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Boundary value problem for a loaded fractional diffusion equation

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Abstract: In this paper we consider a boundary value problem for a loaded fractional diffusion equation. The loaded term has the form of the Riemann-Liouville fractional derivative or integral. The BVP is considered in the open right upper quadrant. The problem is reduced to an integral equation that, in some cases, belongs to the pseudo-Volterra type, and its solvability depends on the order of differentiation in the loaded term and the behavior of the support line of the load in a neighborhood of the origin. All these cases are considered. In particular, we establish sufficient conditions for the unique solvability of the problem. Moreover, we give an example showing that violation of these conditions can lead to nonuniqueness of the solution.

Key words: Fractional diffusion equation, loaded equation, fractional derivative, integral equation, Wright function

1. Introduction

Consider the equation

$$\left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial x^2}\right) u(x,y) + \lambda \left[D_{0y}^{\sigma} u(x,y)\right]_{x=\gamma(y)} = f(x,y), \quad (0 < \alpha \le 1, \ \sigma \le \alpha) \tag{1.1}$$

where D_{0y}^{α} (as well as D_{0y}^{σ}) stands for the Riemann-Liouvlle fractional derivative (D_{0y}^{σ} may be a derivative or integral) with respect to y of order $\alpha \in (0,1)$ ($\sigma \le \alpha$) with the origin at the point $x = 0$; $\gamma(y)$ is a continuous function that is positive for $y > 0$.

Operators of fractional order integration and differentiation are defined as follows (see, e.g., [14])

$$D_{0y}^{\nu} u(x,y) = \frac{1}{\Gamma(-\nu)} \int_0^y u(t)(y-t)^{-\nu-1}\, dt \qquad (\nu < 0); \qquad D_{0y}^{0} u(x,y) = u(x,y); \tag{1.2}$$

and

$$D_{0y}^{\nu} u(x,y) = \frac{\partial}{\partial y} D_{0y}^{\nu-1} u(x,y) \qquad (0 < \nu \le 1).$$

Over the past three decades, fractional diffusion equations have been the subject of intense research attention, starting from the works [6, 29]. Papers [1, 3, 4, 11, 12, 16, 20–24] give an idea of the variety of

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approaches to the study of these equations. Overviews can be found in [4]. The monographs [5, 7, 18] reflect basic approaches and contain numerous bibliographies concerning the issue.

Equation (1.1) belongs to the class of loaded equations [15]. Of interest are boundary value problems for the loaded heat equation, when the loaded term is represented in the form of a fractional derivative [8, 9, 25]. In [25] the loaded term is the trace of a fractional derivative on the manifold $x = t^{\omega}$. It is proved that there is continuity in the order of the fractional derivative. In [9] the loaded term is represented in the form of a fractional derivative with respect to the time variable. In [8] it is shown that the existence and uniqueness of solutions to the integral equation depend on the order of the fractional derivative in the loaded term. Integral equations of Volterra type also arise in the study of boundary value problems in degenerate domains [2, 10].

In the present paper, we solve a boundary value problem in the right upper quadrant for the equation (1.1). The problem is reduced to an integral equation that can belong to the pseudo-Volterra type. We investigate its solvability that essentially depends on σ and the behavior of $\gamma(y)$ in a neighborhood of the point $y = 0$. The main results are given in the form of two theorem that establish sufficient conditions for the unique solvability of the problem, in different cases. In addition, we give an example showing that violation of these conditions can lead to the nonuniqueness of the solution.

The article is structured as follows. In Section 2, we formulate the problem we are going to solve. Section 3 contains notations and several auxiliary assertions. In Section 4, the problem is reduced to an integral equation. In Sections 5 and 6, we consider various cases of the obtained integral equation and formulate the corresponding results on the solvability of the problem. Section 7 gives an example of a nontrivial solution to the homogeneous problem.

2. Section

In what follows Ω denotes a domain that coincides with the open right upper quadrant, i.e.

$$\Omega = \{(x, y) : \, x > 0, \; y > 0\}.$$

We call a function $u(x, y)$ a regular solution of the equation (1.1) in the domain Ω if $y^{1-\mu} u(x, y) \in C\left(\overline{\Omega}\right)$ for some $\mu > 0$; in Ω, $u(x, y)$ has continuous derivatives with respect to x of the first and second order; the function $D_{0y}^{\alpha-1} u(x, y)$ is continuously differentiable as function of y for a fixed x at interior points of Ω and is continuous up to the positive semiaxis $\{0 < x, \; y = 0\}$; and $u(x, y)$ satisfies the equation (1.1) at all points of Ω.

Problem 2.1 *Find a regular solution of the equation (1.1) in the domain Ω satisfying the conditions*

$$\lim_{y \to 0} D_{0y}^{\alpha-1} u(x, y) = \tau(x) \qquad (x > 0) \tag{2.1}$$

and

$$u(0, y) = \varphi(y) \qquad (y > 0) \tag{2.2}$$

where $\tau(x)$ and $\varphi(y)$ are given continuous functions.

3. Preliminaries

Set

$$G_{\mu}(x, s, y) = \frac{y^{\mu-1}}{2} \left[\Phi\left(-\beta, \mu; -\frac{|x-s|}{y^{\beta}}\right) - \Phi\left(-\beta, \mu; -\frac{x+s}{y^{\beta}}\right) \right]. \tag{3.1}$$

Here and in what follows,

$$\beta = \frac{\alpha}{2} \tag{3.2}$$

and Φ is the Wright function ([27, 28]),

$$\Phi\left(a,b;\, z\right) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(ak+b)} \qquad (a > -1).$$

Also, we will use the letter C to denote positive constants, which are different in different cases, indicating in parentheses the parameters on which they depend, if necessary, $C = C(\alpha, \beta, ...)$.

It is known that ([19, 28])

$$\Phi(-\beta, \mu; -t) = C\, t^{\rho\left(\frac{1}{2}-\mu\right)} \exp\left(-\frac{(\beta t)^{\rho}}{\rho-1}\right)\left[1 + O\left(\frac{1}{t^{\rho}}\right)\right] \quad (t \to \infty) \tag{3.3}$$

where $\beta \in (0,1)$, $\rho^{-1} = 1-\beta$,

$$\frac{d}{dx}\,\Phi(-\beta,\, \mu;\, x) = \Phi(-\beta,\, \mu-\beta;\, x), \tag{3.4}$$

$$D_{0y}^{\nu}\left[y^{\mu-1}\Phi\left(-\beta,\, \mu;\, -\frac{c}{y^{\beta}}\right)\right] = y^{\mu-\nu-1}\Phi\left(-\beta,\, \mu-\nu;\, -\frac{c}{y^{\beta}}\right) \quad (c>0), \tag{3.5}$$

and

$$\int_0^{\infty} \Phi\left(-\beta,\, \mu;\, -x\right)\, dx = \frac{1}{\Gamma(\beta+\mu)}. \tag{3.6}$$

The asymptotic expansion (3.3) gives that

$$\left|y^{\mu-1}\Phi\left(-\beta,\, \mu;\, -\frac{x}{y^{\beta}}\right)\right| \le C\, x^{-\theta} y^{\beta\theta+\mu-1}, \tag{3.7}$$

where

$$\theta \ge \begin{cases} 0, & (-\mu) \notin \mathbb{N} \cup \{0\}, \\ -1, & (-\mu) \in \mathbb{N} \cup \{0\}. \end{cases}$$

The formulas (3.4), (3.5), and (3.6) yield that

$$D_{0y}^{\nu} G_{\mu}(x,s,y) = G_{\mu-\nu}(x,s,y), \quad \lim_{\varepsilon\to 0+}\left[\frac{\partial}{\partial x} G_{\mu}(x,s,y)\right]_{s=x-\varepsilon}^{s=x+\varepsilon} = \frac{y^{\mu-\beta-1}}{\Gamma(\mu-\beta)}, \tag{3.8}$$

$$\left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial x^2}\right) G_{\mu}(x,s,y) = 0, \quad \left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial s^2}\right) G_{\mu}(x,s,y) = 0 \quad (x \ne s), \tag{3.9}$$

and

$$\int_0^{\infty} G_{\mu}(x,s,y)ds = y^{\mu+\beta-1}\left[\frac{1}{\Gamma(\mu+\beta)} - \Phi\left(-\beta,\, \mu+\beta;\, -\frac{x}{y^{\beta}}\right)\right]. \tag{3.10}$$

Let us recall the form of solutions to the problem (2.1) and (2.2) for the equation (1.1) without the loaded term, i.e. for the equation

$$\left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial x^2}\right) u(x,y) = f(x,y). \tag{3.11}$$

Lemma 3.1 *Let*

$$\tau(x) \in C[0,\infty), \qquad y^{1-\mu}\varphi(y) \in C[0,\infty), \qquad y^{1-\mu} f(x,y) \in C(\overline{\Omega}), \tag{3.12}$$

$$\lim_{x\to\infty} \tau(x) \exp\left(-\omega x^{\frac{2}{2-\alpha}}\right) = 0, \qquad \lim_{x\to\infty} y^{1-\mu} f(x,y) \exp\left(-\omega x^{\frac{2}{2-\alpha}}\right) = 0, \tag{3.13}$$

for some $\mu > 0$ *and every* $\omega > 0$, *and let the function* $f(x,y)$ *can be represented in the form*

$$f(x,y) = D_{0y}^{-\delta} f^*(x,y) \tag{3.14}$$

for some $\delta > \beta$ *and* $y^{1-\mu} f^*(x,y) \in C(\overline{\Omega})$.

Then there exists a unique solution of the problem (3.11), (2.1), and (2.2); it has the form

$$u(x,y) = \int_0^y \int_0^\infty f(s,t)\, G_\beta(x,s,y-t)\, ds\, dt + \tag{3.15}$$

$$+ \int_0^y \varphi(t)\, G_0(x,0,y-t)\, dt + \int_0^\infty \tau(s)\, G_\beta(x,s,y)\, ds,$$

where $G_\mu(x,s,y)$ *is defined by (3.1).*

Proof The fact that a regular solution to the problem (3.11), (2.1), and (2.2) has the form (3.15), can be adapted from [17, Theorem 8.1]. In particular, this presentation proves the uniqueness of the problem. A direct verification shows that the conditions (3.12), (3.13), and (3.14) are sufficient for the function (3.15) to be a solution of (3.11), (2.1), and (2.2). For example, by (3.8), (3.9), and (3.14), it is easy to see that

$$\frac{\partial^2}{\partial x^2} \int_0^y \int_0^\infty f(s,t)\, G_\beta(x,s,y-t)\, ds\, dt = \frac{\partial}{\partial x} \int_0^y \int_0^\infty f^*(s,t)\, \frac{\partial}{\partial x} G_{\beta+\mu}(x,s,y-t)\, ds\, dt =$$

$$= -\int_0^y \frac{(y-t)^{\mu-1}}{\Gamma(\mu)} f^*(x,t)\, ds\, dt + \int_0^y \int_0^\infty f^*(s,t) G_{\mu-\beta}(x,s,y-t)\, ds\, dt =$$

$$= -f(x,y) + D_{0y}^{\alpha} \int_0^y \int_0^\infty f(s,t) G_\beta(x,s,y-t)\, ds\, dt.$$

This proves that

$$\left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial x^2}\right) \int_0^y \int_0^\infty f(s,t)\, G_\beta(x,s,y-t)\, ds\, dt = f(x,y).$$

□

Remark 3.2 *It should be noted that condition (3.14) is superfluous if the function* $f(x,y)$ *does not depend on* x. *Indeed, if* $f(x,y) = f(y)$ *then, by (3.10), we have*

$$\int_0^y \int_0^\infty f(s,t)\, G_\beta(x,s,y-t)\, ds\, dt = D_{0y}^{-\alpha} f(y) - \int_0^y f(t)\, (y-t)^{\alpha-1}\, \Phi\left(-\beta,\, \alpha;\, -\frac{x}{(y-t)^\beta}\right) dt.$$

and

$$\left(D_{0y}^{\alpha}-\frac{\partial^2}{\partial x^2}\right)\int_0^y f(t)\,(y-t)^{\alpha-1}\,\Phi\left(-\beta,\,\alpha;\,-\frac{x}{(y-t)^{\beta}}\right)dt=0.$$

Consequently,

$$\left(D_{0y}^{\alpha}-\frac{\partial^2}{\partial x^2}\right)\int_0^y\int_0^{\infty} f(s)\,G_{\beta}(x,s,y-t)\,ds\,dt=f(y).$$

4. Reduction to integral equation

Let $g(y)$ denote the loaded term in (1.1), namely

$$g(y)=\left[D_{0y}^{\sigma}u(x,y)\right]_{x=\gamma(y)} \tag{4.1}$$

By Lemma 3.1, the solution of the equation

$$\left(D_{0y}^{\alpha}-\frac{\partial^2}{\partial x^2}\right)u(x,y)=f(x,y)-\lambda g(y)$$

that satisfies (2.1) and (2.2) can be written in the form

$$u(x,y)=F(x,y)-\lambda\int_0^y\int_0^{\infty} g(t)\,G_{\beta}(x,s,y-t)\,ds\,dt. \tag{4.2}$$

where

$$F(x,y)=\int_0^y\int_0^{\infty} f(s,t)\,G_{\beta}(x,s,y-t)\,ds\,dt+$$
$$+\int_0^y\varphi(t)\,G_0(x,0,y-t)\,dt+\int_0^{\infty}\tau(s)\,G_{\beta}(x,s,y)\,ds.$$

By (1.2), (3.2), and (3.10), the equality (4.2) is equivalent to

$$u(x,y)=F(x,y)-\lambda D_{0y}^{-\alpha}g(y)+\lambda\int_0^y g(t)\,(y-t)^{\alpha-1}\Phi\left(-\beta,\,\alpha;\,-\frac{x}{(y-t)^{\beta}}\right)dt. \tag{4.3}$$

Formulas (3.5) and (4.1) give

$$g(y)=h(y)-\lambda D_{0y}^{\sigma-\alpha}g(y)+\lambda\int_0^y g(t)\,(y-t)^{\alpha-\sigma-1}\Phi\left(-\beta,\,\alpha-\sigma;\,-\frac{\gamma(y)}{(y-t)^{\beta}}\right)dt, \tag{4.4}$$

where

$$h(y)=\left[D_{0y}^{\sigma-\alpha}F(x,y)\right]_{x=\gamma(y)}.$$

Moreover, if the conditions (3.12), (3.13), and (3.14) are satisfied, then

$$y^{1-\mu}h(y)\in C[0,\infty)\qquad\text{for some}\quad\mu>0. \tag{4.5}$$

The formula (4.4) is valid for every σ.

It should be noted that, in some cases, the integral equation (4.4) does not belong to the Volterra type, and its solvability depends on σ and the behavior of $\gamma(y)$ in a neighborhood of the point $y=0$. We next consider these cases. In particular, we investigate separately the cases $\sigma<\alpha$ and $\sigma=\alpha$.

5. The case $\sigma < \alpha$

It follows from (1.2) and (4.4) that $g(y)$ is a solution of the integral equation

$$g(y) + \lambda \int_0^y g(t)\, H(y-t)\, dt = h(y), \tag{5.1}$$

where

$$H(y) = y^{\alpha-\sigma-1} \left[\frac{1}{\Gamma(\alpha-\sigma)} - \Phi\left(-\beta,\, \alpha-\sigma;\, -\frac{\gamma(y)}{y^\beta}\right)\right].$$

Moreover, the inequality (3.7) gives that

$$|H(y)| \le C y^{\alpha-\sigma-1}.$$

This means that the integral equation (5.1) has a solution that is unique. Moreover, it is easy to check that this solution is continuous for $y > 0$ and has a singularity at the point $y = 0$, the same as the function $h(y)$ has. Therefore, $y^{1-\mu} g(y) \in C[0,\infty)$ for some $\mu > 0$.

Thus, taking into account Lemma 3.1 and Remark 3.2, we can formulate the result obtained in the considered case.

Theorem 5.1 *Let $\sigma < \alpha$ and the conditions of Lemma 3.1 imposed on $f(x,y)$, $\tau(x)$, and $\varphi(y)$ be satisfied. Then the problem (1.1), (2.1), and (2.2) has a unique regular solution that can be given by (4.2), where $g(y)$ is the solution of (5.1).*

6. The case $\sigma = \alpha$

Assuming that $\lambda \ne -1$, from (4.4), we get

$$g(y) = h(y) + \frac{\lambda}{1+\lambda} \int_0^y \frac{g(t)}{y-t} \Phi\left(-\beta,\, 0;\, -\frac{\gamma(y)}{(y-t)^\beta}\right) dt. \tag{6.1}$$

By (3.7) we have

$$\frac{1}{y-t} \Phi\left(-\beta,\, 0;\, -\frac{\gamma(y)}{(y-t)^\beta}\right) \le C \left[\gamma(y)\right]^{-\theta} (y-t)^{\beta\theta-1} \qquad (\theta \ge -1). \tag{6.2}$$

(Notice that $\Phi(-\beta,\, \delta;\, -x)$ is positive for $x > 0$ and $\delta \ge 0$, [13, 26].)

Take $\theta = -1$ in (6.2) and let

$$\gamma(y) \ge C y^\delta. \tag{6.3}$$

Then, it follows from (6.2) that

$$\frac{1}{y-t} \Phi\left(-\beta,\, 0;\, -\frac{\gamma(y)}{(y-t)^\beta}\right) \le C y^{-\delta} (y-t)^{\beta-1}. \tag{6.4}$$

(Note that, according to our convention above, in the last two inequalities, the letter C denotes different constants.)

Let K denote the integral operator in (6.1), i.e.

$$K\,g(y) = \int_0^y \frac{g(t)}{y-t}\Phi\left(-\beta,\,0;\,-\frac{\gamma(y)}{(y-t)^\beta}\right)\,dt. \tag{6.5}$$

By (1.2), (4.5), and (6.4), we get

$$K\,h(y) \le C y^{-\delta} D_{0y}^{-\beta} y^{\mu-1} \le C\frac{\Gamma(\mu)}{\Gamma(\mu+\beta)}\cdot y^{\mu+\beta-\delta-1}$$

and, consequently,

$$K^n\,h(y) \le C^n \left(y^{-\delta}D_{0y}^{-\beta}\right)^n y^{\mu-1} \le C^n \prod_{k=0}^{n-1} \frac{\Gamma(\mu+k(\beta-\delta))}{\Gamma(\mu+\beta+k(\beta-\delta))}\cdot y^{\mu+n(\beta-\delta)-1}$$

This means that if $\delta<\beta$ then the series

$$\sum_{n=0}^{\infty}\frac{\lambda^n}{(1+\lambda)^n}K^n\,h(y)$$

is convergent, and its sum defines a unique solution of (6.1).

Thus, the result for this case can be formulated as follows.

Theorem 6.1 *Let $\sigma=\alpha$, $\lambda\ne-1$, and (3.12), (3.13), and (3.14) be satisfied. If $\gamma(y)$ satisfies (6.3) for some $\delta<\beta$, then the problem (1.1), (2.1), and (2.2) has a unique regular solution that is given by (4.2), where $g(y)$ can be found as a solution of (6.1).*

7. Spectral case

By (1.2), (3.5), and (6.5), it is easy to check that

$$K\,y^\varepsilon = \int_0^y \frac{t^\varepsilon}{y-t}\Phi\left(-\beta,\,0;\,-\frac{\gamma(y)}{(y-t)^\beta}\right)\,dt = \int_0^y \frac{(y-t)^\varepsilon}{t}\Phi\left(-\beta,\,0;\,-\frac{\gamma(y)}{t^\beta}\right)\,dt =$$

$$= \Gamma(\varepsilon+1)\left[D_{0t}^{-\varepsilon-1}\frac{1}{t}\Phi\left(-\beta,\,0;\,-\frac{\gamma(y)}{t^\beta}\right)\right]_{t=y} = \Gamma(\varepsilon+1)\,y^\varepsilon\Phi\left(-\beta,\,\varepsilon+1;\,-\frac{\gamma(y)}{y^\beta}\right).$$

Letting $\sigma=\alpha$ and

$$\gamma(y) = Ay^\beta \qquad (A>0),$$

we get that the function

$$g(y) = y^\varepsilon$$

is a solution of the homogeneous equation

$$g(y) = \frac{\lambda}{1+\lambda}\int_0^y \frac{g(t)}{y-t}\Phi\left(-\beta,\,0;\,-\frac{\gamma(y)}{(y-t)^\beta}\right)\,dt$$

where λ is defined by the equality

$$1 = \frac{\lambda}{1+\lambda}\Gamma(\varepsilon+1)\,\Phi\left(-\beta,\,\varepsilon+1;\,-A\right)$$

or

$$\lambda = \left[\Gamma(\varepsilon+1)\,\Phi\left(-\beta,\,\varepsilon+1;\,-A\right)-1\right]^{-1}. \tag{7.1}$$

It is known ([18, Lemma 2.2.4]) that

$$\frac{d}{dz}\Phi\left(-\beta,\,\delta;\,-z\right) < 0 \qquad (z>0,\ \delta\geq\beta)$$

and, moreover,

$$\Phi\left(-\beta,\,\delta;\,0\right) = \frac{1}{\Gamma(\delta)}.$$

This yields that there exists $\lambda_0 < 0$ such that for any $\lambda < \lambda_0$ one can find ε and A that turn (7.1) into equality.

Thus, by (4.3), we get that, for λ, ε, and A satisfying (7.1), the function

$$u(x,y) = y^{\varepsilon+\alpha}\left[\frac{1}{\Gamma(\varepsilon+\alpha+1)} - \Phi\left(-\beta,\,\varepsilon+\alpha+1;\,-\frac{x}{y^{\beta}}\right)\right]$$

is a solution of the homogeneous problem

$$\left(D_{0y}^{\alpha} - \frac{\partial^2}{\partial x^2}\right)u(x,y) + \lambda\left[D_{0y}^{\alpha}u(x,y)\right]_{x=Ay^{\varepsilon}} = 0,$$

$$\lim_{y\to 0} D_{0y}^{\alpha-1}u(x,y) = 0, \qquad u(0,y) = 0.$$

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