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On the monoid of partial isometries of a cycle graph

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Abstract: In this paper we consider the monoid \( DPC_n \) of all partial isometries of an \( n \)-cycle graph \( C_n \). We show that \( DPC_n \) is the submonoid of the monoid of all oriented partial permutations on an \( n \)-chain whose elements are precisely all restrictions of a dihedral group of order \( 2n \). Our main aim is to exhibit a presentation of \( DPC_n \). We also describe Green’s relations of \( DPC_n \) and calculate its cardinality and rank.

Key words: Transformations, orientation, partial isometries, cycle graphs, rank, presentations

1. Introduction
Let \( \Omega \) be a finite set. As usual, let us denote by \( \mathcal{PT}(\Omega) \) the monoid (under composition) of all partial transformations on \( \Omega \), by \( \mathcal{T}(\Omega) \) the submonoid of \( \mathcal{PT}(\Omega) \) of all full transformations on \( \Omega \), by \( \mathcal{I}(\Omega) \) the symmetric inverse monoid on \( \Omega \), i.e. the inverse submonoid of \( \mathcal{PT}(\Omega) \) of all partial permutations on \( \Omega \), and by \( \mathcal{S}(\Omega) \) the symmetric group on \( \Omega \), i.e. the subgroup of \( \mathcal{PT}(\Omega) \) of all permutations on \( \Omega \).

Recall that the rank of a (finite) monoid \( M \) is the minimum size of all (finite) generating sets of \( M \), i.e. the minimum of the set \( \{ |X| : X \subseteq M \text{ and } X \text{ generates } M \} \).

Let \( \Omega \) be a finite set with at least 3 elements. It is well-known that \( \mathcal{S}(\Omega) \) has rank 2 (as a semigroup, a monoid, or a group) and \( \mathcal{T}(\Omega) \), \( \mathcal{I}(\Omega) \), and \( \mathcal{PT}(\Omega) \) have ranks 3, 3, and 4, respectively. The survey [13] presents these results and similar ones for other classes of transformation monoids, in particular, for monoids of order-preserving transformations and for some of their extensions. For example, the rank of the extensively studied monoid of all order-preserving transformations of an \( n \)-chain is \( n \), which was proved by Gomes and Howie [23] in 1992. More recently, for instance, the papers [5, 16, 17, 19, 21] are dedicated to the computation of the ranks of certain classes of transformation semigroups or monoids.

A monoid presentation is an ordered pair \( \langle A \mid R \rangle \), where \( A \) is a set, often called an alphabet, and \( R \subseteq A^* \times A^* \) is a set of relations of the free monoid \( A^* \) generated by \( A \). A monoid \( M \) is said to be defined by a presentation \( \langle A \mid R \rangle \) if \( M \) is isomorphic to \( A^*/\rho_R \), where \( \rho_R \) denotes the smallest congruence on \( A^* \) containing \( R \).

Given a finite monoid, it is clear that we can always exhibit a presentation for it, at worst by enumerating all elements from its multiplication table, but clearly this is of no interest, in general. So, by determining a

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presentation for a finite monoid, we mean to find in some sense a nice presentation (e.g., with a small number of generators and relations).

A presentation for the symmetric group $S(\Omega)$ was determined by Moore [29] over a century ago (1897). For the full transformation monoid $T(\Omega)$, a presentation was given in 1958 by Aizenštat [1] in terms of a certain type of two-generator presentation for the symmetric group $S(\Omega)$, plus an extra generator and seven more relations. Presentations for the partial transformation monoid $PT(\Omega)$ and for the symmetric inverse monoid $I(\Omega)$ were found by Popova [31] in 1961. In 1962, Aizenštat [2] and Popova [32] exhibited presentations for the monoids of all order-preserving transformations and of all order-preserving partial transformations of a finite chain, respectively, and from the Sixties to the present day, several authors obtained presentations for many classes of monoids. See also [33], the survey [13], and, for example, [8–12, 14, 20, 25].

Now, let $G = (V, E)$ be a finite simple connected graph, where $V$ is the set of vertices and $E$ is the list of edges. The (geodesic) distance between two vertices $x$ and $y$ of $G$, denoted by $d_G(x, y)$, is the length of a shortest path between $x$ and $y$, i.e. the number of edges in a shortest path between $x$ and $y$.

Let $\alpha \in PT(V)$. We say that $\alpha$ is a partial isometry or distance preserving partial transformation of $G$ if

$$d_G(x\alpha, y\alpha) = d_G(x, y)$$

for all $x, y \in \text{Dom}(\alpha)$. Denote by $DP(G)$ the subset of $PT(V)$ of all partial isometries of $G$. Clearly, $DP(G)$ is a submonoid of $PT(V)$. Moreover, as a consequence of the property

$$d_G(x, y) = 0 \quad \text{if and only if} \quad x = y$$

for all $x, y \in V$, it immediately follows that $DP(G) \subseteq I(V)$. Furthermore, $DP(G)$ is an inverse submonoid of $I(V)$ (see [18]).

Observe that if $G = (V, E)$ is a complete graph, i.e. $E = \{\{x, y\} : x, y \in V, x \neq y\}$, then $DP(G) = I(V)$. On the other hand, for $n \geq 2$, consider the undirected path graph $P_n$ with $n$ vertices, i.e.

$$P_n = (\{1, \ldots, n\}, \{\{i, i + 1\} : i = 1, \ldots, n - 1\}).$$

Then, obviously, $DP(P_n)$ coincides with the monoid

$$DP_n = \{\alpha \in I(\{1, 2, \ldots, n\}) : |i\alpha - j\alpha| = |i - j| \quad \text{for all} \quad i, j \in \text{Dom}(\alpha)\}$$

of all partial isometries on $\{1, 2, \ldots, n\}$.

The study of partial isometries on $\{1, 2, \ldots, n\}$ was initiated by Al-Kharousi et al. in [3, 4]. The first of these two papers is dedicated to investigating some combinatorial properties of the monoid $DP_n$ and of its submonoid $ODP_n$ of all order-preserving (considering the usual order of $\mathbb{N}$) partial isometries, in particular, their cardinalities. The second paper presents the study of some of their algebraic properties, namely Green’s structure and ranks. Presentations for both the monoids $DP_n$ and $ODP_n$ were given by the first author and Quinteiro in [20]. Moreover, for $2 \leq r \leq n - 1$, Bugay et al. in [6] obtained the ranks of the subsemigroups $DP_{n,r} = \{\alpha \in DP_n : |\text{Im}(\alpha)| \leq r\}$ of $DP_n$ and $ODP_{n,r} = \{\alpha \in ODP_n : |\text{Im}(\alpha)| \leq r\}$ of $ODP_n$.

The monoid $DP_{n}^*$ of all partial isometries of a star graph with $n$ vertices ($n \geq 1$) was considered by the authors in [18]. They determined the rank and size of $DP_{n}^*$ and described its Green’s relations. A presentation for $DP_{n}^*$ was also exhibited in [18].
Now, for \( n \geq 3 \), consider the cycle graph

\[
C_n = \{\{1,2, \ldots ,n\}, \{i, i+1\} : i = 1, 2, \ldots , n-1\} \cup \{\{1, n\}\}
\]

with \( n \) vertices. Notice that cycle graphs and cycle subgraphs play a fundamental role in Graph Theory.

This paper is devoted to studying the monoid \( DPC_n \) of all partial isometries of \( C_n \), which from now on we denote simply by \( DPC_n \). Observe that \( DPC_n \) is an inverse submonoid of the symmetric inverse monoid \( I_n \).

In Section 2, we start by giving a key characterization of \( DPC_n \), which allows for significantly simpler proofs of various results presented later. Also in this section, a description of the Green’s relations of \( DPC_n \) is given and the rank and the cardinality of \( DPC_n \) are calculated. Finally, in Section 3, we determine a presentation for the monoid \( DPC_n \) on \( n + 2 \) generators, from which we deduce another presentation for \( DPC_n \) on 3 generators.

For general background and standard notations, we refer to Howie’s book [24] for Semigroup Theory, and [34] for Graph Theory.

We would like to point out that we made use of computational tools, namely GAP\(^*\)[22].

2. Some properties of \( DPC_n \)

We begin this section by introducing some concepts and notations.

For \( n \in \mathbb{N} \), let \( \Omega_n \) be a set with \( n \) elements. In general, without loss of generality, \( \Omega_n \) is considered the chain \( \Omega_n = \{1 < 2 < \cdots < n\} \) and \( PT(\Omega_n), I(\Omega_n) \) and \( S(\Omega_n) \) are denoted simply by \( PT_n, I_n \) and \( S_n \), respectively. For any \( \alpha \in PT_n \), the domain and the image sets of \( \alpha \) are denoted by \( \text{Dom}(\alpha) \) and \( \text{Im}(\alpha) \), respectively. Also, the cardinality of the set \( \text{Im}(\alpha) \) is called the rank of \( \alpha \).

A partial transformation \( \alpha \in PT_n \) is called order-preserving [order-reversing] if \( x \leq y \) implies \( x\alpha \leq y\alpha \) [\( x\alpha \geq y\alpha \)], for all \( x, y \in \text{Dom}(\alpha) \). It is clear that the product of two order-preserving or of two order-reversing transformations is order-preserving and the product of an order-preserving transformation by an order-reversing transformation, or vice-versa, is order-reversing. We denote by \( POD_n \) the submonoid of \( PT_n \) whose elements are all order-preserving or order-reversing transformations.

Let \( s = (a_1, a_2, \ldots , a_t) \) be a sequence of \( t \) \((t \geq 0)\) elements from the chain \( \Omega_n \). We say that \( s \) is cyclic [anticyclic] if there exists no more than one index \( i \in \{1, \ldots , t\} \) such that \( a_i > a_{i+1} \) \((a_i < a_{i+1})\), where \( a_{t+1} \) denotes \( a_1 \). Notice that, the sequence \( s \) is cyclic [anticyclic] if and only if \( s \) is empty or there exists \( i \in \{0, 1, \ldots , t-1\} \) such that \( a_{i+1} \leq a_{i+2} \leq \cdots \leq a_t \leq \cdots \leq a_i \) \((a_{i+1} \geq a_{i+2} \geq \cdots \geq a_t \geq a_{i+1} \geq \cdots \geq a_i)\) (the index \( i \in \{0, 1, \ldots , t-1\} \) is unique unless \( s \) is constant and \( t \geq 2 \)). We also say that \( s \) is oriented if \( s \) is cyclic or \( s \) is anticyclic (see, for example, [7, 26, 28]). Given a partial transformation \( \alpha \in PT_n \) such that \( \text{Dom}(\alpha) = \{a_1 < \cdots < a_t\} \) with \( t \geq 0 \), we say that \( \alpha \) is orientation-preserving [orientation-reversing, oriented] if the sequence of its images \( (a_1\alpha , \ldots , a_t\alpha) \) is cyclic [anticyclic, oriented]. It is easy to show that the product of two orientation-preserving or of two orientation-reversing transformations is orientation-preserving and the product of an orientation-preserving transformation by an orientation-reversing transformation, or vice-versa, is orientation-reversing. We denote by \( POR_n \) the submonoid of \( PT_n \) of all oriented transformations.

Notice that \( POD_n \cap I_n \) and \( POR_n \cap I_n \) are inverse submonoids of \( I_n \).

\(^*\)https://www.gap-system.org
Let us consider the following permutations of $\Omega_n$ (for $n \geq 2$) of order $n$ and 2, respectively:

$$g = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}$$ and $$h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}.$$  

It is clear that $g, h \in \mathcal{P} \mathcal{O} \mathcal{R}_n \cap \mathcal{T}_n$. Moreover, for $n \geq 3$, $g$ together with $h$ generate the well-known dihedral group $D_{2n}$ of order $2n$ (considered a subgroup of $S_n$). In fact, for $n \geq 3$,

$$D_{2n} = \langle g, h \mid g^n = 1, h^2 = 1, hg = g^{n-1}h \rangle = \{1, g, g^2, \ldots, g^{n-1}, h, hg, hg^2, \ldots, hg^{n-1}\}$$

and we have

$$g^k = \begin{pmatrix} 1 & 2 & \cdots & n-k & n-k+1 & \cdots & n \\ 1+k & 2+k & \cdots & n & 1 & \cdots & k \end{pmatrix},$$

i.e. $ig^k = \{i+k \mid i \leq k \leq n-k \}$

and

$$hg^k = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ k & \cdots & 1 & n & \cdots & k+1 \end{pmatrix},$$

i.e. $ihg^k = \{k-i+1 \mid k+i-1 \leq i \leq n \}$

for $0 \leq k \leq n-1$. Observe that, for $n \in \{1, 2\}$, the dihedral group $D_{2n} = \langle g, h \mid g^n = 1, h^2 = 1, hg = g^{n-1}h \rangle$ of order $2n$ (also known as the Klein four-group for $n = 2$) cannot be considered a subgroup of $S_n$. Denote also by $C_n$ the cyclic group of order $n$ generated by $g$, i.e. $C_n = \{1, g, g^2, \ldots, g^{n-1}\}$.

Until the end of this paper, we will consider $n \geq 3$. Moreover, for convenience, we will denote $\alpha \in \mathcal{P} \mathcal{T}_n$ with $\text{Dom}(\alpha) = \{i_1, \ldots, i_k\}$ ($k \geq 1$) by $\alpha = \begin{pmatrix} i_1 & \cdots & i_k \\ i_1 \alpha & \cdots & i_k \alpha \end{pmatrix}$.

Now, notice that,

$$d_{C_n}(x, y) = \min\{|x-y|, n-|x-y|\} = \begin{cases} |x-y| & \text{if } |x-y| \leq \frac{n}{2} \\ n-|x-y| & \text{if } |x-y| > \frac{n}{2} \end{cases},$$

and so $0 \leq d_{C_n}(x, y) \leq \frac{n}{2}$ for all $x, y \in \{1, 2, \ldots, n\}$.

From now on, for any two vertices $x$ and $y$ of $C_n$, we denote the distance $d_{C_n}(x, y)$ simply by $d(x, y)$.

Observe for $x, y \in \Omega_n$ that

$$d(x, y) = \frac{n}{2} \iff |x-y| = \frac{n}{2} \iff n-|x-y| = \frac{n}{2} \iff |x-y| = n-|x-y|,$$

in which case $n$ is even, and

$$|\{z \in \{1, 2, \ldots, n\} : d(x, z) = d\}| = \begin{cases} 1 & \text{if } d = \frac{n}{2} \\ \frac{1}{2} & \text{if } d < \frac{n}{2} \end{cases} \quad (2.1)$$

for all $1 \leq d \leq \frac{n}{2}$. Moreover, it is a routine matter to show that

$$D = \{z \in \{1, 2, \ldots, n\} : d(x, z) = d\} = \{z \in \{1, 2, \ldots, n\} : d(y, z) = d'\}$$

implies

$$d(x, y) = \begin{cases} 0 & (\text{i.e. } x = y) \text{ if } |D| = 1 \\ \frac{n}{2} & \text{if } |D| = 2 \end{cases} \quad (2.2)$$
Therefore, we may conclude immediately that:

Recall that $\mathcal{DP}_n$ is an inverse submonoid of $\mathcal{POD}_n \cap I_n$. This is an easy fact to prove and was observed by Al-Kharousi et al. in [3, 4]. A similar result is also valid for $\mathcal{DPC}_n$ and $\mathcal{POR}_n \cap I_n$, as we will deduce below.

First, notice that it is easy to show that both permutations $g$ and $h$ of $\Omega_n$ belong to $\mathcal{DPC}_n$ and so the dihedral group $D_{2n}$ is contained in $\mathcal{DPC}_n$. Furthermore, as we prove next, the elements of $\mathcal{DPC}_n$ are precisely the restrictions of the permutations of the dihedral group $D_{2n}$. This is a key characterization of $\mathcal{DPC}_n$ that will allow us to prove in a simpler way some of the results that we present later in this paper. Observe that

$$\alpha = \sigma|_{\text{Dom}(\alpha)} \iff \alpha = \text{id}_{\text{Dom}(\alpha)} \sigma \iff \alpha = \sigma \text{id}_{\text{Im}(\alpha)},$$

for any $\alpha \in \mathcal{PT}_n$ and $\sigma \in \mathcal{I}_n$, where $\sigma|_{\text{Dom}(\alpha)}$ denotes the restriction mapping of $\sigma$ to $\text{Dom}(\alpha)$ and $\text{id}_U$, with $U \subseteq \Omega_n$, denotes the restriction map of the identity mapping $\text{id}$ of $\Omega_n$ to $U$.

**Lemma 2.1** For any $\alpha \in \mathcal{PT}_n$, $\alpha \in \mathcal{DPC}_n$ if and only if there exists $\sigma \in D_{2n}$ such that $\alpha = \sigma|_{\text{Dom}(\alpha)}$. Furthermore, for $\alpha \in \mathcal{DPC}_n$:

1. if either $|\text{Dom}(\alpha)| = 1$ or $|\text{Dom}(\alpha)| = 2$ and $d(\min \text{Dom}(\alpha), \max \text{Dom}(\alpha)) = \frac{n}{2}$ (in which case $n$ is even), then there exist exactly two (distinct) permutations $\sigma, \sigma' \in D_{2n}$ such that $\alpha = \sigma|_{\text{Dom}(\alpha)} = \sigma'|_{\text{Dom}(\alpha)}$;

2. if either $|\text{Dom}(\alpha)| = 2$ and $d(\min \text{Dom}(\alpha), \max \text{Dom}(\alpha)) \neq \frac{n}{2}$ or $|\text{Dom}(\alpha)| \geq 3$, then there exists exactly one permutation $\sigma \in D_{2n}$ such that $\alpha = \sigma|_{\text{Dom}(\alpha)}$.

**Proof** For any $\alpha \in \mathcal{PT}_n$, if $\alpha = \sigma|_{\text{Dom}(\alpha)}$, for some $\sigma \in D_{2n}$, then $\alpha \in \mathcal{DPC}_n$ since $D_{2n} \subseteq \mathcal{DPC}_n$ and, clearly, any restriction of an element of $\mathcal{DPC}_n$ also belongs to $\mathcal{DPC}_n$.

Conversely, let us suppose that $\alpha \in \mathcal{DPC}_n$. First, observe that, for each pair $1 \leq i, j \leq n$, there exists a unique $k \in \{0, 1, \ldots, n-1\}$ such that $ig^k = j$ and there exists a unique $\ell \in \{0, 1, \ldots, n-1\}$ such that $ihg^\ell = j$, where $g$ and $h$ are the permutations defined above. In fact, for $1 \leq i, j \leq n$ and $k, \ell \in \{0, 1, \ldots, n-1\}$, it is easy to show that

1. if $i \leq j$ then $ig^k = j$ if and only if $k = j - i$;

2. if $i > j$ then $ig^k = j$ if and only if $k = n + j - i$;

3. if $i + j \leq n$ then $ihg^\ell = j$ if and only if $\ell = i + j - 1$;

4. if $i + j > n$ then $ihg^\ell = j$ if and only if $\ell = i + j - 1 - n$.

Therefore, we may conclude immediately that:

1. any nonempty transformation of $\mathcal{DPC}_n$ has at most two distinct extensions in $D_{2n}$ and, if there are two distinct, one must be an orientation-preserving transformation and the other an orientation-reversing transformation;

2. any transformation of $\mathcal{DPC}_n$ with rank 1 has two distinct extensions in $D_{2n}$ (one is an orientation-preserving transformation and the other is an orientation-reversing transformation).
Notice that, as \( g^n = g^{-n} = 1 \), we also have \( ig^{j-i} = j \) and \( ihg^{i+j-1} = j \), for all \( 1 \leq i, j \leq n \).

Next, suppose that \( \text{Dom}(\alpha) = \{ i_1 < i_2 \} \). Then, there exist \( \sigma \in \mathcal{C}_n \) and \( \xi \in \mathcal{D}_{2n} \setminus \mathcal{C}_n \) (both unique) such that \( i_1 \sigma = i_1 \alpha = i_1 \xi \). Take \( D = \{ z \in \{ 1, 2, \ldots, n \} : d(i_1 \alpha, z) = d(i_1, i_2) \} \). Then \( 1 \leq |D| \leq 2 \) and \( i_2 \alpha, i_2 \sigma, i_2 \xi \in D \).

Suppose that \( i_2 \sigma = i_2 \xi \) and let \( j_1 = i_1 \sigma \) and \( j_2 = i_2 \sigma \). Then \( \sigma = g^{j_1-i_1} = g^{j_2-i_2} \) and \( \xi = hg^{j_1+j_1-1} = hg^{j_2+j_2-1} \). Hence, we have \( j_1 - i_1 = j_2 - i_2 \) or \( j_1 - i_1 = j_2 - i_2 \pm n \) from the first equality, and \( i_1 + j_1 = i_2 + j_2 \) or \( i_1 + j_1 = i_2 + j_2 \pm n \) from the second. Since \( i_1 \neq i_2 \) and \( i_2 - i_1 \neq n \), it is a routine matter to conclude that the only possibility is to have \( i_2 - i_1 = \frac{n}{2} \) (in which case \( n \) is even). Thus, \( d(i_1, i_2) = \frac{n}{2} \). By (2.1), it follows that \( |D| = 1 \) and so \( i_2 \alpha = i_2 \sigma = i_2 \xi \), i.e. \( \alpha \) is extended by both \( \sigma \) and \( \xi \).

If \( i_2 \sigma \neq i_2 \xi \), then \( |D| = 2 \) (whence \( d(i_1, i_2) < \frac{n}{2} \)), and so either \( i_2 \alpha = i_2 \sigma \) or \( i_2 \alpha = i_2 \xi \). In this case, \( \alpha \) is extended by exactly one permutation of \( \mathcal{D}_{2n} \).

Now, suppose that \( \text{Dom}(\alpha) = \{ i_1 < i_2 < \cdots < i_k \} \) for some \( 3 \leq k \leq n - 1 \). Since \( \sum_{p=1}^{k-1} (i_{p+1} - i_p) = i_k - i_1 < n \), then there exists at most one index \( 1 \leq p \leq k - 1 \) such that \( i_{p+1} - i_p \geq \frac{n}{2} \). Therefore, we may take \( i, j \in \text{Dom}(\alpha) \) such that \( i \neq j \) and \( d(i, j) \neq \frac{n}{2} \) and so, as \( \alpha|_{\{i,j\}} \in \mathcal{DPC}_n \), by the above deductions, there exists a unique \( \sigma \in \mathcal{D}_{2n} \) such that \( \sigma|_{\{i,j\}} = \alpha|_{\{i,j\}} \). Let \( \ell \in \text{Dom}(\alpha) \setminus \{i, j\} \). Then

\[
\ell \alpha, \ell \sigma \in \{ z \in \{ 1, 2, \ldots, n \} : d(i \alpha, z) = d(i, \ell) \} \cap \{ z \in \{ 1, 2, \ldots, n \} : d(j \alpha, z) = d(j, \ell) \}.
\]

In order to obtain a contradiction, suppose that \( \ell \alpha \neq \ell \sigma \). Therefore, by (2.1), we have

\[
\{ z \in \{ 1, 2, \ldots, n \} : d(i \alpha, z) = d(i, \ell) \} = \{ \ell \alpha, \ell \sigma \} = \{ z \in \{ 1, 2, \ldots, n \} : d(j \alpha, z) = d(j, \ell) \}
\]

and so, by (2.2), \( d(i, j) = d(i \alpha, j \alpha) = \frac{n}{2} \), which is a contradiction. Hence, \( \ell \alpha = \ell \sigma \). Thus, \( \sigma \) is the unique permutation of \( \mathcal{D}_{2n} \) such that \( \alpha = \sigma|_{\text{Dom}(\alpha)} \), as required.

Bearing in mind the previous lemma, it seems appropriate to designate \( \mathcal{DPC}_n \) by dihedral inverse monoid on \( \Omega_n \).

Since \( \mathcal{D}_{2n} \subseteq \mathcal{POR}_n \cap \mathcal{I}_n \), which contains all the restrictions of its elements, we have immediately the following corollary.

**Corollary 2.2** The monoid \( \mathcal{DPC}_n \) is contained in \( \mathcal{POR}_n \cap \mathcal{I}_n \). \( \square \)

Observe that, as \( \mathcal{D}_{2n} \) is the group of units of \( \mathcal{POR}_n \cap \mathcal{I}_n \) (see [14, 15]), then \( \mathcal{D}_{2n} \) also has to be the group of units of \( \mathcal{DPC}_n \).

Next, recall that, given an inverse submonoid \( M \) of \( \mathcal{I}_n \), it is well known that the Green’s relations \( \mathcal{L}, \mathcal{R}, \) and \( \mathcal{H} \) of \( M \) can be described as follows: for \( \alpha, \beta \in M \),

- \( \alpha L \beta \) if and only if \( \text{Im}(\alpha) = \text{Im}(\beta) \);
- \( \alpha R \beta \) if and only if \( \text{Dom}(\alpha) = \text{Dom}(\beta) \);
- \( \alpha H \beta \) if and only if \( \text{Im}(\alpha) = \text{Im}(\beta) \) and \( \text{Dom}(\alpha) = \text{Dom}(\beta) \).

In \( \mathcal{I}_n \), we also have
• \( \alpha \not\sim \beta \) if and only if \(|\text{Dom}(\alpha)| = |\text{Dom}(\beta)|\) (if and only if \(|\text{Im}(\alpha)| = |\text{Im}(\beta)|\)).

Since \( \mathcal{DPC}_n \) is an inverse submonoid of \( \mathcal{I}_n \), it remains to describe its Green’s relation \( \mathcal{J} \). In fact, it is a routine matter to prove the following proposition.

**Proposition 2.3** Let \( \alpha, \beta \in \mathcal{DPC}_n \). Then \( \alpha \not\sim \beta \) if and only if one of the following properties is satisfied:

1. \(|\text{Dom}(\alpha)| = |\text{Dom}(\beta)| \leq 1\);
2. \(|\text{Dom}(\alpha)| = |\text{Dom}(\beta)| = 2 \) and \( d(i_1, i_2) = d(i'_1, i'_2) \) where \( \text{Dom}(\alpha) = \{i_1, i_2\} \) and \( \text{Dom}(\beta) = \{i'_1, i'_2\} \);
3. \(|\text{Dom}(\alpha)| = |\text{Dom}(\beta)| = k \geq 3 \) and there exists \( \sigma \in \mathcal{D}_{2k} \) such that \( (i'_1 \ i'_2 \ \cdots \ i'_k) = (i_1 \ i_2 \ \cdots \ i_k) \) \( \in \mathcal{DPC}_n \) where \( \text{Dom}(\alpha) = \{i_1 < i_2 < \cdots < i_k\} \) and \( \text{Dom}(\beta) = \{i'_1 < i'_2 < \cdots < i'_k\} \).

An alternative description of \( \mathcal{J} \) can be found in the second author’s MSc thesis [30].

Next, we count the number of elements of \( \mathcal{DPC}_n \).

**Theorem 2.4** One has \( |\mathcal{DPC}_n| = n^{2n+1} - \frac{(-1)^n+5}{4}n^2 - 2n + 1 \).

**Proof** Let \( \mathcal{A}_i = \{\alpha \in \mathcal{DPC}_n : |\text{Dom}(\alpha)| = i\} \) for \( i = 0, 1, \ldots, n \). Since the sets \( \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n \) are pairwise disjoints, we get \( |\mathcal{DPC}_n| = \sum_{i=0}^n |\mathcal{A}_i| \).

Clearly, \( \mathcal{A}_0 = \{\emptyset\} \), where \( \emptyset \) denotes the empty mapping on \( \Omega_n \), and \( \mathcal{A}_1 = \{\{i\} : 1 \leq i, j \leq n\} \), whence \( |\mathcal{A}_0| = 1 \) and \( |\mathcal{A}_1| = n^2 \). Moreover, for \( i \geq 3 \), by Lemma 2.1, we have as many elements in \( \mathcal{A}_i \) as there are restrictions of rank \( i \) of permutations of \( \mathcal{D}_{2n} \), i.e., we have \( \binom{n}{i} \) distinct elements of \( \mathcal{A}_i \) for each permutation of \( \mathcal{D}_{2n} \), whence \( |\mathcal{A}_i| = 2n\binom{n}{i} \). Similarly, for an odd \( n \), by Lemma 2.1, we have \( |\mathcal{A}_2| = 2n\binom{n}{\frac{n}{2}} \). On the other hand, if \( n \) is even, also by Lemma 2.1, we have as many elements in \( \mathcal{A}_2 \) as there are restrictions of rank 2 of permutations of \( \mathcal{D}_{2n} \) minus the number of elements of \( \mathcal{A}_2 \) that have two distinct extensions in \( \mathcal{D}_{2n} \), i.e., \( |\mathcal{A}_2| = 2n\binom{n}{\frac{n}{2}} - |\mathcal{B}_2| \), where

\[
\mathcal{B}_2 = \{\alpha \in \mathcal{DPC}_n : |\text{Dom}(\alpha)| = 2 \text{ and } d(\min \text{Dom}(\alpha), \max \text{Dom}(\alpha)) = \frac{n}{2}\}.
\]

It is easy to check that

\[
|\mathcal{B}_2| = 2\left(\frac{n}{2}\right)^2 = \frac{1}{2}n^2.
\]

Therefore,

\[
|\mathcal{DPC}_n| = \begin{cases} 1 + n^2 + 2n \sum_{i=2}^n \binom{n}{i} & \text{if } n \text{ is odd} \\ 1 + n^2 + 2n \sum_{i=2}^n \binom{n}{i} - \frac{1}{2}n^2 & \text{if } n \text{ is even} \end{cases} = \begin{cases} n^{2n+1} - n^2 - 2n + 1 & \text{if } n \text{ is odd} \\ n^{2n+1} - \frac{3}{2}n^2 - 2n + 1 & \text{if } n \text{ is even}, \end{cases}
\]

as required.

We finish this section by deducing that \( \mathcal{DPC}_n \) has rank 3.
Let 
\[
e_i = \text{id}_{\Omega_n \setminus \{i\}} = \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix} \in \mathcal{DPC}_n,
\]
for \(i = 1, 2, \ldots, n\). Clearly, for \(1 \leq i, j \leq n\), we have \(e_i^2 = e_i\) and \(e_ie_j = \text{id}_{\Omega_n \setminus \{i,j\}} = e_je_i\). More generally, for any \(X \subseteq \Omega_n\), we get \(\Pi_{i \in X} e_i = \text{id}_{\Omega_n \setminus X}\).

Now, take \(\alpha \in \mathcal{DPC}_n\). Then, by Lemma 2.1, \(\alpha = h^i g^j|_{\text{Dom}(\alpha)}\) for some \(i \in \{0, 1\}\) and \(j \in \{0, \ldots, n-1\}\). Hence, \(\alpha = h^i g^j\text{id}_{\text{Im}(\alpha)} = h^i g^j \Pi_{k \in \Omega_n \setminus \text{Im}(\alpha)} e_k\). Therefore, \(\{g, h, e_1, e_2, \ldots, e_n\}\) is a generating set of \(\mathcal{DPC}_n\).

Since \(e_i = g^{n-i} e_n g^i\) for all \(i \in \{1, 2, \ldots, n\}\), it follows that \(\{g, h, e_n\}\) is also a generating set of \(\mathcal{DPC}_n\). As \(\mathcal{D}_2n\) is the group of units of \(\mathcal{DPC}_n\), which is a group with rank 2, the monoid \(\mathcal{DPC}_n\) cannot be generated by less than three elements. So, we have the following theorem.

**Theorem 2.5** The rank of the monoid \(\mathcal{DPC}_n\) is 3. \(\square\)

3. **Presentations for \(\mathcal{DPC}_n\)**

In this section, we aim to determine a presentation for \(\mathcal{DPC}_n\). In fact, we first determine a presentation of \(\mathcal{DPC}_n\) on \(n + 2\) generators and then, by applying Tietze transformations, we deduce a presentation for \(\mathcal{DPC}_n\) on 3 generators.

We begin this section by recalling some notions related to the concept of a monoid presentation.

Let \(A\) be an alphabet and consider the free monoid \(A^*\) generated by \(A\). The elements of \(A\) and of \(A^*\) are called letters and words, respectively. The empty word is denoted by 1 and we write \(A^+\) to express \(A^* \setminus \{1\}\). A pair \((u, v)\) of \(A^* \times A^*\) is called a relation of \(A^*\) and it is usually represented by \(u = v\). To avoid confusion, given \(u, v \in A^*\), we will write \(u \equiv v\) instead of \(u = v\), whenever we want to state precisely that \(u\) and \(v\) are identical words of \(A^*\). A relation \(u = v\) of \(A^*\) is said to be a consequence of \(R\) if \(u \rho_R v\), where \(R \subseteq A^* \times A^*\) is a set of relations and recall that \(\rho_R\) denotes the smallest congruence on \(A^*\) containing \(R\).

Let \(X\) be a generating set of a monoid \(M\) and let \(\phi : A \rightarrow M\) be an injective mapping such that \(A\phi = X\). Let \(\varphi : A^* \rightarrow M\) be the (surjective) homomorphism of monoids that extends \(\phi\) to \(A^*\). We say that \(X\) satisfies (via \(\varphi\)) a relation \(u = v\) of \(A^*\) if \(u\varphi = v\varphi\). For more details see, for example, [27, 33].

A direct method to find a presentation for a monoid is described by the following well-known result (see, for example, [33, Proposition 1.2.3]).

**Proposition 3.1** Let \(M\) be a monoid generated by a set \(X\), let \(A\) be an alphabet and let \(\phi : A \rightarrow M\) be an injective mapping such that \(A\phi = X\). Let \(\varphi : A^* \rightarrow M\) be the (surjective) homomorphism that extends \(\phi\) to \(A^*\) and let \(R \subseteq A^* \times A^*\). Then \((A | R)\) is a presentation for \(M\) if and only if the following two conditions are satisfied:

1. The generating set \(X\) of \(M\) satisfies (via \(\varphi\)) all the relations from \(R\);
2. If \(u, v \in A^*\) are any two words such that the generating set \(X\) of \(M\) satisfies (via \(\varphi\)) the relation \(u = v\) then \(u = v\) is a consequence of \(R\). \(\square\)

Given a presentation for a monoid, another method to find a new presentation consists in applying Tietze transformations. For a monoid presentation \((A | R)\), the four elementary Tietze transformations are:
(T1) Adding a new relation \( u = v \) to \( \langle A \mid R \rangle \), provided that \( u = v \) is a consequence of \( R \);

(T2) Deleting a relation \( u = v \) from \( \langle A \mid R \rangle \), provided that \( u = v \) is a consequence of \( R \setminus \{ u = v \} \);

(T3) Adding a new generating symbol \( b \) and a new relation \( b = w \), where \( w \in A^* \);

(T4) If \( \langle A \mid R \rangle \) possesses a relation of the form \( b = w \), where \( b \in A \), and \( w \in (A\setminus\{b\})^* \), then deleting \( b \) from the list of generating symbols, deleting the relation \( b = w \), and replacing all remaining appearances of \( b \) by \( w \).

The next result is well-known (see, for example, [33]):

**Proposition 3.2** Two finite presentations define the same monoid if and only if one can be obtained from the other by a finite number of elementary Tietze transformations (T1), (T2), (T3), and (T4). \( \square \)

Now, consider the alphabet \( A = \{ g, h, e_1, e_2, \ldots, e_n \} \) and the set \( R \) formed by the following \( \frac{n^2 + 5n + 9 + (-1)^n}{2} \) monoid relations:

\[
(R_1) \quad g^n = 1, \quad h^2 = 1 \text{ and } hg = g^{n-1}h;
\]

\[
(R_2) \quad e_i^2 = e_i \text{ for } 1 \leq i \leq n;
\]

\[
(R_3) \quad e_ie_j = e_je_i \text{ for } 1 \leq i < j \leq n;
\]

\[
(R_4) \quad ge_1 = e_ng \text{ and } ge_{i+1} = e_ig \text{ for } 1 \leq i \leq n - 1;
\]

\[
(R_5) \quad he_i = e_{n-i+1}h \text{ for } 1 \leq i \leq n;
\]

\[
(R_6) \quad hge_3 \cdots e_n = e_2e_3 \cdots e_n \text{ if } n \text{ is odd};
\]

\[
(R_6') \quad hge_2 \cdots e_\frac{n}{2} e_\frac{n+2}{2} \cdots e_n = e_2 \cdots e_\frac{n}{2} e_\frac{n+2}{2} \cdots e_n \text{ and } he_1e_2 \cdots e_n = e_1e_2 \cdots e_n \text{ if } n \text{ is even}.
\]

We aim to show that the monoid \( DPC_n \) is defined by the presentation \( \langle A \mid R \rangle \).

Let \( \phi : A \rightarrow DPC_n \) be the mapping defined by \( g\phi = g \), \( h\phi = h \) and \( e_i\phi = e_i \), for \( 1 \leq i \leq n \), and let \( \varphi : A^* \rightarrow DPC_n \) be the homomorphism of monoids that extends \( \phi \) to \( A^* \). Notice that we are using the same symbols for the letters of the alphabet \( A \) and for the generating set of \( DPC_n \), which simplifies notation and, within the context, will not cause ambiguity.

It is a routine matter to check the following lemma.

**Lemma 3.3** The set of generators \( \{ g, h, e_1, e_2, \ldots, e_n \} \) of \( DPC_n \) satisfies (via \( \varphi \)) all the relations from \( R \). \( \square \)

Observe that this result assures us that, if \( u, v \in A^* \) are two words such that the relation \( u = v \) is a consequence of \( R \), then \( u\varphi = v\varphi \).

Next, in order to prove that any relation satisfied by the generating set of \( DPC_n \) is a consequence of \( R \), we first present a series of three lemmas. In what follows, we denote the congruence \( \rho_R \) of \( A^* \) simply by \( \rho \).
Lemma 3.4 If \( n \) is even, then the relation
\[
h^g 2j - 1 e_1 \cdots e_{j - 1} e_{j + 1} \cdots e_{j + \frac{n}{2} - 1} e_{j + \frac{n}{2} + 1} \cdots e_n = e_1 \cdots e_{j - 1} e_{j + 1} \cdots e_{j + \frac{n}{2} - 1} e_{j + \frac{n}{2} + 1} \cdots e_n
\]
is a consequence of \( R \) for \( 1 \leq j \leq \frac{n}{2} \).

Proof We proceed by induction on \( j \).

Let \( j = 1 \). Then \( h e_1 \cdots e_{\frac{n}{2} - 1} e_{\frac{n}{2} + 1} \cdots e_n = e_1 \cdots e_{\frac{n}{2} - 1} e_{\frac{n}{2} + 1} \cdots e_n \) is a relation of \( R \). Next, suppose that \( h^g 2j - 1 e_1 \cdots e_{j - 1} e_{j + 1} \cdots e_{j + \frac{n}{2} - 1} e_{j + \frac{n}{2} + 1} \cdots e_n \) for some \( 1 \leq j \leq \frac{n}{2} - 1 \). Then
\[
h^g (j + 1) - 1 e_1 \cdots e_{j + 1} e_{j + 2} \cdots e_{j + \frac{n}{2} - 1} e_{j + \frac{n}{2} + 1} \cdots e_n = e_1 \cdots e_{j + 1} e_{j + 2} \cdots e_{j + \frac{n}{2} - 1} e_{j + \frac{n}{2} + 1} \cdots e_n
\]
(by induction hypothesis)
\[
\rho \ h^g 2j e_1 \cdots e_{j + 1} e_{j + 2} \cdots e_{j + \frac{n}{2} - 1} e_{j + \frac{n}{2} + 1} \cdots e_n
\]
(by \( R_4 \))
\[
\rho \ h^g (j + 1) - 1 e_1 \cdots e_{j + 1} e_{j + 2} \cdots e_{j + \frac{n}{2} - 1} e_{j + \frac{n}{2} + 1} \cdots e_n g
\]
(by \( R_1 \) and \( R_3 \))
\[
\rho \ h^g (j + 1) - 1 e_1 \cdots e_{j + 1} e_{j + 2} \cdots e_{j + \frac{n}{2} - 1} e_{j + \frac{n}{2} + 1} \cdots e_n e_1
\]
(by induction hypothesis)
\[
\rho \ h^g (j + 1) - 1 e_1 \cdots e_{j + 1} e_{j + 2} \cdots e_{j + \frac{n}{2} - 1} e_{j + \frac{n}{2} + 1} \cdots e_n e_1
\]
(by \( R_4 \))
\[
\rho \ e_1 \cdots e_{j + 1} e_{j + 2} \cdots e_{j + \frac{n}{2} - 1} e_{j + \frac{n}{2} + 1} \cdots e_n
\]
(by \( R_1 \) and \( R_3 \)),
as required.

Lemma 3.5 The relation \( h^g 2i - 1 e_1 \cdots e_{i - 1} e_{i + 1} \cdots e_n = e_1 \cdots e_{i - 1} e_{i + 1} \cdots e_n \) is a consequence of \( R \), for \( 1 \leq i \leq n \).

Proof We proceed by induction on \( i \).

Let \( i = 1 \). If \( n \) is odd then \( h e_1 e_2 \cdots e_n = e_1 e_2 \cdots e_n \) is a relation of \( R \). So, suppose that \( n \) is even. Then \( h e_1 e_2 e_{\frac{n}{2} + 1} \cdots e_n = e_1 e_2 e_{\frac{n}{2} + 1} \cdots e_n \) is a relation of \( R \), whence
\[
h e_1 e_2 e_{\frac{n}{2} + 1} \cdots e_n e_{\frac{n}{2} + 1} \rho e_1 e_2 e_{\frac{n}{2} + 1} \cdots e_n e_{\frac{n}{2} + 1}
\]
and so \( h e_1 e_2 \cdots e_n = e_1 e_2 \cdots e_n \), by \( R_3 \).

Now, suppose that \( h^g 2i - 1 e_1 \cdots e_{i - 1} e_{i + 1} \cdots e_n \rho e_1 \cdots e_{i - 1} e_{i + 1} \cdots e_n \) for some \( 1 \leq i \leq n - 1 \). Then (with steps similar to the previous proof), we have
\[
h^g (i + 1) - 1 e_1 \cdots e_{i + 1} \cdots e_n \equiv h^g (i + 1) - 1 e_1 \cdots e_{i + 1} \cdots e_n
\]
(by induction hypothesis)
\[
\rho \ h^g (i + 1) - 1 e_1 \cdots e_{i + 1} \cdots e_n = e_1 \cdots e_{i + 1} \cdots e_n
\]
(by \( R_4 \))
\[
\rho \ h^g (i + 1) - 1 e_1 \cdots e_{i + 1} \cdots e_n = e_1 \cdots e_{i + 1} \cdots e_n
\]
(by \( R_1 \) and \( R_3 \))
\[
\rho \ h^g (i + 1) - 1 e_1 \cdots e_{i + 1} \cdots e_n g
\]
(by induction hypothesis)
\[
\rho \ h^g (i + 1) - 1 e_1 \cdots e_{i + 1} \cdots e_n e_1
\]
(by \( R_4 \))
\[
\rho \ h^g (i + 1) - 1 e_1 \cdots e_{i + 1} \cdots e_n e_1
\]
(by \( R_3 \)),
as required.

Lemma 3.6 The relation \( h^g m e_1 e_2 \cdots e_n = e_1 e_2 \cdots e_n \) is a consequence of \( R \) for \( \ell, m \geq 0 \).
**Proof** First, we prove that the relation \( he_1 e_2 \cdots e_n = e_1 e_2 \cdots e_n \) is a consequence of \( R \). Since this relation belongs to \( R \) when \( n \) is even, it remains to show that \( he_1 e_2 \cdots e_n \rho e_1 e_2 \cdots e_n \) when \( n \) is odd.

Suppose that \( n \) is odd. Hence, by \( R_0 \), we have \( hge_1 e_2 \cdots e_n e_1 \rho e_2 e_3 \cdots e_{n+1} \), so \( hge_1 e_2 \cdots e_n \rho e_1 e_2 \cdots e_n \) (by \( R_3 \)), whence \( ge_1 e_2 \cdots e_n \rho he_1 e_2 \cdots e_n \) (by \( R_1 \)) and then \( (ge_1 e_2 \cdots e_n)^n \rho (he_1 e_2 \cdots e_n)^n \). Now, by \( R_4 \) and \( R_3 \), we have \( ge_1 e_2 \cdots e_n \rho e_n g e_2 \cdots e_n \rho e_n e_1 \cdots e_{n-1} g \rho e_1 e_2 \cdots e_n g \) and so, by relations \( R_1, R_3 \), and \( R_2 \), it follows that \( (ge_1 e_2 \cdots e_n)^n \rho g^n(e_1 e_2 \cdots e_n)^n \rho e_1 e_2 \cdots e_n \). On the other hand, by \( R_5 \) and \( R_3 \), we have \( he_1 e_2 \cdots e_n \rho e_n e_1 \cdots e_{n-1} \rho e_1 e_2 \cdots e_{n-1} h \rho e_1 e_2 \cdots e_{n-1} h \), whence \( (he_1 e_2 \cdots e_n)^n \rho h^n(e_1 e_2 \cdots e_n)^n \rho he_1 e_2 \cdots e_n \) by relations \( R_1, R_3 \), and \( R_2 \), since \( n \) is odd. Therefore, \( he_1 e_2 \cdots e_n \rho e_1 e_2 \cdots e_n \).

Secondly, we prove that the relation \( ge_1 e_2 \cdots e_n = e_1 e_2 \cdots e_n \) is a consequence of \( R \). In fact, we have

\[
\begin{align*}
ge_1 e_2 \cdots e_n & \rho ge_1 hge_2 \cdots e_n & \text{(by Lemma 3.5)} \\
n & \rho e_n ghe_2 \cdots e_n & \text{(by } R_4) \\
n & \rho e_n g g ^ {n-1} e_2 \cdots e_n & \text{(by } R_1) \\
n & \rho e_n h e_2 \cdots e_n & \text{(by } R_1) \\
n & \rho h e_1 e_2 \cdots e_n & \text{(by } R_5) \\
n & \rho e_1 e_2 \cdots e_n & \text{(by the first part).}
\end{align*}
\]

Now, clearly, for \( \ell, m \geq 0 \), \( h^\ell g^n e_1 e_2 \cdots e_n \rho e_1 e_2 \cdots e_n \) follows immediately from \( ge_1 e_2 \cdots e_n \rho e_1 e_2 \cdots e_n \) and \( he_1 e_2 \cdots e_n \rho e_1 e_2 \cdots e_n \), which concludes the proof of the lemma.

We are now in a position to prove the following result.

**Theorem 3.7** The monoid \( \mathcal{DPC}_n \) is defined by the presentation \( \langle A \mid R \rangle \) on \( n+2 \) generators.

**Proof** In view of Proposition 3.1 and Lemma 3.3, it remains to prove that any relation satisfied by the generating set \( \{ g, h, e_1, e_2, \ldots, e_n \} \) of \( \mathcal{DPC}_n \) is a consequence of \( R \).

Let \( u, v \in A^* \) be two words such that \( u \varphi = v \varphi \). We aim to show that \( u \rho v \). Take \( \alpha = u \varphi \).

It is clear that relations \( R_1 \) to \( R_5 \) allow us to deduce that \( u \rho h^\ell g^m e_{i_1} \cdots e_{i_k} \rho e_{i_1} \cdots e_{i_k} \) for some \( \ell \in \{ 0, 1 \}, m \in \{ 0, 1, \ldots, n-1 \}, 1 \leq i_1 < \cdots < i_k \leq n \) and \( 0 \leq k \leq n \). Similarly, we have \( v \rho h^{\ell'} g^{m'} e_{i'_1} \cdots e_{i'_{k'}} \rho e_{i'_1} \cdots e_{i'_{k'}} \) for some \( \ell' \in \{ 0, 1 \}, m' \in \{ 0, 1, \ldots, n-1 \}, 1 \leq i'_1 < \cdots < i'_{k'} \leq n \) and \( 0 \leq k' \leq n \).

Since \( \alpha = h^\ell g^m e_{i_1} \cdots e_{i_k} \), it follows that \( \text{Im}(\alpha) = \Omega_n \setminus \{ i_1, \ldots, i_k \} \) and \( \alpha = h^\ell g^m | \text{Dom}(\alpha) \). Similarly, as also \( \alpha = v \varphi \), from \( \alpha = h^{\ell'} g^{m'} e_{i'_1} \cdots e_{i'_{k'}} \), we get \( \text{Im}(\alpha) = \Omega_n \setminus \{ i'_1, \ldots, i'_{k'} \} \) and \( \alpha = h^{\ell'} g^{m'} | \text{Dom}(\alpha) \). Hence, \( k' = k \) and \( \{ i'_1, \ldots, i'_{k'} \} = \{ i_1, \ldots, i_k \} \). Furthermore, if either \( k = n - 2 \) and \( d(\min \text{Dom}(\alpha), \max \text{Dom}(\alpha)) \neq \frac{n}{2} \) or \( k \leq n - 3 \), by Lemma 2.1, we obtain \( \ell' = \ell \) and \( m' = m \), and so \( u \rho h^\ell g^m e_{i_1} \cdots e_{i_k} \rho v \).

If \( h^{\ell'} g^{m'} = h^\ell g^m \) (including as elements of \( \mathcal{D}_{2n} \)) then \( \ell' = \ell \) and \( m' = m \), and so we get again \( u \rho h^\ell g^m e_{i_1} \cdots e_{i_k} \rho v \).

Therefore, let us suppose that \( h^{\ell'} g^{m'} \neq h^\ell g^m \). Hence, by Lemma 2.1, we may conclude that \( \alpha = \emptyset \) or \( \ell' = \ell - 1 \) or \( \ell' = \ell + 1 \). If \( \alpha = \emptyset \), i.e. \( k = n \), then \( u \rho h^\ell g^m e_{i_1} e_2 \cdots e_n \rho e_1 e_2 \cdots e_n \rho h^{\ell'} g^{m'} e_{i_1} e_2 \cdots e_n \rho v \) by Lemma 3.6.

Thus, we may suppose that \( \alpha \neq \emptyset \) and, without loss of generality, also that \( \ell' = \ell + 1 \), i.e. \( \ell = 0 \) and \( \ell' = 1 \). Let \( k = n - 2 \) and admit that \( d(\min \text{Dom}(\alpha), \max \text{Dom}(\alpha)) = \frac{n}{2} \) (in which case \( n \) is even).

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Let \( \alpha = \left( \frac{i_1}{j_1}, \frac{i_2}{j_2} \right) \) with \( 1 \leq i_1 < i_2 \leq n \). Then \( i_2 - i_1 = \frac{n}{2} = d(i_1, i_2) = d(j_1, j_2) = |j_2 - j_1| \), and so \( j_2 \in \{j_1 - \frac{n}{2}, j_1 + \frac{n}{2} \} \). Let \( j = \min \{j_1, j_2\} \) (notice that \( 1 \leq j < \frac{n}{2} \)) and \( i = j\alpha^{-1} \). Hence, \( \text{Im}(\alpha) = \{j, j + \frac{n}{2} \} \) and \( \alpha = g^{n+j-1}|_{\text{Dom}(\alpha)} = hg^{j+1-n}|_{\text{Dom}(\alpha)} \) (cf. proof of Lemma 2.1). So, we have

\[
u \rho g^m e_1 \cdots e_{j-1} e_{j+1} \cdots e_j \frac{n}{2} - 1 e_j + \frac{n}{2} + 1 \cdots e_n \quad \text{and} \quad v \rho h g^{m'} e_1 \cdots e_{j-1} e_{j+1} \cdots e_j \frac{n}{2} - 1 e_j + \frac{n}{2} + 1 \cdots e_n.
\]

and, by Lemma 2.1, \( m = rn + j - i \) for some \( r \in \{0, 1\} \), and \( m' = i + j - 1 - r'n \) for some \( r' \in \{0, 1\} \). Thus, we get

\[
u \rho g^m e_1 \cdots e_{j-1} e_{j+1} \cdots e_j \frac{n}{2} - 1 e_j + \frac{n}{2} + 1 \cdots e_n \quad \text{and} \quad v \rho h g^{m'} e_1 \cdots e_{j-1} e_{j+1} \cdots e_j \frac{n}{2} - 1 e_j + \frac{n}{2} + 1 \cdots e_n.
\]

Finally, consider that \( k = n - 1 \). Let \( i \in \Omega_n \) be such that \( \Omega_n \setminus \{i_1, \ldots, i_{n-1}\} = \{i\} \). Then \( \text{Im}(\alpha) = \{i\} \) and \( \{i_1, \ldots, i_{n-1}\} = \{1, \ldots, i-1, i+1, \ldots, n\} \). Take \( a = i\alpha^{-1} \). Then \( ag^m = i = a g^{m'} \). Since \( ag^m = a + m - rn \) for some \( r \in \{0, 1\} \), and \( a g^{m'} = (n - a + 1) g^{m'} = r'n - a + 1 + m' \) for some \( r' \in \{0, 1\} \), in a similar way to what we proved before, we have

\[
u \rho g^m e_1 \cdots e_{i-1} e_{i+1} \cdots e_n \quad \text{and} \quad v \rho g^{m'} e_1 \cdots e_{i-1} e_{i+1} \cdots e_n.
\]

as required. \( \square \)

Notice that, taking into account the relation \( h^2 = 1 \) of \( R_1 \), we could have taken only half of the relations \( R_5 \), namely the relations \( h e_i = e_{n-i+1} h \) with \( 1 \leq i \leq \lceil \frac{n}{2} \rceil \), where \( \lceil \frac{n}{2} \rceil \) denotes the least integer greater than or equal to \( \frac{n}{2} \).

Our next and final goal is, by using Tietze transformations, to deduce a new presentation on 3 generators from the previous presentation for \( \mathcal{DP}_n \).

Since we have \( e_i = h g^{i-1} e_n h g^{i-1} \) (as transformations) for all \( i \in \{1, 2, \ldots, n\} \), we will proceed as follows: first, by applying T1, we add the relations \( e_i = h g^{i-1} e_n h g^{i-1} \) for \( 1 \leq i \leq n \); secondly, we apply T4 to each of the relations \( e_i = h g^{i-1} e_n h g^{i-1} \) with \( i \in \{1, 2, \ldots, n-1\} \) and, in some cases, by convenience, we also replace \( e_n \) by \( h g^{n-1} e_n h g^{n-1} \); finally, by using the relations \( R_1 \), we simplify the new relations obtained, eliminating the trivial ones or those that are deduced from others. Performing this procedure for each of the sets of relations \( R_1 \) to \( R_6 \), and renaming \( e_n \) by \( e \), we may routinely obtain the following set \( Q \) of \( \frac{n^2 - n + 13 + (-1)^n}{2} \) many monoid relations on the alphabet \( B = \{g, h, e\} \):

\[
(Q_1) \quad g^n = 1, \quad h^2 = 1 \quad \text{and} \quad h g = g^{n-1} h;
\]
\( (Q_2) \) \( e^2 = e \) and \( ghgh = e \);

\( (Q_3) \) \( e^{j-i} e^{n-j+i} = g^{j-i} e^{n-j+i} e \) for \( 1 \leq i < j \leq n \);

\( (Q_4) \) \( h(g(e))^{n-2} e = (e(g))^{n-2} e \) if \( n \) is odd;

\( (Q_5) \) \( h(g(e))^\frac{n}{2} - 1 g(e)^\frac{n}{2} - 2 e = (e(g))^\frac{n}{2} - 1 g(e)^\frac{n}{2} - 2 e \) and \( h(e)^{n-1} e = (e(g))^{n-1} e \) if \( n \) is even.

Notice that, the use of the expressions \( e_i = h(g^{i-1} e_n h g^{i-1}) \) for all \( i \in \{1, 2, \ldots, n\} \), instead of those observed at the end of Section 2, i.e. \( e_i = g^{n-i} e_n g^i \) for all \( i \in \{1, 2, \ldots, n\} \), allowed us to obtain simpler relations.

Now, in view of Proposition 3.2, we have the following theorem.

**Theorem 3.8** The monoid \( DPC_n \) is defined by the presentation \( \langle B \mid Q \rangle \) on 3 generators. \( \square \)

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### References


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[29] Moore EH. Concerning the abstract groups of order $k!$ and $\frac{1}{2}k!$ holohedrically isomorphic with the symmetric and the alternating substitution groups on $k$ letters. Proceedings of the London Mathematical Society 1897; 28: 357-366.


