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Adjunction greatest element to ordered hypersemigroups

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Abstract: As a continuation of the paper “Adjunction Identity to Hypersemigroup” in Turk J Math 2022; 46 (7): 2834–2853, it has been proved here that the adjunction of a greatest element to an ordered hypersemigroup is actually an embedding problem. The concept of pseudoideal has been introduced and has been proved that for each ordered hypersemigroup S an ordered hypersemigroup V having a greatest element (poe -hypersemigroup) can be constructed in such a way that there exists a pseudoideal T of S such that S is isomorphic to T . If S does not have a greatest element, then this can be regarded as the embedding of an ordered hypersemigroup in an ordered semigroup with greatest element.

Key words: poe -hypersemigroup, pseudoideal, embedding, semisimple, ideal, bi-ideal

1. Introduction

Fuchs and Halperin have shown that every regular ring can be embedded in a regular ring with identity [1]. The problem of adjunction identity to semigroups, greatest element to ordered sets has been considered in [2]. In both cases the adjunction has the same meaning: If S (P) is a semigroup (ordered set) without identity (greatest element), the adjunction of an identity to S (P) means that we construct a semigroup (ordered set) V with identity (greatest element) in such a way that there exists an ideal I of V such that $S \cong I$ ($P \cong I$). Later, it has been proved that each ordered semigroup S can be embedded in an ordered semigroup having a greatest element. If S does not have a greatest element, then this is a problem of adjunction greatest element to S [3]. The problem of adjunction identity to hypersemigroups has been considered [7]. As a continuation of the paper in [7], we discuss here the problem of adjunction of a greatest element to an ordered hypersemigroup.

2. Main result

A hypersemigroup is a nonempty set S with an “operation” \circ assigning to each couple (a, b) of S a nonempty subset $a \circ b$ (called hyperoperation as the $a \circ b$ is a subset and not element of S) and an operation $*$ between the nonempty subsets A, B of S such that $A * B = \bigcup \{a \circ b \mid a \in A, b \in B\}$ satisfying the relation $\{a\} * (b \circ c) = (a \circ b) * \{c\}$ for all $a, b, c \in S$ [4]. A hypersemigroup (S, \circ) is called an ordered hypersemigroup if there exists an order relation \leq on S such that $a \leq b$ implies $a \circ c \leq b \circ c$ and $c \circ a \leq c \circ b$ for every $c \in S$; in the sense that for every $u \in a \circ c$ there exists $v \in b \circ c$ such that $u \leq v$ and for every $u \in c \circ a$ there exists $v \in c \circ b$ such that $u \leq v$ [5].

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Definition 2.1 Let (S, \circ, \leq) be an ordered hypersemigroup. A nonempty subset T of S is called pseudoideal of (S, \circ, \leq) if

- (1) $T * T \subseteq T$ and
- (2) if $a \in T$ and $S \ni b \leq a$ then $b \in T$.

Definition 2.2 Two ordered hypersemigroups (S, \circ, \leq) and $(T, \bar{\circ}, \bar{\leq})$ are called isomorphic if there exists a (1-1) mapping f of S onto T such that, for every $a, b \in S$, we have

- (1) $f(a \circ b) \subseteq f(a) \bar{\circ} f(b)$; in the sense that if $u \in a \circ b$, then $f(u) \in f(a) \bar{\circ} f(b)$
- (2) if $a \leq b$, then $f(a) \bar{\leq} f(b)$
- (3) if $a, b \in S$ such that $f(a) \bar{\leq} f(b)$, then $a \leq b$.

Theorem 2.3 Let (S, \circ, \leq) be an ordered hypersemigroup. Then there exists an ordered hypersemigroup V having a greatest element (poe-hypersemigroup) and a pseudoideal T of V such that $S \cong T$.

Proof For an element e not containing in S ((x, x) is, for example, such an element), we consider the set $S \cup \{e\}$. We define an hyperoperation “ $\bar{\circ}$ ” on $S \cup \{e\}$ and an operation “ $\bar{*}$ ” on the set $\mathcal{P}^*(S \cup \{e\})$ of all nonempty subsets of S as follows:

$$\bar{\circ} : (S \cup \{e\}) \times (S \cup \{e\}) \rightarrow \mathcal{P}^*(S \cup \{e\}) \mid (x, y) \rightarrow x \bar{\circ} y \text{ where}$$

$$x \bar{\circ} y = \begin{cases} x \circ y & \text{if } x, y \in S \\ \{e\} & \text{if } x \in S, y = e \\ \{e\} & \text{if } x = e, y \in S \\ \{e\} & \text{if } x = y = e \end{cases}$$

$$\bar{*} : \mathcal{P}^*(S \cup \{e\}) \times \mathcal{P}^*(S \cup \{e\}) \rightarrow \mathcal{P}^*(S \cup \{e\}) \mid (A, B) \rightarrow A \bar{*} B \text{ where}$$

$$A \bar{*} B = \bigcup_{a \in A, b \in B} a \bar{\circ} b$$

$$\text{(For } A = \{x\}, B = \{y\}, \text{ we clearly have } \{x\} \bar{*} \{y\} = \bigcup_{u \in \{x\}, v \in \{y\}} u \bar{\circ} v = \bigcup_{u=x, v=y} u \bar{\circ} v = x \bar{\circ} y)$$

Then $(S \cup \{e\}, \bar{\circ}, \bar{*})$ is a hypersemigroup. In fact:

(A) The operation $\bar{\circ}$ is well defined. Indeed: If $x, y \in S$, then $x \bar{\circ} y = x \circ y \subseteq S \subseteq S \cup \{e\}$. Otherwise, $x \bar{\circ} y = \{e\} \subseteq S \cup \{e\}$. Let $(x, y), (z, t) \in (S \cup \{e\}) \times (S \cup \{e\})$ such that $(x, y) = (z, t)$. Then $x \bar{\circ} y = z \bar{\circ} t$. Indeed: If $x, y \in S$, then $z, t \in S$, $x \bar{\circ} y = x \circ y = z \circ t = z \bar{\circ} t$. If $x \in S, y = e$, then $z \in S, t = e$, $x \bar{\circ} y = \{e\}$ and $z \bar{\circ} t = \{e\}$ and so $x \bar{\circ} y = z \bar{\circ} t$. If $x = e, y \in S$, then $z = e, t \in S$, $x \bar{\circ} y = \{e\}$, $z \bar{\circ} t = \{e\}$ and so $x \bar{\circ} y = z \bar{\circ} t = \{e\}$. If $x = y = e$, then $z = t = e$ and $x \bar{\circ} y = \{e\} = z \bar{\circ} t$.

(B) The operation $\bar{*}$ is well defined. Indeed: Let $A, B \in \mathcal{P}^*(S \cup \{e\})$. Since $\emptyset \neq a \bar{\circ} b \subseteq S \cup \{e\}$ for every $a \in A$ and every $b \in B$, we have $\emptyset \neq A \bar{*} B \subseteq S \cup \{e\}$. Let $(A, B), (C, D) \in \mathcal{P}^*(S \cup \{e\}) \times \mathcal{P}^*(S \cup \{e\})$ such that $(A, B) = (C, D)$. Then $A \bar{*} B = \bigcup_{a \in A, b \in B} a \bar{\circ} b = \bigcup_{a \in C, b \in D} a \bar{\circ} b = C \bar{*} D$.

(C) $\{x\} \bar{*} (y \bar{o} z) = (x \bar{o} y) \bar{*} \{z\}$ for every $x, y, z \in S \cup \{e\}$. Indeed:

We have to check the following two cases:

- (a) $x \in S$, ($y \in S$ or $y = e$), ($z \in S$ or $z = e$) and
- (b) $x = e$, ($y \in S$ or $y = e$), ($z \in S$ or $z = e$).

(1) If $x, y, z \in S$, then $\{x\} \bar{*} (y \bar{o} z) = (x \bar{o} y) \bar{*} \{z\}$; its proof is the same with the proof in [7, p. 2838].

(2) Let $x, y \in S$, $z = e$. Then $\{x\} \bar{*} (y \bar{o} z) = (x \bar{o} y) \bar{*} \{z\}$. Indeed: We have

$$\{x\} \bar{*} (y \bar{o} z) = \{x\} \bar{*} \{e\} = x \bar{o} e = \{e\} \text{ and } (x \bar{o} y) \bar{*} \{z\} = (x \bar{o} y) \bar{*} \{e\}.$$

On the other hand, $(x \bar{o} y) \bar{*} \{e\} = \{e\}$. Indeed: If $t \in (x \bar{o} y) \bar{*} \{e\}$, then $t \in u \bar{o} e$ for some $u \in x \bar{o} y$. Since $x, y \in S$, we have $x \bar{o} y = x \circ y$, then $u \in x \circ y \subseteq S$. Since $u \in S$, we have $u \bar{o} e = \{e\}$, then $t = e$ and so $(x \bar{o} y) \bar{*} \{e\} \subseteq \{e\}$. Let now $t = e$. Take an element $u \in x \bar{o} y$ ($x \bar{o} y \neq \emptyset$). Since $x, y \in S$, we have $x \bar{o} y = x \circ y \subseteq S$. Since $u \in S$, we have $u \bar{o} e = \{e\}$. Then $t = e \in u \bar{o} e = \{u\} \bar{*} \{e\} \subseteq (x \bar{o} y) \bar{*} \{e\}$ and so $\{e\} \subseteq (x \bar{o} y) \bar{*} \{e\}$.

(3) Let $x \in S$, $y = e$, $z \in S$. Then $\{x\} \bar{*} (y \bar{o} z) = (x \bar{o} y) \bar{*} \{z\}$. Indeed: We have

$$\{x\} \bar{*} (y \bar{o} z) = \{x\} \bar{*} (e \bar{o} z) = \{x\} \bar{*} \{e\} = x \bar{o} e = \{e\} \text{ and } (x \bar{o} y) \bar{*} \{z\} = (x \bar{o} e) \bar{*} \{z\} = \{e\} \bar{*} \{z\} = e \bar{o} z = \{e\}.$$

(4) Let $x \in S$, $y = z = e$. Then $\{x\} \bar{*} (y \bar{o} z) = (x \bar{o} y) \bar{*} \{z\}$. Indeed: We have

$$\{x\} \bar{*} (y \bar{o} z) = \{x\} \bar{*} (e \bar{o} e) = \{x\} \bar{*} \{e\} = x \bar{o} e = \{e\} \text{ and } (x \bar{o} y) \bar{*} \{z\} = (x \bar{o} e) \bar{*} \{e\} = \{e\} \bar{*} \{e\} = e \bar{o} e = \{e\}.$$

(5) Let $x = e$, $y \in S$, $z \in S$. Then $\{x\} \bar{*} (y \bar{o} z) = (x \bar{o} y) \bar{*} \{z\}$. Indeed: We have

$$\{x\} \bar{*} (y \bar{o} z) = \{e\} \bar{*} (y \bar{o} z) \text{ and } (x \bar{o} y) \bar{*} \{z\} = (e \bar{o} y) \bar{*} \{z\} = \{e\} \bar{*} \{z\} = e \bar{o} z = \{e\}.$$

On the other hand, $\{e\} \bar{*} (y \bar{o} z) = \{e\}$. Indeed: If $t \in \{e\} \bar{*} (y \bar{o} z)$, then $t \in e \bar{o} u$ for some $u \in y \bar{o} z$. Since $y, z \in S$, we have $y \bar{o} z = y \circ z \subseteq S$. Since $u \in S$, we have $e \bar{o} u = \{e\}$ and so $t = e$. Let now $t = e$. Take an element $u \in y \bar{o} z$ ($y \bar{o} z \neq \emptyset$). Since $y, z \in S$, we have $y \bar{o} z = y \circ z \subseteq S$. Since $u \in S$, we have $e \bar{o} u = \{e\}$. Then we have $t = e \in e \bar{o} u = \{e\} \bar{*} \{u\} \subseteq \{e\} \bar{*} (y \bar{o} z)$ and so $\{e\} \subseteq \{e\} \bar{*} (y \bar{o} z)$.

(6) Let $x = e$, $y \in S$, $z = e$. Then $\{x\} \bar{*} (y \bar{o} z) = (x \bar{o} y) \bar{*} \{z\}$. Indeed: We have

$$\{x\} \bar{*} (y \bar{o} z) = \{e\} \bar{*} (y \bar{o} e) = \{e\} \bar{*} \{e\} = e \bar{o} e = \{e\} \text{ and } (x \bar{o} y) \bar{*} \{z\} = (e \bar{o} y) \bar{*} \{e\} = \{e\} \bar{*} \{e\} = \{e\}.$$

(7) Let $x = e$, $y = e$, $z \in S$. Then $\{x\} \bar{*} (y \bar{o} z) = (x \bar{o} y) \bar{*} \{z\}$. Indeed: We have

$$\{x\} \bar{*} (y \bar{o} z) = \{e\} \bar{*} (e \bar{o} z) = \{e\} \bar{*} \{e\} = e \bar{o} e = \{e\} \text{ and } (x \bar{o} y) \bar{*} \{z\} = (e \bar{o} e) \bar{*} \{z\} = \{e\} \bar{*} \{z\} = e \bar{o} z = \{e\}.$$

(8) Let $x = y = z = e$. Then $\{x\} \bar{*} (y \bar{o} z) = (x \bar{o} y) \bar{*} \{z\}$. Indeed: We have

$$\{x\} \bar{*} (y \bar{o} z) = \{e\} \bar{*} (e \bar{o} e) = \{e\} \bar{*} \{e\} = e \bar{o} e = \{e\} \text{ and } (x \bar{o} y) \bar{*} \{z\} = (e \bar{o} e) \bar{*} \{e\} = \{e\} \bar{*} \{e\} = \{e\}.$$

We endow $S \cup \{e\}$ with the relation \preceq defined by

$$\preceq := \leq \cup \{(x, e) \mid x \in S \cup \{e\}\}.$$

(D) The relation \preceq is an order on $S \cup \{e\}$. Indeed:

It is reflexive: Let $a \in S \cup \{e\}$. If $a \in S$, then $(a, a) \in \leq \subseteq \preceq$; if $a = e$, then $(a, a) \in \{(x, e) \mid x \in S \cup \{e\}\} \subseteq \preceq$.

Thus, we have $(a, a) \in \preceq$ for every $a \in S$ and the relation \preceq is reflexive.

The relation \preceq is symmetric. Indeed: Let $(a, b) \in \preceq$ and $(b, a) \in \preceq$. Then

$$(a, b) \in \leq \text{ or } (a, b) = (x, e) \text{ for some } x \in S \cup \{e\} \text{ and} \\ (b, a) \in \leq \text{ or } (b, a) = (y, e) \text{ for some } y \in S \cup \{e\}.$$

We consider the cases:

- (1) $(a, b) \in \leq$ and $(b, a) \in \leq$
- (2) $(a, b) \in \leq$ and $(b, a) = (y, e)$ for some $y \in S \cup \{e\}$
- (3) $(a, b) = (x, e)$ for some $x \in S \cup \{e\}$ and $(b, a) \in \leq$
- (4) $(a, b) = (x, e)$ for some $x \in S \cup \{e\}$ and $(b, a) = (y, e)$ for some $y \in S \cup \{e\}$.

(1) If $a \leq b$ and $b \leq a$, then $a = b$.

(2) Let $(a, b) \in \leq$ and $(b, a) = (y, e)$ for some $y \in S \cup \{e\}$. Since $(a, b) \in \leq$, we have $a, b \in S$. Since $(y, a) = (y, e)$ for some $y \in S \cup \{e\}$, we have $a = e$. Thus, we have $S \ni a = e$. The case is impossible.

(3) Let $(a, b) = (x, e)$ for some $x \in S \cup \{e\}$ and $(b, a) \in \leq$. Then we have $S \ni b = e$. The case is impossible.

(4) Let $(a, b) = (x, e)$ for some $x \in S \cup \{e\}$ and $(b, a) = (y, e)$ for some $y \in S \cup \{e\}$. Then we have $b = e = a$ and so $a = b$.

The relation \preceq is transitive. Indeed: Let $(a, b) \in \preceq$ and $(b, c) \in \preceq$. Then

$$(a, b) \in \leq \text{ or } (a, b) = (x, e) \text{ for some } x \in S \cup \{e\} \text{ and} \\ (b, c) \in \leq \text{ or } (b, c) = (y, e) \text{ for some } y \in S \cup \{e\}. \text{ We consider the cases:}$$

- (1) $(a, b) \in \leq$ and $(b, c) \in \leq$. Then $(a, c) \in \leq \subseteq \preceq$.
- (2) $(a, b) \in \leq$ and $(b, c) = (y, e)$ for some $y \in S \cup \{e\}$. Since $(b, c) = (y, e)$; $y \in S \cup \{e\}$, we have $c = e$. Then we have $(a, c) = (a, e) \in \{(x, e) \mid x \in S \cup \{e\}\} \subseteq \preceq$.
- (3) Let $(a, b) = (x, e)$ for some $x \in S \cup \{e\}$ and $(b, c) \in \leq$. Then we have $S \ni b = e$. The case is impossible.
- (4) $(a, b) = (x, e)$ for some $x \in S \cup \{e\}$ and $(b, c) = (y, e)$ for some $y \in S \cup \{e\}$. Then we have $(a, c) = (a, e) \in \{(x, e) \mid x \in S \cup \{e\}\} \subseteq \preceq$.

(E) The element e is the greatest element of $S \cup \{e\}$. Indeed: Let $a \in S \cup \{e\}$. Then $(a, e) \in \{(x, e) \mid x \in S \cup \{e\}\} \subseteq \preceq$ and so $(a, e) \in \preceq$ i.e. $a \preceq e$.

(F) $(S \cup \{e\}, \bar{o}, \preceq)$ is a *poe*-hypersemigroup.

Let $a, b \in S \cup \{e\}$ such that $a \preceq b$. Then $a \bar{o} c \preceq b \bar{o} c$ and $c \bar{o} a \preceq c \bar{o} b$ for every $c \in S \cup \{e\}$. Let us prove the first one. The proof of the second is similar.

We have $(a \in S \text{ or } a = e)$, $a \preceq b$, $(c \in S \text{ or } c = e)$. Thus, we have

$$a \in S, a \preceq b, c \in S \\ a \in S, a \preceq b, c = e \\ a = e, a \preceq b, c \in S \\ a = e, a \preceq b, c = e.$$

Thus, we have to check the following cases:

- (1) $a \in S, a \preceq b, c \in S$
- (2) $a \in S, (a, b) = (x, e)$ for some $x \in S \cup \{e\}, c \in S$

- (3) $a \in S, a \leq b, c = e$
- (4) $a \in S, (a, b) = (x, e)$ for some $x \in S \cup \{e\}, c = e$
- (5) $a = e, a \leq b, c \in S$
- (6) $a = e, (a, b) = (x, e)$ for some $x \in S \cup \{e\}, c \in S$
- (7) $a = e, a \leq b, c = e$
- (8) $a = e, (a, b) = (x, e)$ for some $x \in S \cup \{e\}, c = e$.

(1) Let $a \in S, a \leq b, c \in S$ and $u \in a \bar{o} c$. Then there exists $v \in b \bar{o} c$ such that $u \preceq v$. Indeed: Since $a, c \in S$, we have $a \bar{o} c = a \circ c$. Since $a \leq b$, we have $a \circ c \preceq b \circ c$. Since $u \in a \circ c$, there exists $v \in b \circ c$ such that $u \leq v$. Since $b, c \in S$, we have $b \circ c = b \bar{o} c$. Since $u \leq v$, we have $(u, v) \in \leq \subseteq \preceq$. Thus, we have $v \in b \bar{o} c$ and $u \preceq v$.

(2) Let $a \in S, (a, b) = (x, e)$ for some $x \in S \cup \{e\}, c \in S$ and $u \in a \bar{o} c$. Then there exists $v \in b \bar{o} c$ such that $u \preceq v$. Indeed: Since $a, c \in S$, we have $u \in a \bar{o} c = a \circ c \subseteq S$. Since $u \in S$, we have $(u, e) \in \{(x, e) \mid x \in S \cup \{e\}\} \subseteq \preceq$ and so $u \preceq e$. Since $b = e$, we have $b \bar{o} c = e \bar{o} c = \{e\}$. For the element $v := e \in b \bar{o} c$, we have $u \preceq v$.

(3) Let $a \in S, a \leq b, c = e$ and $u \in a \bar{o} c$. Then there exists $v \in b \bar{o} c$ such that $u \preceq v$. Indeed: We have $u \in a \bar{o} c = a \bar{o} e = \{e\}$ and so $u = e$. We also have $b \bar{o} c = b \bar{o} e = \{e\}$ and $e \preceq e$ (since \preceq is reflexive). So, for the element $v := e \in b \bar{o} c$, we have $u \preceq v$.

(4) Let $a \in S, (a, b) = (x, e)$ for some $x \in S \cup \{e\}, c = e$ and $u \in a \bar{o} c$. Then there exists $v \in b \bar{o} c$ such that $u \preceq v$. Indeed: We have $u \in a \bar{o} c = a \bar{o} e = \{e\}$ and so $u = e$. We also have $b \bar{o} c = e \bar{o} c = \{e\}$ and $e \preceq e$ (as \preceq is reflexive). For the element $v := e \in b \bar{o} c$, we have $u \preceq v$.

(5) Let $a = e, a \leq b, c \in S$ and $u \in a \bar{o} c$. Since $a \leq b$, we have $a \in S$. Since $a = e$, we have $e \in S$. The case is impossible.

(6) Let $a = e, (a, b) = (x, e)$ for some $x \in S \cup \{e\}, c \in S$ and $u \in a \bar{o} c$. Then there exists $v \in b \bar{o} c$ such that $u \preceq v$. Indeed: We have $u \in a \bar{o} c = e \bar{o} c = \{e\}$ and so $u = e$. We also have $b \bar{o} c = e \bar{o} c = \{e\}$. For the element $v := e \in b \bar{o} c$, we have $u \preceq v$.

(7) Let $a = e, a \leq b, c = e$ and $u \in a \bar{o} c$. Then there exists $v \in b \bar{o} c$ such that $u \preceq v$. Indeed: We have $u \in a \bar{o} c = e \bar{o} e = \{e\}$ and so $u = e$. We also have $b \bar{o} c = b \bar{o} e = \{e\}$. So, for the element $v := e \in b \bar{o} c$, we have $u \preceq v$.

(8) Let $a = e, (a, b) = (x, e)$ for some $x \in S \cup \{e\}, c = e$ and $u \in a \bar{o} c$. Then there exists $v \in b \bar{o} c$ such that $u \preceq v$. Indeed: We have $u \in a \bar{o} c = e \bar{o} e = \{e\}$ and so $u = e$. We also have $b \bar{o} c = e \bar{o} e = \{e\}$. So, for the element $v := e \in b \bar{o} c$, we have $u \preceq v$.

(G) The ordered hypersemigroups (S, \circ, \leq) and (S, \bar{o}, \preceq) are isomorphic under the identity mapping. Indeed, for the one to one and onto mapping

$$i : (S, \circ, \leq) \rightarrow (S, \bar{o}, \preceq) \mid a \rightarrow i(a) := a$$

and, any $a, b \in S$, we have

- (1) $i(a \circ b) \subseteq i(a) \bar{o} i(b)$; that is if $u \in a \circ b$, then $u \in a \bar{o} b$. This is clear, as $a, b \in S$ implies $a \circ b = a \bar{o} b$.

(2) $a \leq b$ implies $a \preceq b$. Indeed, if $a \leq b$, then $(a, b) \in \subseteq \subseteq \subseteq \cup \{(x, e) \mid x \in S \cup \{e\}\} = \preceq$ i.e. $(a, b) \in \preceq$ and so $a \preceq b$.

(3) if $a, b \in S$ such that $i(a) \preceq i(b)$, then $a \leq b$. Indeed: if $i(a) \preceq i(b)$, then $a \preceq b$ i.e. $(a, b) \in \subseteq \subseteq \cup \{(x, e) \mid c \in S \cup \{e\}\}$. If $(a, b) \in \subseteq$, then $a \leq b$ and the proof is complete. If $(a, b) \in \{(x, e) \mid x \in S \cup \{e\}\}$, then $(a, b) = (x, e)$ for some $x \in S \cup \{e\}$. Then we have $S \ni b = e$ i.e. $e \in S$ and the case is impossible.

(G) S is a pseudoideal of $(S \cup \{e\}, \bar{\circ}, \preceq)$. Indeed, $\emptyset \neq S \subseteq S \cup \{e\}$, $S * S \subseteq S$ and if $a \in S$ and $S \cup \{e\} \ni b \preceq a$, then $b \in S$ (as $b = e$ implies $e = a \in S$ that is impossible). \square

3. Some further results

A *poe*-semigroup (S, \cdot, \preceq) is called regular if $a \leq aea$ for every $a \in S$; intra-regular if $a \leq ea^2e$ for every $a \in S$. It is called right (resp. left) regular if $a \leq a^2e$ (resp. $a \leq ea^2$) for every $a \in S$. A *poe*-semigroup (S, \cdot, \preceq) is called right (resp. left) quasi-regular if $a \leq aeae$ (resp. $a \leq eaea$) for every $a \in S$. It is called semisimple if $a \leq eaeae$ for every $a \in S$.

These concepts can be extended for a *poe*-hypersemigroup (S, \circ, \preceq) in the way indicated below.

Definition 3.1 A *poe*-hypersemigroup (S, \circ, \preceq) is called regular if $\{a\} \leq (a \circ e) * \{a\}$ for any $a \in S$; in the sense that for any $a \in S$ there exists $t \in S$ such that $t \in (a \circ e) * \{a\}$ and $a \leq t$.

To see that Definition 3.1 is correct, we have to prove that it coincides with the definition of a regular *poe*-hypersemigroup. A *poe*-hypersemigroup (S, \circ, \preceq) is called regular if for every $a \in S$ there exists $x \in S$ such that $\{a\} \leq (a \circ x) * \{a\}$ (in the sense that for every $a \in S$ there exist $x, t \in S$ such that $t \in (a \circ x) * \{a\}$ and $a \leq t$) [6].

In this respect, the following proposition holds.

Proposition 3.2 Let (S, \circ, \preceq) is a *poe*-hypersemigroup. The following are equivalent:

- (1) S is regular.
- (2) $\{a\} \leq (a \circ e) * \{a\}$ for every $a \in S$.

Proof First of all, for any nonempty subsets A, B, C of S , $A \preceq B$ implies $A * C \subseteq (B * C)$. Indeed: Let $x \in A * C$. Then $x \in a \circ c$ for some $a \in A$, $c \in C$. Since $a \in A$, there exists $b \in B$ such that $a \leq b$. Then $a \circ c \preceq b \circ c$. Since $x \in a \circ c$, there exists $y \in b \circ c$ such that $x \leq y \in B * C$ and so $x \in (B * C)$.

(1) \implies (2). Let $a \in S$. Since S is regular, there exist $x, t \in S$ such that $t \in (a \circ x) * \{a\}$ and $a \leq t$. Since $x \leq e$, we have $a \circ x \preceq a \circ e$, then $t \in (a \circ x) * \{a\} \subseteq \left[(a \circ e) * \{a\} \right]$. Then $t \leq y$ for some

$y \in (a \circ e) * \{a\} (\subseteq (S * S) * S \subseteq S * S \subseteq S)$, $y \in (a \circ e) * \{a\}$ and $a \leq y$ and property (2) is satisfied.

(2) \implies (1). Let $a \in S$. By (2), there exists $t \in S$ such that $t \in (a \circ e) * \{a\}$ and $a \leq t$. We put $x := e$. Then $x, t \in S$, $t \in (a \circ x) * \{a\}$ and $a \leq t$ and property (1) holds. \square

In a similar way, the following definitions are true.

Definition 3.3 A *poe*-hypersemigroup (S, \circ, \preceq) is called intra-regular if $\{a\} \leq (e \circ a) * (a \circ e)$ for any $a \in S$; in the sense that for any $a \in S$ there exists $t \in S$ such that $t \in (e \circ a) * (a \circ e)$ and $a \leq t$.

Definition 3.4 A poe-hypersemigroup (S, \circ, \leq) is called right regular if $\{a\} \leq \{a\} * (a \circ e)$ ($= (a \circ a) * \{e\}$) for any $a \in S$; in the sense that for any $a \in S$ there exists $t \in S$ such that $t \in \{a\} * (a \circ e)$ and $a \leq t$. It is called left regular if $\{a\} \leq (e \circ a) * \{a\}$ ($= \{e\} * (a \circ a)$) for any $a \in S$; in the sense that for any $a \in S$ there exists $t \in S$ such that $t \in (e \circ a) * \{a\}$ and $a \leq t$.

Definition 3.5 A poe-hypersemigroup (S, \circ, \leq) is called right quasi-regular if $\{a\} \leq (a \circ e) * (a \circ e)$ for every $a \in S$; in the sense that for any $a \in S$ there exists $t \in S$ such that $t \in (a \circ e) * (a \circ e)$ and $a \leq t$. It is called left quasi-regular if $\{a\} \leq (e \circ a) * (e \circ a)$ for any $a \in S$; in the sense that for any $a \in S$ there exists $t \in S$ such that $t \in (e \circ a) * (e \circ a)$ and $a \leq t$.

Definition 3.6 A poe-hypersemigroup (S, \circ, \leq) is called semisimple if $\{a\} \leq (e \circ a) * (e \circ a) * \{e\}$ for any $a \in S$; in the sense that for any $a \in S$ there exists $t \in S$ such that $t \in (e \circ a) * (e \circ a) * \{e\}$ and $a \leq t$.

Proposition 3.7 The poe-hypersemigroup $(S \cup \{e\}, \bar{\circ}, \preceq)$ constructed in Theorem 2.3 is regular and intra-regular.

Proof Let $a \in S \cup \{e\}$. Then there exists $t \in S \cup \{e\}$ such that $t \in (a \bar{\circ} e) \bar{*} \{a\}$ and $a \preceq t$. Indeed: If $a \in S$, then $a \bar{\circ} e = \{e\}$, $(a \bar{\circ} e) \bar{*} \{a\} = \{e\} \bar{*} \{a\} = e \bar{\circ} a = \{e\}$. If $a = e$, then $a \bar{\circ} e = e \bar{\circ} e = \{e\}$, $(a \bar{\circ} e) \bar{*} \{a\} = \{e\} \bar{*} \{a\} = \{e\}$. In each case, we have $e \in (a \bar{\circ} e) \bar{*} \{a\}$ and $a \preceq e$.

Let $a \in S \cup \{e\}$. Then there exists $t \in S \cup \{e\}$ such that $t \in (e \bar{\circ} a) \bar{*} (a \bar{\circ} e)$ and $a \preceq t$. Indeed, for the element $t := e \in S \cup \{e\}$, we have $e \in \{e\} = e \bar{\circ} e = \{e\} \bar{*} \{e\} = (e \bar{\circ} a) \bar{*} (a \bar{\circ} e)$; thus, we have $e \in (e \bar{\circ} a) \bar{*} (a \bar{\circ} e)$ and $a \preceq e$ and so $(S \cup \{e\}, \bar{\circ}, \preceq)$ is intra-regular. \square

Proposition 3.8 The poe-hypersemigroup $(S \cup \{e\}, \bar{\circ}, \preceq)$ is right regular and left regular.

Proof Let $a \in S \cup \{e\}$. Then there exists $t \in S \cup \{e\}$ such that $t \in \{a\} \bar{*} (a \bar{\circ} e)$ and $a \preceq t$. Indeed: for the element $t := e \in S \cup \{e\}$, we have $e \in \{e\} = a \bar{\circ} e = \{a\} \bar{*} \{e\} = \{a\} \bar{*} (a \bar{\circ} e)$; thus, we have $e \in \{a\} \bar{*} (a \bar{\circ} e)$ and $a \preceq e$ and so $(S \cup \{e\}, \bar{\circ}, \preceq)$ is right regular. We also have $e \in \{e\} = e \bar{\circ} a = \{e\} \bar{*} \{a\} = (e \bar{\circ} a) \bar{*} \{a\}$; thus, we have $e \in (e \bar{\circ} a) \bar{*} \{a\}$ and $a \preceq e$ and so $(S \cup \{e\}, \bar{\circ}, \preceq)$ is left regular. \square

Proposition 3.9 The poe-semigroup $(S \cup \{e\}, \bar{\circ}, \preceq)$ is right quasi-regular and left quasi-regular.

Proof Let $a \in S \cup \{e\}$. Then there exists $t \in S \cup \{e\}$ such that $t \in (a \bar{\circ} e) \bar{*} (a \bar{\circ} e)$ and $a \preceq t$. In fact, for the element $t := e \in S \cup \{e\}$, we have $e \in \{e\} = e \bar{\circ} e = \{e\} \bar{*} \{e\} = (a \bar{\circ} e) \bar{*} (a \bar{\circ} e)$ and $a \preceq e$ and so $(S \cup \{e\}, \bar{\circ}, \preceq)$ is right quasi-regular. We also have $e \in \{e\} = e \bar{\circ} e = \{e\} \bar{*} \{e\} = (e \bar{\circ} a) \bar{*} (e \bar{\circ} a)$ and $a \preceq e$ and so $(S \cup \{e\}, \bar{\circ}, \preceq)$ is left quasi-regular. \square

Proposition 3.10 The poe-semigroup $(S \cup \{e\}, \bar{\circ}, \preceq)$ is semisimple.

Proof Let $a \in S \cup \{e\}$. Then there exists $t \in S \cup \{e\}$ such that $t \in (e \bar{\circ} a) \bar{*} (e \bar{\circ} a) \bar{*} \{e\}$ and $a \preceq t$. Indeed, for the element $t := e \in S \cup \{e\}$, we have $e \in (e \bar{\circ} a) \bar{*} (e \bar{\circ} a) \bar{*} \{e\}$ and $a \preceq e$. \square

According to Proposition 3.7, $(S \cup \{e\}, \bar{\circ}, \preceq)$ is intra-regular. This can be also obtained as corollary to the next proposition. To prove it, we need the following lemma.

If $(S, \circ, *, \leq)$ is an ordered hypersemigroup and A, B nonempty subsets of S we write $A \leq B$ if for any $a \in A$ there exists $b \in B$ such that $a \leq b$.

Lemma 3.11 *Let (S, \circ, \leq) be an ordered hypersemigroup. Then we have the following:*

- (a) *For any nonempty subsets A, B, C of S such that $A \leq B$, we have $A * C \leq B * C$ and $C * A \leq C * B$*
- (b) *The operation $*$ is associative (see, for example [4]).*
- (c) *If $A \leq B \leq C$, then $A \leq C$.*

Proof (a) Let $A \leq B$ and $x \in A * C$. Then there exists $y \in B * C$ such that $x \leq y$. Indeed: Since $x \in A * C$, we have $x \in a \circ c$ for some $a \in A, c \in C$. Since $a \in A$, there exists $b \in B$ such that $a \leq b$. Then $a \circ c \leq b \circ c$. Since $x \in a \circ c$, there exists $y \in b \circ c = B * C$ such that $x \leq y$.

(c) If $a \in A$, then there exists $b \in B$ such that $a \leq b$. Since $b \in B$, there exists $c \in C$ such that $b \leq c$. Hence, for any $a \in A$ there exists $c \in C$ such that $a \leq c$ and so $A \leq C$. □

Proposition 3.12 *A poe-hypersemigroup (S, \circ, \leq) that is right regular or left regular is intra-regular.*

Proof Let (S, \circ, \leq) be right regular and $a \in S$. Then we have

$$\begin{aligned} \{a\} &\leq (a \circ a) * \{e\} = \{a\} * \{a\} * \{e\} \leq \{e\} * \{a\} * \{e\} \text{ (since } a \leq e \text{ implies } \{a\} \leq \{e\}) \\ &\leq \{e\} * (\{a\} * \{a\} * \{e\}) * \{e\} = \{e\} * \{a\} * \{a\} * (\{e\} * \{e\}) \\ &\leq \{e\} * \{a\} * \{a\} * \{e\} \text{ (as } \{e\} * \{e\} = e \circ e \leq \{e\}) \\ &= (e \circ a) * (a \circ e) \end{aligned}$$

and so S is intra-regular. □

By Propositions 3.7, 3.8, 3.9, and 3.10, we have the following corollary.

Corollary 3.13 *Each ordered hypersemigroup can be embedded*

- (1) *in a regular poe-hypersemigroup.*
- (2) *in an intra-regular poe-hypersemigroup.*
- (3) *in a right regular (or left regular) poe-hypersemigroup.*
- (4) *in right quasi-regular (or left quasi-regular) poe-hypersemigroup.*
- (5) *in a semisimple poe-hypersemigroup.*

4. Examples

We apply the above results to the following examples.

Remark 4.1 Theorem 2.3 can be also applied to a poe-hypersemigroup and we have the following: If (S, \circ, \leq) is a poe-hypersemigroup, t an element not included in S , $\bar{\circ}$ the hyperoperation and \preceq the order $S \cup \{t\}$ defined in Theorem 2.3, then the set $V := (S \cup \{t\}, \bar{\circ}, \preceq)$ is still a poe-hypersemigroup and S is a pseudoideal of V .

Let us give an example based on the remark.

Example 4.2 We consider the ordered semigroup $S = \{a, b, c\}$ given by Table 1 and Figure 1. From this, the ordered hypersemigroup given by Table 2 and the same figure (Figure 1) can be obtained. Take an element t not included in S and consider the ordered hypersemigroup $S \cup \{t\}$. Then $(S \cup \{t\}, \bar{\circ}, \preceq)$ is a *poe*-hypersemigroup having the S as a pseudoideal. According to Section 2, the ordered hypersemigroup $(S \cup \{e\}, \bar{\circ}, \preceq)$ given by Table 3 and Figure 2 is regular, intra-regular, right regular, left regular, right quasi-regular, left quasi-regular, and semisimple. Independently,

$(S \cup \{e\}, \bar{\circ}, \preceq)$ is regular, that is, $\{a\} \preceq (a \bar{\circ} e) \bar{\circ} \{a\}$ for every $a \in S \cup \{e\}$; in other words, for every $a \in S \cup \{e\}$ there exists $t \in (a \bar{\circ} e) \bar{*} \{a\}$ such that $a \preceq t$. In fact,

$$\begin{aligned} e \in (a \bar{\circ} e) \bar{*} \{a\} &= \{e\} \bar{*} \{a\} = e \bar{\circ} a = \{e\} \text{ and } a \preceq e; \\ e \in (b \bar{\circ} e) \bar{*} \{b\} &= \{e\} \bar{*} \{b\} = e \bar{\circ} b = \{e\} \text{ and } b \preceq e; \\ e \in (c \bar{\circ} e) \bar{*} \{c\} &= \{e\} \bar{*} \{c\} = e \bar{\circ} c = \{e\} \text{ and } c \preceq e; \\ e \in (e \bar{\circ} e) \bar{*} \{e\} &= \{e\} \bar{*} \{e\} = e \bar{\circ} e = \{e\} \text{ and } e \preceq e. \end{aligned}$$

$(S \cup \{e\}, \bar{\circ}, \preceq)$ is intra-regular as

$$\begin{aligned} e \in (e \bar{\circ} a) \bar{*} (a \bar{\circ} e) &= \{e\} \bar{*} \{e\} = e \bar{\circ} e = \{e\} \text{ and } a \preceq e; \\ e \in (e \bar{\circ} b) \bar{*} (b \bar{\circ} e) &= \{e\} \bar{*} \{e\} = \{e\} \text{ and } b \preceq e; \\ e \in (e \bar{\circ} c) \bar{*} (c \bar{\circ} e) &= \{e\} \bar{*} \{e\} = \{e\} \text{ and } c \preceq e; \\ e \in (e \bar{\circ} e) \bar{*} (e \bar{\circ} e) &= \{e\} \bar{*} \{e\} = \{e\} \text{ and } e \preceq e. \end{aligned}$$

$(S \cup \{e\}, \bar{\circ}, \preceq)$ is right regular as

$$\begin{aligned} e \in (a \bar{\circ} a) \bar{*} \{e\} &= \{a\} \bar{*} \{e\} = a \bar{\circ} e = \{e\} \text{ and } a \preceq e; \\ e \in (b \bar{\circ} b) \bar{*} \{e\} &= \{b\} \bar{*} \{e\} = b \bar{\circ} e = \{e\} \text{ and } b \preceq e; \\ e \in (c \bar{\circ} c) \bar{*} \{e\} &= \{a, b, c\} \bar{*} \{e\} = b \bar{\circ} e = \{e\} \text{ and } c \preceq e; \\ e \in (e \bar{\circ} e) \bar{*} \{e\} &= \{e\} \bar{*} \{e\} = e \bar{\circ} e = \{e\} \text{ and } e \preceq e. \end{aligned}$$

$(S \cup \{e\}, \bar{\circ}, \preceq)$ is left regular as

$$\begin{aligned} e \in (e \bar{\circ} a) \bar{*} \{a\} &= \{e\} \bar{*} \{a\} = e \bar{\circ} a = \{e\} \text{ and } a \preceq e; \\ e \in (e \bar{\circ} b) \bar{*} \{b\} &= \{a, b, c\} \bar{*} \{b\} = \{a, b, c\} \text{ and } b \preceq e; \\ e \in (e \bar{\circ} c) \bar{*} \{c\} &= \{a, b, c\} \bar{*} \{c\} = \{a, b, c\} \text{ and } c \preceq e; \\ e \in (e \bar{\circ} e) \bar{*} \{e\} &= \{e\} \bar{*} \{e\} = e \bar{\circ} e = \{e\} \text{ and } e \preceq e. \end{aligned}$$

$(S \cup \{e\}, \bar{\circ}, \preceq)$ is right quasi-regular as

$$\begin{aligned} e \in (a \bar{\circ} e) \bar{*} (a \bar{\circ} e) &= \{e\} \bar{*} \{e\} = e \bar{\circ} e = \{e\} \text{ and } a \preceq e; \\ e \in (b \bar{\circ} e) \bar{*} (b \bar{\circ} e) &= \{e\} \bar{*} \{e\} = \{e\} \text{ and } b \preceq e; \\ e \in (c \bar{\circ} e) \bar{*} (c \bar{\circ} e) &= \{e\} \bar{*} \{e\} = \{e\} \text{ and } c \preceq e; \\ e \in (e \bar{\circ} e) \bar{*} (e \bar{\circ} e) &= \{e\} \text{ and } e \preceq e. \end{aligned}$$

$(S \cup \{e\}, \bar{\circ}, \preceq)$ is left quasi-regular as

$$\begin{aligned} e \in (e \bar{\circ} a) \bar{*} (e \bar{\circ} a) &= \{e\} \bar{*} \{e\} = \{e\} \text{ and } a \preceq e. \\ e \in (e \bar{\circ} b) \bar{*} (e \bar{\circ} b) &= \{e\} \bar{*} \{e\} = \{e\} \text{ and } b \preceq e. \\ e \in (e \bar{\circ} c) \bar{*} (e \bar{\circ} c) &= \{e\} \bar{*} \{e\} = \{e\} \text{ and } c \preceq e. \end{aligned}$$

$$e \in (e \bar{o} e) \bar{*} (e \bar{o} e) = \{e\} \bar{*} \{e\} = \{e\} \text{ and } e \preceq e.$$

$(S \cup \{e\}, \bar{o}, \preceq)$ is semisimple as

$$e \in (e \bar{o} a) \bar{*} (e \bar{o} a) \bar{*} \{e\} = \{e\} \bar{*} \{e\} = \{e\} \text{ and } a \preceq e.$$

$$e \in (e \bar{o} b) \bar{*} (e \bar{o} b) \bar{*} \{e\} = \{e\} \bar{*} \{e\} = \{e\} \text{ and } b \preceq e.$$

$$e \in (e \bar{o} c) \bar{*} (e \bar{o} c) \bar{*} \{e\} = \{e\} \bar{*} \{e\} = \{e\} \text{ and } c \preceq e.$$

$$e \in (e \bar{o} e) \bar{*} (e \bar{o} e) \bar{*} \{e\} = \{e\} \bar{*} \{e\} = \{e\} \text{ and } e \preceq e.$$

S is a pseudoideal of $S \cup \{e\}$. Indeed:

$$\begin{aligned} S * S &= \{a, b, c\} * \{a, b, c\} = a \circ a \cup a \circ b \cup a \circ c \cup b \circ a \cup b \circ b \cup b \circ c \cup c \circ a \cup c \circ b \cup c \circ c \\ &= \{a, b, c\} \subseteq S \end{aligned}$$

and if $a \in S$ and $S \cup \{e\} \ni b \preceq a$, then $b \in S$ (as $b = e$ implies $e \preceq a$ and so $e = a \in S$ that is impossible).

Table 1: The multiplication of the ordered semigroup of Example 4.2.

\cdot	a	b	c
a	a	a	c
b	a	b	c
c	a	c	c

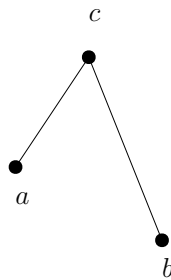


Figure 1: The order of Example 4.2.

Table 2: The hyperoperation of (S, \circ, \preceq) of Example 4.2.

\circ	a	b	c
a	$\{a\}$	$\{a\}$	$\{a, b, c\}$
b	$\{a\}$	$\{b\}$	$\{a, b, c\}$
c	$\{a\}$	$\{a, b, c\}$	$\{a, b, c\}$

Example 4.3 (see also [3]) We consider the ordered semigroup $S = \{a, b, c, d, e, f, g\}$ given by Table 4 and Figure 3. From this, in the way indicated in [5], the ordered hypersemigroup defined by Table 5 and the same figure (Figure 3) can be obtained. If h is an element not containing in S , then the Table 6 and Figure 4 define a *poe*-hypersemigroup that is regular, intra-regular, right (left) regular, right (left) quasi-regular, and semisimple. In Theorem 2.3, we assume that S is an ordered hypersemigroup. In case of *poe*-hypersemigroups, one can continue the process given in Theorem 2.3 for countable many steps as it is shown in Figure 5.

Table 3: The hyperoperation of $(S \cup \{e\}, \bar{\circ}, \preceq)$ of Example 4.2.

$\bar{\circ}$	a	b	c	e
a	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{e\}$
b	$\{a\}$	$\{b\}$	$\{a, b, c\}$	$\{e\}$
c	$\{a\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{e\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$

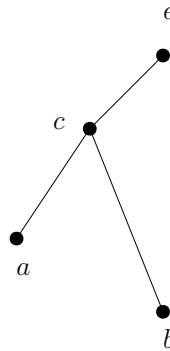


Figure 2: The order $S \cup \{e\}$ of Example 4.2.

Independently,

$(S \cup \{h\}, \bar{\circ}, \preceq)$ is regular, that is $\{a\} \preceq (a \bar{\circ} h) \bar{*} \{a\}$ for every $a \in S \cup \{h\}$; in other words, for every $a \in S \cup \{h\}$ there exists $t \in (a \bar{\circ} h) \bar{*} \{a\}$ such that $a \preceq t$. Indeed, we have

$$\begin{aligned}
 h \in (a \bar{\circ} h) \bar{*} \{a\} &= \{h\} \bar{*} \{a\} = h \bar{\circ} a = \{h\} \text{ and } a \preceq h \\
 h \in (b \bar{\circ} h) \bar{*} \{b\} &= \{h\} \bar{*} \{b\} = h \bar{\circ} b = \{h\} \text{ and } b \preceq h \\
 h \in (c \bar{\circ} h) \bar{*} \{c\} &= \{h\} \bar{*} \{c\} = h \bar{\circ} c = \{h\} \text{ and } c \preceq h \\
 h \in (d \bar{\circ} h) \bar{*} \{d\} &= \{h\} \bar{*} \{d\} = h \bar{\circ} d = \{h\} \text{ and } d \preceq h \\
 h \in (h \bar{\circ} h) \bar{*} \{e\} &= \{h\} \bar{*} \{e\} = h \bar{\circ} e = \{h\} \text{ and } e \preceq h \\
 h \in (f \bar{\circ} h) \bar{*} \{f\} &= \{h\} \bar{*} \{f\} = h \bar{\circ} f = \{h\} \text{ and } f \preceq h \\
 h \in (g \bar{\circ} h) \bar{*} \{g\} &= \{h\} \bar{*} \{g\} = h \bar{\circ} g = \{h\} \text{ and } g \preceq h \\
 h \in (h \bar{\circ} h) \bar{*} \{h\} &= \{h\} \bar{*} \{h\} = h \bar{\circ} h = \{h\} \text{ and } h \preceq h.
 \end{aligned}$$

$(S \cup \{h\}, \bar{\circ}, \preceq)$ is intraregular, that is $\{a\} \preceq (h \bar{\circ} a) \bar{*} (a \bar{\circ} h)$ for every $a \in S \cup \{h\}$; in other words, for every $a \in S \cup \{h\}$ there exists $t \in (h \bar{\circ} a) \bar{*} (a \bar{\circ} h)$ such that $a \preceq t$. Indeed, we have

$$\begin{aligned}
 h \in (h \bar{\circ} a) \bar{*} (a \bar{\circ} h) &= \{h\} \bar{\circ} \{h\} = h \bar{\circ} h = \{h\} \text{ and } a \preceq h \\
 h \in (h \bar{\circ} b) \bar{*} (b \bar{\circ} h) &= \{h\} \bar{\circ} \{h\} = h \bar{\circ} h = \{h\} \text{ and } b \preceq h \\
 h \in (h \bar{\circ} c) \bar{*} (c \bar{\circ} h) &= \{h\} \bar{\circ} \{h\} = h \bar{\circ} h = \{h\} \text{ and } c \preceq h \\
 h \in (h \bar{\circ} d) \bar{*} (d \bar{\circ} h) &= \{h\} \bar{\circ} \{h\} = \{h\} \text{ and } d \preceq h \\
 h \in (h \bar{\circ} e) \bar{*} (e \bar{\circ} h) &= \{h\} \bar{\circ} \{h\} = \{h\} \text{ and } e \preceq h \\
 h \in (h \bar{\circ} f) \bar{*} (f \bar{\circ} h) &= \{h\} \bar{\circ} \{h\} = \{h\} \text{ and } f \preceq h \\
 h \in (h \bar{\circ} g) \bar{*} (g \bar{\circ} h) &= \{h\} \bar{\circ} \{h\} = \{h\} \text{ and } g \preceq h \\
 h \in (h \bar{\circ} h) \bar{*} (h \bar{\circ} h) &= \{h\} \bar{\circ} \{h\} = \{h\} \text{ and } h \preceq h.
 \end{aligned}$$

$(S \cup \{h\}, \bar{\circ}, \preceq)$ is right regular, that is $\{a\} \preceq (a \bar{\circ} a) \bar{*} \{h\}$ for every $a \in S \cup \{h\}$; in other words, for every

$a \in S \cup \{h\}$ there exists $t \in (a \bar{\circ} a) * \{h\}$ such that $a \preceq t$. Indeed, we have

$$\begin{aligned} h \in (a \bar{\circ} a) * \{h\} &= \{a\} * \{h\} = a \bar{\circ} h = \{h\} \text{ and } a \preceq h \\ h \in (b \bar{\circ} b) * \{h\} &= \{a\} * \{h\} = a \bar{\circ} h = \{h\} \text{ and } b \preceq h \\ h \in (c \bar{\circ} c) * \{h\} &= \{a\} * \{h\} = \{h\} \text{ and } c \preceq h \\ h \in (d \bar{\circ} d) * \{h\} &= \{a\} * \{h\} = \{h\} \text{ and } d \preceq d \\ h \in (e \bar{\circ} e) * \{h\} &= \{a\} * \{h\} = \{h\} \text{ and } e \preceq h \\ h \in (f \bar{\circ} f) * \{h\} &= \{f\} * \{h\} = f \bar{\circ} h = \{h\} \text{ and } f \preceq h \\ h \in (g \bar{\circ} g) * \{h\} &= \{a\} * \{h\} = a \bar{\circ} h = \{h\} \text{ and } g \preceq h \\ h \in (h \bar{\circ} h) * \{h\} &= \{h\} * \{h\} = h \bar{\circ} h = \{h\} \text{ and } h \preceq h. \end{aligned}$$

$(S \cup \{h\}, \bar{\circ}, \preceq)$ is left regular, that is $\{a\} \preceq \{h\} * (a \bar{\circ} a)$ for every $a \in S \cup \{h\}$; that is for every $a \in S \cup \{h\}$ there exists $t \in \{h\} * (a \bar{\circ} a)$ such that $a \preceq t$. Indeed, we have

$$\begin{aligned} h \in \{h\} * (a \bar{\circ} a) &= \{h\} * \{a\} = h \bar{\circ} a = \{h\} \text{ and } a \preceq h \\ h \in \{h\} * (b \bar{\circ} b) &= \{h\} * \{a\} = \{h\} \text{ and } b \preceq h \\ c \in \{h\} * (c \bar{\circ} c) &= \{h\} * \{a\} = \{h\} \text{ and } c \preceq h \\ h \in \{h\} * (d \bar{\circ} d) &= \{h\} * \{a\} = \{h\} \text{ and } d \preceq h \\ h \in \{h\} * (e \bar{\circ} e) &= \{h\} * \{a\} = \{h\} \text{ and } e \preceq h \\ h \in \{h\} * (f \bar{\circ} f) &= \{h\} * \{f\} = h \bar{\circ} f = h \text{ and } f \preceq h \\ h \in \{h\} * (g \bar{\circ} g) &= \{h\} * \{a\} = \{h\} \text{ and } g \preceq h \\ h \in \{h\} * (h \bar{\circ} h) &= \{h\} * \{h\} = h \bar{\circ} h = \{h\} \text{ and } h \preceq h. \end{aligned}$$

$(S \cup \{h\}, \bar{\circ}, \preceq)$ is right quasi-regular, that is $\{a\} \preceq (a \bar{\circ} h) * (a * h)$ for every $a \in S \cup \{h\}$. Indeed, we have

$$\begin{aligned} \{a\} \preceq (a \bar{\circ} h) \bar{\circ} (a \bar{\circ} h) &= \{h\} * \{h\} = h \bar{\circ} h = \{h\}, \{b\} \preceq (b \bar{\circ} h) \bar{\circ} (b \bar{\circ} h) = \{h\} * \{h\} = \{h\} \\ \{c\} \preceq (c \bar{\circ} h) \bar{\circ} (c \bar{\circ} h) &= \{h\} * \{h\} = \{h\}, \{d\} \preceq (d \bar{\circ} h) \bar{\circ} (h \bar{\circ} d) = \{h\} * \{h\} = \{h\} \\ \{e\} \preceq (e \bar{\circ} h) \bar{\circ} (e \bar{\circ} h) &= \{h\} * \{h\} = \{h\}, \{f\} \preceq (f \bar{\circ} h) \bar{\circ} (f \bar{\circ} h) = \{h\} * \{h\} = \{h\} \\ \{g\} \preceq (g \bar{\circ} h) \bar{\circ} (g \bar{\circ} h) &= \{h\} * \{h\} = \{h\}, \{h\} \preceq (h \bar{\circ} h) \bar{\circ} (h \bar{\circ} h) = \{h\} * \{h\} = \{h\} \end{aligned}$$

$(S \cup \{h\}, \bar{\circ}, \preceq)$ is left quasi-regular, that is $\{a\} \preceq (h \bar{\circ} a) * (h \bar{\circ} a)$ for every $a \in S \cup \{h\}$. Indeed, we have

$$\begin{aligned} \{a\} \preceq (h \bar{\circ} a) \bar{\circ} (h \bar{\circ} a) &= \{h\} * \{h\} = \{h\}, \{b\} \preceq (h \bar{\circ} b) \bar{\circ} (h \bar{\circ} b) = \{h\} * \{h\} = \{h\} \\ \{c\} \preceq (h \bar{\circ} c) \bar{\circ} (h \bar{\circ} c) &= \{h\} * \{h\} = \{h\}, \{d\} \preceq (h \bar{\circ} d) \bar{\circ} (h \bar{\circ} d) = \{h\} * \{h\} = \{h\} \\ \{e\} \preceq (h \bar{\circ} e) \bar{\circ} (h \bar{\circ} e) &= \{h\} * \{h\} = \{h\}, \{f\} \preceq (h \bar{\circ} f) \bar{\circ} (h \bar{\circ} f) = \{h\} * \{h\} = \{h\} \\ \{g\} \preceq (h \bar{\circ} g) \bar{\circ} (h \bar{\circ} g) &= \{h\} * \{h\} = \{h\}, \{h\} \preceq (h \bar{\circ} h) \bar{\circ} (h \bar{\circ} h) = \{h\} * \{h\} = \{h\} \end{aligned}$$

$(S \cup \{h\}, \bar{\circ}, \preceq)$ is semisimple, that is $\{a\} \preceq (h \bar{\circ} a) * (h \bar{\circ} a)$ for every $a \in S \cup \{h\}$. Indeed, for every $x \in S$, we have

$$(x \bar{\circ} h) * (x \bar{\circ} h) = \{h\}; \text{ and } a \leq h, b \leq h, c \leq h, d \leq h, e \leq h, f \leq h, g \leq h, h \leq h.$$

For the definitions of intra-regular, right (left) regular, right (left) quasi-regular, and semisimple ordered hypersemigroups, we refer to [6]

Recall that the ordered hypersemigroup (S, \cdot, \leq) given by Table 4 and Figure 3 is

- (a) not regular as, for example, $\nexists x \in S$ such that $\{b\} \leq (b \circ x) * \{b\}$
- (b) not intra-regular as, for example, $\nexists x, y \in S$ such that $\{c\} \leq (x \circ x) * (c \circ y)$

- (c) not right regular, as $\nexists x \in S$ such that $\{f\} \leq (f \circ f) * \{x\}$
- (d) not left regular, as $\nexists x \in S$ such that $\{f\} \leq \{x\} * (a \circ a)$
- (e) not right quasi-regular, as $\nexists x, y \in S$ such that $\{e\} \leq (e \circ x) * (e \circ y)$
- (f) not left quasi-regular, as $\nexists x, y \in S$ such that $\{e\} \leq (x \circ e) * (y \circ y)$
- (g) not semisimple, as $\nexists x, y, z \in S$ such that $\{f\} \leq (x \circ f) * (y \circ f) * \{z\}$.

Table 4: The multiplication of the ordered semigroup S of Example 4.3.

\cdot	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	a	a	a	a	a	a
c	a	a	a	a	a	a	a
d	a	a	a	a	a	a	a
e	a	a	a	a	a	a	b
f	a	b	c	d	e	f	a
g	a	a	a	a	a	a	a

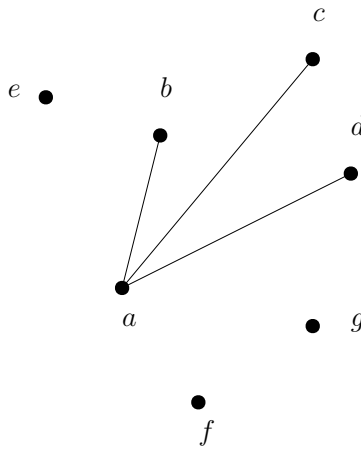


Figure 3: The order of S of Example 4.3.

Table 5: The hyperoperation of S of Example 4.3.

\circ	a	b	c	d	e	f	g
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
e	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, b\}$
f	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$	$\{e\}$	$\{f\}$	$\{a\}$
g	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$

According to Remark 4.1, we can continue this process for countable many steps, the resulting figure is the following:

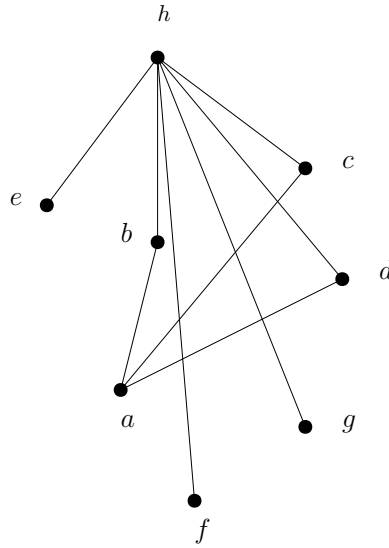


Figure 4: The order of $S \cup \{h\}$ of Example 4.3.

Table 6: The hyperoperation of $S \cup \{h\}$ of Example 4.3.

\bar{o}	a	b	c	d	e	f	g	h
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{h\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{h\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{h\}$
d	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{h\}$
e	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{h\}$
f	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$	$\{e\}$	$\{f\}$	$\{a\}$	$\{h\}$
g	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{h\}$
h	$\{h\}$	$\{h\}$	$\{h\}$	$\{h\}$	$\{h\}$	$\{h\}$	$\{h\}$	$\{h\}$

5. Pseudoideal and ideals of $S \cup \{e\}$

Proposition 5.1 *If (S, \circ, \leq) is an ordered hypersemigroup and T is a pseudoideal of S , then T is a pseudoideal of $(S \cup \{e\}, \bar{o}, \preceq)$.*

Proof Since $T * T \subseteq T$, we have $T \bar{*} T \subseteq T$. Indeed: Let $x \in T \bar{*} T$. Then $x \in a \bar{o} b$ for some $a, b \in T$. Since $a, b \in T \subseteq S$, we have $a \bar{o} b = a \circ b$. Then $x \in a \circ b = \{a\} * \{b\} \subseteq T * T \subseteq T$ and so $x \in T$.

Let now $a \in T$ and $S \cup \{e\} \ni b \preceq a$. Then $b \in T$. Indeed: We have

$a \in T, (b \in S \text{ or } b = e), b \preceq a$; that is we have the following two cases:

- (a) $a \in T, b \in S, (b \leq a \text{ or } (b, a) = (x, e) \text{ for some } x \in S \cup \{e\})$
- (b) $a \in T, b = e, (b \leq a \text{ or } (b, a) = (x, e) \text{ for some } x \in S \cup \{e\})$

So we have to check the following:

- (1) $a \in T, b \in S, b \leq a$
- (2) $a \in T, b \in S, (b, a) = (x, e)$ for some $x \in S$
- (3) $a \in T, b \in S, (b, a) = (e, e)$

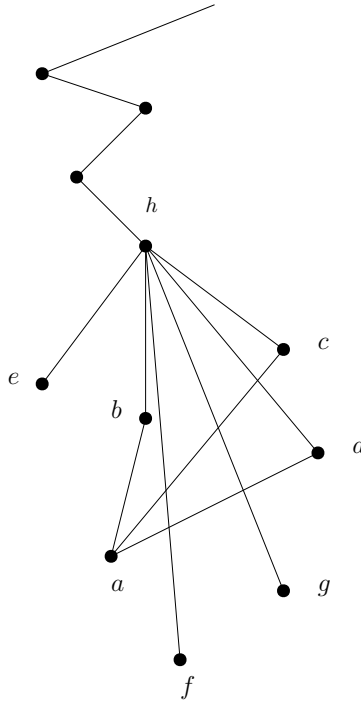


Figure 5: Theorem 2.3 holds for a *poe*-hypersemigroup as well and the process given in that theorem can be continued for countable many sets leading to the order of Figure 5.

- (4) $a \in T, b = e, b \leq a$
- (5) $a \in T, b = e, (b, a) = (x, e)$ for some $x \in S$
- (6) $a \in T, b = e, (b, a) = (e, e)$.

- (1) If $a \in T, b \in S, b \leq a$ then, since T is a pseudoideal of (S, \circ, \leq) , we have $b \in T$.
- (2) Let $a \in T, b \in S, (b, a) = (x, e)$ for some $x \in S$. Since $T \subseteq S$, we have $a \in S$. Since $(b, a) = (x, e)$ for some $x \in S$, we have $a = e$. Thus, we have $e \in S$. The case is impossible.
- (3) Let $a \in T, b \in S, (b, a) = (e, e)$. Then we have $S \ni b = e$. The case is impossible.
- (4) Let $a \in T, b = e, b \leq a$. Then we have $S \ni b = e$. The case is impossible.
- (5) Let $a \in T, b = e, (b, a) = (x, e)$ for some $x \in S$. Then $e = b = x \in S$. The case is impossible.
- (6) Let $a \in T, b = e, (b, a) = (e, e)$. Then we have $T \ni a = e = b$ and so $b \in T$. □

Proposition 5.2 Let (S, \circ, \leq) be an ordered hypersemigroup. If A is an ideal of (S, \circ, \leq) , then $A \cup \{e\}$ is an ideal of $(S \cup \{e\}, \bar{\circ})$ but it is not an ideal of $(S \cup \{e\}, \bar{\circ} \preceq)$.

Proof We have $(A \cup \{e\}) \bar{*} (S \cup \{e\}) \subseteq A \cup \{e\}$. Indeed: Let $t \in (A \cup \{e\}) \bar{*} (S \cup \{e\})$. Then $t \in x \bar{\circ} y$ for some $x \in A \cup \{e\}$ and $y \in S \cup \{e\}$. We consider the cases:

- (1) $x \in A, y \in S$
- (2) $x \in A, y = e$
- (3) $x = e, y \in S$

(4) $x = y = e$.

(1) Let $x \in A, y \in S$. Since $x, y \in S$, we have $x \bar{\circ} y = x \circ y$. We have $t \in x \circ y = \{x\} \bar{*} \{y\} \subseteq A * S \subseteq A$ and so $t \in A \subseteq A \cup \{e\}$.

(2) Let $x \in A, y = e$. Then $t \in x \bar{\circ} y = x \bar{\circ} e = \{e\}$ and so $t = e \in A \cup \{e\}$.

(3) Let $x = e, y \in S$. Then $t \in x \bar{\circ} y = e \bar{\circ} y = \{e\}$ and so $t \in A \cup \{e\}$.

(4) Let $x = y = e$. Then $t \in x \bar{\circ} y = e \bar{\circ} e = \{e\}$ and so $t \in A \cup \{e\}$.

Similarly, $(S \cup \{e\}) \bar{*} (A \cup \{e\}) \subseteq A \cup \{e\}$.

We consider the ordered hypersemigroup $S = \{a, b, c, d, e, f, g\}$ given by Table 5 and Figure 3. The set $\{a, b, e\}$ is an ideal of $S = \{a, b, c, d, e, f, g\}$. Indeed: We have

$$\begin{aligned} \{a, b, e\} * \{a, b, c, d, e, f, g\} &= (a \circ a) \cup (a \circ b) \cup (a \circ c) \cup (a \circ d) \cup (a \circ e) \cup (a \circ f) \cup (a \circ g) \\ &\quad \cup (b \circ a) \cup (b \circ b) \cup (b \circ c) \cup (b \circ d) \cup (b \circ e) \cup (b \circ f) \cup (b \circ g) \\ &\quad \cup (e \circ a) \cup (e \circ b) \cup (e \circ c) \cup (e \circ d) \cup (e \circ e) \cup (e \circ f) \cup (e \circ g) \\ &= \{a\} \cup \{b\} = \{a, b\} \subseteq \{a, b, e\}, \end{aligned}$$

similarly, $\{a, b, c, d, e, f, g\} * \{a, b, e\} = \{a\} \cup \{a, b\} \cup \{e\} \subseteq \{a, b, e\}$ and if $x \in \{a, b, e\}$ and $\{a, b, c, d, e, f, g\} \ni y \leq x$, then $y \in \{a, b, e\}$, but $\{a, b, e, h\}$ is not an ideal of $S \cup \{h\}$ as $f \in S \cup \{h\}, f \leq h$ and $f \notin \{a, b, e, h\}$. \square

Proposition 5.3 *Let (S, \circ, \leq) be an ordered hypersemigroup. If B is a bi-ideal of (S, \circ, \leq) , then $B \cup \{e\}$ is a bi-ideal of $(S \cup \{e\}, \bar{\circ})$, but it is not an ideal of $(S \cup \{e\}, \bar{\circ}, \preceq)$.*

Proof We have $(B \cup \{e\}) \bar{*} (S \cup \{e\}) \bar{*} (B \cup \{e\}) \subseteq B \cup \{e\}$. In fact:

Let $t \in (B \cup \{e\}) \bar{*} (S \cup \{e\}) \bar{*} (B \cup \{e\})$. We have $t \in u \bar{\circ} v$ for some $u \in (B \cup \{e\}) \bar{*} (S \cup \{e\}), v \in B \cup \{e\}$ and $u \in x \bar{\circ} y$ for some $x \in B \cup \{e\}, y \in S \cup \{e\}$. We have the cases:

- (a) $x \in B, (y \in S \text{ or } y = e), (v \in B \text{ or } v = e)$
- (b) $x = e, (y \in S \text{ or } y = e), (v \in B \text{ or } v = e)$.

So we have to check the following:

- (1) $x \in B, y \in S, v \in B$
- (2) $x \in B, y \in S, v = e$
- (3) $x \in B, y = e, v \in B$
- (4) $x \in B, y = e, v = e$
- (5) $x = e, y \in S, v \in B$
- (6) $x = e, y \in S, v = e$
- (7) $x = e, y = e, v \in B$
- (8) $x = y = v = e$.

(1) Let $x \in B, y \in S, v \in B$. We have $t \in u \bar{\circ} v = \{u\} \bar{*} \{v\} \subseteq (x \bar{\circ} y) \bar{*} \{v\}$. We also have $(x \bar{\circ} y) \bar{*} \{v\} \subseteq (x \circ y) * \{v\}$. Indeed: Let $t \in (x \bar{\circ} y) \bar{*} \{v\}$. Then $t \in a \bar{\circ} v$ for some $a \in x \bar{\circ} y$. Since $x, y \in S$, we have $a \in x \bar{\circ} y = x \circ y \subseteq S$. Since $a, v \in S$, we have $a \bar{\circ} v = a \circ v$. Thus, we have $t \in a \circ v = \{a\} * \{v\} \subseteq (x \circ y) * \{v\}$

and so $(x \bar{\circ} y) \bar{*} \{v\} \subseteq (x \circ y) * \{v\}$. Hence, we have $t \in (x \circ y) * \{v\} = \{x\} * \{y\} * \{v\} \subseteq B * S * B \subseteq B$ and so $t \in B \subseteq B \cup \{e\}$.

(2) Let $x \in B, y \in S, v = e$. Since $t \in u \bar{\circ} v = t \in u \bar{\circ} e = \{e\}$, we have $t = e \in B \cup \{e\}$.

(3) Let $x \in B, y = e, v \in B$. We have

$$t \in u \bar{\circ} v = \{u\} \bar{*} \{v\} \subseteq (x \bar{\circ} y) \bar{*} \{v\} = (x \bar{\circ} e) \bar{*} \{v\} = \{e\} \bar{*} \{v\} = e \bar{\circ} v = \{e\}$$

and so $t = e \in B \cup \{e\}$.

(4) Let $x \in B, y = e, v = e$. We have

$$t \in u \bar{\circ} v = \{u\} \bar{*} \{v\} \subseteq (x \bar{\circ} y) \bar{*} \{e\} = (x \bar{\circ} e) \bar{*} \{e\} = \{e\} \bar{*} \{e\} = e \bar{\circ} e = \{e\}$$

and so $t = e \in B \cup \{e\}$.

(5) Let $x = e, y \in S, v \in B$. We have

$$t \in u \bar{\circ} v = \{u\} \bar{*} \{v\} \subseteq (x \bar{\circ} y) \bar{*} \{v\} = (e \bar{\circ} y) \bar{*} \{v\} = \{e\} \bar{*} \{v\} = e \bar{\circ} v = \{e\}$$

and so $t \in B \cup \{e\}$.

(6) Let $x = e, y \in S, v = e$. We have

$$t \in u \bar{\circ} v = \{u\} \bar{*} \{v\} \subseteq (x \bar{\circ} y) \bar{*} \{v\} = (e \bar{\circ} y) \bar{*} \{e\} = \{e\} \bar{\circ} \{e\} = e \bar{\circ} e = \{e\}$$

and so $t \in B \cup \{e\}$.

(7) Let $x = e, y = e, v \in B$. We have

$$t \in u \bar{\circ} v = \{u\} \bar{*} \{v\} \subseteq (x \bar{\circ} y) \bar{*} \{v\} = (e \bar{\circ} e) \bar{*} \{v\} = \{e\} \bar{*} \{v\} = e \bar{\circ} v = \{e\}$$

and so $t \in B \cup \{e\}$.

(8) Let $t = y = v = e$. We have

$$t \in u \bar{\circ} v = \{u\} \bar{*} \{v\} \subseteq (x \bar{\circ} y) \bar{*} \{v\} = (e \bar{\circ} e) \bar{*} \{e\} = \{e\} \bar{*} \{e\} = e \bar{\circ} e = \{e\}$$

and so $t \in B \cup \{e\}$.

We consider the ordered hypersemigroup $S = \{a, b, c, d, e, f, g\}$ given by Table 5 and Figure 3. The set $\{a, b, e\}$ (as an ideal) is a bi-ideal of $S = \{a, b, c, d, e, f, g\}$, but, as we have already seen, $f \in S \cup \{h\}$, $f \leq h$ and $f \notin \{a, b, e, h\}$ and so $\{a, b, e, h\}$ is not a bi-ideal of $S \cup \{h\}$.

The set $\{a, d, e, g\}$ is a bi-ideal of $\{a, b, c, d, e, f, g\}$ as

$$\{a, d, e, g\} * \{a, b, c, d, e, f, g\} * \{a, d, e, g\} = \{a, b\} * \{a, b, c, d, e, f, g\} = \{a\} \subseteq \{a, d, e, g\},$$

$x \in \{a, d, e, g\}$ and $\{a, b, c, d, e, f, g\} \ni y \leq x$ implies $x \in \{a, d, e, g\}$, but $\{a, d, e, g, h\}$ is not a bi-ideal of $\{a, b, c, d, e, f, g, h\}$ as $h \in \{a, d, e, g, h\}$ and $\{a, b, c, d, e, f, g, h\} \ni f \leq h$, but $f \notin \{a, d, e, g, h\}$. \square

Proposition 5.4 *Let (S, \circ, \leq) be an ordered hypersemigroup. If Q is a quasi-ideal of (S, \circ, \leq) , then $Q \cup \{e\}$ is a quasi-ideal of $(S \cup \{e\}, \bar{\circ})$, but it is not a quasi-ideal of $(S \cup \{e\}, \bar{\circ}, \leq)$.*

Proof We have $\left((Q \cup \{e\}) \bar{*} (S \cup \{e\}) \right) \cap \left((S \cup \{e\}) \bar{*} (Q \cup \{e\}) \right) \subseteq Q \cup \{e\}$. Indeed:

Let $t \in \left((Q \cup \{e\}) \bar{*} (S \cup \{e\}) \right) \cap \left((S \cup \{e\}) \bar{*} (Q \cup \{e\}) \right)$. Then $t \in x \bar{\circ} y$ for some $x \in Q \cup \{e\}$, $y \in S \cup \{e\}$ and $t \in z \bar{\circ} h$ for some $z \in S \cup \{e\}$, $h \in S \cup \{e\}$. We have the cases:

- (a) $x \in Q$, ($y \in S$ or $y = e$), ($z \in S$ or $z = e$), ($h \in Q$ or $h = e$)
- (b) $x = e$, ($y \in S$ or $y = e$), ($z \in S$ or $z = e$), ($h \in Q$ or $h = e$)

So, we have to check the following:

- (1) $x \in Q, y \in S, z \in S, h \in Q$ (2) $x \in Q, y \in S, z \in S, h = e$
- (3) $x \in Q, y \in S, z = e, h \in Q$ (4) $x \in Q, y \in S, z = e, h = e$
- (5) $x \in Q, y = e, z \in S, h \in Q$ (6) $x \in Q, y = e, z \in S, h = e$
- (7) $x \in Q, y = e, z = e, h \in Q$ (8) $x \in Q, y = e, z = e, h = e$
- (9) $x = e, y \in S, z \in S, h \in Q$ (10) $x = e, y \in S, z \in S, h = e$
- (11) $x = e, y \in S, z = e, h \in Q$ (12) $x = e, y \in S, z = e, h = e$
- (13) $x = e, y = e, z \in S, h \in Q$ (14) $x = e, y = e, z \in S, h = e$
- (15) $x = e, y = e, z = e, h \in Q$ (16) $x = e, y = e, z = e, h = e$.

(1) Let $x \in Q, y \in S, z \in S, h \in Q$. Since $x, y \in S$, we have $t \in x \bar{\circ} y = x \circ y = \{x\} * \{y\} \subseteq Q * S$. Since $z, h \in S$, we have $t \in z \bar{\circ} h = z \circ h = \{z\} * \{h\} \subseteq S * Q$. Then we have $t \in (Q * S) \cap (S * Q) \subseteq (Q * S) \cap (S * Q) \subseteq Q$ and so $t \in Q \subseteq Q \cup \{e\}$.

(2) Let $x \in Q, y \in S, z \in S, h = e$. Since $x, y \in S$, we have $t \in x \bar{\circ} y = x \circ y = \{x\} * \{y\} \subseteq Q * S \subseteq S$. Since $z \in S, h = e$, we have $t \in z \bar{\circ} h = z \bar{\circ} e = \{e\}$. Then $S \ni t = e$. The case is impossible.

(3) Let $x \in Q, y \in S, z = e, h \in Q$. Then $t \in x \bar{\circ} y = x \circ y = \{x\} * \{y\} \subseteq Q * S \subseteq S$ and $t \in z \bar{\circ} h = z \bar{\circ} e = \{e\}$. Then we have $S \ni t = e$. The case is impossible.

(4) Let $x \in Q, y \in S, z = e, h = e$. Then $t \in x \bar{\circ} y = x \circ y = \{x\} * \{y\} \subseteq Q * S \subseteq S$ and $t \in z \bar{\circ} h = e \bar{\circ} e = \{e\}$. Then $S \ni t = e$, the case is impossible.

(5) Let $x \in Q, y = e, z \in S, h \in Q$. Then $t \in x \bar{\circ} y = x \bar{\circ} e = \{e\}$, $t \in z \bar{\circ} h = z \circ h = \{z\} * \{h\} \subseteq S * Q \subseteq S$. Then $S \ni t = e$, the case is impossible.

(6) Let $x \in Q, y = e, z \in S, h = e$. Then $t \in x \bar{\circ} y = x \bar{\circ} e = \{e\}$, $t \in z \bar{\circ} h = z \bar{\circ} e = \{e\}$. Then $t = e \in Q \cup \{e\}$.

(7) Let $x \in Q, y = e, z = e, h \in Q$. Then $t \in x \bar{\circ} y = x \bar{\circ} e = \{e\}$, $t \in z \bar{\circ} h = e \bar{\circ} h = \{e\}$. Then $t = e \in Q \cup \{e\}$.

(8) Let $x \in Q, y = e, z = e, h = e$. Then $t \in x \bar{\circ} y = x \bar{\circ} e = \{e\}$, $t \in z \bar{\circ} h = e \bar{\circ} e = \{e\}$. Then $t = e \in Q \cup \{e\}$.

(9) Let $x = e, y \in S, z \in S, h \in Q$. Then $t \in x \bar{\circ} y = e \bar{\circ} y = \{e\}$, $t \in z \bar{\circ} h = z \circ h = \{z\} * \{h\} \subseteq S * Q \subseteq S$. Then $S \ni t = e$, the case is impossible.

(10) Let $x = e, y \in S, z \in S, h = e$. Then $t \in x \bar{\circ} y = e \bar{\circ} y = \{e\}$, $t \in z \bar{\circ} h = z \bar{\circ} e = \{e\}$ and so $t = e \in Q \cup \{e\}$.

(11) Let $x = e, y \in S, z = e, h \in Q$. Then $t \in x \bar{\circ} y = e \bar{\circ} y = \{e\}$, $t \in z \bar{\circ} h = e \bar{\circ} h = \{e\}$ and so $t = e \in Q \cup \{e\}$.

(12) Let $x = e, y \in S, z = e, h = e$. Then $t \in x \bar{\circ} y = e \bar{\circ} y = \{e\}$, $t \in z \bar{\circ} h = e \bar{\circ} e = \{e\}$ and so

$t = e \in Q \cup \{e\}$.

(13) Let $x = e, y = e, z \in S, h \in Q$. Then $t \in x \bar{\circ} y = e \bar{\circ} e = \{e\}, t \in z \bar{\circ} h = z \circ h = \{z\} * \{h\} \subseteq S * Q \subseteq S$ and so $S \ni t = e$, the case is impossible.

(14) Let $x = e, y = e, z \in S, h = e$. Then $t \in x \bar{\circ} y = e \bar{\circ} e = \{e\}, t \in z \bar{\circ} h = z \bar{\circ} e = \{e\}$ and so $t = e \in Q \cup \{e\}$.

(15) Let $x = e, y = e, z = e, h \in Q$. Then $t \in x \bar{\circ} y = e \bar{\circ} e = \{e\}, t \in z \bar{\circ} h = e \bar{\circ} h = \{e\}$ and so $t = e \in Q \cup \{e\}$.

(16) Let $x = e, y = e, z = e, h = e$. Then $t \in x \bar{\circ} y = e \bar{\circ} e = \{e\}, t \in z \bar{\circ} h = e \bar{\circ} e = \{e\}$ and so $t = e \in Q \cup \{e\}$.

We consider the ordered hypersemigroup $S = \{a, b, c, d, e, f, g\}$ given by Table 5 and Figure 3. The set $\{a, b, f, g\}$ is a quasi-ideal of $\{a, b, c, d, e, f, g\}$ as

$$\begin{aligned} (\{a, b, f, g\} * \{a, b, c, d, e, f, g\}) &\cap (\{a, b, c, d, e, f, g\} * \{a, b, f, g\}) \\ &= (\{a, b, c, d, e, f\} \cap \{a, b, f\}) \\ &= \{a, b, c, d, f, e\} \cap \{a, b, f\} \\ &= \{a, b, f\} \subseteq \{a, b, f, g\}; \end{aligned}$$

$x \in \{a, b, f, g\}$ and $\{a, b, c, d, e, f, g\} \ni y \leq x$ implies $y \in \{a, b, f, g\}$, but $\{a, b, f, g, h\}$ is not a quasi-ideal of $\{a, b, c, d, e, f, g, h\}$ as $h \in \{a, b, f, g, h\}$ and $\{a, b, c, d, e, f, g\} \ni h \preceq h$, but $c \notin \{a, b, f, g, h\}$. \square

Proposition 5.5 *Let (S, \circ, \leq) be an ordered hypersemigroup. If A is an interior ideal of (S, \circ, \leq) , then $A \cup \{e\}$ is an interior ideal of $(S \cup \{e\}, \bar{\circ})$, but it is not an interior ideal of $(S \cup \{e\}, \bar{\circ}, \preceq)$.*

Proof We have $(S \cup \{e\}) \bar{*} (A \cup \{e\}) \bar{*} (S \cup \{e\}) \subseteq A \cup \{e\}$. Indeed:

Let $t \in (S \cup \{e\}) \bar{*} (A \cup \{e\}) \bar{*} (S \cup \{e\})$. Then $t \in x \bar{\circ} y$ for some $x \in (S \cup \{e\}) \bar{*} (A \cup \{e\}), y \in S \cup \{e\}$ and $x \in u \bar{\circ} v$ for some $u \in S \cup \{e\}, v \in A \cup \{e\}$. We have the cases:

- (a) $y \in S, (u \in S \text{ or } u = e), (v \in A \text{ or } v = e)$
- (b) $y = e, (u \in S \text{ or } u = e), (v \in A \text{ or } v = e)$.

So we have to check the following:

- (1) $y \in S, u \in S, v \in A$
- (2) $y \in S, u \in S, v = e$
- (3) $y \in S, u = e, v \in A$
- (4) $y \in S, u = e, v = e$
- (5) $y = e, u \in S, v \in A$
- (6) $y = e, u \in S, v = e$
- (7) $y = e, u = e, v \in A$
- (8) $y = e, u = e, v = e$.

(1) Let $y \in S, u \in S, v \in A$. Then we have

$$t \in x \bar{o} y = \{x\} \bar{*} \{y\} \subseteq (u \bar{o} v) \bar{*} \{y\} = \{u\} \bar{*} \{v\} \bar{*} \{y\} \subseteq S * A * S \subseteq A \subseteq A \cup \{e\}.$$

- (2) Let $y \in S, u \in S, v = e$. Then $t \in u \bar{o} v = u \bar{o} e = \{e\} \subseteq A \cup \{e\}$.
- (3) Let $y \in S, u = e, v \in A$. Then $t \in u \bar{o} v = e \bar{o} v = \{e\} \subseteq A \cup \{e\}$.
- (4) Let $y \in S, u = e, v = e$. Then $t \in u \bar{o} v = e \bar{o} e = \{e\} \subseteq A \cup \{e\}$.
- (5) Let $y = e, u \in S, v \in A$. Then $t \in x \bar{o} y = x \bar{o} e = \{e\} \subseteq A \cup \{e\}$.
- (6) let $y = e, u \in S, v = e$. Then $t \in x \bar{o} y = x \bar{o} e = \{e\} \subseteq A \cup \{e\}$.
- (7) Let $y = e, u = e, v \in A$. Then $t \in x \bar{o} y = x \bar{o} e = \{e\} \subseteq A \cup \{e\}$
- (8) Let $y = e, u = e, v = e$. Then $t \in x \bar{o} y = e \bar{o} e = \{e\} \subseteq A \cup \{e\}$.

We consider the ordered hypersemigroup $S = \{a, b, c, d, e, f, g\}$ given by Table 5 and Figure 3. The set $\{a\}$ is an interior ideal element of $S = \{a, b, c, d, e, f, g\}$ as

$$\{a, b, c, d, e, f, g\} * \{a\} = (a \circ a) \cup (b \circ a) \cdots (f \circ a) \cup (g \circ a) = \{a\},$$

$$\{a, b, c, d, e, f, g\} * \{a\} * \{a, b, c, d, e, f, g\} = \{a\},$$

if $x \in \{a\}$ and $\{a, b, c, d, e, f, g\} \ni y \leq a$, then $y = a$.

However, $\{a\} \cup \{h\}$ is not an interior ideal of $\{a, b, c, d, e, f, g, h\}$. Indeed, $\{a, b, c, d, e, f, g, h\} \ni c \leq h$, but $c \notin \{a, h\}$. □

Note Concerning the ordered hypersemigroup (S, \circ, \leq) given by Table 5 and Figure 3, it might be mentioned that

The ideals of (S, \circ, \leq) are the sets: $\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, b, c, d, e\}, \{a, b, c, d, e, f\}, \{a, b, g\}, \{a, b, c, g\}, \{a, b, d, g\}, \{a, b, c, d, g\}, \{a, b, e, g\}, \{a, b, c, e, g\}, \{a, b, d, e, g\}, \{a, b, c, d, e, g\}$ and S (total 22).

The quasi-ideals of (S, \circ, \leq) are the ideals of S plus the sets $\{a, e\}, \{a, c, e\}, \{a, d, e\}, \{a, c, d, e\}, \{a, f\}, \{a, b, f\}, \{a, c, f\}, \{a, b, c, f\}, \{a, d, f\}, \{a, b, d, f\}, \{a, c, d, f\}, \{a, b, c, d, f\}, \{a, e, f\}, \{a, b, e, f\}, \{a, c, e, f\}, \{a, b, c, e, f\}, \{a, d, e, f\}, \{a, b, d, e, f\}, \{a, c, d, e, f\}, \{a, g\}, \{a, c, g\}, \{a, d, g\}, \{a, c, d, g\}, \{a, b, f, g\}, \{a, b, c, f, g\}, \{a, b, d, f, g\}, \{a, b, c, d, f, g\}, \{a, b, e, f, g\}, \{a, b, c, e, f, g\}, \{a, b, d, e, f, g\}$ (total 52).

The bi-ideals of (S, \circ, \leq) are the quasi-ideals of S plus the sets $\{a, e, g\}, \{a, c, e, g\}, \{a, d, e, g\}, \{a, c, d, e, g\}$ (total 56).

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