TÜBİTAK

Turkish Journal of Mathematics

Volume 47 | Number 5

Article 19

1-1-2023

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 1558 – 1572
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doi:10.55730/1300-0098.3448

# What can lattices do for hypersemigroups?

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**Received:** 13.03.2023 • **Accepted/Published Online:** 05.06.2023 • **Final Version:** 18.07.2023

**Abstract:** This is from Birkhoff, the "father of lattice theory" in Trends in Lattice Theory. Van Nostrand 1970: "Lattices can do things for you, no matter what kind of mathematician you are!". The aim of this paper is to show that the $le$-semigroups (lattice ordered semigroups possessing a greatest element) play the main role in studying the ordered hypersemigroups. From many results on lattice ordered semigroups corresponding results on ordered semigroups can be obtained. The converse is also possible but the beauty and simplicity of "order" makes it easier to investigate the lattice ordered semigroup at first. After getting the results on ordered semigroups, by an easy modification corresponding results on ordered hypersemigroups can be obtained.

**Keywords** $le$-semigroup, ordered semigroup, ordered hypersemigroup, ideal element, ideal

## 1. Introduction

Birkhoff, in [1], summarizes his ideas about the role played by lattice theory in mathematics generally. "In general, lattice theory has helped to simplify, unify and generalize many aspects of mathematics, and it has suggested many interesting new problems. Because of its central concept, that of order, intertwines through almost all of mathemetics" [1, p. 1]. "Lattices can do things for you, no matter what kind of mathematician you are!" [1, p. 38]. "The beauty of lattice theory derives in part from the extreme simplicity of its basic concepts: (partial) ordering, least upper and greatest lower bounds. Many of the deepest and most interesting applications of lattice theory concern (partially) ordered mathematical structures having also a binary addition or multiplication: lattice-ordered groups, monoids, vector spaces, rings, and fields (like the real field)" [2, p. iii]. "The concept of a lattice-ordered monoid (or $l$-monoid) arises naturally in ideal theory, where it has roots in the work of Dedekind, and was studied carefully by Krull [R. Dedekind, Ges. Werke, Vol. III, pp. 62–71; W. Krull, S.-B Phys. Med. Soc. zu Erlangen 56 (1924), 47–63" [2, p. 319].

The aim is to show what lattices can do for hypersemigroups. From many results on $poe$-semigroups or $le$-semigroups, especially from the results related to various kind of ideals, bi-ideals, quasi-ideals, results can be proved using sets (instead of elements) or results on structures like regularity, intra-regularity etc., one can shows analogous results for ordered semigroups. The converse is also true, by the beauty and simplicity of the order makes it easier to pass from $le$-semigroups to ordered semigroups. Besides, if we want to check the validity of some of the results on ordered semigroups, we always check their analogous for the $le$-semigroups. The work on ordered semigroups is based on some well-known properties. After one proves that the analogous properties

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2010 *AMS Mathematics Subject Classification:* 06F05, 08A99, 20M75

(lemmas) hold for ordered hypersemigroups, the results on ordered hypersemigroups can be obtained by a very easy way, just putting the operation $*$ instead of the multiplication of the ordered semigroup. Therefore, although many results of the ordered hypersemigroups can be seen as generalization of ordered semigroups, in fact they can be obtained from ordered semigroups by easy modification.

We try to solve the results in each structure in a similar way to be able to compare them, as this is the only way to show their relation to ordered hypersemigroups. We will deal with the basic concepts of ordered semigroups, the regular and the intra-regular ordered semigroups but the same holds for many results on ordered semigroups especially for the results that can be proved using only sets (instead of elements). The intra-regular semigroups are union of simple semigroups [3, Theorem 4.4]. The concept of regular rings has been introduced by Neumann [21]. Kovács characterized the regular rings as rings satisfying the property $A\cap B = AB$ for every right ideal $A$ and every left ideal $B$ of $S$, where $AB$ is the set of all finite sums of the form $\sum a_i b_i$; $a_i \in A$, $b_i \in B$ [20]. Iséki studied the same for semigroups and has shown that a semigroup $S$ is regular if and only of every right ideal $A$ and every left ideal $B$ of $S$ we have $A \cap B = AB$ [4]. The same problem for an ordered semigroup has been given in [12].

## 2. On $le$-semigroups

An ordered groupoid, shortly $po$-groupoid, is a groupoid $S$ with an order relation $\le$ on $S$ such that $a \le b$ implies $ac \le bc$ and $ca \le cb$ for all $c \in S$ [2]. A lattice ordered groupoid (shortly $l$-groupoid) is a groupoid $S$ at the same time a lattice such that $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for all $a, b, c \in S$ [2, 5]. If $S$ is not a lattice but only an upper semilattice, then it is called a $\vee$-groupoid. Every $\vee$-groupoid $(S, \cdot, \le)$ is a $po$-groupoid. Indeed, if $a \le b$ and $c \in S$, then $bc = (a\vee b)c = ac\vee bc \ge ac$ and $cb = c(a\vee b) = ca\vee cb \ge ca$. We call $poe$-groupoid, $\vee e$-groupoid, $le$-groupoid the $po$-groupoid, $\vee$-groupoid or $l$-groupoid, respectively, possessing a greatest element $e$ (: $e \ge a$ for every $a \in S$).

In a $po$-groupoid, an element $a$ is called subidempotent if $a^2 := aa \le a$ [2]. An element $a$ of a $po$-groupoid $S$ is called a right ideal element if $ax \le a$ for all $x \in S$; it is called a left ideal element if $xa \le a$ for all $x \in S$ [2]. An element which is both a right and a left ideal element is called an ideal element. As a consequence, an element $a$ of a $poe$-groupoid $S$ is a right ideal element of $S$ if and only if $ae \le a$; it is a left ideal element of $S$ if and only if $ea \le a$ [6]. This being so, the relation $ae \le a$ (resp. $ea \le a$) defines the right (resp. left) ideal element of a $poe$-groupoid. A $po$-groupoid $S$ is called right (resp. left) duo if the right ideal elements of $S$ are at the same time left (resp. right) ideal elements; that is ideal elements of $S$. It is called duo if the right ideal elements and the left ideal elements of $S$ coincide. A $poe$ ($le$-groupoid) having an associative multiplication is called $poe$ ($le$-semigroup). Clearly, every $\vee e$-semigroup (and so $le$-semigroup) is a $poe$-semigroup. We denote by $r(a)$ the right ideal element of $S$ generated by $a$; that is, $r(a)$ is a right ideal element of $S$ such that $r(a) \ge a$, and if $t$ is a right ideal element of $S$ such that $t \ge a$, then $r(a) \le t$. Denote by $l(a)$ the left ideal element of $S$ generated by $a$. For a $\vee e$-semigroup, we have $l(a) = a \vee ea$ and $r(a) = a \vee ae$. A $poe$-semigroup $S$ is called regular if $a \le aea$ for all $a \in S$ [7]. A $poe$-semigroup $S$ is called intra-regular if, for every $a \in S$, we have $a \le ea^2e$ [6].

**Theorem 2.1** *Let $S$ be a poe-semigroup. If $S$ is regular then, for every right ideal element $a$ and every left ideal element $b$ of $S$, the infimum of the elements $a$ and $b$ exists and this is the $ab$. In particular, if $S$ is an* *le-semigroup satisfying the relation $a \wedge b = ab$ for every right ideal element $a$ and every left ideal element $b$ of $S$, then $S$ is regular.*

**Proof** Let $a$ be a right ideal element and $b$ a left ideal element of $S$. Since $S$ is regular, we have $ab \le (ab)e(ab) \le ae \le a$ and $ab \le (ab)e(ab) \le eb \le b$ and so $ab \le a, b$. Let now $t$ be an element of $S$ such that $t \le a$ and $t \le b$. Since $S$ is regular, we have $t \le tet \le (ae)b \le ab$ and so $ab \ge t$. Thus we have $\inf\{a, b\} = ab$. For the converse statement, let $a \in S$. Since $r(a)$ is a right ideal element and $l(a)$ is a left ideal element of $S$, by hypothesis, we have

$$a \le r(a) \wedge l(a) = r(a)l(a) = (a \vee ae)(a \vee ea) = a^2 \vee aea \vee ae^2a = a^2 \vee aea,$$

then $a^2 \le a^3 \vee aea^2 \le aea$ from which $a \le aea$ and so $S$ is regular. □

**Corollary 2.2** *(see also [7]) An le-semigroup $S$ is regular if and only if for every right ideal element $a$ and every left ideal element $b$ of $S$ we have $a \wedge b = ab$.*

**Corollary 2.3** *An le-semigroup $S$ is regular duo if and only if for every right ideal element $a$ and every left ideal element $b$ of $S$ we have $a \wedge b = ba$.*

**Proof** $\Longrightarrow$. Let $a$ be a right ideal element and $b$ a left ideal element of $S$. Since $S$ is a duo, $a$ is a left ideal element and $b$ is a right ideal element of $S$ as well. Since $S$ is regular, by Corollary 2.2, we have $b \wedge a = ba$ and so $a \wedge b = ba$.
$\Longleftarrow$. Let $a$ be a right ideal element of $S$. Since $e$ is a left ideal element of $S$, by hypothesis, we have $a = a \wedge e = ea$, then $ea \le a$ i.e. $a$ is a left ideal element of $S$ and so $S$ is right duo. Similarly, $S$ is left duo. Let now $a$ be a right ideal element and $b$ a left ideal element of $S$. Since $S$ is a duo, $a$ is a left and $b$ a right ideal element of $S$. By hypothesis, we have $b \wedge a = ab$. Since $a \wedge b = ab$, by Theorem 2.1, $S$ is regular. □

**Theorem 2.4** *Let $S$ be a poe-semigroup. If $S$ is regular and right (resp. left) duo, then $S$ is intra-regular.*

**Proof** Let $S$ be regular and right duo and let $a \in S$. Since $S$ is regular, we have $a \le aea$. The element $ae$ is a right ideal element of $S$ (as $(ae)e = ae^2 \le ae$). Since $S$ is right duo, $ae$ is a left ideal element of $S$ as well, that is $e(ae) \le ae$. Hence we obtain

$$a \le aea \le (aea)e(aea) = (aea)(eae)a \le (aea)(ae)a = aea^2ea \le ea^2e,$$

then $a \le ea^2e$ and so $S$ is intra-regular. Let now $S$ be regular, left duo, and $a \in S$. The element $ea$ is a left ideal element of $S$. Since $S$ is left duo, $ea$ is a right ideal element of $S$ as well i.e. $(ea)e \le ea$. Since $S$ is regular, we have

$$a \le aea \le (aea)e(aea) = a(eae)aea \le a(ea)aea = aea^2ea \le ea^2e,$$

then $a \le ea^2e$ and so $S$ is intra-regular. □

**Theorem 2.5** *Let $S$ be a poe-semigroup. If $S$ is intra-regular then, for every right ideal element $a$ and every left ideal element $b$ of $S$ such that $a \wedge b$ exists we have $a \wedge b \le ba$. In particular, if $S$ is an le-semigroup such that $a \wedge b \le ba$ for every right ideal element $a$ and every left ideal element $b$ of $S$, then $S$ is intra-regular.*

**Proof** Let $S$ be an intra-regular *poe*-semigroup, $a$ a right ideal element, $b$ a left ideal element of $S$ and $a \wedge b$ exists. Since $S$ is intra-regular, we have

$$a \wedge b \leq e(a \wedge b)^2 e = e(a \wedge b)(a \wedge b)e \leq (eb)(ae) \leq ba$$

and so $a \wedge b \leq ba$. For the converse statement, let $S$ be an *le*-semigroup and $a \in S$. By hypothesis, we have

$$a \leq r(a) \wedge l(a) = l(a)r(a) = (a \vee ea)(a \vee ae) = a^2 \vee ea^2 \vee a^2 e \vee ea^2 e.$$

Then

$$a^2 \leq a^3 \vee aea^2 \vee a^3 e \vee aea^2 e \leq ea^2 \vee ea^2 e, \;\; a^2 e \leq (ea^2 \vee ea^2 e)e = ea^2 e \vee ea^2 e^2 = ea^2 e,$$

$$a \leq ea^2 \vee ea^2 e, \;\; a^2 \leq ea^3 \vee ea^2 ea \leq ea^2 e, \;\; ea^2 \leq ea^2 e.$$

Then we have $a \leq ea^2 \vee ea^2 e = ea^2 e$ and so $S$ is intra-regular. □

Thus we have the following theorem.

**Corollary 2.6** *(see also [6]) An le-semigroup $S$ is intra-regular if and only if for every right ideal element $a$ and every left ideal element $b$ of $S$ we have $a \wedge b \leq ba$.*

## 3. On ordered semigroups

When we say ordered groupoid (semigroup) we assume that it does not possess a greatest element. For a subset $A$ of an ordered groupoid $S$, we denote by $(A]$ the subset of $S$ defined by $(A] := \{t \in S \mid t \leq a \text{ for some } a \in A\}$ and we have

(1) $\big((A]\big] = (A]$

(2) if $A$ is a right ideal (or a left ideal) of $S$, then $(A] = A$

(3) $A \subseteq B$ implies $(A] \subseteq (B]$

(4) $(A](B] \subseteq (AB]$

(5) $\Big((A](B]\Big] = (AB]$.

If $S$ is an ordered semigroup, then, in addition to properties (1)–(5), we have

(6) $\Big((A]B(C]\Big] = \Big(A(B]C\Big] = \Big(ABC\Big]$

(7) $(ABCD] = \Big(AB(C]D\Big] = \Big(A(B]CD\Big]$.

Clearly, $(S] = S$ and $A \subseteq (A]$.

Later, we will use the same symbol $(A]$ for ordered hypergroupoids as well.

Let $S$ be an ordered groupoid. A nonempty subset $A$ of $S$ is called right (resp. left) ideal of $S$ [8] if (1) $AS \subseteq A$ (resp. $SA \subseteq A$) and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in S$ i.e. if $(A] = A$. If $A$ is both a right and a left ideal of $S$, then it is called an ideal of $S$. An ordered groupoid $S$ is called right (resp. left) duo if the right (resp. left) ideals of $S$ are at the same time left (resp. right) ideals of $S$; that is ideals of $S$. It is called duo if the right and the left ideals of $S$ are the same. For a subset $A$ of an ordered semigroup $S$ we denote by $R(A)$ the right ideal of $S$ generated by $A$ i.e. the least (with respect to the inclusion relation) right ideal of $S$ containing $A$, and we have

$R(A) = (A \cup AS]$.

Denote by $L(A)$ the left ideal of $S$ generated by $A$ and we have

$L(A) = (A \cup SA]$ [11].

For $A = \emptyset$, we clearly have $R(A) = L(A) = S$.

An ordered semigroup $(S, \cdot, \leq)$ is called regular [10] if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. This is equivalent to saying that $A \subseteq (ASA]$ for every subset $A$ of $S$ or $a \in (aSa]$ for every $a \in S$. An ordered semigroup $(S, \cdot, \leq)$ is called intra-regular [9] if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$. That is to say, $A \subseteq (SA^2S]$ for every subset $A$ of $S$ or $a \in (Sa^2S]$ for every $a \in S$.

*A poe-semigroup $S$ is regular if and only if $a \leq aea$ for every $a \in S$. A poe-semigroup $S$ is intra-regular if and only if $a \leq ea^2e$ for every $a \in S$* (its proof is easy).

**Proposition 3.1** *Let $S$ be a groupoid, $A$ a right ideal of $S$ and $B$ a left ideal of $S$. Then the intersection $A \cap B$ is nonempty.*

**Proof** Let $a \in A$ and $b \in B$ $(A, B \neq \emptyset)$. Since $A$ is a right ideal of $S$, we have $ab \in AS \subseteq A$; since $B$ is a left ideal of $S$, we have $ab \in SB \subseteq B$ and so $ab \in A \cap B$. □

For an ordered semigroup, using the properties (1) to (7), the following Theorems 3.2 to 3.5 can be obtained.

**Theorem 3.2** *(see also [12]) An ordered semigroup $S$ is regular if and only if for every right ideal $A$ and every left ideal $B$ of $S$ we have $A \cap B = (AB]$.*

**Proof** $\Longrightarrow$. Let $A$ be a right ideal and $B$ a left ideal of $S$. Since $S$ is regular, we have

$$\begin{aligned} A \cap B &\subseteq \Big((A \cap B)S(A \cap B)\Big] \overset{(3)}{\subseteq} \Big((AS)B\Big] \overset{(3)}{\subseteq} (AB] \overset{(3)}{\subseteq} (AS] \cap (SB] \\ &\overset{(3)}{\subseteq} (A] \cap (B] \overset{(2)}{=} A \cap B \end{aligned}$$

and so $A \cap B = (AB]$.

$\Longleftarrow$. Let $A$ be a subset of $S$. By hypothesis, we have

$$\begin{aligned} A &\subseteq R(A) \cap L(A) \subseteq \Big(R(A)L(A)\Big] = \Big((A \cup AS](A \cup SA]\Big] \overset{(5)}{=} \Big((A \cup AS)(A \cup SA)\Big] \\ &= (A^2 \cup ASA \cup AS^2A] = (A^2 \cup ASA]. \end{aligned}$$

Then we have $A^2 \subseteq (A^2 \cup ASA](A] \overset{(4)}{\subseteq} (A^3 \cup ASA^2] \subseteq (ASA]$, then $A \overset{(3)}{\subseteq} \Big((ASA] \cup ASA\Big] = \big((ASA]\big] \overset{(1)}{=} (ASA]$ and so $S$ is regular. □

**Theorem 3.3** *An ordered semigroup $S$ is regular duo if and only if for every right ideal $A$ and every left ideal $B$ of $S$ we have $A \cap B = (BA]$.*

**Proof** $\Longrightarrow$. Let $A$ be a right ideal and $B$ a left ideal of $S$. Since $S$ is a duo, $A$ is a left ideal and $B$ is a right ideal of $S$ as well. Since $S$ is regular, by Theorem 3.2, we have $B \cap A = (BA]$.

$\Longleftarrow$. Let $A$ be a right ideal of $S$. Since $S$ is a left ideal of $S$, by hypothesis, we have $A = A \cap S = (SA]$, then $SA \subseteq (SA] \subseteq A$, thus $A$ is a left ideal of $S$ and so $S$ is right duo. Similarly $S$ is left duo. Let now $A$ be a right ideal and $B$ a left ideal of $S$. Since $S$ is a duo, $A$ is a left ideal and $B$ a right ideal of $S$. By hypothesis, we have $B \cap A = (AB]$. By Theorem 3.2, $S$ is regular. □

**Theorem 3.4** *Let $S$ be an ordered semigroup. If $S$ is regular and right (or left) duo, then $S$ is intra-regular.*

**Proof** Let $S$ be regular and right duo and $A$ be a subset of $S$. Since $S$ is regular, we have $A \subseteq (ASA]$. The set $(AS]$ is a right ideal of $S$ (as $(AS]S = (AS](S] \overset{(4)}{\subseteq} (AS^2] \overset{(3)}{\subseteq} (AS]$ and $\big((AS]\big] \overset{(1)}{=} (AS]$). Since $S$ is right duo, $(AS]$ is a left ideal of $S$ as well, that is $S(AS] \subseteq (AS]$. Hence we obtain

$$\begin{aligned} A &\subseteq (ASA] \overset{(3)}{\subseteq} \Big((ASA]S(ASA]\Big] \overset{(6)}{=} \Big((ASA)S(ASA)\Big] = \Big((ASA)S(AS)A\Big] \\ &\overset{(7)}{=} \Big((ASA)S(AS]A\Big] \overset{(3)}{\subseteq} \Big((ASA)(AS]A\Big] \overset{(6)}{=} \Big((ASA)(AS)A\Big] \\ &= (ASA^2SA] \overset{(3)}{\subseteq} (SA^2S], \end{aligned}$$

then $A \subseteq (SA^2S]$ and so $S$ is intra-regular. Let now $S$ be regular and left duo and $A$ be a subset of $S$. The set $(SA]$ is a left ideal and, since $S$ is left duo, it is a right ideal of $S$ as well, that is $(SA]S \subseteq (SA]$. Since $S$ is regular, we have

$$\begin{aligned} A &\subseteq (ASA] \overset{(3)}{\subseteq} \Big((ASA]S(ASA]\Big] \overset{(6)}{=} \Big((ASA)S(ASA)\Big] = \Big(A(SA)S(ASA)\Big] \\ &\overset{(7)}{=} \Big(A(SA]S(ASA)\Big] \overset{(3)}{=} \Big(A(SA](ASA)\Big] \overset{(6)}{=} \Big(A(SA)(ASA)\Big] \\ &= (ASA^2SA] \overset{(3)}{\subseteq} (SA^2S], \end{aligned}$$

then $A \subseteq (SA^2S]$ and so $S$ is intra-regular. □

**Theorem 3.5** *An ordered semigroup $S$ is intra-regular if and only if for every right ideal $A$ and every left ideal $B$ of $S$, we have $A \cap B \subseteq (BA]$.*

**Proof** $\Longrightarrow$. Let $A$ be a right ideal and $B$ a left ideal of $S$. Since $S$ is intra-regular, we have

$$A \cap B \subseteq \Big(S(A\cap B)^2S\Big] = \Big(S(A\cap B)(A\cap B)S\Big] \overset{(3)}{\subseteq} \Big((SB)(AS)\Big] \overset{(3)}{\subseteq} (BA]$$

and so $A \cap B \subseteq (BA]$.
$\Longleftarrow$. Let $A \subseteq S$. By hypothesis, we have

$$\begin{aligned} A &\subseteq R(A) \cap L(A) = \Big(L(A)R(A)\Big] = \Big((A \cup SA](A \cup AS]\Big] \overset{(5)}{=} \Big((A \cup SA)(A \cup AS\Big] \\ &= (A^2 \cup SA^2 \cup A^2S \cup SA^2S]. \end{aligned}$$

Then

$$A^2 \subseteq (A](A^2 \cup SA^2 \cup A^2S \cup SA^2S] \overset{(4)}{\subseteq} (A^3 \cup ASA^2 \cup A^3S \cup ASA^2S] \overset{(3)}{\subseteq} (SA^2 \cup SA^2S],$$

$$A^2S \subseteq (SA^2 \cup SA^2S](S] \overset{(4)}{\subseteq} (SA^2S \cup SA^2S^2] = (SA^2S],$$

$$A \subseteq \Big((SA^2 \cup SA^2S]\Big] \overset{(1)}{=} (SA^2 \cup SA^2S].$$

Then

$$A^2 \subseteq (SA^2 \cup SA^2S](A] \overset{(4)}{\subseteq} (SA^3 \cup SA^2SA] \subseteq (SA^2S],$$

and thus

$$SA^2 \subseteq (S](SA^2S] \overset{(4)}{\subseteq} (S^2A^2S] \overset{(3)}{\subseteq} (SA^2S].$$

Hence we have

$$A \subseteq (SA^2 \cup SA^2S] \overset{(3)}{\subseteq} \Big((SA^2S] \cup SA^2S\Big] = \Big((SA^2S]\Big] \overset{(1)}{=} (SA^2S],$$

thus $A \subseteq (SA^2S]$ and so $S$ is intra-regular. □

## 4. On ordered hypersemigroups

Denote by $\mathcal{P}^*(S)$ the set of all nonempty subsets of $S$. An hypergroupoid is a nonempty set $S$ with an "operation" $\circ : S \times S \to \mathcal{P}^*(S) \mid (a,b) \to a \circ b$ on $S$ (called hyperoperation as it assigns to each couple $(a,b)$ of elements of $S$ a nonempty subset instead of an element of $S$) and an operation

$$* : \mathcal{P}^*(S) \times \mathcal{P}^*(S) \to \mathcal{P}^*(S) \mid (A,B) \to A * B := \bigcup_{a \in A, b \in B} a \circ b$$

on $\mathcal{P}^*(S)$ (induced by the hyperoperation $\circ$) [13]; we denote it by $(S,\circ)$. If $A,B,C,D$ are subsets of $S$ such that $A \neq \emptyset$, $C \neq \emptyset$ and $A \subseteq B$, $C \subseteq D$, then $A * C \subseteq B * D$ and $C * A \subseteq D * B$. It is easy to see that

$$x \circ y = \{x\} * \{y\} \text{ for every } x, y \in S.$$

From the definition of $*$, we have the following:

(1) if $x \in A * B$, then $x \in a \circ b$ for some $a \in A$, $b \in B$ and

(2) if $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$; that is $a \circ b \subseteq A * B$ for every $a \in A$, $b \in B$.

An hypergroupoid is called hypersemigroup if $(a \circ b) * \{c\} = \{a\} * (b \circ c)$ for every $a,b,c \in S$ [13]. An hypergroupoid $(S,\circ)$ is called an ordered hypergroupoid if there exists an order relation $\leq$ on $S$ such that $a \leq b$ implies $a \circ c \preceq b \circ c$ and $c \circ a \preceq c \circ b$ for every $c \in S$; in the sense that for every $c \in S$ and every $u \in a \circ c$ there exists $v \in b \circ c$ such that $u \leq v$ and for every $u \in c \circ a$ there exists $v \in c \circ b$ such that $u \leq v$. A nonempty subset $A$ of an ordered hypergroupoid $S$ is called a right (resp. left) ideal of $S$ if (1) $A * S \subseteq A$ (resp. $S * A \subseteq A$) and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$. An hypergroupoid $S$ is called right (resp. left) duo if the right ideals of $S$ are at the same time left (resp. right) ideals of $S$, that is ideals of $S$. It is called duo if the right and the left ideals of $S$ coincide. For a nonempty subset $A$ of an hypersemigroup $S$, we denote by $R(A)$ (resp. $L(A)$) the right (resp. left) ideal of $S$ generated by $A$ and we have $R(A) = \Big(A \cup (A * S)\Big]$ and $L(A) = \big(A \cup (S * A)\big]$ [18].

The concept of regular (intra-regular) ordered semigroup can be naturally extended to an ordered hypersemigroup as follows: An ordered hypersemigroup $(S,\circ,\leq)$ is called regular if for every $a \in S$ there exists $x \in S$ such that $\{a\} \preceq (a \circ x) * \{a\}$; in the sense that for every $a \in S$ there exist $x, t \in S$ such that $t \in (a \circ x) * \{a\}$ and $a \leq t$. An ordered hypersemigroup $(S, \circ, \leq)$ is called intra-regular if for every $a \in S$ there exist $x, y \in S$ such that $\{a\} \preceq (x \circ a) * (a \circ y)$; in the sense that for every $a \in S$ there exist $x, y, t \in S$ such that $t \in (x \circ a) * (a \circ y)$ and $a \leq t$ (see [18]).

Properties (1)–(3) of ordered semigroup (of Section 3), for ordered hypersemigroups also hold (as the operation $*$ does not play any role in them). Our aim is to prove the properties (4)–(7) for an ordered hypersemigroup. To prove them, we need the following four lemmas:

**Lemma 4.1** *(see also [14]). For any nonempty subsets $A, B, C$ of an hypersemigroup $S$, we have $(A*B)*C = A*(B*C)$.*

**Proof** A better proof than that one given in [14] is as follows: If $x \in (A*B)*C$, then $x \in u \circ v$ for some $u \in A*B$, $v \in C$, $u \in a \circ b$ for some $a \in A$, $b \in B$, thus $x \in u \circ v = \{u\} * \{v\} \subseteq (a \circ b) * \{v\} = \{a\} * (b \circ v) \subseteq A * (B * C)$ and so $(A*B)*C \subseteq A*(B*C)$. If $x \in A*(B*C)$, then $x \in a \circ u$ for some $a \in A$, $u \in B*C$, $u \in b \circ c$ for some $b \in B$, $c \in C$, then $x \in \{a\} * \{u\} \subseteq \{a\} * (b \circ c) = (a \circ b) * \{c\} \subseteq (A * B) * C$ and so $A * (B * C) \subseteq (A * B) * C$. □

Therefore, the operation $*$ is associative and we can write expressions like $A * B * C * \cdots * F$ without using parentheses.

**Lemma 4.2** *(see also [18]) If $S$ is an ordered hypergroupoid, $a \leq b$ and $c \leq d$, then $a \circ c \preceq b \circ d$ (i.e. for every $x \in a \circ c$ there exists $y \in b \circ d$ such that $x \leq y$).*

We write, for short, $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.

**Lemma 4.3** *Let $S$ be an ordered hypergroupoid and $A, B, C$ nonempty subsets of S.*
*If $A \preceq B$, then $A * C \subseteq (B * C]$ and $C * A \subseteq (C * B]$.*

**Proof** Let $A \preceq B$ and $x \in A * C$. Then $x \in a \circ c$ for some $a \in A$, $c \in C$ and $a \leq b$ for some $b \in B$. Since $a \leq b$, we have $a \circ c \preceq b \circ c$. Since $x \in a \circ c$, there exists $y \in b \circ c$ such that $x \leq y$. We have $x \leq y \in b \circ c \subseteq B*C$ and so $x \in (B * C]$. The proof that $A \preceq B$ implies $C * A \subseteq (C * B]$ is similar. □

**Lemma 4.4** *(see [13, 14]) If S is an hypergroupoid and $A, B, C$ nonempty subsets of $S$, then we have*
*$(A \cup B) * C = (A * C) \cup (B * C)$ and $A \cup (B * C) = (A \cup B) * (A \cup C)$.*

We are ready now to prove the analogous of properties (4)–(7) for an ordered hypersemigroup. We give them in the following Propositions 4.5 to 4.8.

**Proposition 4.5** *[15, 18] Corresponds to property (4). For an ordered hypergroupoid $S$ and nonempty subsets $A, B$ of $S$, we have*

$$(A] * (B] \subseteq (A * B].$$

**Proposition 4.6** *[16, 18] Corresponds to property (5). For an ordered hypergroupoid $S$ and nonempty subsets $A, B$ of $S$, we have*

$$\Big((A] * (B]\Big] = (A * B].$$

**Proposition 4.7** *Corresponds to property (6). If $S$ is an ordered hypersemigroup then, for any nonempty subsets $A, B, C$ of $S$, we have*

$$\Big((A] * B * (C]\Big] = \Big(A * (B] * C\Big] = (A * B * C].$$

**Proof** Since $A \subseteq (A]$ and $C \subseteq (C]$, we have $(A * B * C] \subseteq \Big((A] * B * (C]\Big]$. Let now $t \in \Big((A] * B * (C]\Big]$. Then $t \leq x$ for some $x \in (A] * B * (C]$, $x \in y \circ z$ for some $y \in (A] * B$, $z \in (C]$, $y \in w \circ b$ for some $w \in (A]$, $b \in B$, $w \leq a$ for some $a \in A$, $z \leq c$ for some $c \in C$. Then we have

$$t \leq x \in y \circ z = \{y\} * \{z\} \subseteq (w \circ b) * \{z\}.$$

Since $w \leq a$, we have $w \circ b \preceq a \circ b$ and, by Lemma 4.3, we have

$$(w \circ b) * \{z\} \subseteq \Big((a \circ b) * \{z\}\Big] = \Big(\{a\} * (b \circ z)\Big].$$

Since $z \leq c$, we have $b \circ z \preceq b \circ c$ again, by Lemma 4.3, $\{a\} * (b \circ z) \subseteq \Big(\{a\} * (b \circ c)\Big]$. Hence we obtain

$$\begin{aligned} t \in \Big((w \circ b) * \{z\}\Big] &\subseteq \Big(\Big(\{a\} * (b \circ z)\Big]\Big] = \Big(\{a\} * (b \circ z)\Big] \subseteq \Big(\Big(\{a\} * (b \circ c]\Big]\Big] \\ &= \Big(\{a\} * (b \circ c]\Big] \subseteq (A * B * C] \end{aligned}$$

and so $\Big((A] * B * (C]\Big] \subseteq (A * B * C]$.

For the proof of the second equality see [16]. □

**Proposition 4.8** *Corresponds to property (7). If $S$ is an ordered hypersemigroup then, for any nonempty subsets $A, B, C, D$ of $S$, we have*

$$(A * B * C * D] = \Big(A * B * (C] * D\Big] = \Big(A * (B] * C * D\Big].$$

**Proof** Since $C \subseteq (C]$, we have $(A * B * C * D] = \Big(A * B * (C] * D\Big]$. Let now $t \in \Big(A * B * (C] * D\Big]$. Then $t \leq x$ for some $x \in A * B * (C] * D$, $x \in u \circ v$ for some $u \in A * B$, $v \in (C] * D$, $u \in a \circ b$ for some $a \in A$, $b \in B$, $v \in h \circ d$ for some $h \in (C]$, $d \in D$, $h \leq c$ for some $c \in C$. Then we have

$$t \leq x \in u \circ v = \{u\} * \{v\} \subseteq (a \circ b) * (h \circ d).$$

Since $h \leq c$, we have $h \circ d \preceq c \circ d$. By Lemma 4.3, we have

$$(a \circ b) * (h \circ d) \subseteq \Big((a \circ b) * (c \circ d)\Big] \subseteq (A * B * C * D].$$

Then

$$t \in \Big((a \circ b) * (h \circ d)\Big] \subseteq \Big((A * B * C * D]\Big] = (A * B * C * D]$$

and so $\Big(A * B * (C] * D\Big] \subseteq (A * B * C * D]$.

We have $\Big(A * (B] * C * D\Big] = (A * B * C * D]$. Indeed: Clearly, $(A * B * C * D] \subseteq \Big(A * (B] * C * D\Big]$. Let now $t \in \Big(A * (B] * C * D\Big]$. Then $t \le x$ for some $x \in A * (B] * C * D$, $x \in u \circ v$ for some $u \in A * (B]$, $v \in C * D$, $u \in a \circ w$ for some $a \in A$, $w \in (B]$, $v \in c \circ d$ for some $c \in C$, $d \in D$, $w \le b$ for some $b \in B$. Thus we have

$$t \le x \in u \circ v = \{u\} * \{v\} \subseteq (a \circ w) * (c \circ d).$$

Since $w \le b$, we have $a \circ w \preceq a \circ b$. Then, by Lemma 4.3,

$$(a \circ w) * (c \circ d) \subseteq \Big((a \circ b) * (c \circ d)\Big] \subseteq (A * B * C * D].$$

We have $t \le x \in (A * B * C * D]$ and so $t \in \Big((A * B * C * D]\Big] = (A * B * C * D]$. □

Using only the definitions given at the beginning of Section 4, exactly as in ordered semigroups, the following hold: An ordered hypersemigroup $S$ is regular if and only if $A \subseteq (A * S * A]$ for every nonempty subset $A$ of $S$. An ordered hypersemigroup $S$ is intra-regular if and only if $A \subseteq (S * A * A * S]$ for every nonempty subset $A$ of $S$.

Propositions 4.5 to 4.8 play the main role in the investigation as they are the only part in which the order of the hypersemigroup plays a role. Using these propositions, the results of the previous section on ordered semigroups hold for ordered hypersemigroups just replacing, in proofs, the multiplication "$\cdot$" of the ordered semigroup by the operation "$*$" of the ordered hypersemigroup.

Before we go further, we need the following lemma.

**Lemma 4.9** *[14] If $S$ is an hypergroupoid, $A$ is a right ideal and $B$ a left ideal of S, then the intersection $A \cap B$ is nonempty.*

**Proof** For the sake of completeness, we will give its proof. Take $a \in A$, $b \in B$ $(A, B \neq \emptyset)$. Then $a \circ b \subseteq A * S \cap S * B \subseteq A \cap B$ and so $a \circ b \subseteq A \cap B$. Since $a \circ b \neq \emptyset$, $A \cap B \neq \emptyset$. □

The theorem that corresponds to Theorem 3.2 is the following. Propositions 4.5 and 4.6 (also properties (1)–(3) and Lemma 4.9) are needed in its proof.

**Theorem 4.10** *(see also [18]) An ordered hypersemigroup $S$ is regular if and only if for every right ideal $A$ and every left ideal $B$ of $S$ we have $A \cap B = (A * B]$.*

The theorem that corresponds to Theorem 3.3 is the following.

**Theorem 4.11** *An ordered hypersemigroup $S$ is regular duo if and only if for eery right ideal $A$ and every left ideal $B$ of $S$ we have $A \cap B = (B * A]$.*

The theorem that corresponds to Theorem 3.4 is the following. We will give its proof to be able to compare it with the proof of Theorem 3.4 and see that both are the same. Propositions 4.5, 4.7, 4.8 (also properties (1),(3) and Lemma 4.1) have been used in it.

**Theorem 4.12** *If $S$ is a regular and right (or left) duo ordered hypersemigroup, then $S$ is intra-regular.*

**Proof** Let $S$ be regular and right duo and $A$ be a nonempty subset of $S$. Since $S$ is regular, we have $A \subseteq (A*S*A]$. The set $(A*S]$ is a right ideal of $S$ (as $(A*S]*S = (A*S]*(S] \subseteq (A*S^2] \subseteq (A*S]$ (by Proposition 4.5) and $\big((A*S]\big] = (A*S]\big)$. Since $S$ is right duo, $(A*S]$ is a left ideal of $S$ as well, that is $S*(A*S] \subseteq (A*S]$. Hence we obtain

$$\begin{aligned} A &\subseteq (A*S*A] \subseteq \Big((A*S*A]*S*(A*S*A]\Big] \\ &= \Big((A*S*A)*S*(A*S*A)\Big] \text{ (by Proposition 4.7)} \\ &= \Big((A*S*A)*S*(A*S)*A\Big] \text{ (by Lemma 4.1)} \\ &= \Big((A*S*A)*S*(A*S]*A\Big] \text{ (by Proposition 4.8)} \\ &\subseteq \Big((A*S*A)*(A*S]*A\Big] = \Big((A*S*A)*(A*S)*A\Big] \text{ (by Proposition 4.7)} \\ &= (A*S*A^2*S*A] \subseteq (S*A^2*S], \end{aligned}$$

then $A \subseteq (S*A^2*S]$ and so $S$ is intra-regular. Let now $S$ is regular and left duo and $A$ be a nonempty subset of $S$. The set $(S*A]$ is a left ideal and, since $S$ is left duo, it is a right ideal of $S$ as well, that is $(S*A]*S \subseteq (S*A]$. Since $S$ is regular, we have

$$\begin{aligned} A &\subseteq (A*S*A] \subseteq \Big((A*S*A]*S*(A*S*A]\Big] \\ &= \Big((A*S*A)*S*(A*S*A)\Big] \text{ (by Proposition 4.7)} \\ &= \Big(A*(S*A)*S*(A*S*A)\Big] \text{ (by Lemma 4.1)} \\ &= \Big(A*(S*A]*S*(A*S*A)\Big] \text{ (by Proposition 4.8)} \\ &= \Big(A*(S*A]*(A*S*A)\Big] = \Big(A*(S*A)*(A*S*A)\Big] \text{ (by Proposition 4.7)} \\ &= (A*S*A^2*S*A] \subseteq (S*A^2*S], \end{aligned}$$

then $A \subseteq (S*A^2*S]$ and so $S$ is intra-regular.

□

Using Propositions 4.5 and 4.6 (also properties (1),(3) and Lemma 4.9), we can prove the following theorem that corresponds to Theorem 3.5.

**Theorem 4.13** *(see also [18]) An ordered hypersemigroup $S$ is intra-regular if and only if for every right ideal $A$ and every left ideal $B$ of $S$, we have $A \cap B \subseteq (B*A]$.*

Exactly as in Theorem 4.10, in the $\Rightarrow$-part of the proof of Theorem 4.13, the set $A \cap B$ should be nonempty (this is not the case for an ordered semigroup); and it is nonempty by Lemma 4.9.

We apply the above results to the following example.

**Example 4.14** Consider the $le$-semigroup $S=\{a,b,c,d,e\}$ given by Table 1 and Figure 1. This is a regular (as $x \le xex \ \forall x$), duo, and intra-regular (as $x \le ex^2e \ \forall x$) $le$-semigroup. The right ideal elements of $S$ are the elements $a,b,d$ and $e$ and coincide with the left ideal elements of $S$. The results of Section 2 can be applied. As an ordered semigroup, this is clearly regular and intra-regular; the right ideals of $(S,\cdot,\le)$ are the sets $\{a\}$, $\{a,b\}$, $\{a,b,c,d\}$ and $S$ and coincide with the left ideals of $S$, so it is a duo. The results of Section 3 can be applied. From the ordered semigroup $(S,\cdot,\le)$, in the way indicated in [17], the ordered hypersemigroup $(S,\circ,\le)$ given by Table 2 and the same figure (Figure 1) can be obtained. This is a regular, duo, and intra-regular ordered hypersemigroup. The results of Section 4 can be applied.

The ordered hypersemigroup $(S,\circ,\le)$ is regular, as

$a \in (a\circ a) * \{a\} = \{a\} * \{a\} = a \circ a = \{a\}$ and $a \le a$,

$b \in (b\circ b) * \{b\} = \{a,b\} * \{b\} = \bigcup\limits_{x\in\{a,b\}} x\circ b = (a\circ b)\cup(b\circ b) = \{a\}\cup\{a,b\} = \{a,b\}$ and $b \le b$,

$c \in (c\circ c) * \{c\} = \{a,c\} * \{c\} = \bigcup\limits_{x\in\{a,c\}} x\circ c = (a\circ c)\cup(c\circ c) = \{a\}\cup\{a,c\} = \{a,c\}$ and $c \le c$,

$d \in (d\circ d) * \{d\} = \{a,b,c,d\} * \{d\} = \bigcup\limits_{x\in\{a,b,c,d\}} x\circ d = (a\circ d)\cup(b\circ d)\cup(c\circ d)\cup(d\circ d) = \{a,b,c,d\}$

and $d \le d$,

$e \in (e\circ e) * \{e\} = S * \{e\} = S$ and $e \le e$.

The ordered hypersemigroup $(S,\circ,\le)$ is intra-regular as

$a \in (a\circ a) * (a\circ a) = \{a\} * \{a\} = \{a\}$ and $a \le a$,

$b \in (b\circ b) * (b\circ b) = \{a,b\} * \{a,b\} = \bigcup\limits_{x\in\{a,b\},y\in\{a,b\}} x\circ y = a\circ a \cup b\circ a \cup a\circ b \cup b\circ b = \{a,b\}$ and $b \le b$,

$c \in (c\circ c) * (c\circ c) = \{a,c\} * \{a,c\} = \bigcup\limits_{x,y\in\{a,c\}} x\circ y = a\circ a \cup c\circ a \cup a\circ c \cup c\circ c = \{a,c\}$ and $c \le c$,

$d \in (d\circ d) * (d\circ d) = \{a,b,c,d\} * \{a,b,c,d\} = \bigcup\limits_{x,y\in\{a,b,c,d\}} x\circ y = \{a,b,c,d\}$ and $d \le d$,

$e \in (e\circ e) * (e\circ e) = S * S = S$ and $e \le e$.

The ideals of $(S,\circ,\le)$ coincide with the ideals of $(S,\cdot,\le)$ [17]. The sets $\{a\}$, $\{a,b\}$, $\{a,b,c,d\}$ and $S$ are the right at the same time the left ideals of $(S,\circ,\le)$ and so $(S,\circ,\le)$ is a duo. One can also check it independently. Consider the set

$$\begin{aligned}\mathcal{P}^*(S) \;=\; &\Big\{\{a\},\{b\},\{c\},\{d\},\{e\},\{a,b\},\{a,c\},\{a,d\},\{a,e\},\{b,c\},\{b,d\},\{b,e\},\{c,d\},\\ &\{c,e\},\{d,e\},\{a,b,c\},\{a,b,d\},\{a,b,e\},\{a,c,d\},\{a,c,e\},\{a,d,e\},\{b,c,d\},\\ &\{b,c,e\},\{b,d,e\},\{c,d,e\},\{a,b,c,d\},\{a,b,c,e\},\{a,b,d,e\},\{a,c,d,e\},\\ &\{b,c,d,e\},S\Big\}\end{aligned}$$

and check each of its elements. The sets $\{a\}$, $\{a,b\}$, $\{a,b,c,d\}$ and $S$ are the only ideals of $(S,\circ,\le)$.

If we consider the $le$-semigroup $(S,\cdot,\le)$ given by Table 3 and Figure 1, this is intra-regular (as $x \le ex^2e$ for every $x\in S$), not regular ($c \not\le cec$) and not duo. The right ideal elements of $S$ are the elements $a$ and $e$, and the left ideal elements of $S$ are the elements $a,b,d,e$. As an ordered semigroup, this is clearly intra-regular and not regular. The right ideals of $(S,\cdot,\le)$ are the sets $\{a\}$ and $S$, and the left ideal elements of $(S,\cdot,\le)$ are the sets $\{a\}$, $\{a,b\}$, $\{a,b,c\}$, $\{a,b,c,d\}$, and $S$. From this ordered semigroup, in the way indicated in [17], the ordered hypersemigroup $(S,\circ,\le)$ given by Table 4 and Figure 1 can be obtained. This is an intra-regular ordered hypersemigroup as

$$a \in (a\circ a) * (a\circ a) = \{a\} * \{a\} = a\circ a = \{a\} \text{ and } a \le a,$$
$$b \in (b\circ b) * (b\circ b) = \{a,b\} * \{a,b\} = (a\circ a)\cup(b\circ a)\cup(a\circ b)\cup(b\circ b) = \{a,b\} \text{ and } b \le b,$$
$$c \in (c\circ c) * (c\circ e) = \{a,b\} * \{a,b,c,d,e\} = \{a,b,c,d,e\} \text{ and } c \le c,$$
$$d \in (e\circ d) * (d\circ e) = \{a,b\} * \{a,b,c,d,e\} = \{a,b,c,d,e\} \text{ and } d \le d,$$
$$e \in (e\circ e) * (e\circ e) = S * S = S \text{ and } e \le e.$$

The right ideals of $(S,\circ,\le)$ are the sets $\{a\}$ and $S$, and the left ideal of $(S,\circ,\le)$ are the sets $\{a\}$, $\{a,b\}$, $\{a,b,c\}$, $\{a,b,c,d\}$ and $S$ (see [17]). One can also check it independently. Theorem 2.5, Corollary 2.6, Theorem 3.5, and Theorem 4.13 can be applied.

**Table 1.** The multiplication of the $le$-semigroup of Example 4.14.

| $\cdot$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ | $b$ | $b$ | $b$ |
| $c$ | $a$ | $b$ | $c$ | $d$ | $d$ |
| $d$ | $a$ | $b$ | $d$ | $d$ | $d$ |
| $e$ | $a$ | $b$ | $d$ | $d$ | $e$ |

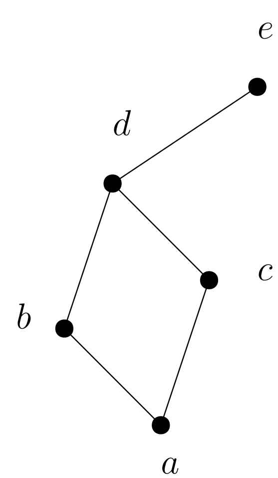


**Figure 1.** The order of Example 4.14.

**Table 2.** The hyperoperation of the example 4.14.

| $\circ$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ |
| $b$ | $\{a\}$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b\}$ |
| $c$ | $\{a\}$ | $\{a,b\}$ | $\{a,c\}$ | $\{a,b,c,d\}$ | $\{a,b,c,d\}$ |
| $d$ | $\{a\}$ | $\{a,b\}$ | $\{a,b,c,d\}$ | $\{a,b,c,d\}$ | $\{a,b,c,d\}$ |
| $e$ | $\{a\}$ | $\{a,b\}$ | $\{a,b,c,d\}$ | $\{a,b,c,d\}$ | $\{a,b,c,d,e\}$ |

**Table 3.** The multiplication of Table 3 of the example 4.14.

| $\cdot$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ | $b$ | $b$ | $e$ |
| $c$ | $a$ | $b$ | $b$ | $b$ | $e$ |
| $d$ | $a$ | $b$ | $b$ | $b$ | $e$ |
| $e$ | $a$ | $b$ | $b$ | $b$ | $e$ |

**Table 4.** The hyperoperation of Table 4 of the example 4.14.

| $\circ$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ |
| $b$ | $\{a\}$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b,c,d,e\}$ |
| $c$ | $\{a\}$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b,c,d,e\}$ |
| $d$ | $\{a\}$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b,c,d,e\}$ |
| $e$ | $\{a\}$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b,c,d,e\}$ |

**Note:** The sets in the proofs of Sections 3 and 4 show the pointless character of the results that is in the same spirit with the abstract formulation of general topology (the so-called topology without points) initiated by Koutský, Nöbeling [19, 22] and, even earlier, by Chittenden, Terasaka, Nakamura, Monteiro and Ribeiro; and provides a further indication that the results of Sections 3 and 4 are based on $le$-semigroups.

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