)^{\frac{n}{2}-2}e$ and $h(eg)^{n-1}e = (eg)^{n-1}e$ if n is even.

Notice that, the use of the expressions $e_i = hg^{i-1}e_nhg^{i-1}$ for all $i \in \{1, 2, \ldots, n\}$, instead of those observed at the end of Section 2, i.e. $e_i = g^{n-i}e_ng^i$ for all $i \in \{1, 2, \ldots, n\}$, allowed us to obtain simpler relations.

Now, in view of Proposition 3.2, we have the following theorem.

Theorem 3.8 *The monoid $\mathcal{DPC}_n$ is defined by the presentation $\langle B \mid Q\rangle$ on 3 generators.* □

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