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Liouville-type theorem for one-dimensional porous medium systems with sources

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Abstract: In this paper, we are concerned with the one-dimensional porous medium system with sources

$$\begin{cases} u_t - (u^m)_{xx} = a_{11}u^p + a_{12}u^r v^{r+m}, & (x, t) \in J \times I \subset \mathbb{R} \times \mathbb{R} \\ v_t - (v^m)_{xx} = a_{21}u^{r+m}v^r + a_{22}v^p, & (x, t) \in J \times I \subset \mathbb{R} \times \mathbb{R}, \end{cases}$$

where $p = 2r + m$, $m > 1$, $r > 0$. Under the conditions $a_{12} \geq 0$, $a_{21} \geq 0$, $a_{11} > 0$, and $a_{22} > 0$, we prove that the system does not possess any nontrivial nonnegative weak solution.

Key words: Liouville-type theorem, Porous medium system with sources, Nonexistence result

1. Introduction

In recent years, the Liouville properties for elliptic and parabolic equations/systems have been much investigated and emerged as one of the most powerful tools in the study of initial and boundary value problems. From Liouville-type theorems, one can deduce a variety of results on qualitative properties of solutions such as: universal, pointwise, a priori estimates of local solutions; universal and singularity estimates; decay estimates; blow-up rate of solutions, see [25, 26, 29] and references therein.

Let us first go back to the pioneering work of Gidas and Spruck [14] where the existence and nonexistence of a positive solution to the Lane-Emden equation

$$-\Delta u = u^p \text{ in } \mathbb{R}^N$$

was completely established. The optimal range of the exponent p for the nonexistence of positive solutions is $1 < p < p_s(N) := \frac{N+2}{N-2}$. However, a similar question for the Lane-Emden system

$$\begin{cases} -\Delta u = v^p \text{ in } \mathbb{R}^N \\ -\Delta v = u^q \text{ in } \mathbb{R}^N \end{cases}$$

has not been completely solved. It is conjectured that the Lane-Emden system has no positive solution if and only if

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}.$$

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This conjecture has been confirmed in the low dimensions $N \leq 4$, see [19, 31, 32, 34]. In higher dimensions $N \geq 5$, it has not been solved yet, see [34].

We next consider the parabolic model

$$u_t - \Delta u = u^p \text{ in } \mathbb{R}^N \times I \subset \mathbb{R}^N \times \mathbb{R}, \tag{1.1}$$

which has been extensively studied by many mathematicians. The well-known Fujita result ensures the nonexistence of nontrivial nonnegative supersolutions in $\mathbb{R}^N \times (0, +\infty)$ of (1.1) provided that $1 < p \leq \frac{N+2}{N}$, see [10], [20, Sec. 26], and [2, 15, 23] for generalized models. In the supercritical case $p > \frac{N+2}{N}$, problem (1.1) possesses, see [16, Example 1], a nonnegative supersolution in $\mathbb{R}^N \times \mathbb{R}$ of the form

$$u(x, t) = \begin{cases} kt^{-\frac{1}{p-1}} e^{-\gamma \frac{1+|x|^2}{t}} & \text{if } t > 0, x \in \mathbb{R}^N \\ 0 & \text{if } t \leq 0, x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where k, γ are suitably chosen, see also [8] for the system. The complete classification of solutions to (1.1) is one of the most interesting and challenging problems, see [21, 27]. Very recently, the optimal Liouville type theorem for solutions of (1.1) in any dimension has been completely proved by Quittner [28]. We also refer to papers of Bidaut-Véron [4], Quittner [27] and the references [3, 9, 24, 26, 29] for related results.

We now consider the quasilinear parabolic equation

$$u_t - \Delta u^m = u^p \text{ in } \mathbb{R}^N \times I \subset \mathbb{R}^N \times \mathbb{R} \tag{1.3}$$

with $m > 1$. For this equation, some local solvability and general regularity results of solutions of (1.3) can be found in [1, 6, 11–13, 30, 33, 35]. In particular, the authors in [13, 30] proved that when $p \leq m + \frac{2}{N}$, the solution u of (1.3) in $\mathbb{R}^N \times (0, +\infty)$ with bounded, continuous initial data $u_0 \not\equiv 0$ does not exist globally and blow up in a finite time, i.e. there is $T > 0$ such that

$$\sup_{x \in \mathbb{R}^N} u(x, t) \rightarrow +\infty \text{ as } t \rightarrow T.$$

Under the same condition $p \leq m + \frac{2}{N}$, it was also established in [6] that any solution u of (1.3) in $\mathbb{R}^N \times (0, T)$ satisfies the blow up estimate

$$u(x, t) \leq C(N, m, p) \left(t^{-\frac{1}{p-1}} + (T - t)^{-\frac{1}{p-1}} \right). \tag{1.4}$$

Furthermore, the range of the exponent p is extended in [1] such that the estimate (1.4) is still true. The extended range is $p < p_0(m, N)$ where $p_0(m, N)$ is explicitly given by

$$p_0(m, N) = \begin{cases} \frac{N(N+2)}{2(N-1)^2} (1 + \theta + \sqrt{1 + 2\theta}) & \text{if } N \geq 2, \\ \infty & \text{if } N = 1 \end{cases}$$

with $\theta = \frac{(N-1)(m-1)}{N}$. Remark that when $m = 1$,

$$p_0(1, N) = p_B(N) = \begin{cases} \frac{N(N+2)}{2(N-1)^2} & \text{if } N \geq 2, \\ \infty & \text{if } N = 1, \end{cases}$$

which is called the Bidaux-Véron exponent.

The approach in [1] is based on establishing a Liouville type theorem on the whole space. In fact, the authors showed that the equation (1.3) has no nontrivial nonnegative weak solution in the whole space $\mathbb{R}^N \times \mathbb{R}$ when $m < p < p_0(m, N)$. This nonexistence results in the full range $m < p < mp_S$ is still left open. Among other things, it was shown in [1] that:

Theorem A. *Let $1 < m < p$. Then, the equation $u_t - (u^m)_{xx} = u^p$ has no nontrivial nonnegative weak solution in $\mathbb{R} \times \mathbb{R}$.*

Let us next consider the semilinear cooperative parabolic system

$$\begin{cases} u_t - \Delta u = a_{11}u^p + a_{12}u^r v^{r+1}, (x, t) \in \Omega \times I \subset \mathbb{R}^N \times \mathbb{R} \\ v_t - \Delta v = a_{21}u^{r+1}v^r + a_{22}v^p, (x, t) \in \Omega \times I \subset \mathbb{R}^N \times \mathbb{R}, \end{cases} \tag{1.5}$$

where $p = 2r + 1$ and $r > 0$. This system has been studied in [7, 22, 27] in any dimension. Some Liouville type theorems were established in the case $a_{12} = a_{21}$ in [22] and in the general case $a_{12} \neq a_{21}$ in [7]. In particular, the following result was proved in [27], see also [28].

Theorem B. *Let $N = 1$ and $a_{12}, a_{21} \geq 0; a_{11}, a_{22} > 0$. Then, the system (1.5) has no nontrivial nonnegative solution in $\mathbb{R} \times \mathbb{R}$.*

The main tools in [27] are scaling argument and energy estimates. Noticing that, by a simple scaling, one can reduce the system (1.5) to a parabolic system with gradient structure as in [27].

Our purpose in this paper is to study the following porous medium system

$$\begin{cases} u_t - (u^m)_{xx} = a_{11}u^p + a_{12}u^r v^{r+m}, (x, t) \in J \times I \subset \mathbb{R} \times \mathbb{R} \\ v_t - (v^m)_{xx} = a_{21}u^{r+m}v^r + a_{22}v^p, (x, t) \in J \times I \subset \mathbb{R} \times \mathbb{R}, \end{cases} \tag{1.6}$$

where the exponents p, m, r and the coefficients a_{ij} satisfy

$$\begin{aligned} m > 1, r > 0, p = 2r + m, \\ a_{12}, a_{21} \geq 0; a_{11}, a_{22} > 0. \end{aligned} \tag{1.7}$$

As mentioned above, there are many contributions to the porous medium equations. Nevertheless, to the best of our knowledge, there are a few results on porous medium system with sources, see [5, 17, 18, 36] for the global existence and boundedness of solutions. In this paper, we propose to study the Liouville properties of the porous medium system with sources. The first attempt is to establish a nonexistence result for the one-dimensional porous medium system with sources (1.6). As in [1], by a nonnegative weak solution (u, v) of (1.6) we mean $u, v \in C(J \times I)$, $u, v \geq 0$, satisfying (1.6) in the distributional sense.

Our main result in this paper is the following.

Theorem 1.1 *Under the assumption (1.7), the system (1.6) does not possess any nontrivial nonnegative weak solution in $\mathbb{R} \times \mathbb{R}$.*

In order to prove Theorem 1.1, we shall develop the idea in [7, 22] where the main tool is a combination of the Bochner formula, nonlinear integral estimates, the scaling invariance argument, and some idea from [4, 14]. Remark that our proof is not straightforward in comparison with that for the case $m = 1$ in [7, 22]. The main difficulty arising in the proof is the presence of a quasilinear term ($m > 1$). Some key estimates for the

semilinear case ($m = 1$) do not work for the case ($m > 1$) and the nonlinear integral estimates become more delicate.

Before closing the introduction, we present a consequence of Theorem 1.1 without proof since it is totally similar to [1, Theorem 2.1], see also [7, Proposition 1].

Proposition 1.2 *Let $p > m > 1$ and $J \subset \mathbb{R}$. Suppose that (1.7) holds. If (u, v) is a nonnegative weak solution of (1.6) in $J \times (0, T)$, then there holds*

$$u(x, t) + v(x, t) \leq C(t^{-\frac{1}{p-1}} + (T - t)^{-\frac{1}{p-1}} + \text{dist}^{-\frac{2}{p-m}}(x, J)), \quad x \in J, t \in (0, T).$$

The rest of this paper is devoted to the proof of Theorem 1.1.

2. Proof of Theorem 1.1

For the sake of simplicity, we denote by \int the integral $\int_{(-1,1) \times (-1,1)} dxdt$. Denote by C a generic positive constant whose value may change from line to line.

In order to prove Theorem 1.1, it is sufficient to prove that the following system

$$\begin{cases} (u^{\frac{1}{m}})_t - u_{xx} = a_{11}u^{\frac{p}{m}} + a_{12}u^{\frac{r}{m}}v^{\frac{r+m}{m}}, & (x, t) \in J \times I \subset \mathbb{R} \times \mathbb{R} \\ (v^{\frac{1}{m}})_t - v_{xx} = a_{21}u^{\frac{r+m}{m}}v^{\frac{r}{m}} + a_{22}v^{\frac{p}{m}}, & (x, t) \in J \times I \subset \mathbb{R} \times \mathbb{R}. \end{cases} \tag{2.1}$$

has no nontrivial nonnegative weak solution. In what follows, we shall prove this assertion.

An integral estimate, which plays a crucial role in the proof of Theorem 1.1, is given in the following lemma.

Lemma 2.1 *Assume that (1.7) holds. Let (u, v) be a positive regular solution of (2.1) on $(-1, 1) \times (-1, 1)$. Fix $\chi \in C_0^\infty((-1, 1) \times (-1, 1))$ and put*

$$I := a_{21} \int \chi u^{-1-\frac{1}{m}} |u_x|^4 + a_{12} \int \chi v^{-1-\frac{1}{m}} |v_x|^4$$

and

$$L := a_{21} \int \chi u^{1-\frac{1}{m}} (a_{11}u^{\frac{p}{m}} + a_{12}u^{\frac{r}{m}}v^{\frac{r}{m}+1})^2 + a_{12} \int \chi v^{1-\frac{1}{m}} (a_{21}u^{\frac{r}{m}+1}v^{\frac{r}{m}} + a_{22}v^{\frac{p}{m}})^2.$$

Then, there exists $C > 0$ independent of u, v and χ such that

$$\begin{aligned} I + L \leq & C \int \chi \left(|(u^{\frac{1}{m}})_t| u^{-\frac{1}{m}} |u_x|^2 + |(v^{\frac{1}{m}})_t| v^{-\frac{1}{m}} |v_x|^2 \right) \\ & + C \int u^{1-\frac{1}{m}} |\chi_x u_x| \left(u^{\frac{p}{m}} + u^{\frac{r}{m}} v^{\frac{r}{m}+1} + |(u^{\frac{1}{m}})_t| + u^{-1} |u_x|^2 \right) \\ & + C \int v^{1-\frac{1}{m}} |\chi_x v_x| \left(v^{\frac{p}{m}} + v^{\frac{r}{m}} u^{\frac{r}{m}+1} + |(v^{\frac{1}{m}})_t| + v^{-1} |v_x|^2 \right) \\ & + C \int |\chi_{xx}| \left(u^{1-\frac{1}{m}} |u_x|^2 + v^{1-\frac{1}{m}} |v_x|^2 \right) + C \int |\chi_t| \left(u^{\frac{p}{m}+1} + v^{\frac{p}{m}+1} \right) \\ & + C \int \chi \left(u^{1-\frac{1}{m}} |(u^{\frac{1}{m}})_t|^2 + v^{1-\frac{1}{m}} |(v^{\frac{1}{m}})_t|^2 \right). \end{aligned} \tag{2.2}$$

Proof Define

$$I_1 = \int \chi u^{-1-\frac{1}{m}} |u_x|^4, I_2 = \int \chi v^{-1-\frac{1}{m}} |v_x|^4$$

and $J = a_{21}J_1 + a_{12}J_2$ with

$$J_1 = \int \chi u^{-\frac{1}{m}} |u_x|^2 u_{xx}, J_2 = \int \chi v^{-\frac{1}{m}} |v_x|^2 v_{xx}.$$

Then, we have

$$I = a_{21}I_1 + a_{12}I_2.$$

Applying [1, Lemma 4.1] with $q = 1 - \frac{1}{m}$, $N = 1$ and $k \in \mathbb{R}$, we obtain

$$\begin{aligned} & \left(-\frac{k}{m} + \frac{m-1}{2m^2}\right) I_1 + 3 \left(k - \frac{m-1}{2m}\right) J_1 \\ & \leq \frac{1}{2} \int u^{1-\frac{1}{m}} |u_x|^2 \chi_{xx} + \int u^{1-\frac{1}{m}} \left(u_{xx} + \left(\frac{m-1}{m} - k\right) u^{-1} |u_x|^2\right) u_x \chi_x \end{aligned}$$

and

$$\begin{aligned} & \left(-\frac{k}{m} + \frac{m-1}{2m^2}\right) I_2 + 3 \left(k - \frac{m-1}{2m}\right) J_2 \\ & \leq \frac{1}{2} \int v^{1-\frac{1}{m}} |v_x|^2 \chi_{xx} + \int v^{1-\frac{1}{m}} \left(v_{xx} + \left(\frac{m-1}{m} - k\right) v^{-1} |v_x|^2\right) v_x \chi_x. \end{aligned}$$

Multiplying the first inequality by a_{21} , the second one by a_{12} , we deduce that

$$\begin{aligned} & \left(-\frac{k}{m} + \frac{m-1}{2m^2}\right) I + 3 \left(k - \frac{m-1}{2m}\right) J \leq C \int \left(u^{1-\frac{1}{m}} |u_x|^2 + v^{1-\frac{1}{m}} |v_x|^2\right) |\chi_{xx}| \\ & \quad + C \int u^{1-\frac{1}{m}} \left| \left(u_{xx} + \left(\frac{m-1}{m} - k\right) u^{-1} |u_x|^2\right) u_x \chi_x \right| \\ & \quad + C \int v^{1-\frac{1}{m}} \left| \left(v_{xx} + \left(\frac{m-1}{m} - k\right) v^{-1} |v_x|^2\right) v_x \chi_x \right|. \end{aligned} \tag{2.3}$$

Since (u, v) is a positive regular solution of (2.1), we have

$$\begin{aligned} -J & = a_{21} \int \chi u^{-\frac{1}{m}} |u_x|^2 (-u_{xx}) + a_{12} \int \chi v^{-\frac{1}{m}} |v_x|^2 (-v_{xx}) \\ & = a_{21} \int \chi a_{11} u^{\frac{p-1}{m}} |u_x|^2 + a_{12} \int \chi a_{22} v^{\frac{p-1}{m}} |v_x|^2 \\ & \quad + a_{12} a_{21} \left(\int \chi |u_x|^2 u^{\frac{r-1}{m}} v^{\frac{r}{m}+1} + \int \chi |v_x|^2 v^{\frac{r-1}{m}} u^{\frac{r}{m}+1} \right) \\ & \quad - a_{21} \int \chi u^{-\frac{1}{m}} |u_x|^2 (u^{\frac{1}{m}})_t - a_{12} \int \chi v^{-\frac{1}{m}} |v_x|^2 (v^{\frac{1}{m}})_t. \end{aligned} \tag{2.4}$$

It follows from an integration by parts that

$$\begin{aligned} \int \chi |u_x|^2 u^{\frac{r-1}{m}} v^{\frac{r}{m}+1} &= \frac{m}{r+m-1} \int \chi v^{\frac{r}{m}+1} u_x \left(u^{\frac{r+m-1}{m}} \right)_x \\ &= -\frac{m}{r+m-1} \int \chi u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} u_{xx} - \frac{r+m}{r+m-1} \int \chi u^{\frac{r+m-1}{m}} v^{\frac{r}{m}} u_x v_x \\ &\quad - \frac{m}{r+m-1} \int u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} \chi_x u_x \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \int \chi |v_x|^2 v^{\frac{r-1}{m}} u^{\frac{r}{m}+1} &= \frac{m}{r+m-1} \int \chi u^{\frac{r}{m}+1} v_x \left(v^{\frac{r+m-1}{m}} \right)_x \\ &= -\frac{m}{r+m-1} \int \chi v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} v_{xx} - \frac{r+m}{r+m-1} \int \chi v^{\frac{r+m-1}{m}} u^{\frac{r}{m}} v_x u_x \\ &\quad - \frac{m}{r+m-1} \int v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} \chi_x v_x. \end{aligned} \tag{2.6}$$

By combining (2.5) and (2.6), we arrive at

$$\begin{aligned} \int \chi |u_x|^2 u^{\frac{r-1}{m}} v^{\frac{r}{m}+1} + \int \chi |v_x|^2 v^{\frac{r-1}{m}} u^{\frac{r}{m}+1} \\ &= -\frac{m}{r+m-1} \int \chi u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} u_{xx} - \frac{m}{r+m-1} \int \chi v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} v_{xx} \\ &\quad - \frac{r+m}{r+m-1} \int \chi \left(u^{\frac{r+m-1}{m}} v^{\frac{r}{m}} + v^{\frac{r+m-1}{m}} u^{\frac{r}{m}} \right) u_x v_x \\ &\quad - \frac{m}{r+m-1} \int u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} \chi_x u_x - \frac{m}{r+m-1} \int v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} \chi_x v_x. \end{aligned} \tag{2.7}$$

On the other hand, by using the Young inequality and some elementary computations, it holds

$$2\left(u^{\frac{r+m-1}{m}} v^{\frac{r}{m}} + v^{\frac{r+m-1}{m}} u^{\frac{r}{m}}\right) u_x v_x \leq |u_x|^2 \left(u^{\frac{r-1}{m}} v^{\frac{r}{m}+1} + u^{\frac{p-1}{m}}\right) + |v_x|^2 \left(v^{\frac{r-1}{m}} u^{\frac{r}{m}+1} + v^{\frac{p-1}{m}}\right).$$

This together with (2.7) implies that

$$\begin{aligned} &\left(1 + \frac{r+m}{2(r+m-1)}\right) \left(\int \chi |u_x|^2 u^{\frac{r-1}{m}} v^{\frac{r}{m}+1} + \int \chi |v_x|^2 v^{\frac{r-1}{m}} u^{\frac{r}{m}+1}\right) \\ &+ \frac{r+m}{2(r+m-1)} \left(\int \chi (|u_x|^2 u^{\frac{p-1}{m}} + |v_x|^2 v^{\frac{p-1}{m}})\right) \\ &\geq -\frac{m}{r+m-1} \int \chi u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} u_{xx} - \frac{m}{(r+m-1)} \int \chi v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} v_{xx} \\ &\quad - \frac{m}{r+m-1} \int u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} \chi_x u_x - \frac{m}{r+m-1} \int v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} \chi_x v_x. \end{aligned}$$

Consequently,

$$\begin{aligned}
 & \int \chi |u_x|^2 u^{\frac{r-1}{m}} v^{\frac{r}{m}+1} + \int \chi |v_x|^2 v^{\frac{r-1}{m}} u^{\frac{r}{m}+1} \\
 & + \frac{r+m}{3(r+m)-2} \left(\int \chi (|u_x|^2 u^{\frac{p-1}{m}} + |v_x|^2 v^{\frac{p-1}{m}}) \right) \\
 & \geq -\frac{2m}{3(r+m)-2} \left(\int \chi u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} u_{xx} + \int \chi v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} v_{xx} \right) \\
 & - \frac{2m}{3(r+m)-2} \left(\int u^{\frac{r+m-1}{m}} v^{\frac{r}{m}+1} \chi_x u_x + \int v^{\frac{r+m-1}{m}} u^{\frac{r}{m}+1} \chi_x v_x \right).
 \end{aligned} \tag{2.8}$$

We next use an integration by parts to get

$$\begin{aligned}
 \int \chi u^{\frac{p-1}{m}} |u_x|^2 &= \frac{m}{p+m-1} \int \chi u_x \left(u^{\frac{p-1}{m}+1} \right)_x \\
 &= -\frac{m}{p+m-1} \int \chi u^{\frac{p-1}{m}+1} u_{xx} - \frac{m}{p+m-1} \int u^{\frac{p-1}{m}+1} \chi_x u_x
 \end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
 \int \chi v^{\frac{p-1}{m}} |v_x|^2 &= \frac{m}{p+m-1} \int \chi v_x \left(v^{\frac{p-1}{m}+1} \right)_x \\
 &= -\frac{m}{p+m-1} \int \chi v^{\frac{p-1}{m}+1} v_{xx} - \frac{m}{p+m-1} \int v^{\frac{p-1}{m}+1} \chi_x v_x.
 \end{aligned} \tag{2.10}$$

By taking into account (2.4), (2.8), (2.9), (2.10) and the fact that $\frac{m}{p+m-1} < \frac{r+m}{3(r+m)-2}$, we have

$$\begin{aligned}
 & - \left(1 + \left(\frac{a_{12}a_{21}}{a_{11}} + \frac{a_{12}a_{21}}{a_{22}} \right) \frac{r+m}{3(r+m)-2} \right) J \\
 & \geq \frac{r+m}{3(m+r)-2} a_{21} \int \chi (a_{11} u^{\frac{p-1}{m}+1} + a_{12} u^{1+\frac{r-1}{m}} v^{\frac{r}{m}+1}) (-u_{xx}) \\
 & + \frac{r+m}{3(m+r)-2} a_{12} \int \chi (a_{21} v^{1+\frac{r-1}{m}} u^{\frac{r}{m}+1} + a_{22} v^{\frac{p-1}{m}+1}) (-v_{xx}) \\
 & - C \int \left(\chi |(u^{\frac{1}{m}})_t| u^{-\frac{1}{m}} |u_x|^2 + \chi |(v^{\frac{1}{m}})_t| v^{-\frac{1}{m}} |v_x|^2 \right) \\
 & - C \int \left((u^{\frac{p-1}{m}+1} + u^{\frac{r-1}{m}+1} v^{\frac{r}{m}+1}) |\chi_x u_x| + (v^{\frac{p-1}{m}+1} + v^{\frac{r-1}{m}+1} u^{\frac{r}{m}+1}) |\chi_x v_x| \right).
 \end{aligned} \tag{2.11}$$

In (2.11) we use

$$\begin{cases} -u_{xx} = a_{11} u^{\frac{p}{m}} + a_{12} u^{\frac{r}{m}} v^{\frac{r}{m}+1} - (u^{\frac{1}{m}})_t \\ -v_{xx} = a_{21} v^{\frac{r}{m}} u^{\frac{r}{m}+1} + a_{22} v^{\frac{p}{m}} - (v^{\frac{1}{m}})_t \end{cases}$$

and an integrating by parts in t to obtain

$$\begin{aligned}
 -J &\geq \varepsilon \left(a_{21} \int \chi u^{1-\frac{1}{m}} (a_{11} u^{\frac{p}{m}} + a_{12} u^{\frac{r}{m}} v^{\frac{r}{m}+1})^2 + a_{12} \int \chi v^{1-\frac{1}{m}} (a_{21} v^{\frac{r}{m}} u^{\frac{r}{m}+1} + a_{22} v^{\frac{p}{m}})^2 \right) \\
 &\quad - C \int \left(\chi |(u^{\frac{1}{m}})_t| u^{-\frac{1}{m}} |u_x|^2 + \chi |(v^{\frac{1}{m}})_t| v^{-\frac{1}{m}} |v_x|^2 \right) \\
 &\quad - C \int \left((u^{\frac{p-1}{m}+1} + u^{\frac{r-1}{m}+1} v^{\frac{r}{m}+1}) |\chi_x u_x| + (v^{\frac{p-1}{m}+1} + v^{\frac{r-1}{m}+1} u^{\frac{r}{m}+1}) |\chi_x v_x| \right) \\
 &\quad - C \int |\chi_t| (u^{\frac{p}{m}+1} + v^{\frac{p}{m}+1}) \\
 &\quad - C \int \chi (u^{1+\frac{r-1}{m}} v^{\frac{r}{m}+1} (u^{\frac{1}{m}})_t + v^{1+\frac{r-1}{m}} u^{\frac{r}{m}+1} (v^{\frac{1}{m}})_t).
 \end{aligned} \tag{2.12}$$

Here $\varepsilon > 0$ is some small positive constant and is independent of u, v and χ . Applying an integration by parts, the last term in (2.12) becomes

$$\int \chi (u^{1+\frac{r-1}{m}} v^{\frac{r}{m}+1} (u^{\frac{1}{m}})_t + v^{1+\frac{r-1}{m}} u^{\frac{r}{m}+1} (v^{\frac{1}{m}})_t) = -\frac{1}{r+1} \int \chi_t u^{\frac{r}{m}+1} v^{\frac{r}{m}+1}.$$

This equality combined with (2.12) and the Young inequality $u^{\frac{r}{m}+1} v^{\frac{r}{m}+1} \leq C(u^{\frac{p}{m}+1} + v^{\frac{p}{m}+1})$ yield

$$\begin{aligned}
 -J &\geq \varepsilon L - C \int \left(\chi |(u^{\frac{1}{m}})_t| u^{-\frac{1}{m}} |u_x|^2 + \chi |(v^{\frac{1}{m}})_t| v^{-\frac{1}{m}} |v_x|^2 \right) \\
 &\quad - C \int \left((u^{\frac{p-1}{m}+1} + u^{\frac{r-1}{m}+1} v^{\frac{r}{m}+1}) |\chi_x \cdot u_x| + (v^{\frac{p-1}{m}+1} + v^{\frac{r-1}{m}+1} u^{\frac{r}{m}+1}) |\chi_x v_x| \right) \\
 &\quad - C \int |\chi_t| (u^{\frac{p}{m}+1} + v^{\frac{p}{m}+1}).
 \end{aligned} \tag{2.13}$$

By plugging (2.13) into (2.3) and choosing k such that $k - \frac{m-1}{2m} < 0$, we obtain (2.2). Lemma 2.1 is proved. \square

Lemma 2.2 *In addition to (1.7), assume that $a_{12}, a_{21} > 0$. Let (u, v) be a nonnegative weak solution of (2.1) in $(-1, 1) \times (-1, 1)$. Then, there exists $C > 0$ independent of u, v such that*

$$\int_{(-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})} \left(u^{\frac{2p+m-1}{m}} + v^{\frac{2p+m-1}{m}} \right) dxdt \leq C. \tag{2.14}$$

Proof The proof is based on the idea in [1]. We first assume that (u, v) is a positive regular solution of the system (2.1) in $(-1, 1) \times (-1, 1)$.

Let $\xi \in C_0^\infty((-1, 1) \times (-1, 1))$ be a test function such that $\xi = 1$ in $(-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ and $0 \leq \xi \leq 1$. Put $\chi = \xi^{\frac{2(2p+m-1)}{p-m}}$. Then, it is easy to see that

$$\begin{aligned}
 |\chi_x| &\leq C \chi^{\frac{1}{2}}, \\
 |\chi_{xx}| &\leq C \chi^{\frac{p+2m-1}{2p+m-1}}, \\
 |\chi_t| &\leq C \chi^{\frac{p+2m-1}{2p+m-1}}.
 \end{aligned}$$

We use again the notation \int which stands for $\int_{(-1,1)\times(-1,1)} dxdt$ for simplicity. From [1, Formula (4.11)], given any constant $\eta > 0$ and any positive function $w \in C^{2,1}((-1, 1) \times (-1, 1))$, we have

$$\int w^{1-\frac{1}{m}}|w_x|^2 (|\chi_{xx}| + \chi^{-1}|\chi_x|^2 + |\chi_t|) \leq \eta \int \chi \left(w^{-1-\frac{1}{m}}|w_x|^4 + w^{\frac{2p-1}{m}+1} \right) + C(\eta). \tag{2.15}$$

Here, $C(\eta)$ is a positive constant depending on η . Using (2.15) and the Young inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}$, $\varepsilon > 0$, we control the terms corresponding to u on the right hand side of (2.2) as follows

$$\begin{aligned} \int \chi |(u^{\frac{1}{m}})_t| u^{-\frac{1}{m}} |u_x|^2 &\leq \varepsilon \int \chi u^{-1-\frac{1}{m}} |u_x|^4 + \frac{1}{4\varepsilon} \int \chi u^{1-\frac{1}{m}} |(u^{\frac{1}{m}})_t|^2, \\ \int |\chi_x u_x| (u^{\frac{p-1}{m}+1} + u^{\frac{r-1}{m}+1} v^{\frac{r}{m}+1}) &\leq \varepsilon \int \chi u^{1-\frac{1}{m}} (u^{\frac{p}{m}} + u^{\frac{r}{m}} v^{\frac{r}{m}+1})^2 + \frac{1}{4\varepsilon} \int u^{1-\frac{1}{m}} \chi^{-1} |\chi_x u_x|^2 \\ &\leq \varepsilon \int \chi u^{1-\frac{1}{m}} (u^{\frac{p}{m}} + u^{\frac{r}{m}} v^{\frac{r}{m}+1})^2 + \varepsilon \int \chi (u^{-1-\frac{1}{m}} |u_x|^4 + u^{\frac{2p-1}{m}+1}) + C(\varepsilon), \\ \int |\chi_x u_x| u^{-\frac{1}{m}} |u_x|^2 &\leq \varepsilon \int \chi u^{-1-\frac{1}{m}} |u_x|^4 + \frac{1}{4\varepsilon} \int u^{1-\frac{1}{m}} \chi^{-1} |\chi_x u_x|^2 \\ &\leq \varepsilon \int \chi u^{-1-\frac{1}{m}} |u_x|^4 + \varepsilon \int \chi (u^{-1-\frac{1}{m}} |u_x|^4 + u^{\frac{2p-1}{m}+1}) + C(\varepsilon), \\ \int u^{1-\frac{1}{m}} |\chi_x u_x| |(u^{\frac{1}{m}})_t| &\leq \int u^{1-\frac{1}{m}} \chi |(u^{\frac{1}{m}})_t|^2 + \frac{1}{4} \int u^{1-\frac{1}{m}} \chi^{-1} |\chi_x u_x|^2 \\ &\leq \int u^{1-\frac{1}{m}} \chi |(u^{\frac{1}{m}})_t|^2 + \varepsilon \int \chi (u^{-1-\frac{1}{m}} |u_x|^4 + u^{\frac{2p-1}{m}+1}) + C(\varepsilon), \\ \int u^{1-\frac{1}{m}} |\chi_{xx}| |u_x|^2 &\leq \varepsilon \int \chi (u^{-1-\frac{1}{m}} |u_x|^4 + u^{\frac{2p-1}{m}+1}) + C(\varepsilon), \\ \int |\chi_t| u^{\frac{p}{m}+1} &\leq \varepsilon \int \chi u^{\frac{2p-1}{m}+1} + C(\varepsilon) \int \chi^{-\frac{p+m}{p-1}} |\chi_t|^{\frac{2p+m-1}{p-1}} \\ &\leq \varepsilon \int \chi u^{\frac{2p-1}{m}+1} + C(\varepsilon). \end{aligned} \tag{2.16}$$

Similarly, we obtain the estimates corresponding to v . Combining these estimates and Lemma 2.1, we arrive at

$$\begin{aligned} I + L &\leq \varepsilon C \int \chi u^{1-\frac{1}{m}} \left((u^{\frac{p}{m}} + u^{\frac{r}{m}} v^{\frac{r}{m}+1})^2 + (v^{\frac{p}{m}} + v^{\frac{r}{m}} u^{\frac{r}{m}+1})^2 + u^{-2}|u_x|^4 + v^{-2}|v_x|^4 \right) \\ &\quad + C \left(1 + \frac{1}{\varepsilon} \right) \int \chi \left(u^{1-\frac{1}{m}} |(u^{\frac{1}{m}})_t|^2 + v^{1-\frac{1}{m}} |(v^{\frac{1}{m}})_t|^2 \right) + C(\varepsilon). \end{aligned}$$

This inequality and the assumption that $a_{ij} > 0$ for all $i, j = 1, 2$ give

$$I + L \leq \varepsilon C(I + L) + C \left(1 + \frac{1}{\varepsilon} \right) \int \chi \left(u^{1-\frac{1}{m}} |(u^{\frac{1}{m}})_t|^2 + v^{1-\frac{1}{m}} |(v^{\frac{1}{m}})_t|^2 \right) + C(\varepsilon). \tag{2.17}$$

We next have

$$\begin{aligned}
 & \int \chi \left(a_{21} u^{1-\frac{1}{m}} |(u^{\frac{1}{m}})_t|^2 + a_{12} v^{1-\frac{1}{m}} |(v^{\frac{1}{m}})_t|^2 \right) \\
 &= \int \chi a_{21} u^{1-\frac{1}{m}} (u^{\frac{1}{m}})_t \left(u_{xx} + a_{11} u^{\frac{p}{m}} + a_{12} u^{\frac{r}{m}} v^{\frac{r}{m}+1} \right) \\
 & \quad + \int \chi a_{12} v^{1-\frac{1}{m}} (v^{\frac{1}{m}})_t \left(v_{xx} + a_{22} v^{\frac{p}{m}} + a_{21} u^{\frac{r}{m}+1} v^{\frac{r}{m}} \right) \\
 &= \frac{1}{m} \int \chi a_{21} u_t (u_{xx} + a_{11} u^{\frac{p}{m}} + a_{12} u^{\frac{r}{m}} v^{\frac{r}{m}+1}) + \frac{1}{m} \int \chi a_{12} v_t (v_{xx} + a_{22} v^{\frac{p}{m}} + a_{21} u^{\frac{r}{m}+1} v^{\frac{r}{m}}) \\
 &= -\frac{1}{2m} \int \chi \partial_t (a_{21} |u_x|^2 + a_{12} |v_x|^2) + \frac{1}{p+m} \int \chi \partial_t (a_{21} a_{11} u^{\frac{p}{m}+1} + a_{12} a_{22} v^{\frac{p}{m}+1}) \\
 & \quad + \frac{a_{21} a_{12}}{r+m} \int \chi \partial_t (u^{\frac{r}{m}+1} v^{\frac{r}{m}+1}) - \frac{1}{m} \int (a_{21} u_t \chi_x u_x + a_{12} v_t \chi_x v_x).
 \end{aligned}$$

Integrating by parts in t and using again the Young inequality, we arrive at

$$\begin{aligned}
 & \int \chi \left(a_{21} u^{1-\frac{1}{m}} |(u^{\frac{1}{m}})_t|^2 + a_{12} v^{1-\frac{1}{m}} |(v^{\frac{1}{m}})_t|^2 \right) \\
 &= \frac{1}{2m} \int \chi_t (a_{21} |u_x|^2 + a_{12} |v_x|^2) - \frac{1}{p+m} \int \chi_t (a_{21} a_{11} u^{\frac{p}{m}+1} + a_{12} a_{22} v^{\frac{p}{m}+1}) \\
 & \quad - \frac{a_{12} a_{21}}{r+m} \int \chi_t u^{\frac{r}{m}+1} v^{\frac{r}{m}+1} - \frac{1}{m} \int (a_{21} u_t \chi_x u_x + a_{12} v_t \chi_x v_x) \\
 &\leq C \int |\chi_t| (|u_x|^2 + |v_x|^2 + u^{\frac{p}{m}+1} + v^{\frac{p}{m}+1}) + \frac{1}{2} \int \chi (a_{21} u^{1-\frac{1}{m}} (u^{\frac{1}{m}})_t^2 + a_{12} v^{1-\frac{1}{m}} (v^{\frac{1}{m}})_t^2) \\
 & \quad + C \int \chi^{-1} |\chi_x|^2 (u^{1-\frac{1}{m}} |u_x|^2 + v^{1-\frac{1}{m}} |v_x|^2).
 \end{aligned}$$

Consequently, this estimate yields

$$\begin{aligned}
 \int \chi \left(a_{21} u^{1-\frac{1}{m}} (u^{\frac{1}{m}})_t^2 + a_{12} v^{1-\frac{1}{m}} (v^{\frac{1}{m}})_t^2 \right) &\leq C \int |\chi_t| (|u_x|^2 + |v_x|^2 + u^{\frac{p}{m}+1} + v^{\frac{p}{m}+1}) \\
 & \quad + C \int \chi^{-1} |\chi_x|^2 (u^{1-\frac{1}{m}} |u_x|^2 + v^{1-\frac{1}{m}} |v_x|^2). \tag{2.18}
 \end{aligned}$$

Using $a_{12}, a_{21} > 0$, for any $\eta > 0$, we deduce from (2.15), (2.16), (2.18) and the Young inequality that

$$\int \chi \left(a_{21} u^{1-\frac{1}{m}} (u^{\frac{1}{m}})_t^2 + a_{12} v^{1-\frac{1}{m}} (v^{\frac{1}{m}})_t^2 \right) \leq C\eta(I + L) + C(\eta). \tag{2.19}$$

Therefore, it follows from (2.17) and (2.19) that

$$I + L \leq C\varepsilon(I + L) + C(\varepsilon) + C\left(1 + \frac{1}{\varepsilon}\right) (C\eta(I + L) + C(\eta)).$$

By taking $\eta = \varepsilon^2$ and choosing ε sufficiently small, we obtain $I + L \leq C$ and this ends proof of Lemma for positive regular solutions.

We now consider the case where (u, v) is a nonnegative weak solutions of the system in $(-1, 1) \times (-1, 1)$. By using the similar argument as in [33, Proposition 2.4], there is a sequence $(u_k, v_k)_k$ of positive regular solutions of the system in $(-1, 1) \times (-1, 1)$ such that $(u_k, v_k) \rightarrow (u, v)$ locally uniformly in $(-1, 1) \times (-1, 1)$. By combining the above argument and the Fatou lemma, we get

$$\int_{(-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})} (u^{\frac{2p+m-1}{m}} + v^{\frac{2p+m-1}{m}}) dxdt \leq \liminf_{k \rightarrow +\infty} \int_{(-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})} (u_k^{\frac{2p+m-1}{m}} + v_k^{\frac{2p+m-1}{m}}) dxdt \leq C.$$

Lemma 2.2 is proved. □

With the above preparation at hand, we are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Theorem 1.1 is followed from Theorem A if $a_{12} = 0$ or $a_{21} = 0$ since the system is reduced to scalar equation. We then assume that $a_{12} > 0, a_{21} > 0$. Suppose that (u, v) is a nonnegative weak solution of (2.1) in $\mathbb{R} \times \mathbb{R}$. We are going to prove that $u \equiv v \equiv 0$.

For any $R > 0$, by the scaling invariance of the system (2.1), we rescale

$$u_R(x, t) = R^{\frac{2m}{p-m}} u\left(Rx, R^{\frac{2(p-1)}{p-m}} t\right), \quad v_R(x, t) = R^{\frac{2m}{p-m}} v\left(Rx, R^{\frac{2(p-1)}{p-m}} t\right).$$

Then, it is not difficult to check that (u_R, v_R) is also a nonnegative weak solution to (2.1). By Lemma 2.2, we have

$$\begin{aligned} & \int_{|y| < \frac{1}{2}R} \int_{|s| < \frac{1}{2}R} \frac{2(p-1)}{p-m} \left(u^{\frac{2p+m-1}{m}} + v^{\frac{2p+m-1}{m}} \right) (y, s) dy ds \\ &= R^{1 + \frac{2(p-1)}{p-m} - \frac{2(2p+m-1)}{p-m}} \int_{|x| < \frac{1}{2}} \int_{|t| < \frac{1}{2}} \left(u_R^{\frac{2p+m-1}{m}} + v_R^{\frac{2p+m-1}{m}} \right) (x, t) dx dt \\ &\leq CR^{1 + \frac{2(p-1)}{p-m} - \frac{2(2p+m-1)}{p-m}} = CR^{-\frac{p+3m}{p-m}}. \end{aligned}$$

Letting $R \rightarrow \infty$ and noting that the exponent on the right hand side is negative, we deduce that $u \equiv v \equiv 0$. The proof is complete.

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