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Numerical radius, Berezin number, and Berezin norm inequalities for sums of operators

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Abstract: The purpose of this article is to explore various inequalities pertaining to the numerical radius of operators in a Hilbert space. Additionally, we present several bounds for the Berezin number and Berezin norm of operators that act on a reproducing kernel Hilbert space. Finally, we establish a necessary and sufficient condition for the triangle inequality related to the Berezin number to hold.

Key words: Sum, inequality, numerical radius, Berezin number, Berezin norm

1. Introduction

In recent years, there has been significant research into the applications of numerical radius and Berezin numbers of operators across various fields such as quantum computing, engineering, quantum mechanics, differential equations, numerical analysis, etc. Additionally, there has been growing interest in the study of quadratic forms within Hilbert spaces, particularly in relation to the field of values or numerical range. This concept was first introduced by Toeplitz in [34] for matrices, and later, Wintner investigated the connection between the field of values and the convex hull of the spectrum of Hilbert space operators in [35].

The article aims to investigate different inequalities related to the numerical radius of operators in a Hilbert space. Furthermore, it provides various bounds for the Berezin number and Berezin norm of operators that operate on a reproducing kernel Hilbert space.

We need to recall several facts and notations. Firstly, let us denote the algebra of all linear and bounded operators on a complex Hilbert space $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ by $\mathcal{B}(\mathbb{H})$. Additionally, we use the norm associated with $\langle\cdot,\cdot\rangle$ and denote it by $\|\cdot\|$. If $\dim\mathcal{H} = d$, then $\mathcal{B}(\mathbb{H})$ is identified with the matrix algebra $\mathbb{M}_d$ of every $d\times d$ matrix with entries in $\mathbb{C}$, where $\mathbb{C}$ is the field of the complex numbers. Let $\mathbb{S}^1$ stand for the unit sphere of $\mathbb{H}$, i.e.

$$\mathbb{S}^1 := \left\{x \in \mathbb{H};\ \|x\| = \sqrt{\langle x, x\rangle} = 1\right\}.$$

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Let $Q \in \mathcal{B}(\mathbb{H})$. The numerical radius and the operator norm of Q, denoted respectively by $\omega(Q)$ and $\|Q\|$, are given by

$$\omega(Q) = \sup_{x\in\mathbb{S}^1} \left|\langle Qx, x\rangle\right| \quad \text{and} \quad \|Q\| = \sup_{x\in\mathbb{S}^1} \|Qx\|.$$

It is well-known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathbb{H})$ which is equivalent to $\|\cdot\|$. Indeed, the following inequalities

$$\frac{\|Q\|}{2} \le \omega(Q) \le \|Q\|,$$

hold for every $Q \in \mathcal{B}(\mathbb{H})$. Notice that if Q is normal (that is, $Q^*Q = QQ^*$), the equality $\omega(Q) = \|Q\|$ holds. It can be shown that $\omega(\cdot)$ is neither a submultiplicative nor a multiplicative norm on $\mathcal{B}(\mathbb{H})$ (see for instance, [18]).

However, the power inequality holds for $\omega(\cdot)$, i.e. for every positive integer n and all $Q \in \mathcal{B}(\mathbb{H})$, we have

$$\omega(Q^n) \le \omega(Q)^n. \tag{1.1}$$

Although (1.1) was conjectured by Halmos and answered by Berger (for more details and references see [4, 18]), the reverse inequality of (1.1) may not hold in general. This can be demonstrated by considering the following nilpotent matrix: $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_2$, then we see that $\omega(A^2) = 0$, but $\omega(A) = \frac{1}{2}$. To delve deeper into the topic of numerical radius of Hilbert space operators and explore its various applications, interested readers are encouraged to refer to [5, 15] and the references cited therein.

An operator $Q \in \mathcal{B}(\mathbb{H})$ is called positive (we write $Q \ge 0$) if $\langle Qx, x\rangle \ge 0$ for every $x \in \mathcal{H}$. If $Q \ge 0$, then there exists a unique bounded positive linear operator, denoted by $Q^{1/2}$, which satisfies the property $Q = (Q^{1/2})^2$. The absolute value of an operator $Q \in \mathcal{B}(\mathbb{H})$, denoted by $|Q|$, is defined as $|Q| = (Q^*Q)^{1/2}$. Clearly $|Q| \ge 0$.

We mention here that Kittaneh showed respectively in [24] and [25] that the following inequalities

$$\omega(Q) \le \frac{1}{2}\big\||Q| + |Q^*|\big\|$$

and

$$\frac{1}{4}\big\||Q|^2 + |Q^*|^2\big\| \le \omega^2(Q) \le \frac{1}{2}\big\||Q|^2 + |Q^*|^2\big\|. \tag{1.2}$$

hold for every $Q \in \mathcal{B}(\mathbb{H})$.

For $Q, W \in \mathcal{B}(\mathbb{H})$, we found

$$\omega^r(W^*Q) \le \frac{1}{2}\big\||Q|^{2r} + |W|^{2r}\big\|, \quad (r \ge 2). \tag{1.3}$$

This represents an inequality given by Dragomir in [12]. Next, we present an improvement of the above inequality for $r = 2$, given by Kittaneh and Moradi in [26]:

$$\omega^2(W^*Q) \le \frac{1}{6}\big\||Q|^4 + |W|^4\big\| + \frac{1}{3}\omega(W^*Q)\big\||Q|^2 + |W|^2\big\| \le \frac{1}{2}\big\||Q|^4 + |W|^4\big\|. \tag{1.4}$$

Additional results related to the numerical radius of Hilbert space operators can be found in [6, 11, 29, 30] and their respective reference lists.

Let us now introduce the concept of reproducing kernel Hilbert spaces. Consider a nonempty set Ω, and let $\mathcal{F}(\Omega,\mathbb{C})$ denote the set of all complex-valued functions defined on Ω. We say that a set $\mathbb{H}_\Omega \subseteq \mathcal{F}(\Omega,\mathbb{C})$ is a reproducing kernel Hilbert space on Ω (RKHS for short) if $\mathbb{H}_\Omega$ is a Hilbert space and for each $\alpha \in \Omega$, the map $E_\alpha : \mathbb{H}_\Omega \to \mathbb{C}$ defined as $E_\alpha(f) = f(\alpha)$ for all $f \in \mathbb{H}_\Omega$ is bounded. By applying the well-known Riesz representation theorem, we infer that for every $\alpha \in \Omega$, there exists a unique $k_\alpha \in \mathbb{H}_\Omega$ satisfying $f(\alpha) = E_\alpha(f) = \langle f, k_\alpha\rangle$ for every $f \in \mathbb{H}_\Omega$. Note that k_α is called the reproducing kernel for α. Moreover, the reproducing kernel of $\mathbb{H}_\Omega$ is the following set $\{k_\alpha ;\ \alpha \in \Omega\}$. Let $\widehat{k}_\alpha = \frac{k_\alpha}{\|k_\alpha\|}$ denote the normalized reproducing kernel of $\mathbb{H}_\Omega$. It is worth noting that $\{\widehat{k}_\alpha : \alpha \in \Omega\}$ is dense in $\mathbb{H}_\Omega$. Let $Q \in \mathcal{B}(\mathbb{H}_\Omega)$. The Berezin transform (or Berezin symbol) of Q is the function $\widetilde{Q} : \Omega \to \mathbb{C}$ such that $\widetilde{Q}(\alpha) = \langle Q\widehat{k}_\alpha, \widehat{k}_\alpha\rangle$ for all $\alpha \in \Omega$ (see [7, 8]). Berezin transforms play a fundamental role in the theory of operators, more specifically, in the study of Hankel operators, composition operators and Toeplitz operators. In particular, the Berezin symbol uniquely determines the operator, i.e. two operators $Q, W \in \mathcal{B}(\mathbb{H}_\Omega)$ are equal if and only if , $\widetilde{Q}(\alpha) = \widetilde{W}(\alpha)$ for all $\alpha \in \Omega$. For more information regarding Berezin symbols, see [3, 21, 22, 28] and the references therein. Another related and important concept in operator theory is the Berezin number of an operator $Q \in \mathcal{B}(\mathbb{H}_\Omega)$, denoted by $\mathbf{ber}(Q)$, and given by

$$\mathbf{ber}(Q) := \sup_{\alpha\in\Omega} |\widetilde{Q}(\alpha)| = \sup_{\alpha\in\Omega} \left|\langle Q\widehat{k}_\alpha, \widehat{k}_\alpha\rangle\right|.$$

It is easy to see that $\mathbf{ber}(\cdot)$ defines a norm on $\mathcal{B}(\mathbb{H}_\Omega)$. Let $Q \in \mathcal{B}(\mathbb{H}_\Omega)$. It is clear that $0 \le \mathbf{ber}(Q) \le \omega(Q) \le \|Q\|$. Furthermore, if $\mathbf{ber}(Q) \neq 0$, then for any positive integer n we have

$$\mathbf{ber}(Q^n) \le \left(\frac{\omega(Q)}{\mathbf{ber}(Q)}\right)^n \mathbf{ber}(Q)^n,$$

(see [16]). It is clear that $\mathbf{ber}(I_\Omega) = 1$, where I_Ω denotes the identity operator on $\mathbb{H}_\Omega$. Further, if $Q \in \mathcal{B}(\mathbb{H}_\Omega)$, then it can be observed that

$$\left|\langle Qk_\alpha, k_\alpha\rangle\right| \le \mathbf{ber}(Q)\|k_\alpha\|^2$$

for every reproducing kernel k_α. Several studies have been recently carried out on the Berezin number of operators. For instance, one may refer to [16, 17, 19, 20, 33] and the references therein.

Some other norms on $\mathcal{B}(\mathbb{H}_\Omega)$ are given by

$$\|Q\|_{\mathbf{ber}} := \sup\left\{|\langle Q\widehat{k}_\alpha, \widehat{k}_\beta\rangle|\,;\ \alpha, \beta \in \Omega\right\}$$

and

$$\|Q\|_{\widetilde{\mathbf{ber}}} := \sup_{\alpha\in\Omega} \|Q\widehat{k}_\alpha\|,$$

where $Q \in \mathcal{B}(\mathbb{H}_\Omega)$ and $\widehat{k}_\alpha$, $\widehat{k}_\beta$ are two normalized reproducing kernels of $\mathbb{H}_\Omega$ (see [20]). We mention that, in general, neither $\|\cdot\|_{\mathbf{ber}}$ nor $\|\cdot\|_{\widetilde{\mathbf{ber}}}$ verifies the submultiplicative property (see the recent work [10]). In addition, the equality $\|Q\|_{\mathbf{ber}} = \|Q\|_{\widetilde{\mathbf{ber}}}$ fails to hold in general for $Q \in \mathcal{B}(\mathbb{H}_\Omega)$ (see also [10]). A very important observation is that

$$\mathbf{ber}(Q) \le \|Q\|_{\mathbf{ber}} \le \|Q\|_{\widetilde{\mathbf{ber}}} \le \|Q\| \tag{1.5}$$

for all $Q \in \mathcal{B}(\mathbb{H}_\Omega)$. Note that, in general, the inequalities in (1.5) are strict. It has been recently proven in [9] that if $Q \ge 0$, then

$$\mathbf{ber}(Q) = \|Q\|_{\mathbf{ber}}. \tag{1.6}$$

Remark 1.1 *It is well known that the equality $\omega(Q) = \|Q\|$ holds for every self-adjoint operator Q. However,* (1.6) *may not be correct, in general, for self-adjoint operators (see [9]).*

The next inequality has been recently established in (1.5) and provides an improvement of the inequality $\mathbf{ber}(Q) \le \|Q\|$.

$$\mathbf{ber}^2(Q) \le \frac{1}{2}\left\||Q|^2 + |Q^*|^2\right\|_{\mathbf{ber}}, \quad \forall Q \in \mathcal{B}(\mathbb{H}_\Omega). \tag{1.7}$$

The goal of this article is to explore inequalities associated with the numerical radius of operators in a Hilbert space, and to provide bounds for the Berezin number and Berezin norm of operators that operate on a reproducing kernel Hilbert space. The article is structured as follows: Section 2 presents some well-known lemmas that will be used throughout the paper. In Section 3, we establish an inequality that improves the Cauchy-Schwarz inequality and generalize a result of Dragomir. We also derive several inequalities for the numerical radius of operators in a Hilbert space. Section 4 presents various results concerning the Berezin number and Berezin norm. Section 5 contains additional results related to the Berezin number, and we conclude the article by providing a necessary and sufficient condition for the equality of the triangle inequality related to the Berezin number.

The results of this study are expected to be of interest to researchers in the field of Hilbert spaces and operator theory. The various inequalities derived for the numerical radius of operators and the bounds established for the Berezin number and Berezin norm contribute to the advancement of Hilbert space theory and may inspire further research.

2. Preliminary

In this section, we provide some preliminary results that are essential to our analysis. We begin by introducing some well-known facts and lemmas that will be repeatedly used throughout the paper. Specifically, we work with a complex Hilbert space $\mathbb{H}$ equipped with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$.

In [26], a new refinement of the Cauchy-Schwarz inequality was presented:

$$|\langle x, y\rangle| \le \sqrt{\frac{1}{2}(\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2) + |\langle x, y\rangle|\|x\|\|y\|} \le \|x\|\|y\|, \tag{2.1}$$

where $x, y \in \mathbb{H}$. This inequality provides a more accurate bound than some previously established numerical radius inequalities for Hilbert space operators. Another inequality of similar form was introduced by Alomari [1]:

$$\left|\langle x, y\rangle\right|^2 \le \nu\|x\|^2\|y\|^2 + (1-\nu)\left|\langle x, y\rangle\right|\|x\|\|y\| \le \|x\|^2\|y\|^2, \tag{2.2}$$

where $x, y \in \mathbb{H}$ and $\nu \in [0,1]$. These refined inequalities are crucial tools for our analysis in this paper.

In [27], Kouba noted, using Bessel's inequality, that if $\mathbb{H}$ is a real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$, then for all $x, y, u, v \in \mathbb{H}$ with $\|u\| = \|v\| = 1$, we have

$$\langle u, x\rangle^2 + \langle v, x\rangle^2 \leq \|x\|^2 \left(1 + |\langle u, v\rangle|\right) \tag{2.3}$$

and

$$\|z\|^2 \langle x, y\rangle^2 + \|y\|^2 \langle x, z\rangle^2 \leq \|x\|^2 \|y\| \|z\| \left(\|y\| \|z\| + |\langle y, z\rangle|\right). \tag{2.4}$$

When $\mathbb{H}$ is a complex vector space equipped with an inner product, Dragomir proved, in [11], the following:

Lemma 2.1 *For any $x, y, z \in \mathbb{H}$, we have*

$$\left|\langle x, y\rangle\right|^2 + \left|\langle x, z\rangle\right|^2 \leq \|x\|^2 \Big(\max\left\{\|y\|^2, \|z\|^2\right\} + \left|\langle y, z\rangle\right|\Big). \tag{2.5}$$

It is easy to see that for $\|y\| = \|z\| = 1$ in (2.5) we deduce inequality (2.3). Next, we propose a generalization of the result given by Dragomir.

The classical Schwarz inequality for positive operators is given by:

$$|\langle Qx, y\rangle|^2 \leq \langle Qx, x\rangle \langle Qy, y\rangle \tag{2.6}$$

for any positive operator $Q \in \mathcal{B}(\mathbb{H})$ and for any vectors $x, y \in \mathbb{H}$. Kato [23] established a companion of the Schwarz inequality (2.6), which asserts:

$$|\langle Qx, y\rangle|^2 \leq \langle |Q|^{2\theta} x, x\rangle \langle |Q^*|^{2(1-\theta)} y, y\rangle \tag{2.7}$$

for every operator $Q \in \mathcal{B}(\mathbb{H})$, for any vectors $x, y \in \mathbb{H}$, and $\theta \in [0,1]$. For $\theta = \frac{1}{2}$, we obtain a result attributed to Halmos [18, pp. 75-76]; thus,

$$|\langle Qx, x\rangle| \leq \sqrt{\langle |Q|\, x, x\rangle \, \langle |Q^*|\, x, x\rangle} \tag{2.8}$$

for every $Q \in \mathcal{B}(\mathbb{H})$ and for all $x, y \in \mathbb{H}$.

The inequality in the following lemma deals with positive operators and is known as the McCarthy inequality:

Lemma 2.2 *([14, Theorem 1.4]) Let $Q \in \mathcal{B}(\mathbb{H})$ be a positive operator and $x \in \mathbb{S}^1$. Then, for every $r \geq 1$ we have*

$$\langle Qx, x\rangle^r \leq \langle Q^r x, x\rangle.$$

If $0 \leq r \leq 1$, then the above inequality is reversed.

3. Numerical radius inequalities

In this section, we will discuss numerical radius inequalities concerning bounded linear operators on a complex Hilbert space. Our main focus will be on proving our first result, which involves a lemma that extends a known inequality referred to as (2.1). This inequality is crucial in demonstrating our first result and can be expressed as follows:

Lemma 3.1 *Let $x, y \in \mathbb{H}$ and $\nu \in [0,1]$. Then*

$$|\langle x,y\rangle|^2 \leq \gamma_\nu \|x\|^2\|y\|^2 + \Gamma_\nu |\langle x,y\rangle| \|x\|\|y\| \leq \|x\|^2\|y\|^2, \tag{3.1}$$

for any $x, y \in \mathbb{H}$, where $\gamma_\nu = \min\{\nu, 1-\nu\}$ and $\Gamma_\nu = \max\{\nu, 1-\nu\}$.

Proof We study two cases. I) For $\nu \in [0, \frac{1}{2}]$, we have $\nu \leq 1-\nu$, so $\gamma_\nu = \nu$ and $\Gamma_\nu = 1-\nu$; thus, the first part of inequality (3.1) becomes

$$\big|\langle x,y\rangle\big|^2 \leq \nu\|x\|^2\|y\|^2 + (1-\nu)\big|\langle x,y\rangle\big|\|x\|\|y\|.$$

This is equivalent to

$$\big|\langle x,y\rangle\big|^2 \leq \nu\|x\|\|y\| \left(\|x\|\|y\| - \big|\langle x,y\rangle\big|\right) + \big|\langle x,y\rangle\big|\|x\|\|y\|,$$

which is true from the Cauchy–Schwarz inequality, $|\langle x,y\rangle| \leq \|x\|\|y\|$.

II) For $\nu \in [\frac{1}{2}, 1]$, we have $\nu \geq 1-\nu$, so $\gamma_\nu = 1-\nu$ and $\Gamma_\nu = \nu$; thus, the first part of inequality (3.1) becomes

$$\begin{aligned}\big|\langle x,y\rangle\big|^2 &\leq (1-\nu)\|x\|^2\|y\|^2 + \nu\big|\langle x,y\rangle\big|\|x\|\|y\| \\ &= (1-\nu)\|x\|\|y\| \left(\|x\|\|y\| - \big|\langle x,y\rangle\big|\right) + (2\nu-1)\big|\langle x,y\rangle\big|\|x\|\|y\|.\end{aligned}$$

This is true because $|\langle x,y\rangle| \leq \|x\|\|y\|$ and $2\nu \geq 1$. Next, for any $x, y \in \mathbb{H}$ and $\nu \in [0,1]$, we use the following inequality:

$$\gamma_\nu\|x\|^2\|y\|^2 + \Gamma_\nu|\langle x,y\rangle|\|x\|\|y\| \leq \nu\|x\|^2\|y\|^2 + (1-\nu)\big|\langle x,y\rangle\big|\|x\|\|y\|. \tag{3.2}$$

This is true, because it is equivalent with

$$0 \leq (\nu - \gamma_\nu)\|x\|\|y\|(\|x\|\|y\| - \big|\langle x,y\rangle\big|)$$

and by using inequality (2.2), we deduce that the second inequality of the statement is true. □

Remark. Note that inequality (3.1) can be written as:

$$|\langle x,y\rangle| \leq \sqrt{\gamma_\nu\|x\|^2\|y\|^2 + \Gamma_\nu|\langle x,y\rangle|\|x\|\|y\|} \leq \|x\|\|y\|, \tag{3.3}$$

for any $x, y \in \mathbb{H}$ and $\nu \in [0,1]$, which is another improvement on the Cauchy–Schwarz inequality. From relation (3.2) we can say that the first part of inequality (3.1) is better than (2.2).

Theorem 3.2 *Let $Q, W \in \mathcal{B}(\mathbb{H})$ and $\nu \in [0,1]$. Then the inequality*

$$\omega^2(W^*Q) \leq \frac{\gamma_\nu}{2}\big\||Q|^4 + |W|^4\big\| + \frac{\Gamma_\nu}{2}\omega(W^*Q)\big\||Q|^2 + |W|^2\big\| \leq \frac{1}{2}\big\||Q|^4 + |W|^4\big\|$$

holds, where $\gamma_\nu = \min\{\nu, 1-\nu\}$ and where $\Gamma_\nu = \max\{\nu, 1-\nu\}$.

Proof We take the first inequality from Lemma 3.1,

$$|\langle x,y\rangle|^2 \leq \gamma_\nu\|x\|^2\|y\|^2 + \Gamma_\nu|\langle x,y\rangle|\|x\|\|y\|$$

for any $x, y \in \mathbb{H}$ and $\nu \in [0,1]$. Because we need to apply the Hölder–McCarthy inequality for positive operators, it is easy to see that the operators $|Q|^2$ and $|W|^2$ are positive. Now, we replace x and y by Qx and Wx, in the above inequality, and we assume that $x \in \mathbb{S}^1$, then we obtain

$$\begin{aligned}
|\langle W^*Qx, x\rangle|^2 &\leq \gamma_\nu \|Qx\|^2\|Wx\|^2 + \Gamma_\nu |\langle W^*Qx, x\rangle| \|Qx\|\|Wx\| \\
&\leq \gamma_\nu \langle Qx, Qx\rangle\langle Wx, Wx\rangle + \Gamma_\nu |\langle W^*Qx, x\rangle| \sqrt{\langle Qx, Qx\rangle\langle Wx, Wx\rangle} \\
&= \gamma_\nu \langle |Q|^2x, x\rangle\langle |W|^2x, x\rangle + \Gamma_\nu |\langle W^*Qx, x\rangle| \sqrt{\langle |Q|^2x, x\rangle\langle |W|^2x, x\rangle} \\
&\leq \frac{\gamma_\nu}{4}\left(\langle |Q|^2x, x\rangle + \langle |W|^2x, x\rangle\right)^2 + \frac{\Gamma_\nu}{2}|\langle W^*Qx, x\rangle| \left(\langle |Q|^2x, x\rangle + \langle |W|^2x, x\rangle\right) \\
&\leq \frac{\gamma_\nu}{2}\left(\langle |Q|^2x, x\rangle^2 + \langle |W|^2x, x\rangle^2\right) + \frac{\Gamma_\nu}{2}|\langle W^*Qx, x\rangle| \left(\langle (|Q|^2 + |W|^2)x, x\rangle\right) \\
&\overset{McCarthy}{\leq} \frac{\gamma_\nu}{2}\left(\langle |Q|^4x, x\rangle + \langle |W|^4x, x\rangle\right) + \frac{\Gamma_\nu}{2}|\langle W^*Qx, x\rangle| \left(\langle (|Q|^2 + |W|^2)x, x\rangle\right) \\
&= \frac{\gamma_\nu}{2}\left(\langle (|Q|^4 + |W|^4)x, x\rangle\right) + \frac{\Gamma_\nu}{2}|\langle W^*Qx, x\rangle| \left(\langle (|Q|^2 + |W|^2)x, x\rangle\right).
\end{aligned}$$

Taking the supremum over $x \in \mathbb{S}^1$ in the above inequality, we obtain the first inequality of the statement.

Now, taking into account that

$$\left\||Q|^4 + |W|^4\right\| \geq \omega(S^*Q)\left\||Q|^2 + |W|^2\right\|,$$

from inequality (1.4), we obtain the inequality

$$\begin{aligned}
\frac{\gamma_\nu}{2}\left\||Q|^4 + |W|^4\right\| + \frac{\Gamma_\nu}{2}\omega(W^*Q)\left\||Q|^2 + |W|^2\right\| &\leq \frac{\gamma_\nu}{2}\left\||Q|^4 + |W|^4\right\| + \frac{\Gamma_\nu}{2}\left\||Q|^4 + |W|^4\right\| \\
&= \frac{1}{2}\left\||Q|^4 + |W|^4\right\|.
\end{aligned}$$

Therefore, the second inequality of the statement is true. □

Remark 3.3 *Through various particular cases of ν in Theorem 3.2, we obtain some known results; thus, for $\nu = 1$ in the first inequality of Theorem 3.2, we deduce the second inequality from (1.2), and for $\nu = 0$, we find inequality (1.3) for $r = 2$. For $\nu = \frac{1}{3}$ in Theorem 3.2, we obtain inequality (1.4). If we take $\nu = \frac{1}{2}$ in the first inequality of Theorem 3.2, then we deduce the inequality*

$$\omega^2(W^*Q) \leq \frac{1}{4}\left(\left\||Q|^4 + |W|^4\right\| + \omega(W^*Q)\left\||Q|^2 + |W|^2\right\|\right) \leq \frac{1}{2}\left\||Q|^4 + |W|^4\right\|.$$

If, in Theorem 3.2, we take $W = Q^*$, then we find the following inequality:

$$\omega^2(Q^2) \leq \frac{\gamma_\nu}{2}\left\||Q|^4 + |Q^*|^4\right\| + \frac{\Gamma_\nu}{2}\omega(Q^2)\left\||Q|^2 + |Q^*|^2\right\| \leq \frac{1}{2}\left\||Q|^4 + |Q^*|^4\right\|.$$

Theorem 3.4 *Let $Q \in \mathcal{B}(\mathbb{H})$ and $\nu \in [0,1]$. Then, the inequality*

$$\omega^2(Q) \leq \frac{\gamma_\nu}{2}\left\||Q|^2 + |Q^*|^2\right\| + \frac{\Gamma_\nu}{2}\omega(Q)\left\||Q| + |Q^*|\right\| \tag{3.4}$$

holds.

Proof Applying inequality (2.8), we deduce

$$|\langle Qx,x\rangle| \leq \gamma_\nu \sqrt{\langle |Q|x,x\rangle\langle |Q^*|x,x\rangle} + \Gamma_\nu |\langle Qx,x\rangle|$$

for any $x \in \mathbb{H}$ and $\nu \in [0,1]$. Multiplying the above relation by $\sqrt{\langle |Q|x,x\rangle\langle |Q^*|x,x\rangle}$, and taking into account that $|\langle Qx,x\rangle|^2 \leq |\langle Qx,x\rangle|\sqrt{\langle |Q|x,x\rangle\langle |Q^*|x,x\rangle}$, we find the following inequality:

$$|\langle Qx,x\rangle|^2 \leq \gamma_\nu \langle |Q|x,x\rangle\langle |Q^*|x,x\rangle + \Gamma_\nu |\langle Qx,x\rangle|\sqrt{\langle |Q|x,x\rangle\langle |Q^*|x,x\rangle}. \tag{3.5}$$

Because the operators $|Q|$ and $|Q^*|$ are positive, we can apply the Hölder–McCarthy inequality [13], thus:

$$\langle |Q|x,x\rangle^2 \leq \langle |Q|^2x,x\rangle \quad \text{and} \quad \langle |Q^*|x,x\rangle^2 \leq \langle |Q^*|^2x,x\rangle.$$

Therefore, inequality (3.5) becomes

$$\begin{aligned}|\langle Qx,x\rangle|^2 &\leq \frac{\gamma_\nu}{4}\left(\langle |Q|x,x\rangle + \langle |Q^*|x,x\rangle\right)^2 + \frac{\Gamma_\nu}{2}|\langle Qx,x\rangle|\langle (|Q|+|Q^*|)x,x\rangle \\ &\leq \frac{\gamma_\nu}{2}\left(\langle |Q|x,x\rangle^2 + \langle |Q^*|x,x\rangle^2\right) + \frac{\Gamma_\nu}{2}|\langle Qx,x\rangle|\langle (|Q|+|Q^*|)x,x\rangle \\ &\leq \frac{\gamma_\nu}{2}\langle (|Q|^2+|Q^*|^2)x,x\rangle + \frac{\Gamma_\nu}{2}|\langle Qx,x\rangle|\langle (|Q|+|Q^*|)x,x\rangle.\end{aligned}$$

If we take the supremum over all $x \in \mathbb{S}^1$ in the last bound, then the inequality of the statement is true. □

Remark 3.5 *For $\nu = \frac{1}{3}$ in inequality (3.4), we find an inequality given in [26], namely:*

$$\omega^2(Q) \leq \frac{1}{6}\left\||Q|^2+|Q^*|^2\right\| + \frac{1}{3}\omega(Q)\left\||Q|+|Q^*|\right\|. \tag{3.6}$$

If we take $\nu = \frac{1}{2}$ in (3.4), then we obtain the following sequence of inequalities:

$$\begin{aligned}\omega^2(Q) &\leq \frac{1}{4}\left(\left\||Q|^2+|Q^*|^2\right\| + \omega(Q)\left\||Q|+|Q^*|\right\|\right) \\ &\leq \frac{1}{4}\left(\left\||Q|^2+|Q^*|^2\right\| + \frac{1}{2}\left\||Q|+|Q^*|\right\|^2\right) \\ &\leq \frac{1}{4}\left(\left\||Q|^2+|Q^*|^2\right\| + \left\||Q|^2+|Q^*|^2\right\|\right) \\ &= \frac{1}{2}\left\||Q|^2+|Q^*|^2\right\|.\end{aligned}$$

Consequently, we deduce a refinement of the second inequality of relation (1.2) given by Kittaneh [25].

Next, we present a result which generalizes the inequality (2.1).

Theorem 3.6 *For any $x,y,z \in \mathbb{H}$ and $\nu \in [0,1]$, we have*

$$\nu\left|\langle x,y\rangle\right|^2 + (1-\nu)\left|\langle x,z\rangle\right|^2 \leq \|x\|^2\Big(\max\left\{\nu\|y\|^2,(1-\nu)\|z\|^2\right\} + \sqrt{\nu(1-\nu)}\left|\langle y,z\rangle\right|\Big). \tag{3.7}$$

Proof We first note that

$$\nu\left|\langle x,y\rangle\right|^2+(1-\nu)\left|\langle x,z\rangle\right|^2=\langle \nu\langle x,y\rangle y+(1-\nu)\langle x,z\rangle z,x\rangle^2.$$

Using the Cauchy-Schwarz inequality, $\left|\langle u,x\rangle\right|^2\leq\|u\|^2\|x\|^2$, for $u=\nu\langle x,y\rangle y+(1-\nu)\langle x,z\rangle z$, and from the above equality we deduce that

$$\left(\nu\left|\langle x,y\rangle\right|^2+(1-\nu)\left|\langle x,z\rangle\right|^2\right)^2\leq\|u\|^2\|x\|^2.$$

However, we have

$$\begin{aligned}\|u\|^2&=\nu^2\left|\langle x,y\rangle\right|^2\|y\|^2+(1-\nu)^2\left|\langle x,z\rangle\right|^2\|z\|^2+2\nu(1-\nu)\Re(\langle x,y\rangle\langle y,z\rangle\langle z,x\rangle)\\&\leq\max\left\{\nu\|y\|^2,(1-\nu)\|z\|^2\right\}\left(\nu\left|\langle x,y\rangle\right|^2+(1-\nu)\left|\langle x,z\rangle\right|^2\right)+2\nu(1-\nu)|\langle x,y\rangle||\langle y,z\rangle||\langle z,x\rangle|\\&\leq\max\left\{\nu\|y\|^2,(1-\nu)\|z\|^2\right\}\left(\nu\left|\langle x,y\rangle\right|^2+(1-\nu)\left|\langle x,z\rangle\right|^2\right)+\sqrt{\nu(1-\nu)}\left(\nu\left|\langle x,y\rangle\right|^2+(1-\nu)\left|\langle x,z\rangle\right|^2\right)|\langle y,z\rangle|\\&=\left(\nu\left|\langle x,y\rangle\right|^2+(1-\nu)\left|\langle x,z\rangle\right|^2\right)\left(\max\left\{\nu\|y\|^2,(1-\nu)\|z\|^2\right\}+\sqrt{\nu(1-\nu)}|\langle y,z\rangle|\right).\end{aligned}$$

Therefore, by replacing u, we obtain

$$\begin{aligned}&\left(\nu\left|\langle x,y\rangle\right|^2+(1-\nu)\left|\langle x,z\rangle\right|^2\right)^2\\&\leq\|x\|^2\left(\nu\left|\langle x,y\rangle\right|^2+(1-\nu)\left|\langle x,z\rangle\right|^2\right)\left(\max\left\{\nu\|y\|^2,(1-\nu)\|z\|^2\right\}+\sqrt{\nu(1-\nu)}|\langle y,z\rangle|\right),\end{aligned}$$

which proves the relation of the statement. □

Remark 3.7 *(1) For $\nu=\frac{1}{2}$ in relation (3.7), we prove inequality (2.5).*

(2) If we take two real numbers $a,b>0$ and $\nu=\frac{a}{a+b}$ in Theorem 3.6, then we obtain

$$a\left|\langle x,y\rangle\right|^2+b\left|\langle x,z\rangle\right|^2\leq\|x\|^2\Big(\max\left\{a\|y\|^2,b\|z\|^2\right\}+\sqrt{ab}\left|\langle y,z\rangle\right|\Big).\tag{3.8}$$

By replacing a with $\|z\|^2$ and b with $\|x\|^2$ in relation (3.8), we deduce inequality (2.4), for the case when $\mathbb{H}$ is a complex vector space equipped with an inner product.

Corollary 3.8 *For any $x,y,z\in\mathbb{H}$ and $\nu\in[0,1]$, the following inequality*

$$2\sqrt{\nu(1-\nu)}\left|\langle x,y\rangle\right|\left|\langle x,z\rangle\right|\leq\|x\|^2\Big(\max\left\{\nu\|y\|^2,(1-\nu)\|z\|^2\right\}+\sqrt{\nu(1-\nu)}\left|\langle y,z\rangle\right|\Big)\tag{3.9}$$

holds.

Proof Using the inequality between the geometric mean and the arithmetic mean, we obtain

$$2\sqrt{\nu(1-\nu)}\left|\langle x,y\rangle\right|\left|\langle x,z\rangle\right|\leq\nu\left|\langle x,y\rangle\right|^2+(1-\nu)\left|\langle x,z\rangle\right|^2$$

and from inequality (3.7), we deduce the relation of the statement. □

4. Berezin number and norm inequalities

In this section, we will demonstrate the versatility of the Berezin number and norm in a reproducing Kernel Hilbert space by applying the results from the previous section. Specifically, we will state a theorem that relates the Berezin norm of a convex combination of two bounded linear operators on a reproducing Kernel Hilbert space to their Berezin numbers and inner products, using the outcomes obtained in the preceding section. The theorem is as follows:

Theorem 4.1 *Let $Q, W \in \mathcal{B}(\mathbb{H}_\Omega)$ and $\nu \in [0,1]$. Then, the inequality*

$$\|\nu Q + (1-\nu)W\|_{\boldsymbol{ber}}^2 \leq \frac{1}{2}\left\|\nu|Q|^2 + (1-\nu)|W|^2\right\|_{\boldsymbol{ber}} + \frac{1}{2}\,\boldsymbol{ber}\left(\nu|Q|^2 - (1-\nu)|W|^2\right) + \sqrt{\nu(1-\nu)}\,\boldsymbol{ber}\left(W^*Q\right)$$

holds.

Proof Let $\alpha, \beta \in \Omega$ and $\widehat{k}_\alpha, \widehat{k}_\beta$ be two normalized reproducing kernels of $\mathbb{H}_\Omega$. If we replace x, y, and z by $\widehat{k}_\beta$, $Q\widehat{k}_\alpha$, and $W\widehat{k}_\alpha$, respectively, in (3.7), and we know that $\|\widehat{k}_\beta\| = 1$, then we have

$$\nu\left|\langle Q\widehat{k}_\alpha, \widehat{k}_\beta\rangle\right|^2 + (1-\nu)\left|\langle W\widehat{k}_\alpha, \widehat{k}_\beta\rangle\right|^2 \overset{(3.7)}{\leq} \max\left\{\nu\|Q\widehat{k}_\alpha\|^2, (1-\nu)\|W\widehat{k}_\alpha\|^2\right\} + \sqrt{\nu(1-\nu)}\left|\langle W^*Q\widehat{k}_\alpha, \widehat{k}_\alpha\rangle\right|$$

for any $x, y \in \mathbb{H}$ and $\nu \in [0,1]$. Therefore, we have the following calculations:

$$\begin{aligned}
\left|\left\langle(\nu Q + (1-\nu)W)\widehat{k}_\alpha, \widehat{k}_\beta\right\rangle\right|^2 &\leq \left(\nu\left|\left\langle Q\widehat{k}_\alpha, \widehat{k}_\beta\right\rangle\right| + (1-\nu)\left|\left\langle W\widehat{k}_\alpha, \widehat{k}_\beta\right\rangle\right|\right)^2 \\
&\leq \nu\left|\langle Q\widehat{k}_\alpha, \widehat{k}_\beta\rangle\right|^2 + (1-\nu)\left|\langle W\widehat{k}_\alpha, \widehat{k}_\beta\rangle\right|^2 \\
&\leq \max\left\{\nu\|Q\widehat{k}_\alpha\|^2, (1-\nu)\|W\widehat{k}_\alpha\|^2\right\} + \sqrt{\nu(1-\nu)}\left|\langle W^*Q\widehat{k}_\alpha, \widehat{k}_\alpha\rangle\right| \\
&= \frac{1}{2}\left(\nu\|Q\widehat{k}_\alpha\|^2 + (1-\nu)\|W\widehat{k}_\alpha\|^2 + |\nu\|Q\widehat{k}_\alpha\|^2 - (1-\nu)\|W\widehat{k}_\alpha\|^2|\right) \\
&\quad + \sqrt{\nu(1-\nu)}\left|\langle Q\widehat{k}_\alpha, W\widehat{k}_\alpha\rangle\right. \\
&= \frac{1}{2}\left(\nu\langle Q\widehat{k}_\alpha, Q\widehat{k}_\alpha\rangle + (1-\nu)\langle W\widehat{k}_\alpha, W\widehat{k}_\alpha\rangle + |\nu\langle Q\widehat{k}_\alpha, Q\widehat{k}_\alpha\rangle - (1-\nu)\langle W\widehat{k}_\alpha, W\widehat{k}_\alpha\rangle|\right) \\
&\quad + \sqrt{\nu(1-\nu)}\left|\langle Q\widehat{k}_\alpha, W\widehat{k}_\alpha\rangle\right. \\
&= \frac{1}{2}\left(\left\langle\left(\nu|Q|^2 + (1-\nu)|W|^2\right)\widehat{k}_\alpha, \widehat{k}_\alpha\right\rangle + \left|\left\langle\left(\nu|Q|^2 - (1-\nu)|W|^2\right)\widehat{k}_\alpha, \widehat{k}_\alpha\right\rangle\right) \\
&\quad + \sqrt{\nu(1-\nu)}\left|\left\langle W^*Q\widehat{k}_\alpha, \widehat{k}_\alpha\right\rangle\right|,
\end{aligned}$$

whence,

$$\begin{aligned}
&\left|\left\langle(\nu Q + (1-\nu)W)\widehat{k}_\alpha, \widehat{k}_\beta\right\rangle\right|^2 \\
&\leq \frac{1}{2}\,\mathbf{ber}\left(\nu|Q|^2 + (1-\nu)|W|^2\right) + \frac{1}{2}\,\mathbf{ber}\left(\nu|Q|^2 - (1-\nu)|W|^2\right) + \sqrt{\nu(1-\nu)}\,\mathbf{ber}\left(W^*Q\right) \\
&= \frac{1}{2}\left\|\nu|Q|^2 + (1-\nu)|W|^2\right\|_{\mathbf{ber}} + \frac{1}{2}\,\mathbf{ber}\left(\nu|Q|^2 - (1-\nu)|W|^2\right) + \sqrt{\nu(1-\nu)}\,\mathbf{ber}\left(W^*Q\right),
\end{aligned}$$

where we have used (1.6) in the last equality since $\nu|Q|^2 + (1-\nu)|W|^2 \geq 0$. Therefore, by taking the supremum over all $\alpha, \beta \in \Omega$ in the above inequality, we obtain the desired result. □

Remark 4.2 *For $\nu = \frac{1}{2}$ in the inequality of Theorem 4.1, we deduce*

$$\|Q+W\|_{ber}^2 \le \frac{1}{2}\big\||Q|^2+|W|^2\big\|_{ber} + \frac{1}{2}\,\boldsymbol{ber}\left(|Q|^2-|W|^2\right) + \boldsymbol{ber}\left(W^*Q\right)$$

for all $Q, W \in \mathcal{B}(\mathbb{H}_\Omega)$.

Corollary 4.3 *Let $Q \in \mathcal{B}(\mathbb{H}_\Omega)$ and $\nu \in [0,1]$. Then the inequality*

$$\|\nu Q+(1-\nu)Q^*\|_{ber}^2 \le \frac{1}{2}\big\|\nu|Q|^2+(1-\nu)|Q^*|^2\big\|_{ber} + \frac{1}{2}\,\boldsymbol{ber}\left(\nu|Q|^2-(1-\nu)|Q^*|^2\right) + \sqrt{\nu(1-\nu)}\,\boldsymbol{ber}\left(Q^2\right)$$

holds.

Proof By taking $W = Q^*$ in the inequality of Theorem 4.1, we have

$$\begin{aligned}\|\nu Q+(1-\nu)Q^*\|_{\mathbf{ber}}^2 \le \frac{1}{2}\big\|\nu|Q|^2+(1-\nu)|Q^*|^2\big\|_{\mathbf{ber}} &+ \frac{1}{2}\,\mathbf{ber}\left(\nu|Q|^2-(1-\nu)|Q^*|^2\right)\\ &+ \sqrt{\nu(1-\nu)}\,\mathbf{ber}\left((Q^*)^*Q\right).\end{aligned}$$

Consequently, we obtain the inequality of the statement. □

5. Further results related to the Berezin number

This section contains additional results related to the Berezin number and provides a necessary and sufficient condition for the equality of the triangle inequality related to the Berezin number. This section begins with the following proposition that presents a refinement of the triangle inequality for $\mathbf{ber}(\cdot)$:

Proposition 5.1 *Let $Q, W \in \mathcal{B}(\mathbb{H}_\Omega)$. Then the following inequality*

$$\boldsymbol{ber}\left(Q+W\right) \le 2\int_0^1 \boldsymbol{ber}\Big(\nu Q+(1-\nu)W\Big)d\nu \le \boldsymbol{ber}\left(Q\right)+\boldsymbol{ber}\left(W\right)$$

holds.

Proof Consider the following function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\nu \mapsto \varphi(\nu) := \mathbf{ber}\Big(\nu Q+(1-\nu)W\Big).$$

Let $\nu_1, \nu_2 \in \mathbb{R}$ and $t \in [0,1]$. We see that

$$\begin{aligned}\varphi\Big(t\nu_1+(1-t)\nu_2\Big) &= \mathbf{ber}\Big(\big[t\nu_1+(1-t)\nu_2\big]Q+\big[1-\big(t\nu_1+(1-t)\nu_2\big)\big]\,W\Big)\\ &= \mathbf{ber}\Big(t\nu_1Q+(1-t)\nu_2Q+W-t\nu_1W-(1-t)\nu_2W+tW-tW\Big)\\ &= \mathbf{ber}\Big(t\big[\nu_1Q+(1-\nu_1)W\big]+(1-t)\big[\nu_2Q+(1-\nu_2)W\big]\Big)\\ &\le t\,\mathbf{ber}\Big(\nu_1Q+(1-\nu_1)W\Big)+(1-t)\,\mathbf{ber}\Big(\nu_2Q+(1-\nu_2)W\Big)\\ &= t\varphi(\nu_1)+(1-t)\varphi(\nu_2).\end{aligned}$$

This proves that φ is convex. Hence, an application of the Hermite–Hadamard inequality (see [31, p. 137]) shows that

$$\varphi\left(\frac{0+1}{2}\right) \leq \frac{1}{1-0}\int_0^1 \varphi(\nu)d\nu \leq \frac{\varphi(0)+\varphi(1)}{2}.$$

This implies that

$$\mathbf{ber}\left(\frac{1}{2}Q+\frac{1}{2}W\right) \leq \int_0^1 \mathbf{ber}\left(\nu Q+(1-\nu)W\right)d\nu \leq \frac{\mathbf{ber}\,(Q)+\mathbf{ber}\,(W)}{2}.$$

So, we deduce that

$$\mathbf{ber}\,(Q+W) \leq 2\int_0^1 \mathbf{ber}\left(\nu Q+(1-\nu)W\right)d\nu \leq \mathbf{ber}\,(Q)+\mathbf{ber}\,(W).$$

□

As demonstrated in the following example, the inequality presented in Proposition 5.1 offers a noteworthy enhancement to the triangle inequality associated with $\mathbf{ber}(\cdot)$.

Example 5.2 *Let $\{e_1, e_2\}$ denote the canonical orthonormal basis of $\mathbb{C}^2$ and $\Omega = \{1,2\}$. Let us consider the space $\mathbb{C}^2$ as a RKHS on the set Ω. Then e_1 and e_2 are the kernel functions given by*

$$e_k(l) = \begin{cases} 1, & \text{if } k = l \\ 0, & \text{if } k \neq l \end{cases}$$

for $k,l \in \Omega$ (see [32, pp. 4-5]). Now, we consider the following matrices in $\mathbb{M}_2$: $Q = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ and $W = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. It is not difficult to check that $\boldsymbol{ber}(Q) = 2$, $\boldsymbol{ber}(W) = 1$ and $\boldsymbol{ber}(Q+W) = 2$. Moreover, we see that

$$\int_0^1 \boldsymbol{ber}\left(\nu Q+(1-\nu)W\right)d\nu = \int_0^1 \max\{1, 2\nu\}d\nu = \frac{5}{4}.$$

Hence, we deduce that

$$\boldsymbol{ber}(Q+W) = 2 < 2\int_0^1 \boldsymbol{ber}\left(\nu Q+(1-\nu)W\right)d\nu = \frac{5}{2} < \boldsymbol{ber}(Q)+\boldsymbol{ber}(W) = 3.$$

By proceeding as in the proof of Proposition 5.1, we state without proof the next result.

Proposition 5.3 *Let $Q, W \in \mathcal{B}(\mathbb{H}_\Omega)$. Then*

$$\left\|Q+W\right\|_{\boldsymbol{ber}} \leq 2\int_0^1 \left\|\nu Q+(1-\nu)W\right\|_{\boldsymbol{ber}} d\nu \leq \left\|Q\right\|_{\boldsymbol{ber}} + \left\|W\right\|_{\boldsymbol{ber}}$$

and

$$\left\|Q+W\right\|_{\widetilde{\boldsymbol{ber}}} \leq 2\int_0^1 \left\|\nu Q+(1-\nu)W\right\|_{\widetilde{\boldsymbol{ber}}} d\nu \leq \left\|Q\right\|_{\widetilde{\boldsymbol{ber}}} + \left\|W\right\|_{\widetilde{\boldsymbol{ber}}}.$$

In order to demonstrate our upcoming outcome, it is necessary to revisit the subsequent lemma from [2].

Lemma 5.4 *Let* $Q \in \mathcal{B}(\mathbb{B}_\Omega)$. *Then*

$$\boldsymbol{ber}(Q) = \sup_{\gamma\in\mathbb{R}} \boldsymbol{ber}\Big(\Re(e^{i\gamma}Q)\Big), \quad where \quad \Re(e^{i\gamma}Q) = \frac{e^{i\gamma}Q + e^{-i\gamma}Q^*}{2}.$$

By utilizing Proposition 5.1, we provide a clear expression for $\mathbf{ber}(X)$ in the following outcome.

Theorem 5.5 *Let* $X \in \mathcal{B}(\mathbb{B}_\Omega)$. *Then*

$$\boldsymbol{ber}(X) = \sup_{\gamma\in\mathbb{R}} \int_0^1 \boldsymbol{ber}\left(\nu e^{i\gamma}X + (1-\nu)e^{-i\gamma}X^*\right)d\nu.$$

Proof Let $\gamma \in \mathbb{R}$. By applying Proposition 5.1 with $Q := \frac{e^{i\gamma}X}{2}$ and $W := \frac{e^{-i\gamma}X^*}{2}$, we see that

$$\begin{aligned}\mathbf{ber}\left(\frac{e^{i\gamma}X}{2} + \frac{e^{-i\gamma}X^*}{2}\right) &\le 2\int_0^1 \mathbf{ber}\left(\nu\frac{e^{i\gamma}X}{2} + (1-\nu)\frac{e^{-i\gamma}X^*}{2}\right)d\nu \\ &\le \mathbf{ber}\left(\frac{e^{i\gamma}X}{2}\right) + \mathbf{ber}\left(\frac{e^{-i\gamma}X^*}{2}\right).\end{aligned}$$

This implies that

$$\mathbf{ber}\Big(\Re(e^{i\gamma}X)\Big) \le \int_0^1 \mathbf{ber}\left(\nu e^{i\gamma}X + (1-\nu)e^{-i\gamma}X^*\right)d\nu \le \frac{1}{2}\mathbf{ber}\left(X\right) + \frac{1}{2}\mathbf{ber}\left(X^*\right),$$

whence

$$\mathbf{ber}\Big(\Re(e^{i\gamma}X)\Big) \le \int_0^1 \mathbf{ber}\left(\nu e^{i\gamma}X + (1-\nu)e^{-i\gamma}X^*\right)d\nu \le \mathbf{ber}\left(X\right). \tag{5.1}$$

We can obtain the desired result by computing the supremum over $\gamma \in \mathbb{R}$ in equation (5.1) and subsequently applying Lemma 5.4. □

The subsequent theorem presents a condition that is both necessary and sufficient for the triangle inequality related to $\mathbf{ber}(\cdot)$ to be equal.

Theorem 5.6 *Let* $Q, W \in \mathcal{B}(\mathbb{B}_\Omega)$. *Then, the following assertions are equivalent:*

(i) $\boldsymbol{ber}(Q+W) = \boldsymbol{ber}(Q) + \boldsymbol{ber}(W)$.

(ii) There exists a sequence $\{\alpha_n\}$ *in* Ω *such that*

$$\lim_{n\to+\infty}\langle \widehat{k}_{\alpha_n}, Q\widehat{k}_{\alpha_n}\rangle\langle W\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\rangle = \boldsymbol{ber}(Q)\,\boldsymbol{ber}(W), \tag{5.2}$$

where $\widehat{k}_{\alpha_n}$ *is the normalized reproducing kernels of* $\mathbb{H}_\Omega$ *at* α_n *for all* n.

Proof "$(ii) \Rightarrow (i)$": Assume that there exists a sequence $\{\alpha_n\}$ in Ω such that (5.2) is true. If $\mathbf{ber}(Q) = 0$ or $\mathbf{ber}(W) = 0$, then the assertion (i) holds trivially. Assume that $\mathbf{ber}(Q) \neq 0$ and $\mathbf{ber}(W) \neq 0$. We see that

$$\begin{aligned}\left|\langle \widehat{k}_{\alpha_n}, Q\widehat{k}_{\alpha_n}\rangle\langle W\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\rangle\right| &= \left|\langle \widehat{k}_{\alpha_n}, Q\widehat{k}_{\alpha_n}\rangle\right|\left|\langle W\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\rangle\right| \\ &\leq \mathbf{ber}(Q)\left|\langle W\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\rangle\right| \\ &\leq \mathbf{ber}(Q)\,\mathbf{ber}(W).\end{aligned}$$

By letting n go to $+\infty$ and then using (5.2), we deduce that

$$\lim_{n\to+\infty} \mathbf{ber}(Q)\left|\langle W\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\rangle\right| = \mathbf{ber}(Q)\,\mathbf{ber}(W).$$

This yields that

$$\lim_{n\to+\infty} \left|\langle W\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\rangle\right| = \mathbf{ber}(W). \tag{5.3}$$

Similarly, we prove that

$$\lim_{n\to+\infty} \left|\langle Q\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\rangle\right| = \mathbf{ber}(Q). \tag{5.4}$$

On the other hand, for every $n \in \mathbb{N}$, we see that

$$\begin{aligned}\left|\left\langle Q\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\right\rangle\right|^2 + \left|\left\langle W\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\right\rangle\right|^2 + 2\mathrm{Re}\left(\langle \widehat{k}_{\alpha_n}, Q\widehat{k}_{\alpha_n}\rangle\langle W\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\rangle\right) &= \left|\left\langle (Q+W)\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\right\rangle\right|^2 \\ &\leq \mathbf{ber}^2(Q+W).\end{aligned}$$

By letting n go to $+\infty$ and then using (5.2), (5.3), and (5.4), we conclude that

$$\Big(\mathbf{ber}(Q) + \mathbf{ber}(W)\Big)^2 \leq \mathbf{ber}^2(Q+W),$$

whence we have $\mathbf{ber}(Q) + \mathbf{ber}(W) \leq \mathbf{ber}(Q+W)$. Hence $\mathbf{ber}(Q+W) = \mathbf{ber}(Q) + \mathbf{ber}(W)$ as desired.

"$(i) \Rightarrow (ii)$": Assume that $\mathbf{ber}(Q+W) = \mathbf{ber}(Q) + \mathbf{ber}(W)$. Then there exists a sequence $\{\alpha_n\}$ in Ω such that

$$\lim_{n\to+\infty} \left\langle Q\widehat{k}_{\alpha_n} + W\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\right\rangle = \mathbf{ber}(Q) + \mathbf{ber}(W), \tag{5.5}$$

where $\widehat{k}_{\alpha_n}$ is the normalized reproducing kernels of $\mathbb{H}_\Omega$ at α_n for all n. Moreover, it is clear that

$$\left|\left\langle Q\widehat{k}_{\alpha_n}\right\rangle\right| \leq \mathbf{ber}(Q) \quad \text{and} \quad \left|\left\langle W\widehat{k}_{\alpha_n}\right\rangle\right| \leq \mathbf{ber}(W),$$

for every $n \in \mathbb{N}$. Therefore, we deduce that

$$\begin{aligned}\left|\left\langle Q\widehat{k}_{\alpha_n} + W\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\right\rangle\right| &\leq \left|\left\langle Q\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\right\rangle\right| + \left|\left\langle W\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\right\rangle\right| \\ &\leq \left|\left\langle Q\widehat{k}_{\alpha_n}, \widehat{k}_{\alpha_n}\right\rangle\right| + \mathbf{ber}(W) \\ &\leq \mathbf{ber}(Q) + \mathbf{ber}(W).\end{aligned}$$

Hence, by letting n go to $+\infty$ and taking (5.5) into consideration, we deduce that

$$\lim_{n\to+\infty}\left|\left\langle Q\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\right\rangle\right| + \mathbf{ber}(W) = \mathbf{ber}(Q) + \mathbf{ber}(W).$$

From this, we conclude that

$$\lim_{n\to+\infty}\left|\left\langle Q\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\right\rangle\right| = \mathbf{ber}(Q). \tag{5.6}$$

By using a similar argument to the one above, we get

$$\lim_{n\to+\infty}\left|\left\langle W\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\right\rangle\right| = \mathbf{ber}(W). \tag{5.7}$$

On the other hand, for every $n\in\mathbb{N}$, we see that

$$\left|\left\langle Q\widehat{k}_{\alpha_n}+W\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\right\rangle\right|^2 = \left|\left\langle Q\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\right\rangle\right|^2 + \left|\left\langle W\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\right\rangle\right|^2 + 2\mathrm{Re}\left(\langle\widehat{k}_{\alpha_n},Q\widehat{k}_{\alpha_n}\rangle\langle W\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\rangle\right).$$

By using (5.5), (5.6), and (5.7), we deduce that

$$\lim_{n\to+\infty}\mathrm{Re}\left(\langle\widehat{k}_{\alpha_n},Q\widehat{k}_{\alpha_n}\rangle\langle W\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\rangle\right) = \mathbf{ber}(Q)\,\mathbf{ber}(W). \tag{5.8}$$

Furthermore, we have

$$\begin{aligned}\left[\mathrm{Re}\left(\langle\widehat{k}_{\alpha_n},Q\widehat{k}_{\alpha_n}\rangle\langle W\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\rangle\right)\right]^2 &\le \left[\mathrm{Re}\left(\langle\widehat{k}_{\alpha_n},Q\widehat{k}_{\alpha_n}\rangle\langle W\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\rangle\right)\right]^2 + \left[\mathrm{Im}\left(\langle\widehat{k}_{\alpha_n},Q\widehat{k}_{\alpha_n}\rangle\langle W\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\rangle\right)\right]^2\\ &= \left|\langle\widehat{k}_{\alpha_n},Q\widehat{k}_{\alpha_n}\rangle\langle W\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\rangle\right|^2\\ &\le \mathbf{ber}^2(Q)\,\mathbf{ber}^2(W),\end{aligned}$$

for every $n\in\mathbb{N}$. So, by using (5.8), we infer that

$$\lim_{n\to+\infty}\mathrm{Im}\left(\langle\widehat{k}_{\alpha_n},Q\widehat{k}_{\alpha_n}\rangle\langle W\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\rangle\right) = 0.$$

Finally, another application of (5.8) shows that

$$\lim_{n\to+\infty}\langle\widehat{k}_{\alpha_n},Q\widehat{k}_{\alpha_n}\rangle\langle W\widehat{k}_{\alpha_n},\widehat{k}_{\alpha_n}\rangle = \mathbf{ber}(Q)\,\mathbf{ber}(W).$$

as desired. Hence, the proof is complete. □

Declarations

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