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The adjoint Reidemeister torsion for compact 3-manifolds admitting a unique decomposition

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Abstract: Let M be a triangulated, oriented, connected compact 3-manifold with a connected nonempty boundary. Such a manifold admits a unique decomposition into $\triangle$-prime 3-manifolds. In this paper, we show that the adjoint Reidemeister torsion has a multiplicative property on the disk sum decomposition of compact 3-manifolds without a corrective term.

Key words: Adjoint Reidemeister torsion, compact 3-manifolds, disk sum

1. Introduction

In this paper we focus on the adjoint Reidemeister torsions of triangulated, oriented, connected compact 3-manifolds with connected nonempty boundaries. Let $\mathcal{X}$ denote the class of 3-manifolds M with connected nonempty boundary such that every 2-sphere in M bounds a 3-cell. Let M and M' be two manifolds in the class $\mathcal{X}$. Then the disk sum (also called boundary connected sum) $M \triangle M'$ can be formed by pasting a 2-cell on the boundary of M to a 2-cell on the boundary of M'. The operation of disk sum $\triangle$ is well-defined, associative, and commutative up to homeomorphism. A manifold $M \in \mathcal{X}$ is called $\triangle$-prime if it is not a 3-cell, and whenever $M \cong P \triangle P'$, either P or P' is a 3-cell. In [5], Gross proved the following decomposition theorem:

Theorem 1.1 ([5]) *For any 3-manifold M (different from a 3-cell) with connected nonempty boundary, there is an isomorphism*

$$M \cong M_1 \triangle M_2 \triangle \dots \triangle M_k,$$

where the summands M_i are $\triangle$-prime 3-manifolds, and they are uniquely determined up to order and homeomorphism.

In 1935, Reidemeister introduced a new invariant, called now *Reidemeister torsion*, to classify 3-dimensional lens spaces (up to PL equivalence) [10]. Later, Franz classified higher dimensional lens spaces by extending the notion of this invariant [3]. In 1969, Kirby and Siebenmann showed that Reidemeister torsion is a topological invariant for manifolds [4]. The invariance for arbitrary simplicial complexes was proved by Chapman [2] and thus the classification of lens spaces of Reidemeister and Franz was shown to be a topological

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invariant. In 1961, Milnor disproved Hauptvermutung by using this invariant. He constructed two homeomorphic but combinatorially distinct finite simplicial complexes. Then he described Reidemeister torsion with the Alexander polynomial which plays an important role in knot theory and links [6, 7].

The twisted chain complex of a 3–manifold M with a connected nonempty boundary is never acyclic, so its first twisted homology group does not vanish. Hence, the computation of the adjoint Reidemeister torsion of M via $\triangle$-prime 3-manifolds involves a corrective term $\mathbb{T}(\mathcal{H}_*)$ coming from the homologies as a factor of the adjoint Reidemeister torsion of M.

Throughout this paper G denotes a complex reductive algebraic group $SL_n(\mathbb{C})$ or $PSL_n(\mathbb{C})$ and $\mathfrak{g}$ is the Lie algebra of G. Let K be a cell-decomposition of finite CW-complex X and $\rho : \pi_1(X) \to G$ be any representation. We denote the twisted chain complex by $C_*(K;\mathfrak{g}_{\mathrm{Ad}_\rho})$, the subspaces of cycles $\mathrm{Ker}\{\partial_p \otimes \mathrm{id} : C_p(K;\mathfrak{g}_{\mathrm{Ad}_\rho}) \to C_{p-1}(K;\mathfrak{g}_{\mathrm{Ad}_\rho})\}$ by $Z_p(K;\mathfrak{g}_{\mathrm{Ad}_\rho})$, the subspaces of boundaries $\mathrm{Im}\{\partial_{p+1} \otimes \mathrm{id} : C_{p+1}(K;\mathfrak{g}_{\mathrm{Ad}_\rho}) \to C_p(K;\mathfrak{g}_{\mathrm{Ad}_\rho})\}$ by $B_p(K;\mathfrak{g}_{\mathrm{Ad}_\rho})$, and the twisted p-th homology group of $C_*(K;\mathfrak{g}_{\mathrm{Ad}_\rho})$ by $H_p(K;\mathfrak{g}_{\mathrm{Ad}_\rho}) = Z_p(K;\mathfrak{g}_{\mathrm{Ad}_\rho})/B_p(K;\mathfrak{g}_{\mathrm{Ad}_\rho})$. For a given basis $\mathbf{h}_p^M$ of $H_p(M;\mathfrak{g}_{\mathrm{Ad}_\rho})$, we denote by $\mathbb{T}_\rho(M,\{\mathbf{h}_p^M\}_{p=0}^3)$ the adjoint Reidemeister torsion of M twisted by a representation $\rho : \pi_1(M) \to G$. In this paper, we show that the adjoint Reidemeister torsion of M has multiplicativity property without a corrective term. More precisely, we establish a multiplicative adjoint Reidemeister torsion formula for M in terms of the adjoint Reidemeister torsions of $\triangle$-prime 3-manifolds in the unique disk sum decomposition:

Theorem 1.2 *Let* $M = \overset{n}{\underset{i=1}{\triangle}}(M_i)$. *For each* $j \in \{1,\dots,n-1\}$, *we denote the disks on the boundaries of* M_j *and* M_{j+1} *that are identified in the construction of* M *by* $\mathbb{D}_j^2$. *Assume that interiors of* $\mathbb{D}_j^2$ *'s are pairwise disjoint in* M. *Let* $\varrho : \pi_1(M) \to G$ *be a given representation with the restrictions* $\varrho_{|_{\pi_1(M_i)}} = \psi_i$. *For a given basis* $\mathbf{h}_p^M$ *of* $H_p(M;\mathfrak{g}_{\mathrm{Ad}_\varrho})$ *and a basis* $\mathbf{h}_0^{\mathbb{D}_j^2}$ *of* $H_0(\mathbb{D}_j^2;\mathfrak{g}_{\mathrm{Ad}_{\varrho_{|_{\mathbb{D}_j^2}}}})$, *there exists a basis* $\mathbf{h}_p^{M_i}$ *of* $H_p(M_i;\mathfrak{g}_{\mathrm{Ad}_{\psi_{M_i}}})$ *for each* $i \in \{1,\dots,n\}$ *such that the following formula holds*

$$\mathbb{T}_\varrho(M,\{\mathbf{h}_p^M\}_{p=0}^3) = \frac{\prod_{i=1}^{n} \mathbb{T}_{\psi_i}(M_i,\{\mathbf{h}_p^{M_i}\}_{p=0}^3)}{\prod_{j=1}^{n-1} \mathbb{T}_{\varrho_{|_{\mathbb{D}_j^2}}}(\mathbb{D}_j^2,\{\mathbf{h}_0^{\mathbb{D}_j^2}\})}.$$

Moreover, if we choose $\mathbf{h}_0^{\mathbb{D}_j^2} = f_*^j(\varphi_0(\mathbf{c}_0))$ *for each* $j \in \{1,\dots,n-1\}$, *then we obtain*

$$\mathbb{T}_\varrho(M,\{\mathbf{h}_p^M\}_{p=0}^3) = \prod_{i=1}^{n} \mathbb{T}_{\psi_i}(M_i,\{\mathbf{h}_p^{M_i}\}_{p=0}^3).$$

Here, f_*^j *is the map induced by the simple homotopy equivalence* $f^j : \{*\} \to \mathbb{D}_j^2$, $\varphi_0 : Z_0(\{*\};\mathrm{Ad}_\varphi) \to H_0(\{*\};\mathrm{Ad}_\varphi)$ *is the natural projection, and* $\mathbf{c}_0^j$ *is the geometric basis of* $C_0(\{*\};\mathrm{Ad}_\varphi)$.

In the theorem above, by interiors of $\mathbb{D}_j^2$'s being disjoint we mean that the interiors of the images of inclusions $\imath(\mathbb{D}_j^2)$ and $\imath(\mathbb{D}_{j+1}^2)$ in M_j are pairwise disjoint whenever $j \in \{1,\dots,n-2\}$ and the point $*$ is in the intersection of all $\mathbb{D}_j^2$'s.

2. The adjoint Reidemeister torsion

Let X be a CW-complex with dimension n and let $\widetilde{X}$ denote its universal covering. Let us denote the nondegenerate Killing form on $\mathfrak{g}$ by $\mathcal{B}$ which is defined by $\mathcal{B}(A,B) = 4\cdot\mathrm{Trace}(AB)$. For a representation $\rho : \pi_1(X) \to G$, consider the action of $\pi_1(X)$ on $\mathfrak{g}$ via the adjoint of ρ. Let $\mathbb{Z}[\pi_1(X)]$ be the integral group ring.

Let K be a cell-decomposition of X and $\widetilde{K}$ be a lifting of K. By using the cellular chain complex $C_*(\widetilde{K};\mathbb{Z})$, one can define the twisted chain complex as follows

$$C_*(K;\mathfrak{g}_{\mathrm{Ad}_\rho}) := C_*(\widetilde{K};\mathbb{Z}) \otimes \mathfrak{g}/\sim, \tag{2.1}$$

where $\sigma\otimes t \sim \gamma\cdot\sigma\otimes\gamma\cdot t$ for every $\gamma\in\pi_1(X)$, $\pi_1(X)$ acts on $\widetilde{X}$ by deck transformations, and the action of $\pi_1(X)$ on $\mathfrak{g}$ is the adjoint action.

Let $\{e_1^p,\dots,e_{m_p}^p\}$ be the set of p-cells of K giving us a $\mathbb{Z}$-basis for $C_p(K;\mathbb{Z})$. Choose a lift $\widetilde{e}_j^p$ of e_j^p for $j=1,\dots,m_p$. Then we get a $\mathbb{Z}[\pi_1(X)]$-basis $c_p = \{\widetilde{e}_j^p\}_{j=1}^{m_p}$ for $C_p(\widetilde{K};\mathbb{Z})$. Suppose that $\mathcal{A} = \{\mathfrak{a}_k\}_{k=1}^{\dim\mathfrak{g}}$ is a $\mathcal{B}$-orthonormal basis of $\mathfrak{g}$. Then $\mathbf{c}_p = c_p\otimes_\rho\mathcal{A}$ is a *geometric basis* for $C_p(K;\mathfrak{g}_{\mathrm{Ad}_\rho})$.

Consider the following chain complex

$$C_* := C_*(K;\mathfrak{g}_{\mathrm{Ad}_\rho}) = (0\to C_n(K;\mathfrak{g}_{\mathrm{Ad}_\rho})\to C_{n-1}(K;\mathfrak{g}_{\mathrm{Ad}_\rho})\to\cdots\to C_0(K;\mathfrak{g}_{\mathrm{Ad}_\rho})\to 0).$$

For $p\in\{0,\dots,n\}$, let

$$B_p(C_*) = \mathrm{Im}\{\partial_{p+1} : C_{p+1}\to C_p\},$$

$$Z_p(C_*) = \mathrm{Ker}\{\partial_p : C_p \to C_{p-1}\},$$

and $H_p(C_*) = Z_p(C_*)/B_p(C_*)$ be p-th homology group of the chain complex $C_*(K;\mathfrak{g}_{\mathrm{Ad}_\rho})$, which has the structure of a $\mathbb{C}$-vector space. Then we have the following short exact sequences

$$0\longrightarrow Z_p(C_*)\overset{\imath}{\hookrightarrow} C_p(C_*)\overset{\partial_p}{\longrightarrow} B_{p-1}(C_*)\longrightarrow 0, \tag{2.2}$$

$$0\longrightarrow B_p(C_*)\overset{\imath}{\hookrightarrow} Z_p(C_*)\overset{\varphi_p}{\longrightarrow} H_p(C_*)\longrightarrow 0. \tag{2.3}$$

Here, $\imath$ and φ_p are the inclusion and the natural projection, respectively.

Let $s_p : B_{p-1}(C_*)\to C_p(C_*)$, $\ell_p : H_p(C_*)\to Z_p(C_*)$ be sections of $\partial_p : C_p(C_*)\to B_{p-1}(C_*)$, $\varphi_p : Z_p(C_*)\to H_p(C_*)$, respectively. By Splitting Lemma, the short exact sequences (2.2) and (2.3) yield

$$C_p(C_*) = B_p(C_*)\oplus\ell_p(H_p(C_*))\oplus s_p(B_{p-1}(C_*)). \tag{2.4}$$

If $\mathbf{b_p}$ and $\mathbf{h_p}$ are respectively bases of $B_p(C_*)$, and $H_p(C_*)$, then by equation (2.4) the following disjoint union

$$\mathbf{b}_p\sqcup\ell_p(\mathbf{h}_p)\sqcup s_p(\mathbf{b}_{p-1})$$

becomes a new basis for $C_p(C_*)$.

Definition 2.1 *The adjoint Reidemeister torsion of a chain complex $C_*(K;\mathfrak{g}_{\mathrm{Ad}_\rho})$ is defined as the following alternating product*

$$\mathbb{T}(C_*(K;\mathfrak{g}_{\mathrm{Ad}_\rho}),\{\mathbf{c}_p\}_{p=0}^n,\{\mathbf{h}_p\}_{p=0}^n)=\prod_{p=0}^n[\mathbf{b}_p\sqcup\ell_p(\mathbf{h}_p)\sqcup s_p(\mathbf{b}_{p-1}),\mathbf{c}_p]^{(-1)^{(p+1)}}\in\mathbb{C}^*/\{\pm1\},$$

where $[\mathbf{e}_p,\mathbf{f}_p]$ is the determinant of the transition matrix from basis $\mathbf{f}_p$ to $\mathbf{e}_p$ of $C_p(C_*)$.

Throughout this paper the adjoint Reidemeister torsion lives in $\mathbb{C}^*/\{\pm1\}$.

The adjoint Reidemeister torsion of a chain complex $C_*(K;\mathfrak{g}_{\mathrm{Ad}_\rho})$ depends on the choice of the homology bases $\mathbf{h}_p$, but it is independent of the conjugacy class of ρ, the choice of the lifts $\widetilde{e}_j^p$ of the cells e_j^p, and the basis $\mathcal{A}$ (since $\mathcal{A}$ is orthonormal with respect to Killing form $\mathcal{B}$) by [9, Section 0.2, Remarks a1, a2 in p.10]. Furthermore, it is independent of the bases $\mathbf{b}_p$, the sections s_p,ℓ_p due to [7, Section 3, p.365].

Besides, by [7] the adjoint Reidemeister torsion of M is invariant under subdivisions, hence it defines an invariant of manifolds with dimension less than or equal to 3. More precisely,

Definition 2.2 *Let M be a smooth compact n-manifold with a triangulation K, where $n\leq3$. For a given representation $\rho:\pi_1(M)\to G$ and given bases $\{\mathbf{h}_p\}_{p=0}^n$ of homologies, the adjoint Reidemeister torsion of M can be defined as*

$$\mathbb{T}_\rho(M,\{\mathbf{h}_p\}_{p=0}^n)=\mathbb{T}(C_*(K;\mathfrak{g}_{\mathrm{Ad}_\rho}),\{\mathbf{c}_p\}_{p=0}^n,\{\mathbf{h}_p\}_{p=0}^n).$$

The adjoint Reidemiester torsion is independent of the triangulation K of M.

Mayer-Vietoris sequence is one of the useful tools to compute the adjoint Reidemeister torsion. The following theorem is due to [9].

Theorem 2.3 *Assume that X is a compact CW-complex with subcomplexes $X_1,\ X_2\subset X$ so that $X=X_1\cup X_2$ and $Y=X_1\cap X_2$. Let $Y_1,\dots,Y_k$ be the connected components of Y. For $\nu=1,2,$ consider the inclusions*

$$Y\overset{i_\nu}{\hookrightarrow}X_\nu\overset{j_\nu}{\hookrightarrow}X.$$

For $\nu=1,2,$ and $\mu=1,\dots,k,$ let $\rho:\pi_1(X)\to G$ be a representation with the restrictions $\rho_{|_{X_\nu}}:\pi_1(X_\nu)\to G,$ $\rho_{|_{Y_\mu}}:\pi_1(Y_\mu)\to G$. Then

(i) The following sequence is short-exact

$$0\to\bigoplus_\mu C_*(Y_\mu;\mathfrak{g}_{\mathrm{Ad}_{\rho_{|_{Y_\mu}}}})\overset{(i_1)_\#\oplus(i_2)_\#}{\longrightarrow}C_*(X_1;\mathfrak{g}_{\mathrm{Ad}_{\rho_{|_{X_1}}}})\oplus C_*(X_2;\mathfrak{g}_{\mathrm{Ad}_{\rho_{|_{X_2}}}})\overset{(j_1)_\#-(j_2)_\#}{\longrightarrow}C_*(X;\mathfrak{g}_{\mathrm{Ad}_\rho})\to0.$$

(ii) Corresponding to the sequence in (i), there is a Mayer-Vietoris long exact sequence in homology with twisted coefficients

$$\mathcal{H}_*:\cdots\longrightarrow\bigoplus_\mu H_i(Y_\mu;\mathfrak{g}_{\mathrm{Ad}_{\rho_{|_{Y_\mu}}}})\overset{i_1^*\oplus i_2^*}{\longrightarrow}H_i(X_1;\mathfrak{g}_{\mathrm{Ad}_{\rho_{|_{X_1}}}})\oplus H_i(X_2;\mathfrak{g}_{\mathrm{Ad}_{\rho_{|_{X_2}}}})\overset{j_1^*-j_2^*}{\longrightarrow}$$

$$H_i(X;\mathfrak{g}_{\mathrm{Ad}_\rho})\longrightarrow\bigoplus_\mu H_{i-1}(Y_\mu;\mathfrak{g}_{\mathrm{Ad}_{\rho_{|_{Y_\mu}}}})\longrightarrow\cdots\qquad(2.5)$$

(iii) Choose a basis for each of these homology groups such as $\mathbf{h}_i^X$ *for* $H_i(X;\mathfrak{g}_{\mathrm{Ad}_\rho})$, $\mathbf{h}_i^{X_1}$ *for* $H_i(X_1;\mathfrak{g}_{\mathrm{Ad}_{\rho_{|X_1}}})$, $\mathbf{h}_i^{X_2}$ *for* $H_i(X_2;\mathfrak{g}_{\mathrm{Ad}_{\rho_{|X_2}}})$, *and* $\mathbf{h}_i^{Y_\mu}$ *for* $H_i(Y_\mu;\mathfrak{g}_{\mathrm{Ad}_{\rho_{|Y_\mu}}})$. *The long exact sequence (2.5) can be viewed as a chain complex. Indeed, the following formula holds*

$$\mathbb{T}_{\rho_{|X_1}}(X_1,\{\mathbf{h}_i^{X_1}\})\,\mathbb{T}_{\rho_{|X_2}}(X_2,\{\mathbf{h}_i^{X_2}\}) = \mathbb{T}_\rho(X,\{\mathbf{h}_i^X\})\prod_{\mu=1}^{k}\mathbb{T}_{\rho_{|Y_\mu}}(Y_\mu,\{\mathbf{h}_i^{Y_\mu}\}) \times \mathbb{T}(\mathcal{H}_*,\{\mathbf{h}_{**}\}).$$

The proof of Theorem 2.3 can be found in [7, Theorem 3.2] or in [9, Section 0.4, Proposition 0.11].

Notation 2.4 *The torsion* $\mathbb{T}(\mathcal{H}_*,\{\mathbf{h}_{**}\})$ *is called the corrective term. The same terminology is also used in [1].*

We have the following lemma.

Lemma 2.5 *If* d *is the dimension of the CW-complex* X, *then the corrective term in Theorem 2.3 satisfies*

$$\mathbb{T}\left(\mathcal{H}_*,\{\mathbf{h}_p\}_{p=0}^{3d+2},\{0\}_{p=0}^{3d+2}\right) = \prod_{p=0}^{3d+2}\left[\mathbf{h}'_p,\mathbf{h}_p\right]^{(-1)^{(p+1)}}. \tag{2.6}$$

Here, $\mathbf{h}'_p = \mathbf{b}_p \sqcup s_p(\mathbf{b}_{p-1})$.

Proof First, we denote the vector spaces in the long exact sequence (2.5) (from right to left) by $C_p(\mathcal{H}_*)$ for $p \in \{0,\dots,3d+2\}$. Then we consider the short exact sequences

$$0 \to Z_p(\mathcal{H}_*) \hookrightarrow C_p(\mathcal{H}_*) \stackrel{\partial_p}{\twoheadrightarrow} B_{p-1}(\mathcal{H}_*) \to 0, \tag{2.7}$$

$$0 \to B_p(\mathcal{H}_*) \hookrightarrow Z_p(\mathcal{H}_*) \stackrel{\varphi_p}{\twoheadrightarrow} H_p(\mathcal{H}_*) \to 0. \tag{2.8}$$

For each p, let us consider the isomorphism $s_p : B_{p-1}(\mathcal{H}_*) \to s_p(B_{p-1}(\mathcal{H}_*))$ obtained by the First Isomorphism Theorem as a section of $C_p(\mathcal{H}_*) \to B_{p-1}(\mathcal{H}_*)$. Using the exactness of $\mathcal{H}_*$ in the short exact sequence (2.8), we obtain

$$B_p(\mathcal{H}_*) = Z_p(\mathcal{H}_*).$$

Hence, the sequence (2.7) becomes

$$0 \to B_p(\mathcal{H}_*) \hookrightarrow C_p(\mathcal{H}_*) \twoheadrightarrow B_{p-1}(\mathcal{H}_*) \to 0. \tag{2.9}$$

Applying the Splitting Lemma for the sequence (2.9), we have

$$C_p(\mathcal{H}_*) = B_p(\mathcal{H}_*) \oplus s_p(B_{p-1}(\mathcal{H}_*)). \tag{2.10}$$

Thus, the Reidemeister torsion of $\mathcal{H}_*$ satisfies the following formula

$$\mathbb{T}\left(\mathcal{H}_*,\{\mathbf{h}_p\}_{p=0}^{3d+2},\{0\}_{p=0}^{3d+2}\right) = \prod_{p=0}^{3d+2}\left[\mathbf{h}'_p,\mathbf{h}_p\right]^{(-1)^{(p+1)}}. \tag{2.11}$$

Here, $\mathbf{h}'_p$ denotes the new basis $\mathbf{b}_p \sqcup s_p(\mathbf{b}_{p-1})$ of $C_p(\mathcal{H}_*)$ for all p. □

For further information and applications, we refer to [6, 9, 11, 12] and the references therein.

3. Main results

To prove Theorem 1.2, we need the following auxiliary results. The adjoint Reidemeister torsion is a simple homotopy invariant (due to Remark 2.8 (a) of the preprint by Porti: Reidemeister torsion, hyperbolic three-manifolds, and character varieties, 2016, arXiv:1511.00400). As a special case, we consider a closed disk $\mathbb{D}^2$. Note that $\mathbb{D}^2$ is simple homotopy equivalent to the point $\{*\}$. Since $\pi_1(\mathbb{D}^2)$ is trivial, any representation $\rho : \pi_1(\mathbb{D}^2) \to G$ is trivial. Hence, $\mathbb{D}^2$ is a special complex by [7, Definition in Section 12.3]. Let $f : \{*\} \to \mathbb{D}^2$ be a simple homotopy equivalence. Set $\varphi = \rho \circ f_\# : \pi_1(\{*\}) \to G$. Then it follows from [7, Lemma 12.5] that

$$\mathbb{T}_\rho(\mathbb{D}^2, \{\mathbf{h}_0^{\mathbb{D}^2}\}) = \mathbb{T}_\varphi(\{*\}, \{\mathbf{h}_0^{\{*\}}\}) \tag{3.1}$$

for $f_*(\mathbf{h}_0^{\mathbb{D}^2}) = \mathbf{h}_0^{\{*\}}$.

Let $K = \widetilde{K} = \{e_0\}$ denote the single 0-cell of $\{*\}$. As $\mathbb{Z}[\pi_1(\{*\})] = \mathbb{Z}[< e_0 >] \cong \mathbb{Z}$,

$$C_0(K; \mathrm{Ad}_\varphi) = C_0(\widetilde{K}; \mathbb{Z}) \otimes \mathfrak{g}/\sim \,\cong \mathfrak{g}. \tag{3.2}$$

Fix a $\mathcal{B}$-orthonormal $\mathbb{C}$-basis $\mathcal{A} = \{\mathfrak{a}_i\}_{i=1}^{\dim \mathfrak{g}}$ of $\mathfrak{g}$. Then $\mathbf{c}_0 = \{e_0 \otimes \mathfrak{a}_k\}_{k=1}^{\dim \mathfrak{g}}$ is the geometric basis of $C_0(K; \mathrm{Ad}_\varphi)$. Consider the following chain complex

$$C_* := C_*(K; \mathrm{Ad}_\varphi) = (0 \xrightarrow{\partial_1} C_0(K; \mathrm{Ad}_\varphi) \xrightarrow{\partial_0} 0). \tag{3.3}$$

Since the following equalities hold

$$B_0(C_*) = \mathrm{Im}\{\partial_1 : C_1(C_*) \to C_0(C_*)\} = 0,$$
$$Z_0(C_*) = \mathrm{Ker}\{\partial_0 : C_0(C_*) \to C_{-1}(C_*)\} = C_0(K; \mathrm{Ad}_\varphi),$$

the 0-th homology of $\{*\}$ twisted by φ can be given as follows

$$H_0(\{*\}; \mathrm{Ad}_\varphi) = H_0(C_*) = Z_0(C_*)/B_0(C_*) \cong C_0(K; \mathrm{Ad}_\varphi) \cong \mathfrak{g}.$$

Then there are the following short exact sequences

$$0 \longrightarrow Z_0(C_*) \overset{\imath}{\hookrightarrow} C_0(C_*) \xrightarrow{\partial_0} B_{-1}(C_*) \longrightarrow 0, \tag{3.4}$$

$$0 \longrightarrow B_0(C_*) \overset{\imath}{\hookrightarrow} Z_0(C_*) \xrightarrow{\varphi_0} H_0(C_*) \longrightarrow 0. \tag{3.5}$$

Here, $\imath$ and φ_0 are the inclusion and the natural projection, respectively. Let $s_0 : B_{-1}(C_*) \to C_0(C_*)$ and $\ell_0 : H_0(C_*) \to Z_0(C_*)$ be sections of the homomorphisms $\partial_0 : C_0(C_*) \to B_{-1}(C_*)$, $\varphi_0 : Z_0(C_*) \to H_0(C_*)$, respectively. Since $B_0(C_*) = B_{-1}(C_*) = \{0\}$, the homomorphism φ_0 becomes an isomorphism, so that the section ℓ_0 is the inverse of this isomorphism. Moreover, we have

$$C_0(C_*) = \ell_0(H_0(C_*)). \tag{3.6}$$

Let $\mathbf{h}_0^{\{*\}}$ be an arbitrary basis of $H_0(\{*\};\mathrm{Ad}_\varphi)$. From equation (3.6) it follows

$$\mathbb{T}_\rho(\{*\},\{\mathbf{h}_0^{\{*\}}\}) = \left[\ell_0(\mathbf{h}_0^{\{*\}}),\mathbf{c}_0\right]. \tag{3.7}$$

The following lemma is evident from equations (3.1) and (3.7).

Lemma 3.1 *If $\mathbf{h}_0^{\mathbb{D}^2}$ is a basis of $H_0(\mathbb{D}^2;\mathrm{Ad}_\rho)$ which is the image of the basis $\mathbf{h}_0^{\{*\}} = \varphi_0(\mathbf{c}_0)$ of $H_0(\{*\};\mathrm{Ad}_\varphi)$ under f_*, then we get*

$$\mathbb{T}_\rho(\mathbb{D}^2,\{\mathbf{h}_0^{\mathbb{D}^2}\}) = \left[\ell_0(\mathbf{h}_0^{\{*\}}),\mathbf{c}_0\right] = [\ell_0(\varphi_0(\mathbf{c}_0)),\mathbf{c}_0] = [\mathbf{c}_0,\mathbf{c}_0] = 1.$$

3.1. The Proof of Theorem 1.2

Let G be a complex reductive algebraic group $SL_n(\mathbb{C})$ or $PSL_n(\mathbb{C})$. Let $M_1, M_2 \in \mathcal{X}$. We first focus on the case where M is the disk sum of M_1 and M_2

$$M = M_1 \triangle M_2.$$

Cleary, $M \in \mathcal{X}$. From the Seifert-Van Kampen's theorem it follows

$$\pi_1(M) = \pi_1(M_1) * \pi_1(M_2).$$

Let $\varrho : \pi_1(M) \to G$ be a given representation with the restrictions $\varrho_{|_{\pi_1(M_1)}} = \psi_1 : \pi_1(M_1) \to G$ and $\varrho_{|_{\pi_1(M_2)}} = \psi_2 : \pi_1(M_2) \to G$. Moreover, we consider the restriction $\varrho_{|_{\mathbb{D}^2}}$ of the representation ϱ to $\pi_1(\mathbb{D}^2)$. We abuse the notation and denote the triangulations of respective manifolds by $\mathbb{D}^2$, M_1, M_2, and M. By Theorem 2.3 (i), the following sequence is short-exact

$$0 \to C_*(\mathbb{D}^2;\mathfrak{g}_{\mathrm{Ad}_{\varrho_{|_{\mathbb{D}^2}}}}) \to C_*(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}}) \oplus C_*(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}}) \to C_*(M;\mathfrak{g}_{\mathrm{Ad}_{\varrho}}) \to 0. \tag{3.8}$$

It is well–known that for any 3–manifold N with nonempty boundary, one has $\dim(H_1(N;\mathfrak{g}_{\mathrm{Ad}_\rho})) \geq 1$, where $\rho : \pi_1(N) \to G$ is a representation. Thus, $C_*(N;\mathfrak{g}_{\mathrm{Ad}_\rho})$ is never acyclic. Hence, associated to the short exact sequence (3.8), by Theorem 2.3, there is a Mayer-Vietoris long exact sequence in homology with twisted coefficients

$$\mathcal{H}_* : \quad 0 \xrightarrow{\partial_3'} H_3(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}}) \oplus H_3(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}}) \xrightarrow{\partial_3} H_3(M;\mathfrak{g}_{\mathrm{Ad}_{\varrho}}) \xrightarrow{\partial_3''}$$

$$0 \xrightarrow{\partial_2'} H_2(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}}) \oplus H_2(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}}) \xrightarrow{\partial_2} H_2(M;\mathfrak{g}_{\mathrm{Ad}_{\varrho}}) \xrightarrow{\partial_2''}$$

$$0 \xrightarrow{\partial_1'} H_1(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}}) \oplus H_1(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}}) \xrightarrow{\partial_1} H_1(M;\mathfrak{g}_{\mathrm{Ad}_{\varrho}}) \xrightarrow{\partial_1''}$$

$$H_0(\mathbb{D}^2;\mathfrak{g}_{\mathrm{Ad}_{\varrho_{|_{\mathbb{D}^2}}}}) \xrightarrow{\partial_0'} H_0(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}}) \oplus H_0(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}}) \xrightarrow{\partial_0} H_0(M;\mathfrak{g}_{\mathrm{Ad}_{\varrho}}) \xrightarrow{\partial_0''} 0.$$

From the exactness of $\mathcal{H}_*$ and the First Isomorphism Theorem it follows that ∂_1'' is the zero map and for each $i \in \{1,2,3\}$, the following isomorphism holds

$$H_i(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}}) \oplus H_i(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}}) \overset{\partial_i}{\cong} H_i(M;\mathfrak{g}_{\mathrm{Ad}_\varrho}).$$

Let $\mathbf{h}_p^M$ and $\mathbf{h}_0^{\mathbb{D}^2}$ be given bases of $H_p(M;\mathfrak{g}_{\mathrm{Ad}_\varrho})$ and $H_0(\mathbb{D}^2;\mathfrak{g}_{\mathrm{Ad}_{\varrho|_{\mathbb{D}^2}}})$. By Lemma 2.5, the Reidemeister torsion of $\mathcal{H}_*$ satisfies the following formula

$$\mathbb{T}\left(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^{11}, \{0\}_{p=0}^{11}\right) = \prod_{p=0}^{11} \left[\mathbf{h}_p', \mathbf{h}_p\right]^{(-1)^{(p+1)}}, \tag{3.9}$$

where $\mathbf{h}_p'$ is the new basis $\mathbf{b}_p \sqcup s_p(\mathbf{b}_{p-1})$ of $C_p(\mathcal{H}_*)$ for all p. Since the Reidemeister torsion does not depend on bases $\mathbf{b}_p$ and sections s_p, we can chose the suitable bases $\mathbf{b}_p$ and sections s_p to show that the existence of the homology bases $\mathbf{h}_p^{M_1}$, $\mathbf{h}_p^{M_2}$ in which the corrective term in equation (3.9) is equal to 1.

- We consider the first part of the long exact sequence $\mathcal{H}_*$:

$$0 \overset{\partial_1''}{\to} H_0(\mathbb{D}^2;\mathfrak{g}_{\mathrm{Ad}_{\varrho|_{\mathbb{D}^2}}}) \overset{\partial_0'}{\to} H_0(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}}) \oplus H_0(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}}) \overset{\partial_0}{\to} H_0(M;\mathfrak{g}_{\mathrm{Ad}_\varrho}) \overset{\partial_0''}{\to} 0.$$

First, we use equation (2.10) for the vector space $C_0(\mathcal{H}_*) = H_0(M;\mathfrak{g}_{\mathrm{Ad}_\varrho})$. Since $\mathrm{Im}\,\partial_0'' = \{0\}$, we get

$$C_0(\mathcal{H}_*) = \mathrm{Im}\,\partial_0 \oplus s_0(\mathrm{Im}\,\partial_0'') = \mathrm{Im}\,\partial_0. \tag{3.10}$$

If we choose the basis $\mathbf{h}^{\mathrm{Im}\,\partial_0}$ of $\mathrm{Im}\,\partial_0$ as $\mathbf{h}_0^M$, then $\mathbf{h}_0^M$ becomes the new basis $\mathbf{h}_0'$ of $C_0(\mathcal{H}_*)$ by equation (3.10). Since $\mathbf{h}_0^M$ is also the given basis $\mathbf{h}_0$ of $C_0(\mathcal{H}_*)$, the following equality holds

$$[\mathbf{h}_0', \mathbf{h}_0] = 1. \tag{3.11}$$

Let us consider equation (2.10) for $C_1(\mathcal{H}_*) = H_0(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}}) \oplus H_0(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}})$. Then we obtain

$$C_1(\mathcal{H}_*) = \mathrm{Im}\,\partial_0' \oplus s_1(\mathrm{Im}\,\partial_0). \tag{3.12}$$

Recall that the basis $\mathbf{h}^{\mathrm{Im}\,\partial_0}$ of $\mathrm{Im}\,\partial_0$ was chosen as $\mathbf{h}_0^M$ in the previous step. Note also that $\mathrm{Im}\,\partial_0'$ is isomorphic to $H_0(\mathbb{D}^2;\mathfrak{g}_{\mathrm{Ad}_{\varrho|_{\mathbb{D}^2}}})$, so that we can take the basis $\mathbf{h}^{\mathrm{Im}\,\partial_0'}$ of $\mathrm{Im}\,\partial_0'$ as $\partial_0'(\mathbf{h}_0^{\mathbb{D}^2})$. By equation (3.12), we get the new basis for $C_1(\mathcal{H}_*)$ as follows

$$\mathbf{h}_1' = \left\{\partial_0'(\mathbf{h}_0^{\mathbb{D}^2}), s_1(\mathbf{h}_0^M)\right\}.$$

Let n_0^M, $n_0^{M_\ell}$, and $n_0^{\mathbb{D}^2}$ denote respectively the dimensions of spaces $H_0(M;\mathfrak{g}_{\mathrm{Ad}_\varrho})$, $H_0(M_\ell;\mathfrak{g}_{\mathrm{Ad}_{\psi_\ell}})$, and $H_0(\mathbb{D}^2;\mathfrak{g}_{\mathrm{Ad}_{\varrho|_{\mathbb{D}^2}}})$ for $\ell = 1,2$. Since $H_0(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}})$ and $H_0(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}})$ are subspaces of $C_1(\mathcal{H}_*)$, we have

$$n_0^{M_1} + n_0^{M_2} = n_0^M + n_0^{\mathbb{D}^2} = \dim(C_1(\mathcal{H}_*)).$$

Recall that $\mathbf{h}_0^M = \left\{\mathbf{h}_{0,j}^M\right\}_{j=1}^{n_0^M}$ is the given basis of $H_0(M; \mathfrak{g}_{\mathrm{Ad}_\varrho})$. For $i \in \{1, 2, \dots, n_0^{M_1} + n_0^{M_2}\}$, there are nonzero vectors $(a_{i,1}, a_{i,2}, \cdots, a_{i,n_0^{M_1}+n_0^{M_2}})$ such that

$$\left\{\sum_{j=1}^{n_0^{\mathbb{D}^2}} a_{i,j}\, \partial_0'(\mathbf{h}_0^{\mathbb{D}^2}) + \sum_{j=n_0^{\mathbb{D}^2}+1}^{n_0^M + n_0^{\mathbb{D}^2}} a_{i,j}\, s_1\left(\mathbf{h}_{0,j}^M\right)\right\}_{i=1}^{n_0^{M_1}},$$

$$\left\{\sum_{j=1}^{n_0^{\mathbb{D}^2}} a_{i,j}\, \partial_0'(\mathbf{h}_0^{\mathbb{D}^2}) + \sum_{j=n_0^{\mathbb{D}^2}+1}^{n_0^M + n_0^{\mathbb{D}^2}} a_{i,j}\, s_1\left(\mathbf{h}_{0,j}^M\right)\right\}_{i=n_0^{M_1}+1}^{n_0^{M_1}+n_0^{M_2}}$$

are bases of $H_0(M_1; \mathfrak{g}_{\mathrm{Ad}_{\psi_1}})$ and $H_0(M_2; \mathfrak{g}_{\mathrm{Ad}_{\psi_2}})$, respectively. Moreover, $A = [a_{i,j}]$ is an invertible matrix of size $(n_0^{M_1} + n_0^{M_2}) \times (n_0^{M_1} + n_0^{M_2})$.

Let $\mathbf{h}_{0,i} = \sum_{j=1}^{n_0^{\mathbb{D}^2}} a_{i,j}\, \partial_0'(\mathbf{h}_0^{\mathbb{D}^2}) + \sum_{j=n_0^{\mathbb{D}^2}+1}^{n_0^M+n_0^{\mathbb{D}^2}} a_{i,j}\, s_1\left(\mathbf{h}_{0,j}^M\right)$ for each $i \in \{1, \dots, n_0^{M_1} + n_0^{M_2}\}$. Let us take the bases of $H_0(M_1; \mathfrak{g}_{\mathrm{Ad}_{\psi_1}})$ and $H_0(M_2; \mathfrak{g}_{\mathrm{Ad}_{\psi_2}})$ as follows

$$\mathbf{h}_0^{M_1} = \left\{(\det A)^{-1}\mathbf{h}_{0,1}\right\} \sqcup \{\mathbf{h}_{0,i}\}_{i=2}^{n_0^{M_1}},$$

$$\mathbf{h}_0^{M_2} = \{\mathbf{h}_{0,i}\}_{i=n_0^{M_1}+1}^{n_0^{M_1}+n_0^{M_2}}.$$

Then $\mathbf{h}_1 = \{\mathbf{h}_0^{M_1}, \mathbf{h}_0^{M_2}\}$ becomes the initial basis of $C_1(\mathcal{H}_*)$, and we have

$$[\mathbf{h}_1', \mathbf{h}_1] = 1. \tag{3.13}$$

Considering the space $C_2(\mathcal{H}_*) = H_0(\mathbb{D}^2; \mathfrak{g}_{\mathrm{Ad}_{\varrho|_{\mathbb{D}^2}}})$ in equation (2.10) and using the fact that $\operatorname{Im} \partial_1'' = \{0\}$, we can write the space $C_2(\mathcal{H}_*)$ as follows

$$C_2(\mathcal{H}_*) = \operatorname{Im} \partial_1'' \oplus s_2(\operatorname{Im} \partial_0') = s_2(\operatorname{Im} \partial_0'). \tag{3.14}$$

By equation (3.14), $s_2(\partial_0'(\mathbf{h}_0^{\mathbb{D}^2})) = \mathbf{h}_0^{\mathbb{D}^2}$ becomes the new basis $\mathbf{h}_2'$ of $C_2(\mathcal{H}_*)$. Since $\mathbf{h}_0^{\mathbb{D}^2}$ is also the given basis $\mathbf{h}_2$ of $C_2(\mathcal{H}_*)$, we get

$$[\mathbf{h}_2', \mathbf{h}_2] = 1. \tag{3.15}$$

- Let us consider the second part of the sequence $\mathcal{H}_*$ for $i = 1, 2, 3$

$$0 \xrightarrow{\partial_i'} H_i(M_1; \mathfrak{g}_{\mathrm{Ad}_{\psi_1}}) \oplus H_i(M_2; \mathfrak{g}_{\mathrm{Ad}_{\psi_2}}) \xrightarrow{\partial_i} H_i(M; \mathfrak{g}_{\mathrm{Ad}_\varrho}) \xrightarrow{\partial_i''} 0. \tag{3.16}$$

Now we denote the vector spaces in the short exact sequence (3.16) (from right to left) as $C_{3i}(\mathcal{H}_*)$, $C_{3i+1}(\mathcal{H}_*)$ and $C_{3i+2}(\mathcal{H}_*)$ for $i = 1, 2, 3$.

Note that the spaces $C_{3i+2}(\mathcal{H}_*)$ are equal to $\{0\}$. If we use the convention $1 \cdot 0 = 0$ for each $i \in \{1, 2, 3\}$. Then we get

$$[\mathbf{h}_{3i+2}', \mathbf{h}_{3i+2}] = 1. \tag{3.17}$$

By the exactness of $\mathcal{H}_*$, we get the following isomorphism:

$$H_i(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}}) \oplus H_i(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}}) \overset{\partial_i}{\cong} H_i(M;\mathfrak{g}_{\mathrm{Ad}_{\varrho}})$$

for $i \in \{1,2,3\}$.

We use equation (2.10) for the space $C_{3i}(\mathcal{H}_*) = H_i(M;\mathfrak{g}_{\mathrm{Ad}_{\varrho}})$. Since $\operatorname{Im}\partial_i'' = \{0\}$, the following equality holds

$$C_{3i}(\mathcal{H}_*) = \operatorname{Im}\partial_i \oplus s_{3i}(\operatorname{Im}\partial_i'') = \operatorname{Im}\partial_i. \tag{3.18}$$

Since $\operatorname{Im}\partial_i$ equals to $H_i(M;\mathfrak{g}_{\mathrm{Ad}_{\varrho}})$, we can take the basis $\mathbf{h}^{\operatorname{Im}\partial_i}$ of $\operatorname{Im}\partial_i$ as $\mathbf{h}_i^M$. By equation (3.18), $\mathbf{h}_i^M$ becomes the new basis $\mathbf{h}_{3i}'$ of $C_{3i}(\mathcal{H}_*)$. As $\mathbf{h}_i^M$ is also the given basis $\mathbf{h}_{3i}$ of $C_{3i}(\mathcal{H}_*)$, the following equation holds

$$[\mathbf{h}_{3i}', \mathbf{h}_{3i}] = 1. \tag{3.19}$$

Considering equation (2.10) for $C_{3i+1}(\mathcal{H}_*) = H_i(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}}) \oplus H_i(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}})$ and using the fact that $\operatorname{Im}\partial_i' = \{0\}$, we obtain

$$C_{3i+1}(\mathcal{H}_*) = \operatorname{Im}\partial_i' \oplus s_{3i+1}(\operatorname{Im}\partial_i) = s_{3i+1}(\operatorname{Im}\partial_i). \tag{3.20}$$

Since $H_i(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}}) \oplus H_i(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}})$ and $H_i(M;\mathfrak{g}_{\mathrm{Ad}_{\varrho}})$ are isomorphic, the section s_{3i+1} can be considered as the inverse of the isomorphism ∂_i. In the previous step, the basis $\mathbf{h}^{\operatorname{Im}\partial_i}$ of $\operatorname{Im}\partial_i$ was chosen as $\mathbf{h}_i^M$. By equation (3.20), $s_{3i+1}(\mathbf{h}_i^M)$ becomes the new basis $\mathbf{h}_{3i+1}'$ of $C_{3i+1}(\mathcal{H}_*)$.

Let n_i^M and $n_i^{M_\ell}$ denote respectively the dimensions of spaces $H_i(M;\mathfrak{g}_{\mathrm{Ad}_{\varrho}})$ and $H_i(M_\ell;\mathfrak{g}_{\mathrm{Ad}_{\psi_\ell}})$ for $\ell = 1,2$. Note that $H_i(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}})$ and $H_i(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}})$ are subspaces of $C_{3i+1}(\mathcal{H}_*)$ and $\mathbf{h}_i^M = \left\{\mathbf{h}_{i,j}^M\right\}_{j=1}^{n_i^M}$ is the given basis of $H_i(M;\mathfrak{g}_{\mathrm{Ad}_{\varrho}})$. Moreover, the exactness of $\mathcal{H}_*$ yields

$$n_i^{M_1} + n_i^{M_2} = n_i^M = \dim(C_{3i+1}(\mathcal{H}_*)).$$

There are nonzero vectors $(a_{k,1}, a_{k,2}, \cdots, a_{k,n_i^M})$ for $k \in \{1,2,\ldots,n_i^M\}$ such that

$$\left\{\sum_{j=1}^{n_i^M} a_{k,j}\, s_{3i+1}\left(\mathbf{h}_{i,j}^M\right)\right\}_{k=1}^{n_i^{M_1}}, \left\{\sum_{j=1}^{n_i^M} a_{k,j}\, s_{3i+1}\left(\mathbf{h}_{i,j}^M\right)\right\}_{k=n_i^{M_1}+1}^{n_i^M}$$

are bases of $H_i(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}})$ and $H_i(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}})$, respectively. Moreover, $A = [a_{i,j}]$ is an invertible matrix of size $(n_i^M \times n_i^M)$.

We denote the basis element $\sum_{j=1}^{n_i^M} a_{k,j}\, s_{3i+1}\left(\mathbf{h}_{i,j}^M\right)$ by $\mathbf{h}_{3i+1,k}$ for each $k \in \{1,\ldots,n_i^M\}$. Let us take the bases of $H_i(M_1;\mathfrak{g}_{\mathrm{Ad}_{\psi_1}})$ and $H_i(M_2;\mathfrak{g}_{\mathrm{Ad}_{\psi_2}})$ as follows

$$\mathbf{h}_{3i+1}^{M_1} = \left\{(\det A)^{-1}\mathbf{h}_{3i+1,1}\right\} \sqcup \left\{\mathbf{h}_{3i+1,k}\right\}_{k=2}^{n_i^{M_1}}$$

$$\mathbf{h}_{3i+1}^{M_2} = \left\{\mathbf{h}_{3i+1,k}\right\}_{k=n_i^{M_1}+1}^{n_i^M}.$$

Then $\mathbf{h}_{3i+1} = \{\mathbf{h}^{M_1}_{3i+1}, \mathbf{h}^{M_2}_{3i+1}\}$ becomes the initial basis of $C_{3i+1}(\mathcal{H}_*)$ and we have

$$[\mathbf{h}'_{3i+1}, \mathbf{h}_{3i+1}] = 1. \tag{3.21}$$

Equations (3.11), (3.13), (3.15), (3.19), (3.21) yield

$$\mathbb{T}(\mathcal{H}_*, \{\mathbf{h}_p\}_{p=0}^{11}, \{0\}_{p=0}^{11}) = \prod_{p=0}^{11} \left[\mathbf{h}'_p, \mathbf{h}_p\right]^{(-1)^{(p+1)}} = 1. \tag{3.22}$$

For $p = 0, 1, 2, 3,$ we can choose the compatible geometric bases $\mathbf{c}_0^{\mathbb{D}^2}$, $\mathbf{c}_p^{M_1}$, $\mathbf{c}_p^{M_2}$, and $\mathbf{c}_p^{M}$ for the chain complexes $C_*(\mathbb{D}^2; \mathfrak{g}_{\mathrm{Ad}_{\varrho_{|_{\mathbb{D}^2}}}})$, $C_*(M_1; \mathfrak{g}_{\mathrm{Ad}_{\psi_1}})$, $C_*(M_2; \mathfrak{g}_{\mathrm{Ad}_{\psi_2}})$, and $C_*(M; \mathfrak{g}_{\mathrm{Ad}_{\varrho}})$ as in the proof of Theorem 2.3 (see, [9, Section 0.4, Proposition 0.11 (iii)]). Hence, by Theorem 2.3 and equation (3.22), the following formula is valid

$$\mathbb{T}_{\varrho}(M, \{\mathbf{h}_p^M\}_{p=0}^3) = \frac{\mathbb{T}_{\psi_1}(M_1, \{\mathbf{h}_p^{M_1}\}_{p=0}^3)\, \mathbb{T}_{\psi_2}(M_2, \{\mathbf{h}_p^{M_2}\}_{p=0}^3)}{\mathbb{T}_{\varrho_{|_{\mathbb{D}^2}}}(\mathbb{D}^2, \{\mathbf{h}_0^{\mathbb{D}^2}\})}. \tag{3.23}$$

If we choose the basis $\mathbf{h}_0^{\mathbb{D}^2}$ of $H_0(\mathbb{D}^2; \mathfrak{g}_{\mathrm{Ad}_{\varrho_{|_{\mathbb{D}^2}}}})$ as $f_*(\varphi_0(\mathbf{c}_0))$. Then by Lemma 3.1 and equation (3.23), we have

$$\mathbb{T}_{\varrho}(M, \{\mathbf{h}_p^M\}_{p=0}^3) = \mathbb{T}_{\psi_1}(M_1, \{\mathbf{h}_p^{M_1}\}_{p=0}^3)\, \mathbb{T}_{\psi_2}(M_2, \{\mathbf{h}_p^{M_2}\}_{p=0}^3). \tag{3.24}$$

Applying equations (3.23) and (3.24) inductively, the proof of Theorem 1.2 follows. Note that we can do these inductions since the interiors of the images of inclusions $\imath(\mathbb{D}^2_j)$ and $\imath(\mathbb{D}^2_{j+1})$ in M_j are pairwise disjoint for each $j \in \{1, \ldots, n-2\}$.

Remark 3.2 *In 1993, Müller defines the Reidemeister torsion of compact smooth manifolds for unimodular representations to show that the Reidemeister torsion and the analytic torsion of closed 3-manifolds coincide. Following [8, Definition 1.23], a different choice of the lift* $\widetilde{K}$ *of* X *corresponds to a change of basis in the vector space* $C_n(K; \mathfrak{g}_{\mathrm{Ad}_\rho})$ *by a unimodular matrix. If we repeat the same process in Theorem 1.2 for unimodular representations, then the resulting formula changes at most by sign. Therefore, the formula in Theorem 1.2 is valid after taking absolute values.*

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