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Higher topological complexity of a map

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Abstract: The higher topological complexity of a space X, $\mathrm{TC}_r(X)$, $r = 2, 3, \ldots$, and the topological complexity of a map f, $\mathrm{TC}(f)$, have been introduced by Rudyak and Pavešić, respectively, as natural extensions of Farber's topological complexity of a space. In this paper we introduce a notion of higher topological complexity of a map f, $\mathrm{TC}_{r,s}(f)$, for $1 \leq s \leq r \geq 2$, which simultaneously extends Rudyak's and Pavešić's notions. Our unified concept is relevant in the r-multitasking motion planning problem associated to a robot devise when the forward kinematics map plays a role in s prescribed stages of the motion task. We study the homotopy invariance and the behavior of $\mathrm{TC}_{r,s}$ under products and compositions of maps, as well as the dependence of $\mathrm{TC}_{r,s}$ on r and s. We draw general estimates for $\mathrm{TC}_{r,s}(f\colon X \to Y)$ in terms of categorical invariants associated to X, Y and f. In particular, we describe within one the value of $\mathrm{TC}_{r,s}$ in the case of the nontrivial double covering over real projective spaces, as well as for their complex counterparts.

Key words: Higher topological complexity, sectional category

1. Introduction

In this article "space" means a topological space, and by a "map" we will always mean a continuous map. Fibrations are taken in the Hurewicz sense.

Consider an autonomous robot devise $\mathcal{A}$ performing on a known work space $\mathcal{W}$. The fundamental problem in geometric motion planning ([9]) is to find a suitable (safe, efficient, optimal) path taking $\mathcal{A}$ from a given initial configuration to a goal configuration. Here, the term configuration refers to a complete specification of every parameter in the robot's geometry at allowable (collision-free) states. If $\mathcal{C}$ stands for the space of all possible configurations of $\mathcal{A}$, the robot operation usually comes in the form of a forward kinematics map $F\colon \mathcal{C} \to \mathcal{W}$ where, for a configuration $q \in \mathcal{C}$, $F(q)$ encodes the corresponding effect of the robot in the work space.

In practice, motion tasks may involve constraints both on q and on $F(q)$. In such a context, we are interested in a hybrid multitasking version of the motion planning problem. Given a reference configuration q_0 of $\mathcal{A}$ and a tuple

$$(q_1, q_2, \ldots, q_s, e_1, e_2, \ldots, e_\ell) \in \mathcal{C}^s \times \mathcal{W}^\ell \tag{1.1}$$

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of s desired configurations and ℓ desired effects of the robot, the goal is to describe a solving r-multiplath γ, namely, a family of paths $\gamma_1, \gamma_2, \ldots, \gamma_r$ in $\mathcal{C}$ with $r = s + \ell$, all starting at q_0, such that:

- for $1 \leq i \leq s$, γ_i ends at q_i;
- for $1 \leq j \leq \ell$, γ_{s+j} ends at a point in the inverse image $F^{-1}(\{e_j\})$.

The model we propose in Section 3 is intended to study the topological instabilities in the resulting motion planning problem. This is done through the introduction of a numerical invariant that measures the minimal number of robust-to-noise instructions needed to solve, in a global manner, the r-multitasking motion problem above.

Our work is motivated by Rudyak's r-th sequential topological complexity $\mathrm{TC}_r(X)$ of a space X, developed in [1, 14], and by Pavešić's topological complexity $\mathrm{TC}(f)$ of a map $f\colon X \to Y$, developed in [13]. We define the (r,s)-higher topological complexity $\mathrm{TC}_{r,s}(f)$ of f for integers $r \geq 2$ and $1 \leq s \leq r$. Here, the parameter s stands for the number of tasks for which the forward kinematic map must be taken into account, while $\ell := r - s$ is the number of configurations in (1.1) above. Rudyak's and Pavešić's invariants are recovered with $f = 1_X$ and $(r,s) = (2,1)$, respectively.

We note that a previous version of the higher version $\mathrm{TC}_{r,r}(f)$ appeared in the paper [8].

In addition to its relevance in the multitasking problem for the forward kinematics map, the parameter s in our invariant $\mathrm{TC}_{r,s}(f)$ plays a subtle role within more theoretical issues. For starters, our invariant is sensitive to the numbers $r-s$ (of configurations) and s (of effect tasks), a fact reflected in part by the fairly regular monotonic behavior

$$\mathrm{TC}_{r,s}(f) \leq \min\{\mathrm{TC}_{r+1,s}(f), \mathrm{TC}_{r+1,s+1}(f)\}$$

(see Proposition 3.5). For instance, for the double covering map $p_n\colon S^n \to \mathbb{R}P^n$, we show

$$\mathrm{TC}_{r,s}(p_n) = r + s(n-1) + \varepsilon_{r,s,n},$$

where* $\varepsilon_{r,s,n} \in \{0,1\}$. On the other hand, the well known fact that Rudyak's $\mathrm{TC}_r(Y)$ of an H-space Y agrees with the Lusternik-Schnirelmann category of Y^{r-1} is encoded by $\mathrm{TC}_{r,r-1}(f)$ for any fibration over Y (Corollary 3.38). In general, the use of the biparameter (r,s) allows us to get a discrimination of the topological properties of a space Y in a manner which is finer than that provided by the several higher topological complexities $\mathrm{TC}_r(Y)$. For instance, Rudyak's monotonic behavior $\mathrm{TC}_r(Y) \leq \mathrm{TC}_{r+1}(Y)$ is refined by the inequalities

$$\mathrm{TC}_r(Y) \leq \mathrm{TC}_{r,r}(f) \leq \mathrm{TC}_{r+1,r}(f) \leq \mathrm{TC}_{r+1}(Y),$$

valid for any fibration $f\colon X \to Y$ (Remark 3.40).

We provide estimates for $\mathrm{TC}_{r,s}(f)$ for a general map $f\colon X \to Y$, possibly failing to be a fibration. As a way of illustration, Propositions 3.6 and 3.42 yield

$$\max\{\mathrm{sec}^{1\times f^s}(e_r^X), \mathrm{sec}(f^s), \mathrm{nil}\,(\mathrm{Ker}((\Delta_{r-s}, {}^s f)^*))\} \leq \mathrm{TC}_{r,s}(f) \leq \mathrm{sec}(f^s) \cdot \mathrm{sec}^{1\times f^s}(e_r^X).$$

We also study the homotopy invariance of $\mathrm{TC}_{r,s}$ together with its behavior under composition of maps. In fact, virtually all properties developed in [13] for the case $(r,s) = (2,1)$ are extended here to the higher TC realm.

*The precise value of $\varepsilon_{r,s,n}$ is given in Section 4 for $r = s$ (any n), and for $n \in \{1,3,7\}$ (any r and s).

Yet, unlike Pavešić's approach, we work with the standard (and better suited for actual applications) definition of the sectional number of a map in terms of open coverings (reviewed in Section 2).

Rudyak and Soumen [15] have recently introduced a notion of higher topological complexity $\mathrm{TC}_r^{RS}(f)$ of a map f. Their concept is compared to ours. For instance, in Corollary 3.25, we obtain that the equalities $\mathrm{TC}_r^{RS}(f)=\mathrm{TC}_{r,r}(f)=\mathrm{TC}_{r,r-1}(f)$ hold for any fibration f admiting a section. Additionally, we show that, for any map f (possibly not a fibration), Rudyak and Soumen's $\mathrm{TC}_r^{RS}(f)$ is in fact a generalization of Murillo-Wu's notion of topological complexity of f (Proposition 3.10), and that, under special conditions, our $\mathrm{TC}_{r,s}(f)$ with large s unifies previous notions of topological complexity (Corollary 3.25).

2. Preliminaries on sectional numbers

Given a map $f : X \to Y$ and a subset A of Y, we say that a map $s : A \to X$ is a local section of f if $f \circ s = incl_A$, and a local homotopy section of f if $f \circ s \simeq incl_A$, where $incl_A : A \to Y$ is the inclusion map. The sectional number $\sec(f)$ is the least integer m such that Y can be covered by m open subsets each of which admits a local section of f. We set $\sec(f)=\infty$ if no such m exists. Likewise, the sectional category $\mathrm{secat}(f)$ is the least integer m such that Y can be covered by m open subsets each of which admits a local homotopy section of f. Again, we set $\mathrm{secat}(f)=\infty$ if no such m exists. See [2].

Note that f is forced to be surjective whenever $\sec(f)<\infty$. Furthermore, the inequality $\mathrm{secat}(f) \leq \sec(f)$ holds for any map f. Additionally, from the homotopy lifting property, a homotopy section of a fibration can be replaced by a strict section. In particular, $\mathrm{secat}(f)=\sec(f)$ when f is a fibration.

For $f : X \to Y$ and $g : Y \to Z$, we define the sectional number $\sec^g(f)$ as the least integer n for which Y admits a covering by n open sets U_i such that over each U_i there is a map $s_i : U_i \to X$ with $g \circ f \circ s_i = g_{|U_i}$. Likewise, the sectional category $\mathrm{secat}^g(f)$ is the least integer n for which Y admits a covering by n open sets U_i such that over each U_i there is a map $s_i : U_i \to X$ with $g \circ f \circ s_i \simeq g_{|U_i}$. As reviewed at the end of this section, the invariant $\mathrm{secat}^g(f)$ is studied by Murillo and Wu in [10]. The following fact is straightforward to prove:

Lemma 2.1 *Let $f : X \to Y$, $g : Y \to Z$ and $\varphi : Z \to W$ be arbitrary maps. We have*

$$sec^{\varphi\circ g}(f) \leq sec^g(f) \leq sec(f).$$

Recall the pathspace construction from [7, p. 407]. For a map $f : X \to Y$, consider the space

$$E_f = \{(x,\gamma) \in X \times PY \mid \ \gamma(0)=f(x)\},$$

where $PY = Y^I$ is the space of all paths $[0,1] \to Y$. The map

$$\rho_f : E_f \to Y,\ (x,\gamma) \mapsto \rho_f(x,\gamma)=\gamma(1)$$

is a fibration. Furthermore, the projection onto the first coordinate $E_f \to X,\ (x,\gamma) \mapsto x$ is a homotopy equivalence with homotopy inverse $c : X \to E_f$ given by $x \mapsto (x,\gamma_{f(x)})$, where $\gamma_{f(x)}$ is the constant path at $f(x)$. This renders the factorization

$$\left(X \stackrel{f}{\to} Y\right) = \left(X \stackrel{c}{\to} E_f \stackrel{\rho_f}{\to} Y\right),$$

a composition of a homotopy equivalence followed by a fibration. Furthermore, f is a fibration if and only f admits a lifting function, i.e. a map $\Gamma\colon E_f \to PX$ such that, for each $(x,\gamma) \in E_f$, we have

$$\Gamma(x,\gamma)(0) = x \quad \text{and} \quad f \circ \Gamma(x,\gamma) = \gamma. \tag{2.1}$$

By a quasi pullback we mean a strictly commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\varphi'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\varphi"'] & Y \end{tikzcd} \tag{2.2}$$

such that, for any strictly commutative diagram as the one on the left hand-side of (2.3), there exists a (not necessarily unique) map $h : Z \to X'$ that renders a strictly commutative diagram as the one on the right hand-side of (2.3).

$$\begin{tikzcd} Z \arrow[rr, bend left, "\beta"] \arrow[dr, bend right, "\alpha"'] & & X \arrow[d, "f"] \\ & Y' \arrow[r, "\varphi"'] & Y \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "h"] \arrow[rr, bend left, "\beta"] \arrow[dr, bend right, "\alpha"'] & X' \arrow[r, "\varphi'"] \arrow[d, "f'"] & X \\ & Y' & \end{tikzcd} \tag{2.3}$$

Note that such a condition amounts to saying that X' contains the canonical pullback $Y' \times_Y X$ determined by f and φ as a retract in a way that is compatible with the mappings into X and Y'.

For convenience, we record the following standard properties, most of which appear in chapter 4 of [19]:

Lemma 2.2

1. *If (2.2) is a quasi pullback, then*
$$sec(f') \leq sec(f).$$

2. *For a map $f : X \to Y$,*
$$secat(\rho_f) = secat(f).$$

3. *If $f, g : X \to Y$ are homotopic maps (which we shall denote by $f \simeq g$), then*
$$secat(f) = secat(g).$$

4. *If $f : X \to Y$ and $g : Y \to Z$ are maps, then*
$$sec(g) \cdot sec^g(f) \geq sec(g \circ f) \geq \max\{sec(g), sec^g(f)\}.$$
In particular, $sec(g \circ f) = sec^g(f)$ provided g admits a section.

5. *If $p : E \to B$ is a fibration, then*
$$\mathit{sec}(p) \leq \mathit{cat}(B).$$
In particular, $\mathit{secat}(f) \leq \mathit{cat}(Y)$ for any map $f : X \to Y$.

6. *If $f : X \to Y$ is null-homotopic, then*
$$\mathit{secat}(f) = \mathit{cat}(Y).$$

7. *(cf. [16, Proposition 20, p. 83]) Let $f : X \to Y$ be a map with Y normal. If $\{C_1, \ldots, C_k\}$ and $\{D_1, \ldots, D_\ell\}$ are open coverings of Y such that on each $C_i \cap D_j$ there exists a section of f, then*
$$\mathit{sec}(f) \leq k + \ell - 1.$$

8. *For a space Z and a map $f : X \to Y$,*
$$\operatorname{sec}(1_Z \times f) = \operatorname{sec}(f) \quad \textit{and} \quad \operatorname{secat}(1_Z \times f) = \operatorname{secat}(f).$$

The sectional number of the canonical pullback $\varphi^*(p) : K \times_B E \to K$ on the left hand-side of (2.4) below, denoted by $\operatorname{sec}_\varphi(p)$, is called relative sectional number.

$$\begin{tikzcd} K \times_B E \arrow[r] \arrow[d, "\varphi^*(p)"'] & E \arrow[d, "p"] \\ K \arrow[r, "\varphi"'] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "f"'] & W \arrow[d, "h"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \tag{2.4}$$

Lemma 2.3 *The inequalities $\mathit{sec}_g(h) \leq \mathit{sec}^g(f) \leq \mathit{sec}(f)$ hold for any commutative square as the one on the right hand-side of (2.4). If the square is a quasi pullback, then in fact $\mathit{sec}_g(h) = \mathit{sec}^g(f) = \mathit{sec}(f)$.*

Proof For $U \subset Y$ and $s : U \to X$ satisfying $g \circ f \circ s = g_{|U}$, the map $\sigma : U \to W$ given by $\sigma = \varphi \circ s$ defines a lift of $g_{|U}$ through h. The first inequality asserted in the lemma then follows by observing (see [19, Proposition 4.5.16]) that $\operatorname{sec}_\varphi(p)$ can be defined in terms of open covers $\{U_i\}$ of K such that each element of the cover admits a lift $\sigma_i : U_i \to E$ of $\varphi_{|U_i}$ through p, i.e. $p \circ \sigma_i = \varphi_{|U_i}$. The second inequality in the lemma comes from Lemma 2.1. The proof is complete by noticing that $\operatorname{sec}(f) \leq \operatorname{sec}_g(h)$ when the given square is a quasi pullback. Indeed, the quasi pullback hypothesis implies that any lift $\sigma \colon U \to W$ of $g_{|U}$ through h can be lifted through φ to a local section $U \to X$ of f. □

Remark 2.4 *Note that, when p is a fibration, $\mathit{sec}_\varphi(p)$ can be defined in terms of open covers $\{U_i\}$ of K such that each element of the cover admits a homotopic lift $\sigma_i : U_i \to E$ of $\varphi_{|U_i}$ through p, i.e. $p \circ \sigma_i \simeq \varphi_{|U_i}$.*

We close the section by indicating how the sectional numbers we have just formalized capture the different versions in the literature of (topological) complexity of a map $f \colon X \to Y$. Let $e_2^X : PX \to X \times X$ be the double-evaluation fibration given by $e_2^X(\gamma) = (\gamma(0), \gamma(1))$.

- The *complexity* of f, $\mathrm{cx}(f)$, introduced by Pavešić in [11] (see also [12]), is the sectional number

$$\sec(PX \xrightarrow{e_2^X} X \times X \xrightarrow{1_X \times f} X \times Y).$$

  When f is a fibration between ANRs spaces, the number $\mathrm{cx}(f)$ coincides with the notion of topological complexity $\mathrm{TC}(f)$ studied in [13]. The complexity $\mathrm{cx}(f)$ has recently been used in [18, 20].

- A different approach was taken by Murillo and Wu in [10]. Their topological complexity of f, which we denote by $\mathrm{TC}^{MW}(f)$, is given by

$$\mathrm{TC}^{MW}(f) = \mathrm{secat}^{f\times f}(e_2^X),$$

  i.e. the least integer n such that $X \times X$ can be covered by n open sets $\{U_i\}_{i=1}^n$ on each of which there is a map $s_i : U_i \to PX$ satisfying $(f \times f) \circ e_2^X \circ s_i \simeq (f \times f)_{|U_i}$. Their naive or strict topological complexity of f, which we denote by $\mathrm{tc}^{MW}(f)$, is defined analogously, except that one now requires each of the maps $s_i : U_i \to PX$ to satisfy the stronger condition $(f \times f) \circ e_2^X \circ s_i = (f \times f)_{|U_i}$. In other words,

$$\mathrm{tc}^{MW}(f) = \sec^{f\times f}(e_2^X).$$

  As shown in [10], the inequality $\mathrm{TC}^{MW}(f) \leq \mathrm{tc}^{MW}(f)$ holds for any map f, while in fact $\mathrm{TC}^{MW}(f) = \mathrm{tc}^{MW}(f)$ when f is a fibration.

- As detailed in Subsection 3.1, relative sectional numbers are closely related to Rudyak-Soumen's quasistrong sectional category of a map. In fact, by extending ideas in Scott's study of the relative sectional number

$$\sec_{f\times f}(e_2^Y)$$

  ([17, Definition 3.1]), we show that Rudyak-Soumen's higher TC is in fact a generalization of Murillo-Wu's TC of a map. See Proposition 3.10 below.

3. Higher topological complexity

For $r \geq 2$, let J_r be the wedge of r closed intervals $[0,1]_i$, $i = 1, \ldots, r$, where the zero points $0_i \in [0,1]_i$ are identified. For a space X, let X^{J_r} denote the space of maps $\gamma : J_r \to X$ with the compact-open topology. Consider the fibration[†]

$$e_r^X : X^{J_r} \to X^r,\ e_r(\gamma) = (\gamma(1_1), \ldots, \gamma(1_r)), \tag{3.1}$$

where $1_i \in [0,1]_i$. Here we regard X^{J_r} as the space of ordered r-multipaths in X all whose components have a common starting point. From [14], the *r-th higher topological complexity* $\mathrm{TC}_r(X)$ of X is the sectional number of the fibration (3.1). In other words, the r-th higher topological complexity of X is the smallest positive integer $\mathrm{TC}_r(X) = k$ for which the product X^r is covered by k open subsets $X^r = U_1 \cup \cdots \cup U_k$ such that, for any $i = 1, 2, \ldots, k$, there exists a local section $s_i : U_i \to X^{J_r}$ of e_r^X over U_i (i.e., $e_r^X \circ s_i = incl_{U_i}$).

Let $f : X \to Y$ be a map, and let

$$e_{r,s}^f : X^{J_r} \to X^{r-s} \times Y^s,\ e_{r,s}^f = (1_{X^{r-s}} \times f^s) \circ e_r^X,$$

[†]Since PX is homeomorphic to X^{J_2}, the notation e_r^X is compatible with the use of e_2^X in the previous section.

for $1 \leq s \leq r$. For example, $e^f_{r,r-1} = (1_X \times f^{r-1}) \circ e^X_r$ and $e^f_{r,r} = f^r \circ e^X_r$.

Definition 3.1

1. *The strong (r,s)-th higher topological complexity of a map $f\colon X \to Y$, denoted by $TC_{r,s}(f)$, is the sectional number $sec(e^f_{r,s})$ of the map $e^f_{r,s}$, that is, the least integer m such that the cartesian product $X^{r-s} \times Y^s$ can be covered by m open subsets U_i such that, for any $i = 1, 2, \ldots, m$, there exists a local section $s_i : U_i \to X^{J_r}$ of $e^f_{r,s}$, so $e^f_{r,s} \circ s_i = incl_{U_i}$. If no such m exists we set $TC_{r,s}(f) = \infty$.*

2. *The homotopy (r,s)-th higher topological complexity of the map f, denoted by $HTC_{r,s}(f)$, is the sectional category $secat(e^f_{r,s})$ of the map $e^f_{r,s}$, that is, the least integer m such that the cartesian product $X^{r-s} \times Y^s$ can be covered with m open subsets U_i such that, for any $i = 1, 2, \ldots, m$, there exists a local homotopy section $s_i : U_i \to X^{J_r}$ of $e^f_{r,s}$, so $e^f_{r,s} \circ s_i \simeq incl_{U_i}$. If no such m exists we set $HTC_{r,s}(f) = \infty$.*

Note that f is forced to be surjective whenever $\mathrm{TC}_{r,s}(f) < \infty$. The strong form of the higher TC of a map is best suited for applications. Accordingly, $\mathrm{TC}_{r,s}(f)$ will be the main focus in this work.

Remark 3.2 *For $r \geq 2$, consider the evaluation fibration $e'_r : PX \to X^r$ given by*

$$e'_r(\gamma) = \left(\gamma(0), \gamma\left(\frac{1}{r-1}\right), \ldots, \gamma\left(\frac{r-2}{r-1}\right), \gamma(1)\right).$$

We have commutative diagrams

$$\begin{tikzcd} PX \arrow[rr, "\varphi"] \arrow[dr, "e'_r"'] & & X^{J_r} \arrow[dl, "e^X_r"] \\ & X^r & \end{tikzcd} \qquad \begin{tikzcd} X^{J_r} \arrow[rr, "\psi"] \arrow[dr, "e^X_r"'] & & PX \arrow[dl, "e'_r"] \\ & X^r, & \end{tikzcd}$$

where $\varphi(\gamma) = (\gamma_1, \ldots, \gamma_r)$ and $\psi(\alpha_1, \ldots, \alpha_r) = \alpha_1 \cdot (\overline{\alpha}_1 \cdot \alpha_2) \cdot (\overline{\alpha}_2 \cdot \alpha_3) \cdots (\overline{\alpha}_{r-1} \cdot \alpha_r)$. Here $\alpha \cdot \beta$ stands for the concatenation of α and β, $\overline{\alpha}(t) = \alpha(1-t)$ is the path α traversed in opposite direction and, for $i = 1, 2, \ldots, r$,

$$\gamma_i(t) = \gamma\left(\frac{(i-1)\cdot t}{r-1}\right).$$

Therefore

$$TC_{r,s}(f) = sec\left((1_{X^{r-s}} \times f^s) \circ e'_r\right) \quad \text{and} \quad HTC_{r,s}(f) = secat\left((1_{X^{r-s}} \times f^s) \circ e'_r\right), \tag{3.2}$$

*which explains the use of the name "*sequential *topological complexity" as an alternative for "*higher *topological complexity".*

Remark 3.3 *As an abuse of notation, when using the "sequential" setting, we will keep writing $e^f_{r,s}$ for the map $(1_{X^{r-s}} \times f^s) \circ e'_r$ in (3.2).*

More generally, we can use other evaluation maps to define $TC_{r,s}$ *and* $HTC_{r,s}$. *For instance, let* G_r *be any connected graph where* r *ordered distinct vertices* $v_1, \ldots, v_r$ *have been selected, and consider the evaluation map* $e_{G_r} : X^{G_r} \to X^r$, $e_{G_r}(\gamma) = (\gamma(v_1), \ldots, \gamma(v_r))$. *Then, as explained in [1, pages 2106–2107], there are commutative diagrams*

$$\begin{tikzcd} X^{J_r} \arrow[rr] \arrow[dr, "e_r^X"'] & & X^{G_r} \arrow[dl, "e_{G_r}"] \\ & X^r & \end{tikzcd} \qquad \begin{tikzcd} X^{G_r} \arrow[rr] \arrow[dr, "e_{G_r}"'] & & PX \arrow[dl, "e'_r"] \\ & X^r & \end{tikzcd}$$

which, together with (3.2), yield

$$TC_{r,s}(f) = sec\left((1_{X^{r-s}} \times f^s) \times e_{G_r}\right),$$
$$HTC_{r,s}(f) = secat\left((1_{X^{r-s}} \times f^s) \times e_{G_r}\right).$$

Remark 3.4

1. *By definition, the higher topological complexity* $TC_{r,s}(1_X)$ *of the identity map* $1_X : X \to X$ *coincides with the higher topological complexity* $TC_r(X)$, *i.e.,* $TC_{r,s}(1_X) = TC_r(X)$, *for any* $s \in \{1, \ldots, r\}$.

2. *Note that* $HTC_{r,s}(f) \le TC_{r,s}(f)$ *for any map* f. *Moreover, it is easy to see that* f *is a fibration if and only if* $e^f_{r,s}$ *is a fibration (for instance, use Remark 3.2 in the proof of [13, Lemma 4.1]). Therefore, we immediately obtain* $TC_{r,s}(f) = HTC_{r,s}(f)$ *for any fibration* f.

The following result generalizes [14, Proposition 3.3].

Proposition 3.5 *For any map* $f : X \to Y$ *and any* $s = 1, 2, \ldots, r$,

$$TC_{r,s}(f) \le \min\{TC_{r+1,s}(f), TC_{r+1,s+1}(f)\}.$$

Proof Define $\mu_{r,s} : X^{J_{r+1}} \to X^{J_r}$ as the map which forgets the $(r+1-s)$-th path, for any $s \in \{1, \ldots, r\}$. Explicitly, the $(r+1)$-tuple $\gamma = (\gamma_1, \ldots, \gamma_{r-s}, \gamma_{r+1-s}, \gamma_{r+2-s}, \ldots, \gamma_{r+1})$ of paths γ_i in X is sent under $\mu_{r,s}$ to the r-tuple $\gamma = (\gamma_1, \ldots, \gamma_{r-s}, \gamma_{r+2-s}, \ldots, \gamma_{r+1})$ of paths. Choose $a \in X$ and consider the subspace inclusion $\varphi_a \times 1_{Y^s} : X^{r-s} \times Y^s \hookrightarrow X^{r+1-s} \times Y^s$, where

$$\varphi_a : X^{r-s} \hookrightarrow X^{r-s+1},\ \varphi_a(x_1, \ldots, x_{r-s}) = (x_1, \ldots, x_{r-s}, a).$$

If $r = s$, we think of X^{r-s} as the single-point space $\{a\}$, and ignore it in any cartesian product. Take an open cover $U_1, \ldots, U_m$ of $X^{r+1-s} \times Y^s$ such that each U_i has a local section $\sigma_i : U_i \to X^{J_{r+1}}$ of $e^f_{r+1,s}$ for $i = 1, \ldots, m$, and put

$$V_i = U_i \cap \left(X^{r-s} \times Y^s\right).$$

Then a local section $s_i : V_i \to X^{J_r}$ of $e^f_{r,s}$ is given by

$$V_i \hookrightarrow U_i \xrightarrow{\sigma_i} X^{J_{r+1}} \xrightarrow{\mu_{r,s}} X^{J_r}.$$

This yields $\mathrm{TC}_{r,s}(f) \leq \mathrm{TC}_{r+1,s}(f)$.

On the other hand, choose an element $b \in Y$ and consider the subspace inclusion $1_{X^{r-s}} \times i_b \colon X^{r-s} \times Y^s \hookrightarrow X^{r-s} \times Y^{s+1}$, where

$$i_b : Y^s \to Y^{s+1},\ i_b(z) = (b, z).$$

As before, take an open cover $U_1, \ldots, U_m$ of $X^{r-s} \times Y^{s+1}$ such that each U_i has a local section $\sigma_i \colon U_i \to X^{J_{r+1}}$ of $e^f_{r+1,s+1}$ for $i = 1, \ldots, m$, and put

$$V_i = U_i \cap \left(X^{r-s} \times Y^s\right).$$

Then a local section $s_i : V_i \to X^{J_r}$ of $e^f_{r,s}$ is given by

$$V_i \hookrightarrow U_i \xrightarrow{\sigma_i} X^{J_{r+1}} \xrightarrow{\mu_{r,s}} X^{J_r}.$$

We thus get $\mathrm{TC}_{r,s}(f) \leq \mathrm{TC}_{r+1,s+1}(f)$. □

Proposition 3.6 *For a map $f : X \to Y$, we have*

$$TC_{r,s}(f) \geq \begin{cases} \max\{sec(f^s),\ sec^{1_{X^{r-s}} \times f^s}(e^X_r),\ cat(X^{r-s-1} \times Y^s)\}, & \textit{for } s < r; \\ \max\{sec(f^r),\ sec^{f^r}(e^X_r),\ TC_r(Y)\}, & \textit{for } s = r. \end{cases}$$

Proof Item (4) of Lemma 2.2, yields

$$\begin{aligned} \mathrm{TC}_{r,s}(f) &= \mathrm{sec}(X^{J_r} \xrightarrow{e^X_r} X^r \xrightarrow{1_{X^{r-s}} \times f^s} X^{r-s} \times Y^s) \\ &\geq \max\{\mathrm{sec}(1_{X^{r-s}} \times f^s), \mathrm{sec}^{(1_{X^{r-s}} \times f^s)}(e^X_r)\} \\ &= \max\{\mathrm{sec}(f^s), \mathrm{sec}^{(1_{X^{r-s}} \times f^s)}(e^X_r)\}, \end{aligned}$$

where the last equality comes from item (8) of Lemma 2.2.

For $s < r$, consider the canonical pullback

$$\begin{tikzcd} (i_a)^*(e^f_{r,s}) \arrow[r] \arrow[d] & X^{J_r} \arrow[d, "e^f_{r,s}"] \\ X^{r-s-1} \times Y^s \arrow[r, "i_a"'] & X^{r-s} \times Y^s, \end{tikzcd}$$

where $i_a : X^{r-s-1} \times Y^s \hookrightarrow X^{r-s} \times Y^s$ is the subspace inclusion given by $i_a(x, y) = (a, x, y)$, for some fixed $a \in X$. Since $(i_a)^*(e^f_{r,s})$ is contractible, items (1) and (6) of Lemma 2.2 yield $\mathrm{TC}_{r,s}(f) \geq \mathrm{cat}(X^{r-s-1} \times Y^s)$. On the other hand, for $s = r$, the commutative diagram

$$\begin{tikzcd} X^{J_r} \arrow[rr, "f_\#"] \arrow[dr, "e^f_{r,r}"'] & & Y^{J_r} \arrow[dl, "e^Y_r"] \\ & Y^r & \end{tikzcd}$$

yields the inequality $\mathrm{TC}_r(Y) \leq \mathrm{TC}_{r,r}(f)$. □

3.1. Rudyak-Soumen higher TC as a generalization of Murillo-Wu's TC

The quasistrong LS category of a map $f\colon X \to (Y,B)$, $\mathrm{qscat}(f)$, introduced by Rudyak and Soumen in [15, Definition 2.7], is the least integer n such that X can be covered by n open subsets $\{U_i\}_{i=1}^n$ on each of which there is a homotopy $H_i : U_i \times [0,1] \to Y$ satisfying $(H_i)_0 = f|_{U_i}$ and $(H_i)_1(U_i) \subset B$.

For any commutative diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "f"'] & X' \arrow[d, "h"] \\ Y \arrow[r, "g"'] & Z, \end{tikzcd} \tag{3.3}$$

it is easy to see that

$$\mathrm{qscat}(g : Y \to (Z,B)) \leq \mathrm{sec}(f) \cdot \mathrm{qscat}(h : X' \to (Z,B)). \tag{3.4}$$

Furthermore, if Z is path-connected, then

$$\mathrm{qscat}(g : Y \to (Z,B)) \leq \mathrm{cat}(g : Y \to Z),$$

with equality whenever B is contractible.

Proposition 3.7 *Assume (3.3) is a quasi pullback with $h : X' \to Z$ a fibration admiting a section over a subspace B of Z. Then $sec(f) \leq qscat(g : Y \to (Z,B))$.*

Proof Let $\sigma : B \to X'$ be a section of h and $H\colon U \times I \to Z$ be a homotopy with $H_1 = g_{|U}$ and $H_0(U) \subset B$. The outer square in the diagram

$$\begin{tikzcd} U \arrow[r, "\sigma\circ H_0"] \arrow[d, "j_0"'] & X' \arrow[d, "h"] \\ U \times I \arrow[r, "H"'] \arrow[ur, dashed, "G"'] & Z \end{tikzcd}$$

commutes and, since h is a fibration, there is a homotopy $G\colon U \times I \to X'$ that renders the complete diagram commutative. Then the commutative diagram

$$\begin{tikzcd} U \arrow[drr, bend left, "G_1"] \arrow[ddr, bend right] & & \\ & X \arrow[r, "\varphi"] \arrow[d, "f"] & X' \arrow[d, "h"] \\ & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

and the quasi pullback hypothesis yield a section $s\colon U \to X$ of f. □

Taking into account (3.4) we then get:

Corollary 3.8 *Under the conditions in Proposition 3.7, sec*(f) = *qscat*$(g : Y \to (Z, B))$ *provided qscat*$(h : X' \to (Z, B)) = 1$.

Corollary 3.8 implies [4, Proposition 9.18, p. 261] as shown in the next example.

Example 3.9 *Assume (3.3) is a quasi pullback with Z path-connected and $h : X' \to Z$ a null-homotopic fibration. Fix $z_0 \in Z$ and set $B = \{z_0\}$. Then* $\mathrm{qscat}(h : X' \to (Z, B)) = \mathrm{cat}(h : X' \to Z) = 1$, *so that* $\sec(f) = \mathrm{qscat}(g : Y \to (Z, B)) = \mathrm{cat}(g : Y \to Z)$.

We next introduce the two central characters in this subsection.

(A) A notion of higher topological complexity of a map has been introduced in [15] by Rudyak and Soumen as follows. For $r \geq 2$ and a map $f : X \to Y$, the r-higher topological complexity of f (à la Rudyak-Soumen), which we denote $\mathrm{TC}_r^{RS}(f)$, is given by

$$\mathrm{TC}_r^{RS}(f) = \mathrm{qscat}\left(f^r : X^r \to (Y^r, \Delta_r(Y))\right),$$

that is, the least integer n such that X^r can be covered by n open subsets $\{U_i\}_{i=1}^n$ on each of which there is a homotopy $H_i : U_i \times [0, 1] \to Y^r$ satisfying $(H_i)_0 = f^r|_{U_i}$ and $(H_i)_1(U_i) \subseteq \Delta_r(Y)$, where

$$\Delta_r(Y) = \{(y, \ldots, y) \in Y^r : y \in Y\}$$

is the diagonal. More generally, for $2 \leq s \leq r$, set $\Delta_{r,s}(Y) = Y^{r-s} \times \Delta_s(Y)$, and define the (r, s)-th quasistrong higher topological complexity of f, denoted by $\mathrm{qsTC}_{r,s}(f)$, as the least integer n such that X^r can be covered by n open subsets $\{U_i\}_{i=1}^n$ on each of which there is a homotopy $H_i : U_i \times [0, 1] \to Y^r$ satisfying $(H_i)_0 = f^r|_{U_i}$ and $(H_i)_1(U_i) \subseteq \Delta_{r,s}(Y)$, that is,

$$\mathrm{qsTC}_{r,s}(f) = \mathrm{qscat}\left(f^r : X^r \to (Y^r, \Delta_{r,s}(Y))\right).$$

Note that $\mathrm{qsTC}_{r,r}(f) = \mathrm{TC}_r^{RS}(f)$ and $\mathrm{qsTC}_{r,s'}(f) \leq \mathrm{qsTC}_{r,s}(f)$ for any $2 \leq s' \leq s \leq r$.

(B) Here is a natural generalization of Murillo and Wu's complexity reviewed at the end of Section 2. For a map $f : X \to Y$ and $1 \leq s \leq r \geq 2$, consider the diagram

$$\begin{tikzcd} X^{J_r} \arrow[d, "e_r^X"'] & \\ X^r \arrow[r, "1_{X^{r-s}} \times f^s"'] & X^{r-s} \times Y^s. \end{tikzcd}$$

The (r, s)-higher topological complexity of f (à la Murillo-Wu), which we denote by $\mathrm{TC}_{r,s}^{MW}(f)$, is given by

$$\mathrm{TC}_{r,s}^{MW}(f) = \mathrm{secat}^{1_{X^{r-s}} \times f^s}\left(e_r^X\right),$$

i.e. the least integer n such that X^r can be covered by n open sets $\{U_i\}_{i=1}^n$ on each of which there is a map $s_i : U_i \to X^{J_r}$ satisfying $(1_{X^{r-s}} \times f^s) \circ e_r^X \circ s_i \simeq (1_{X^{r-s}} \times f^s)_{|U_i}$. Note that $\mathrm{TC}_{r,s}^{MW}(f)$ coincides with the least integer n such that X^r can be covered by n open sets $\{U_i\}_{i=1}^n$ on each of which there is a map $s_i : U_i \to X$ satisfying

$$(1_{X^{r-s}} \times f^s) \circ \Delta_r^X \circ s_i \simeq (1_{X^{r-s}} \times f^s)_{|U_i},$$

where $\Delta_r^X : X \to X^r$, $x \mapsto (x, \ldots, x)$. Likewise, the naive (r,s)-higher topological complexity of f (à la Murillo-Wu), which we denote by $\mathrm{tc}_{r,s}^{MW}(f)$, is defined analogously, now requiring each of the maps $s_i : U_i \to PX$ to satisfy the stronger condition $(1_{X^{r-s}} \times f^s) \circ e_r^X \circ s_i = (1_{X^{r-s}} \times f^s)_{|U_i}$. In other words,

$$\mathrm{tc}_{r,s}^{MW}(f) = \sec^{1_{X^{r-s}} \times f^s}(e_r^X).$$

Note that the inequality $\mathrm{TC}_{r,s}^{MW}(f) \leq \mathrm{tc}_{r,s}^{MW}(f)$ holds for any map f, while in fact $\mathrm{TC}_{r,s}^{MW}(f) = \mathrm{tc}_{r,s}^{MW}(f)$ when f is a fibration. We will write $\mathrm{TC}_r^{MW}(f) = \mathrm{TC}_{r,r}^{MW}(f)$ and $\mathrm{tc}_r^{MW}(f) = \mathrm{tc}_{r,r}^{MW}(f)$. Of course

$$\mathrm{TC}_2^{MW}(f) = \mathrm{TC}^{MW}(f),$$

the Murillo-Wu's complexity.

The following statement generalizes [17, Theorem 3.4] and solves on the positive the question raised in [15] by Rudyak-Soumen regarding their inequality (3.6).

Proposition 3.10 *For $r \geq 2$ and a map $f : X \to Y$, we have*

$$\mathrm{TC}_r^{RS}(f) = \mathrm{TC}_r^{MW}(f) = \sec_{f^r}(e_r^Y).$$

Proof For $U \subset X^r$ and $\sigma : U \to Y^{J_r}$ satisfying $e_r^Y \circ \sigma = (f^r)_{|U}$, consider the homotopy $H : U \times [0,1] \to Y^r$ given by

$$H(x,t) = \left(\overline{\sigma_1(x)} \cdot \sigma_r(x)(t), \ldots, \overline{\sigma_{r-1}(x)} \cdot \sigma_r(x)(t), f(x_r)\right).$$

Here, for each $x = (x_1, \ldots, x_r) \in U$, $\sigma(x) = (\sigma_1(x), \ldots, \sigma_r(x))$ is an ordered r-multipath in Y—see (3.1). Recall that $\overline{\alpha}(t) = \alpha(1-t)$ is the path α traversed in opposite direction, and that $\alpha \cdot \beta$ stands for the concatenation of α and β. Note that $H_0 = (f^r)_{|U}$ and $H_1(U) \subset \Delta_r(Y)$. This yields $\sec_{f^r}(e_r^Y) \geq \mathrm{TC}_r^{RS}(f)$.

We next argue the inequality $\mathrm{TC}_r^{RS}(f) \geq \mathrm{TC}_r^{MW}(f)$. For $U \subset X^r$ and a homotopy $H : U \times [0,1] \to Y^r$ satisfying $H_0 = (f^r)_{|U}$ and $H_1(U) \subset \Delta_r(Y)$, set

$$\alpha_j(x)(t) := p_j(H(x,t))$$

for each $j = 1, \ldots, r$, $x \in U$ and $t \in [0,1]$, where $p_j : Y^r \to Y$ is the projection to the j-th coordinate. Then the homotopy $G : U \times [0,1] \to Y^r$ given by

$$G(x,t) = \left(\alpha_1(x) \cdot \overline{\alpha_1(x)}(t), \alpha_2(x) \cdot \overline{\alpha_1(x)}(t), \ldots, \alpha_r(x) \cdot \overline{\alpha_1(x)}(t)\right)$$

satisfies $G_0 = (f^r)_{|U}$ and $G_1 = f^r \circ \Delta_r^X \circ \pi_1$, where $\pi_1(x_1, \ldots, x_r) = x_1$. This yields the asserted inequality.

We complete the proof by showing the inequality $\mathrm{TC}_r^{MW}(f) \geq \mathrm{sec}_{f^r}(e_r^Y)$. For $U \subset X^r$ and $s : U \to X$ satisfying $f^r \circ \Delta_r^X \circ s \simeq (f^r)_{|U}$, consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "c_f"] \arrow[d, "\Delta_r^X"'] & Y^{J_r} \arrow[d, "e_r^Y"] \\ X^r \arrow[r, "f^r"'] & Y^r, \end{tikzcd}$$

where $c_f : X \to Y^{J_r}$ is given so that $c_f(x) = \overline{f(x)}$, the constant map. Then the map $\sigma : U \to Y^{J_r}$ given by $\sigma = c_f \circ s$ defines a homotopy lift of $(f^r)_{|U}$ through e_r^Y. The result follows since e_r^Y is a fibration. □

3.2. The $\mathrm{TC}_{r,s}$ input

We start by comparing the generalized Murillo-Rudyak-Soumen-Wu complexity $\mathrm{sec}_{f^r}(e_r^Y)$ to $\mathrm{HTC}_{r,r}(f)$.

Proposition 3.11 *For $r \geq 2$ and a map $f : X \to Y$, we have:*

1. $\mathrm{sec}_{f^r}(e_r^Y) \leq \mathrm{HTC}_{r,r}(f) \leq \mathrm{TC}_{r,r}(f)$.

2. *If f admits a section, then* $\mathrm{sec}_{f^r}(e_r^Y) = \mathrm{HTC}_{r,r}(f) = \mathrm{TC}_{r,r}(f)$.

Proof (1) Choose $U \subset Y^r$ and $s : U \to X^{J_r}$ satisfying $f^r \circ e_r^X \circ s \simeq incl_U$, and consider $V = (f^r)^{-1}(U) \subset X^r$. Then the map $\sigma : V \to Y^{J_r}$ given by $\sigma = f_\# \circ s \circ (f^r)_{|V}$ defines a homotopy lift of $(f^r)_{|V}$ through e_r^Y. This yields the inequality $\mathrm{sec}_{f^r}(e_r^Y) \leq \mathrm{HTC}_{r,r}(f)$; therefore, the proof is complete in view of item (2) in Remark 3.4.

(2) It suffices to show the inequality $\mathrm{TC}_{r,r}(f) \leq \mathrm{sec}_{f^r}(e_r^Y)$ assuming that $s : Y \to X$ is a section of f. Let U be an open subset of X^r, $\sigma : U \to Y^{J_r}$ be a lifting of $(f^r)_{|U}$ through e_r^Y, and consider $V = (s^r)^{-1}(U) \subset Y^r$. Then the map $\rho : V \to X^{J_r}$ given by $\rho = s_\# \circ \sigma \circ (s^r)_{|U}$ defines a local section of $e_{r,r}^f = (f^r) \circ e_r^X$, which yields the asserted inequality. □

Propositions 3.10 and 3.11 immediately yield:

Corollary 3.12 *For $r \geq 2$ and a map $f : X \to Y$ which admits a section, we have*

$$\mathrm{TC}_r^{RS}(f) = \mathrm{TC}_r^{MW}(f) = \mathrm{sec}_{f^r}(e_r^Y) = \mathrm{HTC}_{r,r}(f) = \mathrm{TC}_{r,r}(f).$$

Next we establish general estimates involving our $\mathrm{TC}_{r,s}(-)$ and Rudyak-Soumen's $\mathrm{qsTC}_{r,s}(-)$.

Proposition 3.13 *For any $2 \leq s \leq r$ and any commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "f"'] & W \arrow[d, "h"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

we have

1. $\mathrm{HTC}_{r,s}(f) \cdot \mathrm{secat}\left(f^{r-s}\right) \cdot \mathrm{qsTC}_{r,s}(h) \geq \mathrm{qsTC}_{r,s}(g)$.

2. $\mathrm{HTC}_{r,s}(f) \cdot \mathrm{TC}_s^{RS}(h) \geq \mathrm{secat}(f^s) \cdot \mathrm{TC}_s^{RS}(h) \geq \mathrm{TC}_s^{RS}(g)$.

In particular, $\mathrm{TC}_{r,r}(f) \cdot \mathrm{TC}_r^{RS}(h) \geq \mathrm{HTC}_{r,r}(f) \cdot \mathrm{TC}_r^{RS}(h) \geq \mathrm{TC}_r^{RS}(g)$.

Proof For (1), consider open sets U, A and V, and maps σ, ρ and H satisfying

(i) $U \subset X^{r-s} \times Y^s$, $\sigma : U \to X^{J_r}$ with $e^f_{r,s} \circ \sigma \simeq incl_U$;

(ii) $A \subset Y^r$, $\rho : A \to X^{r-s} \times Y^s$ with $(f^{r-s} \times 1_{Y^s}) \circ \rho \simeq incl_A$;

(iii) $V \subset W^r$, $H : V \times [0,1] \to Z^r$ with $H_0 = h^r|_V$ and $H_1(V) \subset \Delta_{r,s}(Z)$.

(In (ii) we are using the equality $\sec(f^{r-s}) = \sec(f^{r-s} \times 1_{Y^s})$ coming from item (8) of Lemma 2.2.) Consider also the diagram

$$\begin{tikzcd}
 & & & \widetilde{V} \arrow[d, hook] \arrow[rrr, bend left] & & & \\
 & & & X^{J_r} \arrow[dr, "e_r^X"] \arrow[dd, "e^f_{r,s}"'] & & & V \arrow[dl, hook'] \arrow[ddl, "H_0"] \arrow[ddd, "H_1"] \\
 & & & & X^r \arrow[r, "\varphi^r"] \arrow[dl, "1\times f^s"'] \arrow[dddl, "f^r"] & W^r \arrow[d, "h^r"] & \\
 & & & X^{r-s}\times Y^s \arrow[dd, "f^{r-s}\times 1"] & & Z^r & \\
 & & U \arrow[uuur, bend left, "\sigma"] \arrow[ur, hook] & & & & \Delta_{r,s}(Z) \arrow[ul, hook'] \\
 & & & Y^r \arrow[uurr, "g^r"'] & & & \\
 & & A \arrow[uuur, bend left, "\rho"] \arrow[ur, hook] & & & & \\
\widehat{A} \arrow[r, hook] \arrow[uuuuuuurrr, bend left] & \widetilde{A}, \arrow[uuur, bend left, "\rho_|"] \arrow[ur, hook] & & & & &
\end{tikzcd}$$

where $\widetilde{A} = \rho^{-1}(U)$, $\widetilde{V} = \left(\varphi^r \circ e_r^X\right)^{-1}(V)$ and $\widehat{A} = \left(\sigma \circ \rho_|\right)^{-1}(\widetilde{V})$. All regions of the diagram are strictly commutative, except for the three homotopy commutative triangles involving the homotopies in (i), (ii), and (iii). Note that the sets $\widetilde{A}$, $\widetilde{V}$ and $\widehat{A}$ can be empty but, when $\widehat{A} \neq \varnothing$, we can take the homotopy $G : \widehat{A} \times [0,1] \to Z^r$ given by

$$G(y,t) = H\left(\varphi^r \circ e_r^X \circ \sigma \circ \rho(y), t\right).$$

Then $G_0 \simeq g^r|_{\widehat{A}}$ and $G_1(\widehat{A}) \subset \Delta_{r,s}(Z)$. The asserted inequality (1) then follows by observing that, as the sets U, A and V vary over suitable coverings, the resulting sets $\widehat{A}$ cover Y^r.

Regarding (2), the inequality $\mathrm{HTC}_{r,s}(f) \geq \mathrm{secat}(f^s)$ is obvious, and thus, we focus on the second inequality of (2). Consider open sets A and V and maps ρ and H satisfying:

(iv) $A \subset Y^s$, $\rho : A \to X^s$ with $f^s \circ \rho \simeq incl_A$;

(v) $V \subset W^s$, $H : V \times [0,1] \to Z^s$ with $H_0 = h^s|_V$ and $H_1(V) \subset \Delta_s(Z)$.

Consider also the diagram

$$\begin{tikzcd}
& & \widetilde{V} \arrow[rr] \arrow[d, hook] & & V \arrow[dl] \arrow[ddl, "H_0"] \arrow[ddd, "H_1"] \\
& & X^s \arrow[r, "\varphi^s"] \arrow[d, "f^s"] & W^s \arrow[d, "h^s"'] & \\
& & Y^s \arrow[r, "g^s"] & Z^s & \\
\widehat{A} \arrow[r, hook] \arrow[uuurr, bend left] & A \arrow[ur] \arrow[uur, bend left, "\rho"] & & & \Delta_s(Z) \arrow[uul, hook']
\end{tikzcd}$$

where $\widetilde{V} = (\varphi^s)^{-1}(V)$ and $\widehat{A} = \rho^{-1}(\widetilde{V})$. All regions of the diagram are strictly commutative, except for the two homotopy commutative triangles involving the homotopies in (iv) and (v). Note that the sets $\widetilde{V}$ and $\widehat{A}$ can be empty but, when $\widehat{A} \neq \varnothing$, we can take the homotopy $G : \widehat{A} \times [0,1] \to Z^s$ given by

$$G(y,t) = H\left(\varphi^s \circ \rho(y), t\right).$$

Then $G_0 \simeq g^s|_{\widehat{A}}$ and $G_1(\hat{A}) \subset \Delta_s(Z)$. The second inequality in (2) now follows by observing that, as the sets A and V vary over suitable coverings, the resulting sets $\widehat{A}$ cover Y^s. □

3.3. Products

The following result was proved in [16, Proposition 22, p. 84]. It will be used in the proof of Proposition 3.15. Here we agree that a normal space is, by definition, required to be Hausdorff.

Lemma 3.14 *Let $f \times f' : X \times X' \to Y \times Y'$ be the product of two maps $f : X \to Y$ and $f' : X' \to Y'$. If $Y \times Y'$ is normal, then*

$$sec(f \times f') \leq sec(f) + sec(f') - 1.$$

In [1, Proposition 3.11] the authors obtained the subadditivity of TC_r under suitable topological hypothesis. The corresponding property for higher topological complexity of maps is given next.

Proposition 3.15 *Let $f : X \to Y$ and $f' : X' \to Y'$ be two maps. If the cartesian product $(X \times X')^{r-s} \times (Y \times Y')^s$ is normal, then*

$$TC_{r,s}(f \times f') \leq TC_{r,s}(f) + TC_{r,s}(f') - 1.$$

Proof The proof proceeds by analogy with the proof of [1, Proposition 3.11]. Indeed, consider the commutative diagram with horizontal homeomorphisms

$$\begin{array}{ccc} (X\times X')^{J_r} & \xrightarrow{\qquad\varphi\qquad} & X^{J_r}\times X'^{J_r} \\ \Big\downarrow{\scriptstyle e^{f\times f'}_{r,s}} & & \Big\downarrow{\scriptstyle e^{f}_{r,s}\times e^{f'}_{r,s}} \\ (X\times X')^{r-s}\times(Y\times Y')^{s} & \xrightarrow[\psi]{} & (X^{r-s}\times Y^{s})\times\left(X'^{r-s}\times Y'^{s}\right). \end{array}$$

Here $\varphi\left(\gamma: J_r\to X\times X'\right) := \left(p_X\circ\gamma: J_r\to X,\ p_{X'}\circ\gamma: J_r\to X'\right)$, while

$$\psi\left((x_1,x'_1),\dots,((x_{r-s},x'_{r-s}),(y_1,y'_1),\dots,(y_s,y'_s)\right)$$
$$:= ((x_1,\dots,x_{r-s},y_1,\dots,y_s),(x'_1,\dots,x'_{r-s},y'_1\dots,y'_s)),$$

where $x_i\in X$, $y_i\in Y$, $x'_i\in X'$ and $y'_i\in Y'$, and where p_X and $p_{X'}$ are the obvious projections. The desired conclusion then follows from Lemma 3.14. □

3.4. Effect of pre- and postcomposition

We study the effect on the higher topological complexity of maps under pre- and postcomposition.

Lemma 3.16 *Consider the commutative diagram*

$$\begin{array}{ccccc} X' & & X & \xrightarrow{\varphi} & X' \\ \Big\downarrow{\scriptstyle f'} & & {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle f'} \\ Y' & \xrightarrow[\xi]{} & Y & \xrightarrow[\psi]{} & Y'. \end{array}$$

1. *If $\psi\circ\xi\simeq 1_{Y'}$ then $secat(f)\geq secat(f')$.*
2. *If $\psi\circ\xi = 1_{Y'}$ then $sec(f)\geq sec(f')$ (and, of course, $secat(f)\geq secat(f')$).*

Proof Suppose $U\subset Y$ and take $V=\xi^{-1}(U)\subset Y'$. Note that a map $\sigma: U\to X$ yields a map $\delta=\left(V\stackrel{\xi}{\to}U\stackrel{\sigma}{\to}X\stackrel{\varphi}{\to}X'\right)$. If $\psi\circ\xi=1_{Y'}$ ($\psi\circ\xi\simeq 1_{Y'}$, respectively) and $f\circ\sigma=incl_U$ ($f\circ\sigma\simeq incl_U$, respectively), then $f'\circ\delta=incl_V$ ($f'\circ\delta\simeq incl_V$, respectively). □

Proposition 3.17 *Consider the diagram of maps $W\stackrel{h}{\to}X\stackrel{f}{\to}Y\stackrel{g}{\to}Z$.*

(a) If f admits a section (homotopy section, respectively), then

$$TC_{r,s}(f\circ h)\leq TC_{r,s}(h)\ \left(HTC_{r,s}(f\circ h)\leq HTC_{r,s}(h),\ \textit{respectively}\right),\ \textit{for any } s\leq r.$$

(b) If f admits a homotopy section, then

$$HTC_{r,s}(g)\leq HTC_{r,s}(g\circ f),\ \textit{for any } s\leq r; \tag{3.5}$$

$$TC_{r,s}(g)\leq TC_{r,s}(g\circ f),\ \textit{for any } s< r. \tag{3.6}$$

In particular, if f admits a section and $s \leq r$, we get

$$TC_r(Y) \leq HTC_{r,s}(f) \leq TC_{r,s}(f) \leq TC_r(X).$$

Proof We use the sequential setting. Item (a) follows Lemma 3.16 applied to the commutative diagram

$$\begin{tikzcd} PW \arrow[d, "e^{f\circ h}_{r,s}"] & PW \arrow[r, "1"] \arrow[d, "e^{h}_{r,s}"'] & PW \arrow[d, "e^{f\circ h}_{r,s}"] \\ W^{r-s}\times Y^s \arrow[r, "1\times\xi^s"'] & W^{r-s}\times X^s \arrow[r, "1\times f^s"'] & W^{r-s}\times Y^s, \end{tikzcd}$$

where $\xi\colon Y \to X$ is either a section or a homotopy section of f. On the other hand, for item (b), assume only that $\xi : Y \to X$ is a homotopy section to f, and consider the commutative diagram

$$\begin{tikzcd} PY \arrow[d, "e^{g}_{r,s}"] & PX \arrow[r, "f_\#"] \arrow[d, "e^{g\circ f}_{r,s}"'] & PY \arrow[d, "e^{g}_{r,s}"] \\ Y^{r-s}\times Z^s \arrow[r, "\xi^{r-s}\times 1"'] & X^{r-s}\times Z^s \arrow[r, "f^{r-s}\times 1"'] & Y^{r-s}\times Z^s. \end{tikzcd}$$

Since (3.5) follows also from Lemma 3.16, we will focus on (3.6) assuming $s < r$ (in addition to $f\circ\xi \simeq 1_Y$).

Choose a homotopy $H : f\circ\xi \simeq 1_Y$ and suppose we are given an open set $U \subset X^{r-s}\times Z^s$ admitting a local section $\sigma : U \to PX$ of $e^{g\circ f}_{r,s}$. It is then elementary to check that a local section δ of $e^{g}_{r,s}$ on $V := (\xi^{r-s}\times 1_{Z^s})^{-1}(U)$ is given, in terms of concatenation of paths, by the formula

$$\begin{aligned}\delta(v) = &\Big(\overline{H(y_1,-)}\cdot (f\circ\sigma((\xi(y_1),\ldots,\xi(y_{r-s}),z_{r-z+1},\ldots,z_r))\mid_1)\cdot H(y_2,-)\Big)\cdot\\ &\Big(\overline{H(y_2,-)}\cdot (f\circ\sigma((\xi(y_1),\ldots,\xi(y_{r-s}),z_{r-z+1},\ldots,z_r))\mid_2)\cdot H(y_3,-)\Big)\cdot\cdots\cdot\\ &\Big(\overline{H(y_{r-s-1},-)}\cdot (f\circ\sigma((\xi(y_1),\ldots,\xi(y_{r-s}),z_{r-z+1},\ldots,z_r))\mid_{r-s-1})\cdot H(y_{r-s},-)\Big)\cdot\\ &\Big(\overline{H(y_{r-s},-)}\cdot (f\circ\sigma((\xi(y_1),\ldots,\xi(y_{r-s}),z_{r-z+1},\ldots,z_r))\mid_{r-s})\Big)\cdot\\ &\Big(f\circ\sigma((\xi(y_1),\ldots,\xi(y_{r-s}),z_{r-z+1},\ldots,z_r))\mid_{r-s+1}\Big)\cdot\cdots\cdot\\ &\Big(f\circ\sigma((\xi(y_1),\ldots,\xi(y_{r-s}),z_{r-z+1},\ldots,z_r))\mid_{r-1}\Big),\end{aligned}$$

for any $v = (y_1,\ldots,y_{r-s},z_{r-z+1},\ldots,z_r)\in V$. Here, $\overline{\tau}$ is the path τ traversed backwards (see Remark 3.2). Furthermore, $\sigma((\xi(y_1),\ldots,\xi(y_{r-s}),z_{r-z+1},\ldots,z_r))\mid_j$ stands for the restriction of $\sigma((\xi(y_1),\ldots,\xi(y_{r-s}),z_{r-z+1},\ldots,z_r))$ to the segment

$$\left[\frac{j-1}{r-1},\frac{j}{r-1}\right],$$

i.e. $\sigma((\xi(y_1),\ldots,\xi(y_{r-s}),z_{r-z+1},\ldots,z_r))\mid_j(t)$ is given by the formula

$$\sigma((\xi(y_1),\ldots,\xi(y_{r-s}),z_{r-z+1},\ldots,z_r))\left(\frac{t+j-1}{r-1}\right),\quad t\in[0,1],$$

for $j = 1, \ldots, r-1$. Since the sets V cover $Y^{r-s} \times Z^s$ as the sets U cover $X^{r-s} \times Z^s$, we get the desired inequality $\mathrm{TC}_{r,s}(g) \leq \mathrm{TC}_{r,s}(g \circ f)$. $\square$

Remark 3.18 *It is highly illuminating to take a look back to item (b) of Propostion 3.17 and its proof. For starters, it should be stressed that (3.6) involves the* strong *form of the higher TC, even though the hypothesis on f has a homotopy nature. Such a phenomenon works because of the additional hypothesis $s < r$. Indeed, the first four lines in the definition of $\delta(v)$ allow us to incorporate the homotopy H into a pullback-type construction (involving the homotopy section ξ) of the strict section δ out of the strict section σ. Of course, such a trick would not be need if f* had *a strict section, as then (3.6) would be true for any $s \leq r$ (using the "same" argument that proves (3.5)). But then, it is more striking to remark that (3.5) and (3.6) actually have stronger forms when $s = r$, as spelled out next.*

Proposition 3.19 *Let $f : X \to Y$ and $g : Y \to Z$ be maps.*

1. *Independently of whether f admits a (homotopy) section, we have*

$$TC_{r,r}(g) \leq TC_{r,r}(g \circ f) \quad \text{and} \quad HTC_{r,r}(g) \leq HTC_{r,r}(g \circ f).$$

   In particular, $TC_r(Y) \leq HTC_{r,r}(f) \leq TC_{r,r}(f)$.

2. *If f admits a section (homotopy section, respectively), then*

$$TC_{r,r}(g) = TC_{r,r}(g \circ f) \quad (HTC_{r,r}(g) = HTC_{r,r}(g \circ f), \text{respectively}).$$

   In particular $TC_r(Y) = TC_{r,r}(f)$ $\quad$ ($TC_r(Y) = HTC_{r,r}(f)$, respectively).

Proof Working again in the sequential context, item (1) follows immediately by applying Lemma 3.16 to the diagram

$$\begin{tikzcd}
PY \arrow[d, "e^g_{r,r}"] & PX \arrow[r, "f_\#"] \arrow[d, "e^{g\circ f}_{r,r}"'] & PY \arrow[d, "e^g_{r,r}"] \\
Z^r \arrow[r, "1_{Z^r}"'] & Z^r \arrow[r, "1_{Z^r}"'] & Z^r.
\end{tikzcd}$$

Moreover, if f admits a section $\sigma : Y \to X$, then $\mathrm{TC}_{r,r}(g \circ f) \leq \mathrm{TC}_{r,r}(g \circ f \circ \sigma) = \mathrm{TC}_{r,r}(g)$, so in fact $\mathrm{TC}_{r,r}(g) = \mathrm{TC}_{r,r}(g \circ f)$. Likewise, if $\sigma : Y \to X$ is a homotopy section of f, then $\mathrm{HTC}_{r,r}(g \circ f) \leq \mathrm{HTC}_{r,r}(g \circ f \circ \sigma) = \mathrm{HTC}_{r,r}(g)$, so in fact $\mathrm{HTC}_{r,r}(g) = \mathrm{HTC}_{r,r}(g \circ f)$. $\square$

The facts we have discussed in this subsection have a number of interesting corollaries. First, we deduce the following important invariance property, which states that the complexity of the map is not altered by a deformation retraction of the domain.

Corollary 3.20 *If $\rho : X' \to X$ is a deformation retraction, then for any $f : X \to Y$ and any $s \leq r$ we have*

$$TC_{r,s}(f) = TC_{r,s}(f \circ \rho) \quad \text{and} \quad HTC_{r,s}(f) = HTC_{r,s}(f \circ \rho).$$

Proof Let $i : X \hookrightarrow X'$ be the inclusion map, so that $\rho \circ i = 1_X$ and $i \circ \rho \simeq 1_{X'}$. Because ρ admits a section, the case $s = r$ follows from Proposition 3.19. Therefore, we assume $s < r$. Item (b) of Proposition 3.17 implies $\mathrm{TC}_{r,s}(f) \leq \mathrm{TC}_{r,s}(f \circ \rho)$ on the nose, as well as $\mathrm{TC}_{r,s}(f \circ \rho) \leq \mathrm{TC}_{r,s}(f)$, since $(f \circ \rho) \circ i = f$. Similarly, we get the equality $\mathrm{HTC}_{r,s}(f) = \mathrm{HTC}_{r,s}(f \circ \rho)$. $\square$

The following fact (written in the sequential setting) is analogous to [20, Lemma 4.6].

Lemma 3.21 *If $f : X \to Y$ is a fibration and $f' : Y \to Y'$ is a map, then we have the quasi pullback diagram*

$$\begin{tikzcd} PX \arrow[r, "f_\#"] \arrow[d, "e^{f'\circ f}_{r,r-1}"'] & PY \arrow[d, "e^{f'}_{r,r-1}"] \\ X \times Y'^{r-1} \arrow[r, "f \times 1_{Y'^{r-1}}"'] & Y \times Y'^{r-1}. \end{tikzcd}$$

Proof Choose maps β and α that render the commutative diagram

$$\begin{tikzcd} Z \arrow[dr, dashed, "H"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\ & PX \arrow[r, "f_\#"] \arrow[d, "e^{f'\circ f}_{r,r-1}"] & PY \arrow[d, "e^{f'}_{r,r-1}"] \\ & X \times Y'^{r-1} \arrow[r, "f \times 1_{Y'^{r-1}}"'] & Y \times Y'^{r-1}, \end{tikzcd}$$

when the dashed map H is ignored. We need to construct a map H that still fits in the commutative diagram.

Consider the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "p_1 \circ \alpha"] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ Z \times I \arrow[r, "\widehat{\beta}"'] & Y, \end{tikzcd}$$

where p_1 is the projection onto the first coordinate and $\widehat{\beta} : Z \times I \to Y$ is given by $\widehat{\beta}(z,t) = \beta(z)(t)$. Because f is a fibration, there exists $G : Z \times I \to X$ rendering the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "p_1 \circ \alpha"] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ Z \times I \arrow[r, "\hat{\beta}"'] \arrow[ur, "G"'] & Y. \end{tikzcd}$$

It is elementary to check that the map $H : Z \to PX$ given by $H(z)(t) = G(z,t)$ has the required property. $\square$

Just as Proposition 3.19 specializes to $s = r$, the next result specializes to $s = r - 1$ providing a generalization of [20, Proposition 4.7]. The proof follows directly from item (1) of Lemma 2.2 and Lemma 3.21.

Corollary 3.22 *If $f : X \to Y$ is a fibration, then $TC_{r,r-1}(f' \circ f) \leq TC_{r,r-1}(f')$ for any map $f' : Y \to Y'$. In particular, $TC_{r,r-1}(f) \leq TC_r(Y)$.*

In turn, Proposition 3.17 and Corollary 3.22 can be combined to deduce the following important property, which states that the $(r, r-1)$-complexity of a fibration admiting a homotopy section depends only of the complexity of its codomain.

Corollary 3.23 *If $f : X \to Y$ is a fibration that admits a homotopy section, then*

$$TC_{r,r-1}(f) = TC_r(Y).$$

Example 3.24 *For the projection $p_X : X \times F \to X$ we have $TC_{r,r-1}(p_X) = TC_r(X)$.*

Item (2) of Proposition 3.19 together with Corollaries 3.12 and 3.23 yield the following omnibus statement, which comprises the fact that, for large values of s, $\mathrm{TC}_{r,s}$ unifies previous notions of topological complexity.

Corollary 3.25 *If $f : X \to Y$ is a fibration that admits a homotopy section, then for any $r \geq 2$, we have*

$$\mathrm{TC}_r^{RS}(f) = \mathrm{TC}_r^{MW}(f) = \mathrm{sec}_{f^r}(e_r^Y) = \mathrm{HTC}_{r,r}(f) = \mathrm{TC}_{r,r}(f) = TC_{r,r-1}(f) = TC_r(Y).$$

3.5. Homotopy invariance

Recall that two maps $f : X \to Y$ and $f' : X' \to Y$ are said to be *fibre homotopy equivalent* (or FHE-equivalent) if there are commutative diagrams of the form

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[rr, "\varphi"] \arrow[dr, "f'"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

and the maps $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the respective identity map by fibre preserving homotopies.

In [13, Corollary 3.9] the author proved the FHE-invariance of $\mathrm{TC}(f)$. A generalization of the corresponding property for the higher case $\mathrm{TC}_{r,s}(f)$ is given next.

Proposition 3.26 *Given $f : X \to Y$ and $f' : X' \to Y$, assume that there exist fibrewise maps $\psi : X \to X'$ and $\varphi : X' \to X$ that are homotopy inverses of each other. Then*

$$TC_{r,s}(f) = TC_{r,s}(f') \quad \textit{and} \quad HTC_{r,s}(f) = HTC_{r,s}(f'),$$

for any $s \leq r$. In particular, the (r,s)-higher topological complexity is a FHE-invariant.

Proof By Proposition 3.17 and Proposition 3.19 we have

$$\mathrm{TC}_{r,s}(f) \;=\; \mathrm{TC}_{r,s}(f' \circ \psi) \geq \mathrm{TC}_{r,s}(f') = \mathrm{TC}_{r,s}(f \circ \varphi) \geq \mathrm{TC}_{r,s}(f),$$

so $\mathrm{TC}_{r,s}(f) = \mathrm{TC}_{r,s}(f')$. Similarly, we get the equality $\mathrm{HTC}_{r,s}(f) = \mathrm{HTC}_{r,s}(f')$. □

On the other hand, from item (3) of Lemma 2.2 we see that the homotopy higher topological complexity is a homotopy invariant:

Proposition 3.27 *If $f \simeq g$ then $HTC_{r,s}(f) = HTC_{r,s}(g)$, for any $s \leq r$.*

3.6. Upper bounds

In [13, Theorem 3.17], as corrected in version 2 of the Arxiv version, Pavešić presents an upper bound of $\mathrm{TC}(f)$ for any map f. We next generalize such a fact by giving an upper estimate for the $(2s,s)$-higher topological complexity of any map f.

Proposition 3.28 *Let $f : X \to Y$ be a map with X path-connected and $X^s \times Y^s$ normal. We have*

$$TC_{2s,s}(f) \leq cat(X^s) + cat(X^s) \cdot sec(f^s) - 1. \tag{3.7}$$

Proof Fix $x_0 \in X$ and let U be an open subset of X^s so that there exists a homotopy $H : U \times [0,1] \to X^s$ from the inclusion $U \hookrightarrow X^s$ to the constant map to $(x_0,\dots,x_0) \in X^s$. Assume also that $s : V \to X^s$ is a local section of f^s on an open subset V of Y^s. The map $K : (V \cap s^{-1}(U)) \times [0,1] \to X^s$ given by $K(v,t) = H(s(v),t)$ is a homotopy from the restriction of s to the constant map to $(x_0,\dots,x_0)$. Then, in terms of concatenation of paths, the formula

$$\begin{aligned}\delta(u,v) = &\Big[(p_1 \circ H(u,-)) \cdot (p_2 \circ \overline{H(u,-)})\Big] \cdot \Big[(p_2 \circ H(u,-)) \cdot (p_3 \circ \overline{H(u,-)})\Big] \cdot \cdots \cdot \\ &\Big[(p_{s-1} \circ H(u,-)) \cdot (p_s \circ \overline{H(u,-)})\Big] \cdot \Big[(p_s \circ H(u,-)) \cdot (p_1 \circ \overline{K(v,-)})\Big] \cdot \\ &\Big[(p_1 \circ K(v,-)) \cdot (p_2 \circ \overline{K(v,-)})\Big] \cdot \cdots \cdot \Big[(p_{s-1} \circ K(v,-)) \cdot (p_s \circ \overline{K(v,-)})\Big]\end{aligned}$$

defines a local section to $e^f_{2s,s}$ over $U \times (V \cap s^{-1}(U))$. Here, $p_i \colon X^s \to X$ stands for the i-th projection and, as in Remark 3.2, $\overline{\tau}$ stands for the path τ traversed in the opposite direction. The conclusion then follows from item (7) of Lemma 2.2. □

The estimate (3.7) is sharp under special conditions:

Corollary 3.29 *Let $f : X \to Y$ be a map with X contractible and Y^s normal. Then*

$$TC_{2s,s}(f) = sec(f^s).$$

Proof Use Propositions 3.6 and Proposition 3.28. □

Relative sectional numbers $\sec^{-}(-)$ can also be used to draw estimates. Specifically, item (4) of Lemma 2.2 yields:

Proposition 3.30 *For any map $f : X \to Y$, we have*

$$TC_{r,s}(f) \leq sec(f^s) \cdot sec^{1_{X^{r-s}} \times f^s}(e^X_r).$$

Corollary 3.31 *Let $f : X \to Y$ be a map.*

1. *If f admits a section, $TC_{r,s}(f) = sec^{1_{X^{r-s}} \times f^s}(e^X_r)$.*
2. *If X is contractible, $TC_{r,s}(f) = sec(f^s)$.*

Proof Recall the lower estimate $\max\{\mathrm{sec}(f^s), \mathrm{sec}^{1_{X^{r-s}} \times f^s}(e^X_r)\} \leq \mathrm{TC}_{r,s}(f)$ in Proposition 3.6 and the upper estimate $\mathrm{sec}^{1_{X^{r-s}} \times f^s}(e^X_r) \leq \mathrm{TC}_r(X)$ coming from Lemma 2.1. □

Note that item (2) of Corollary 3.31 generalizes Corollary 3.29.

Example 3.32 *Let $f : X \to Y$ be a map. If f admits a section, then*

$$TC_r(Y) = TC_{r,r}(f) = sec^{f^r}(e_r^X).$$

The former equality follows from item (2) of Proposition 3.19.

3.7. Higher complexity of a fibration

We now obtain new estimates for $\mathrm{TC}_{r,s}(f)$ when f is a fibration. Firstly, we restate the definition of $\mathrm{TC}_{r,s}(f)$ (for $s < r$) in more geometric terms. Recall that a *deformation* of $U \subset Z$ in Z to a subset $V \subset Z$ is a map $H : U \times I \to Z$ such that $H(u, 0) = u$ and $H(u, 1) \in V$, for all $u \in U$.

Proposition 3.33 *Let $f : X \to Y$ be a fibration, and let $U \subset X^{r-s} \times Y^s$ with $s < r$. The following statements are equivalent:*

1. *There is a local section $\sigma : U \to X^{J_r}$ for $e^f_{r,s}$.*
2. *U can be deformed in $X^{r-s} \times Y^s$ to the subset*

$$\Delta_f = \{(x, \ldots, x, f(x), \ldots, f(x)) \in X^{r-s} \times Y^s : \ x \in X\}. \tag{3.8}$$

Proof $(1) \Longrightarrow (2)$. The homotopy $H : U \times [0, 1] \to X^{r-s} \times Y^s$ sending (u, t) to

$$\Big(\sigma(u)((1-t)_1), \ldots, \sigma(u)((1-t)_{r-s}), f(\sigma(u)((1-t)_{r-s+1})), \ldots, f(\sigma(u, (1-t)_r))\Big)$$

deforms U in $X^{r-s} \times Y^s$ to Δ_f. Here, for $x \in [0, 1]$, the notation x_i stands for the copy of x lying in the i-th wedge summand of $[0, 1]$ in J_r.

$(2) \Longrightarrow (1)$. Let p_i denote the projection to the i-th factor and choose a lifting function $\Gamma : E_f \to PX$ of the fibration f as in (2.1). Given a deformation $H : U \times [0, 1] \to X^{r-s} \times Y^s$ of U to Δ_f, we define a section $\sigma : U \to X^{J_r}$ for $e^f_{r,s}$ by

$$\sigma(u) = \Big(p_1 \circ \overline{H(u, -)}, \ldots, p_{r-s} \circ \overline{H(u, -)}, \Gamma(*, p_{r-s+1} \circ \overline{H(u, -)}), \ldots, \Gamma(*, p_r \circ \overline{H(u, -)})\Big),$$

where $* = p_1(H(u, 1)) = \cdots = p_{r-s}(H(u, 1))$ (here we use the hypothesis $s < r$) and $\overline{\tau}$ stands for the path τ traversed in the opposite direction. □

Corollary 3.34 *If $f : X \to Y$ is a fibration and $s < r$, then $TC_{r,s}(f)$ equals the minimal number of elements of a covering of $X^{r-s} \times Y^s$ by open sets that can be deformed in $X^{r-s} \times Y^s$ to the set in (3.8).*

Proposition 3.35 *If f is a fibration then:*

1. *$cat(X^{r-s-1} \times Y^s) \le TC_{r,s}(f) \le cat(X^{r-s} \times Y^s)$, for $s < r$.*
2. *$cat(Y^{r-1}) \le TC_{r,r-1}(f) \le \min\{TC_r(Y), cat(X \times Y^{r-1})\}$.*

3. $\max\{sec(f^r), TC_r(Y)\} \leq TC_{r,r}(f) \leq cat(Y^r)$.

Proof Because f is a fibration, the map $e^f_{r,s} : X^{J_r} \to X^{r-s} \times Y^s$ is a fibration too (see item (2) of Remark 3.4). Then, by item (5) of Lemma 2.2, we obtain $\mathrm{TC}_{r,s}(f) = \mathrm{sec}(e^f_{r,s}) \leq \mathrm{cat}\,(X^{r-s} \times Y^s)$ for any $1 \leq s \leq r$.

In addition, from Corollary 3.22, we get that $\mathrm{TC}_{r,r-1}(f) \leq \mathrm{TC}_r(Y)$. All the lower estimates follow from Proposition 3.6. □

The upper estimate $\mathrm{TC}_{r,s}(f) \leq \mathrm{cat}(X^{r-s} \times Y^s)$ for $s \leq r$ in Proposition 3.35 is sharp under special conditions (see Corollary 3.36 below). However, there is room for improvement, as it can be seen from Corollary 3.38 below and, in particular, from Remark 4.1 in the final section of the paper, where the upper estimate in item (2) of Proposition 3.35 becomes sharp due to the TC_r term.

Corollary 3.36 *Let $f : X \to Y$ be a fibration and assume that X is contractible. Then*

$$TC_{r,s}(f) = cat(Y^s) = sec(f^s), \text{ for any } s \leq r.$$

Example 3.37 *If $f : \widetilde{X} \to X$ is the universal covering of a spherical space X, then $TC_{r,s}(f) = cat(X^s) = sec(f^s)$ for $s \leq r$.*

It is well known that, if Y is a topological group, or more generally an H-space, then the r-higher topological complexity of Y coincides with $\mathrm{cat}(Y^{r-1})$. As a consequence:

Corollary 3.38 *Let $f : X \to Y$ be a fibration over an H-space Y. Then*

$$TC_{r,r-1}(f) = cat(Y^{r-1}) = TC_r(Y).$$

Example 3.39 *If $f : X \to Y$ is a fibration with a section, then (3.6), item (4) in Proposition 2.2 and item (2) in Proposition 3.35 yield $TC_{r,r-1}(f) = TC_r(Y) = sec^{1_X \times f^{r-1}}(e^X_r)$.*

Remark 3.40 *Item (2) of Proposition 3.35 together with Propositions 3.5 and 3.6 yield*

$$TC_r(Y) \leq TC_{r,r}(f) \leq TC_{r+1,r}(f) \leq TC_{r+1}(Y),$$

for any fibration $f : X \to Y$.

3.8. Cohomological lower bound

Švarc's cohomological lower bound for the sectional category of a map, a tool widely used in computations, arises as follows. A multiplicative cohomology theory h^* on the homotopy category of pairs of spaces comes equipped with a relative cohomology product

$$\cup : h^*(X, A) \otimes h^*(X, B) \to h^*(X, A \cup B)$$

whenever $A, B \subset X$ are excisive. In our case, A and B will be open sets. On the other hand, consider the index of nilpotence

$$\mathrm{nil}(S) = \min\{n : \text{ every product of } n \text{ elements in } S \text{ vanishes}\}$$

defined for a subset S of a ring R.

Lemma 3.41 **([16, Theorem 4 on p. 73])** *For any map* $f : X \to Y$ *, we have*

$$nil\,(Ker(f^* : h^*(Y) \to h^*(X))) \leq secat(f).$$

In our context:

Proposition 3.42 *For every map* $f : X \to Y$ *and for every multiplicative cohomology theory* h^**, we have*

$$nil\left(Ker((\Delta_{r-s}, {}^s f)^* : h^*(X^{r-s} \times Y^s) \to h^*(X))\right) \leq HTC_{r,s}(f),$$

where $(\Delta_{r-s}, {}^s f) : X \to X^{r-s} \times Y^s$ *is given by* $(\Delta_{r-s}, {}^s f) = (1_{X^{r-s}} \times f^s) \circ \Delta_r$*, with* $\Delta_r : X \to X^r$*,* $x \mapsto (x, \ldots, x)$*, the diagonal map.*

Proof In the sequential context, consider the commutative diagram

$$\begin{tikzcd} PX \arrow[dr, "e^f_{r,s}"'] & & X \arrow[ll, "c"'] \arrow[dl, "(\Delta_{r-s}, {}^s f)"] \\ & X^{r-s} \times Y^s, & \end{tikzcd}$$

where $c : X \to PX$ is the homotopy equivalence given by $c(x) = x$, the constant path at x. The result follows from Lemma 3.41 as $\mathrm{nil}\left(\mathrm{Ker}((e^f_{r,s})^*)\right) = \mathrm{nil}\,(\mathrm{Ker}((\Delta_{r-s}, {}^s f)^*))$. $\square$

Although Proposition 3.42 is formulated in general terms, we will mostly consider cases where the Künneth formula $h^*(X^{r-s} \times Y^s) \cong h^*(X)^{\otimes(r-s)} \otimes h^*(Y)^{\otimes s}$. In such cases, the action of $(\Delta_{r-s}, {}^s f)^*$ on tensors of factors $\alpha_1, \ldots, \alpha_{r-s} \in h^*(X)$ and $\beta_1, \ldots, \beta_s \in h^*(Y)$ is given by the product:

$$(\Delta_{r-s}, {}^s f)^*(\alpha_1 \otimes \cdots \otimes \alpha_{r-s} \otimes \beta_1 \otimes \cdots \otimes \beta_s) = \alpha_1 \cdots \alpha_{r-s} \cdot f^*(\beta_1) \cdots f^*(\beta_s).$$

In concrete cases (e.g., those worked out in Section 4 below) we do not attempt to compute the entire kernel of the homomorphism $(\Delta_{r-s}, {}^s f)^*$, but we rather look for specific elements in the kernel and try to find long nontrivial products.

4. Examples

4.1. The complexity $\mathbf{TC}_{r,s}(p_n : S^n \to \mathbb{R}P^n)$

Recall from [1, Corollary 3.12] the higher topological complexity of the n-th sphere S^n, $n \geq 1$:

$$\mathrm{TC}_r(S^n) = \begin{cases} r, & \text{if } n \text{ is odd,} \\ r+1, & \text{if } n \text{ is even.} \end{cases} \tag{4.1}$$

Consider the usual double covering map $p_n : S^n \to \mathbb{R}P^n$. Since $\mathrm{cat}(S^n) = 2$ and $\mathrm{cat}(\mathbb{R}P^n) = n+1$, Proposition 3.35 and the subadditivity of the Lusternik-Schnirelmann category yield the upper estimate

$$\mathrm{TC}_{r,s}(p_n) \leq sn + r - s + 1, \quad \text{for any } s \leq r. \tag{4.2}$$

For a lower estimate, start by noticing that ${p_n}^* : H^*(\mathbb{R}P^n; \mathbb{Z}_2) \to H^*(S^n; \mathbb{Z}_2)$ is trivial in positive dimensions. Set $\iota \in H^n(S^n; \mathbb{Z}_2)$, the fundamental class of the sphere S^n, and let $\alpha \in H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ be the generator of the cohomology ring $H^*(\mathbb{R}P^n;\mathbb{Z}_2)=\mathbb{Z}_2[\alpha]/\left(\alpha^{n+1}\right)$. Set $v_i=q_i^*\alpha\in H^1((S^n)^{r-s}\times(\mathbb{R}P^n)^s;\mathbb{Z}_2)$, where $q_i:(S^n)^{r-s}\times(\mathbb{R}P^n)^s\to\mathbb{R}P^n$ is the projection onto the i-th factor ($r-s+1\leq i\leq r$). Note that $0\neq v_i\in\operatorname{Ker}(\Delta_{r-s},{}^sp_n)^*$. In fact, the product $\prod_{i=r-s+1}^r v_i^n$ does not vanish so that

$$\mathrm{TC}_{r,s}(p_n)\geq sn+1. \tag{4.3}$$

In particular, (4.2) and (4.3) yield

$$\mathrm{TC}_{r,r}(p_n)=rn+1\quad\text{and}\quad\mathrm{TC}_{r,r-1}(p_n)=(r-1)n+\epsilon_{r-1}, \tag{4.4}$$

where $\epsilon_{r-1}\in\{1,2\}$. Thus, we next assume in addition $r-s\geq 2$. For $i=1,2,\ldots,r-s$, set $u_i=q_i^*\iota\in H^n((S^n)^{r-s}\times(\mathbb{R}P^n)^s;\mathbb{Z}_2)$ and $w_i=u_i+u_{r-s}\in\operatorname{Ker}(\Delta_{r-s},{}^sp_n)^*$, where $q_i:(S^n)^{r-s}\times(\mathbb{R}P^n)^s\to S^n$ is the projection onto the i-th factor.

Then

$$\prod_{i=1}^{r-s-1}w_i\;\cdot\prod_{i=r-s+1}^{r}v_i^n=\sum_{j=1}^{r-s}u_1\cdots\widehat{u_j}\cdots u_{r-s}\;\cdot\prod_{i=r-s+1}^{r}v_i^n\neq 0,$$

so that $\mathrm{TC}_{r,s}(p_n)\geq sn+r-s$, which is a linear improvement over (4.3). Taking into account (4.2), we then see that (4.4) extends to

$$\mathrm{TC}_{r,s}(p_n)=sn+\epsilon_s,\ \text{for any } s\leq r, \tag{4.5}$$

where $\epsilon_s\in\{r-s,r-s+1\}$ and, in fact, $\epsilon_r=1$.

Remark 4.1 *Assume $n\in\{1,3,7\}$, so that $\mathbb{R}P^n$ has the structure of an H-space. Corollary 3.38 then yields*

$$TC_{r,r-1}(p_n)=cat((\mathbb{R}P^n)^{r-1})=TC_r(\mathbb{R}P^n)=(r-1)n+1. \tag{4.6}$$

Note here that $cat(S^n\times(\mathbb{R}P^n)^{r-1})=(r-1)n+2>TC_{r,r-1}(p_n)$, which is relevant for the discussion in the paragraph following the proof of Proposition 3.35. In addition, we note that the following constructions have been done in [3, Section 5]:

- *For $n\in\{1,3,7\}$, an explicit partition of $S^n\times\mathbb{R}P^n$ into $n+1$ subsets, each admitting a section for $e_{2,1}^{p_n}:PS^n\to S^n\times\mathbb{R}P^n$, thus realizing (4.6) when $r=2$.*

- *For general n, an explicit partition of $S^n\times\mathbb{R}P^n$ into $n+2$ subsets, each admitting a section for $e_{2,1}^{p_n}:PS^n\to S^n\times\mathbb{R}P^n$, thus realizing the estimate $\epsilon_{r-1}\leq 2$ in (4.5).*

Similarly, for the standard quotient map $q:S^{2n+1}\to\mathbb{C}P^n$, we obtain the estimate

$$sn+r-s\leq\mathrm{TC}_{r,s}(q)\leq sn+r-s+1,$$

for any $s\leq r$.

4.2. Fibrations over spheres

For a fibration $f : X \to S^n$, (4.1) and item (2) of Proposition 3.35 yield

$$r = \mathrm{cat}((S^n)^{r-1}) \leq \mathrm{TC}_{r,r-1}(f) \leq \mathrm{TC}_r(S^n) \leq r+1.$$

In particular, for n odd, we actually have $\mathrm{TC}_{r,r-1}(f) = \mathrm{TC}_r(S^n) = r$. On the other hand, (4.1), Proposition 3.6 and item (3) of Proposition 3.35 yield

$$r \leq \mathrm{TC}_r(S^n) \leq \mathrm{TC}_{r,r}(f) \leq \mathrm{cat}((S^n)^r) = r+1.$$

In particular, if n is even, we get in fact $\mathrm{TC}_{r,r}(f) = \mathrm{TC}_r(S^n) = r+1$.

5. Conclusion

We introduce a notion of higher topological complexity of a map f, $\mathrm{TC}_{r,s}(f)$, for $1 \leq s \leq r \geq 2$, which simultaneously extends Rudyak's and Pavešić's notions. Our unified concept is relevant in the r-multitasking motion planning problem associated to a robot devise when the forward kinematics map plays a role in s prescribed stages of the motion task. The use of the biparameter (r, s) allows us to get a discrimination of the topological properties of a space Y in a manner which is finer than that provided by the several higher topological complexities $\mathrm{TC}_r(Y)$.

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Conflict of interest

The authors declare that they have no conflict of interest.

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