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# Characterizations and representations of weak core inverses and $m$-weak group inverses

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**Abstract:** In a ring with an involution, we first present some necessary and sufficient conditions for the existence of the $m$-weak group inverse and expression. As an application, we prove that a regular element $a$ is $(m+1)$-weak group invertible if and only if $a^2a^-$ is $m$-weak group invertible, where $a^-$ is an inner inverse of $a$. The relevant results for weak core inverses and for pseudocore inverses are given. In addition, we present some new characterizations of weak core inverses, and also investigate maximal classes of elements determining weak core inverses.

**Key words:** Weak core inverse, $m$-weak group inverse, weak group inverse, pseudocore inverse

## 1. Introduction

Throughout the paper, $R$ is a unitary ring with an involution $*$, $aR = \{ax : x \in R\}$ and $Ra = \{xa : x \in R\}$. We recall that an element $a \in R$ is regular if there is $a^- \in R$ satisfying $aa^-a = a$, in which case, $a^-$ is called an inner inverse (or $\{1\}$-inverse) of $a$. The set of all inner inverses of $a$ is denoted by $a\{1\}$. An element $a \in R$ is said to be $\{1,3\}$-invertible if there is $a^{(1,3)} \in R$ satisfying $aa^{(1,3)}a = a$ and $(aa^{(1,3)})^* = aa^{(1,3)}$, in which case, $a^{(1,3)}$ is called a $\{1,3\}$-inverse of $a$. The symbol $a\{1,3\}$ denotes the set of all $\{1,3\}$-invertible elements of $a$. Also, it was proved in [12] that $a \in R$ is $\{1,3\}$-invertible if and only if $a \in Ra^*a$.

We use $\mathbb{N}$ and $\mathbb{N}^+$ to denote the sets of all nonnegative integers and positive integers, respectively.

Recall that $a \in R$ is said to be Drazin invertible [3] if there is an element $a^D$ (usually called the Drazin inverse of $a$) which is the unique solution to the equations

$$ax^2 = x,\ ax = xa,\ xa^{k+1} = a^k \text{ for some } k \in \mathbb{N}^+.$$

In this case, the smallest positive integer $k$ satisfying the above equations is called the Drazin index of $a$ and denoted by $\mathrm{ind}(a)$. In particular, if $\mathrm{ind}(a) = 1$, then $a$ is said to be group invertible and $a^D$ is called the group inverse of $a$ (written as $a^\#$).

Afterwards, some kinds of new generalized inverses were introduced and investigated, such as core-EP inverses. The core-EP inverse of a complex matrix was introduced by Manjunatha Prasad and Mohana [17]. In 2018, Gao and Chen [9] extended the concept of core-EP inverses of complex matrices to rings and called them pseudocore inverses. An element $a \in R$ is said to be pseudocore invertible if there exist $x \in R$ and $k \in \mathbb{N}^+$

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such that

$$xa^{k+1}=a^k,\ ax^2=x,\ (ax)^*=ax.$$

Such $x$ is unique if it exists, and is called the pseudocore inverse of $a$, denoted by $a^{\textcircled{D}}$. The smallest positive integer $k$ satisfying the equations above is called the pseudocore index of $a$, which coincides with its Drazin index, and still denoted by $\mathrm{ind}(a)$. Some interesting properties and representations of these generalized inverses were investigated, for example, see [1, 4, 11, 15, 24, 30].

The weak group inverse was introduced by Wang and Chen [25] in complex matrices (for some representations, also see [7, 19]), and later was generalized to proper $*$-rings (i.e. $R$ is a proper $*$-ring if $a^*a=0$ implies $a=0$ for any $a\in R$) by Zhou et al. [27]. From [27, Theorem 3.5], it was turned out that each element in a proper $*$-ring has at most one weak group inverse. However, it may not be unique in $R$ (see [27, Remark 3.6]).

**Definition 1.1** [27, Definition 3.1] *Let $a\in R$. Then $a$ is said to be weak group invertible if there exist $x\in R$ and $k\in\mathbb{N}^+$ satisfying*

$$xa^{k+1}=a^k,\ ax^2=x,\ (a^k)^*a^2x=(a^k)^*a.$$

*Any such $x$ is called the weak group inverse of $a$.*

As a common generalization of the pseudocore inverse and the weak group inverse, Zhou et al. [29] proposed the definition of the $m$-weak group inverse in a ring with involution.

**Definition 1.2** [29, Definition 4.1] *Let $m\in\mathbb{N}$. An element $a\in R$ is said to be $m$-weak group invertible if there exist $x\in R$ and $k\in\mathbb{N}^+$ satisfying*

$$xa^{k+1}=a^k,\ ax^2=x,\ (a^k)^*a^{m+1}x=(a^k)^*a^m.$$

*Any such $x$ is called the $m$-weak group inverse of $a$.*

If $k$ is the smallest positive integer such that the above equations hold, then $k$ is called the $m$-weak group index of $a$. If $a$ is $m$-weak group invertible, then $a$ is Drazin invertible and the $m$-weak group index of $a$ is equal to its Drazin index. Therefore, we still use $\mathrm{ind}(a)$ to denote the $m$-weak group index of $a$.

It is worth noting that the definition of the 1-weak group inverse is exactly that of the weak group inverse (see [29, Corollary 4.4]). When the $m$-weak group inverse (resp., weak group inverse) is unique, we use $a^{\textcircled{w}_m}$ (resp., $a^{\textcircled{w}}$) to denote the unique $m$-weak group inverse (resp., weak group inverse) of $a$.

In what follows, the symbols $R^{\{1,3\}}, R^{\#}, R^D, R^{\textcircled{D}}, R^{\textcircled{w}}, R^{\textcircled{w}_m}$ denote the sets of all $\{1,3\}$-invertible, group invertible, Drazin invertible, pseudocore invertible, weak group invertible, and $m$-weak group invertible elements of $R$, respectively.

It was also shown in [29, Corollary 4.3] that $a\in R^{\textcircled{D}}$ if and only if $a\in R^{\textcircled{w}_0}$, in this case, $a$ has at most one 0-weak group inverse and $a^{\textcircled{w}_0}=a^{\textcircled{D}}$. Following [29, Corollary 4.11], if $a\in R^{\textcircled{D}}$, then $a$ has a unique $m$-weak group inverse. Moreover, it was pointed out in [29, Proposition 4.8] that $a^{\textcircled{w}}=(a^{\textcircled{D}})^2a$ when $a\in R^{\textcircled{D}}$. For more details of $m$-weak group inverses, see, e.g., [13, 21].

The weak core inverse is a new type of generalized inverse, which was introduced by Ferreyra et al. [6] for complex matrices. Later, Zhou and Chen [28] generalized this concept to a ring with involution. Following

[28, Proposition 3.1], it was proved that if $a \in R^{\textcircled{w}} \cap R^{\{1,3\}}$, then $a \in R^{\textcircled{D}}$, in which case, $a$ has a unique weak group inverse.

**Definition 1.3** [28, Definition 3.6] *Let $a \in R$. If $a \in R^{\textcircled{w}} \cap R^{\{1,3\}}$, then $a$ is said to be weak core invertible. The unique $x \in R$ satisfying the following equations*

$$xax = x,\ ax = aa^{\textcircled{w}}aa^{(1,3)},\ xa = a^{\textcircled{w}}a$$

*is called the weak core inverse of $a$, denoted by $a^{wC}$.*

We use $R^{wC}$ to denote the set of all weak core invertible elements of $R$. Following [28], the relation

$$R^{wC} \subseteq R^{\textcircled{D}} \subseteq R^{\textcircled{w}} \subseteq R^{\textcircled{w}_m} \subseteq R^{D}\ (m \geq 2)$$

is established. Moreover, $a^{wC} = a^{\textcircled{w}}aa^{(1,3)} = (a^{\textcircled{D}})^2a^2a^{(1,3)}$. Some expressions and properties which can be used in the calculation of weak core inverse for complex matrices can also be found in [8, 20].

The theme of this paper is to investigate some new characterizations and representations of weak core inverses and $m$-weak group inverses in a ring with involution. The motivations are as follows.

It was shown in [9, Theorem 2.3] that $a \in R^{\textcircled{D}}$ with $\mathrm{ind}(a) = k$ if and only if $a \in R^{D}$ with $\mathrm{ind}(a) = k$ and $a^k \in R^{\{1,3\}}$. In addition, Zhou et al. [29, Proposition 3.11] also gave an existence criteria for weak group inverses in $R$. Motivated by these discussions, in Section 3, we first investigate some equivalent conditions for the existence of the $m$-weak group inverse, and improve the relevant result of Zhou et al. As an application, it turned out that $a \in R^{\textcircled{w}_{m+1}}$ if and only if $a^2a^- \in R^{\textcircled{w}_m}$, where $a \in R$ is regular and $a^- \in a\{1\}$. For a regular element $a \in R$ and $b = aa^- \in R$, we prove that $ab \in R^{\textcircled{D}}$ if and only if $ba \in R^{\textcircled{D}}$.

In [27], Zhou et al. characterized the weak group inverse using annihilators in a proper $*$-ring. In [5], Ferreyra et al. investigated maximal classes for complex matrices determining some generalized inverses, such as DMP inverses [16], core-EP inverses and CMP inverses [18]. Later, Zhou and Chen [26] generalized the relevant result of core-EP inverses to a ring with an involution, and obtained maximal classes of elements in a ring determining pseudocore inverses. Inspired by the discussion above, we aim to present some new characterizations of weak core inverses and present maximal classes of elements in a ring related to weak core inverses in Section 4.

## 2. Preliminaries

The right (resp., left) annihilator of $a$ is defined by $a^{\circ} = \{x \in R : ax = 0\}$ (resp., ${}^{\circ}a = \{x \in R : xa = 0\}$). In this section, we give several necessary lemmas.

**Lemma 2.1** [22, Lemma 2.5 ] *Let $a, b \in R$.*

(i) *If $aR \subseteq bR$, then ${}^{\circ}b \subseteq {}^{\circ}a$.*

(ii) *If $Ra \subseteq Rb$, then $b^{\circ} \subseteq a^{\circ}$.*

**Lemma 2.2** [23, Theorem 3.3] *If $a \in R^{D}$, then $a \in R^{\textcircled{D}}$ if and only if $aa^{D} \in R^{\{1,3\}}$. In this case, $aa^{\textcircled{D}} \in (aa^{D})\{1,3\}$ and $a^{\textcircled{D}} = a^{D}(aa^{D})^{(1,3)}$ for any $(aa^{D})^{(1,3)} \in (aa^{D})\{1,3\}$.*

**Lemma 2.3** [9, Lemma 2.1] *Let $a \in R$. If there exists $x \in R$ such that*

$$ax^2 = x, \quad xa^{k+1} = a^k \text{ for some } k \in \mathbb{N}^+,$$

*then $a \in R^D$ with* $\operatorname{ind}(a) \leq k$.

Following [28], Zhou and Chen wrote

$$T_l(a) = \{x \in R : xa^{k+1} = a^k, ax^2 = x \text{ for some } k \in \mathbb{N}^+\},$$

when $a \in R^D$. Moreover, $T_l(a) = \{x \in R : xa^{\operatorname{ind}(a)+1} = a^{\operatorname{ind}(a)}, ax^2 = x\}$ is also established.

**Lemma 2.4** [28, Lemma 2.2] *Let $a \in R^D$, $k_1, \dots, k_n, s_1, \dots, s_n \in \mathbb{N}$ and $x_1, \dots, x_n \in T_l(a)$. If $s_n \neq 0$, then*

$$\prod_{i=1}^{n} a^{k_i} x_i^{s_i} = a^k x_n^s,$$

*where $k = \sum\limits_{i=1}^{n} k_i$ and $s = \sum\limits_{i=1}^{n} s_i$.*

**Lemma 2.5** ([10, Theorem 3.1], core-EP decomposition) *Let $a \in R^{\textcircled{D}}$. Then $a = a_1 + a_2$, where*

(i) $a_1^{\textcircled{\#}}$ *exists.*

(ii) $a_2^m = 0$ *for some* $m \in \mathbb{N}^+$.

(iii) $a_1^* a_2 = a_2 a_1 = 0$.

*In this case, $a_1^{\textcircled{\#}} = a^{\textcircled{D}}$, $a_1^{\#} = (a^{\textcircled{D}})^2 a$, $a_1 = aa^{\textcircled{D}}a$ and $a_2 = a - aa^{\textcircled{D}}a$.*

In what follows, we will restrict $a_1 = aa^{\textcircled{D}}a$ and $a_2 = a - aa^{\textcircled{D}}a$ when $a \in R^{\textcircled{D}}$ following Lemma 2.5.

**Lemma 2.6** [28, Corollary 3.2] *Let $a \in R$. Then $a \in R^{\textcircled{w}} \cap R^{\{1,3\}}$ if and only if $a \in R^{\textcircled{D}}$ and $a_2 \in R^{\{1,3\}}$.*

## 3. Characterizations of $m$-weak group inverses

In this section, we first give some characterizations of $m$-weak group inverses.

**Theorem 3.1** *Let $a \in R^D$ with $\operatorname{ind}(a) = k$ and $m \in \mathbb{N}$. Then the following conditions are equivalent.*

(i) $a \in R^{\textcircled{w}_m}$.

(ii) $(a^k)^* a^m R \subseteq (a^k)^* a^k R$.

(iii) $(aa^D)^* a^m R \subseteq (aa^D)^* (aa^D) R$.

*In this case, $a^D + (a^D)^{m+1} aa^D t(1 - aa^D)$ is an $m$-weak group inverse of $a$, where $(aa^D)^* a^m = (aa^D)^* aa^D t$.*

**Proof** (i)$\Rightarrow$(ii). From Definition 1.2, there exists $x \in R$ such that $(a^k)^* a^m = (a^k)^* a^{m+1} x$. Then it follows that $(a^k)^* a^m = (a^k)^* a^k a^m x^k \in (a^k)^* a^k R$. Hence, $(a^k)^* a^m R \subseteq (a^k)^* a^k R$.

(ii)$\Rightarrow$(i). Since $(a^k)^* a^m R \subseteq (a^k)^* a^k R$, it follows that $(a^k)^* a^m = (a^k)^* a^k t$ for some $t \in R$. Let $x = a^D + (a^D)^{m+1} a^k t(1 - aa^D)$. Then it suffices to prove that $x$ is an $m$-weak group inverse of $a$. By Lemma 2.4, we get that

$$xa^{k+1} = a^D a^{k+1} + (a^D)^{m+1} a^k t(1 - aa^D) a^{k+1} = a^k,$$

$$\begin{aligned} ax^2 &= a\left(a^D + (a^D)^{m+1} a^k t(1 - aa^D)\right)^2 \\ &= a\left((a^D)^2 + (a^D)^{m+2} a^k t(1 - aa^D)\right) \\ &= a^D + (a^D)^{m+1} a^k t(1 - aa^D) = x \end{aligned}$$

and

$$\begin{aligned} (a^k)^* a^{m+1} x &= (a^k)^* a^{m+1}\left(a^D + (a^D)^{m+1} a^k t(1 - aa^D)\right) \\ &= (a^k)^* a^{m+1} a^D + (a^k)^* a^{m+1} (a^D)^{m+1} a^k t(1 - aa^D) \\ &= (a^k)^* a^{m+1} a^D + (a^k)^* a^k t(1 - aa^D) \\ &= (a^k)^* a^{m+1} a^D + (a^k)^* a^m (1 - aa^D) \\ &= (a^k)^* a^m. \end{aligned}$$

Hence, $a$ is $m$-weak group invertible and $x$ is an $m$-weak group inverse of $a$.

(ii)$\Leftrightarrow$(iii) is obvious by $a^k R = aa^D R$. Similarly, it also can be derived that $a^D + (a^D)^{m+1} aa^D t(1 - aa^D)$ is also an $m$-weak group inverse of $a$, where $(aa^D)^* a^m = (aa^D)^* aa^D t$. $\square$

Recall from [14, Definition 2], an element $a \in R$ is left $*$-cancellable if $a^* ax = a^* ay$ implies $ax = ay$. Following [29, Corollary 3.7 and Proposition 4.12], if each idempotent element in $R$ is left $*$-cancellable, then each element in $R$ has at most one $m$-weak group inverse.

**Remark 3.2** *Under the assumption of Theorem 3.1, it was pointed out that $a \in R^{\textcircled{\scriptsize D}}$ if and only if $a^k \in R(a^k)^* a^k$ when $m = 0$, which was first given in [9, Theorem 2.3]. When $m = 1$, it follows that $a \in R^{\textcircled{\scriptsize w}}$ if and only if $(aa^D)^* a \in (aa^D)^* aa^D R$, which is also equivalent to $(a^D)^* a \in (a^D)^* a^D R$. Thus, the condition that each idempotent element in $R$ is left $*$-cancellable of [29, Proposition 3.11] can be dropped, and $a^D + (a^D)^3 t(1 - aa^D)$ is a weak group inverse of $a$, where $(a^D)^* a = (a^D)^* a^D t$.*

**Proposition 3.3** *Let $a \in R^D$ and $m \in \mathbb{N}^+$. Then $a \in R^{\textcircled{\scriptsize w}_m}$ if and only if there exists $t \in R$ such that $(a^D)^* a^m = (a^D)^* a^D t$. If each idempotent element in $R$ is left $*$-cancellable, then $a^{\textcircled{\scriptsize w}_m} = (a^D)^{m+2} t$.*

**Proof** By Theorem 3.1 (i)$\Leftrightarrow$(iii), it is easy to get that $a \in R^{\textcircled{\scriptsize w}_m}$ if and only if there exists $s \in R$ such that $(aa^D)^* a^m = (aa^D)^* aa^D s$, or equivalently, there exists $t \in R$ such that $(a^D)^* a^m = (a^D)^* a^D t$. In this case, $a^{\textcircled{\scriptsize w}_m} = a^D + (a^D)^{m+1} a^D t(1 - aa^D)$.

From $(a^D)^*a^m = (a^D)^*a^Dt$, it follows that $(aa^D)^*aa^Da^{m-1} = (aa^D)^*aa^Da^Dta^D$. Since $aa^D$ is left $*$-cancellable, we obtain $a^m a^D = a^D t a^D$. Hence,

$$\begin{aligned} a^{ⓦ_m} &= a^D + (a^D)^{m+1}a^Dt - (a^D)^{m+1}a^Dta^Da \\ &= a^D + (a^D)^{m+2}t - (a^D)^{m+1}a^m a^D a \\ &= a^D + (a^D)^{m+2}t - a^D = (a^D)^{m+2}t. \end{aligned}$$

□

As an application of Theorem 3.1, we have the following result.

**Theorem 3.4** *Let $a \in R$ be regular with an inner inverse $a^-$ and $m \in \mathbb{N}^+$. Then $a \in R^{ⓦ_{m+1}}$ if and only if $a^2a^- \in R^{ⓦ_m}$. In this case,*

$$a^Daa^- + axa^-(1 - a^Da^2a^-) \text{ is an } m\text{-weak group inverse of } a^2a^-$$

*and*

$$a^D + a^Dya(1 - aa^D) \text{ is an } (m+1)\text{-weak group inverse of } a,$$

*where $x$ is an $(m+1)$-weak group inverse of $a$ and $y$ is an $m$-weak group inverse of $a^2a^-$.*

**Proof** By Cline's formula [2], $a \in R^D$ if and only if $a^2a^- \in R^D$, in this case, $(a^2a^-)^D = a^Daa^-$ and $\mathrm{ind}(a^2a^-) \le \mathrm{ind}(a) + 1$. Then by Theorem 3.1, it follows that when $a \in R^D$, we obtain that

$$\begin{aligned} a \in R^{ⓦ_{m+1}} &\Leftrightarrow (aa^D)^*a^{m+1}R \subseteq (aa^D)^*aa^DR \\ &\Leftrightarrow (aa^D)^*a^{m+1}a^-R \subseteq (aa^D)^*aa^DR \\ &\Leftrightarrow (aa^D)^*(a^2a^-)^mR \subseteq (aa^D)^*aa^DR \\ &\Leftrightarrow \left((a^2a^-)(a^2a^-)^D\right)^*(a^2a^-)^mR \subseteq \left((a^2a^-)(a^2a^-)^D\right)^*(a^2a^-)(a^2a^-)^DR \\ &\Leftrightarrow a^2a^- \in R^{ⓦ_m}. \end{aligned}$$

Now we proceed to present an $m$-weak group inverse of $a^2a^-$. Let $\mathrm{ind}(a) = k$ and $x$ be an $(m+1)$-weak group inverse of $a$. By Lemma 2.4, we get that

$$\begin{aligned} \left((a^2a^-)(a^2a^-)^D\right)^*(a^2a^-)^m &= (aa^Daa^-)^*a^{m+1}a^- \\ &= \left((a^D)^kaa^-\right)^*(a^k)^*a^{m+1}a^- \\ &= \left((a^D)^kaa^-\right)^*(a^k)^*a^{m+2}xa^- \\ &= (aa^Daa^-)^*a^{m+2}xa^- \\ &= \left((a^2a^-)(a^2a^-)^D\right)^*aa^Da^{m+2}xa^- \\ &= \left((a^2a^-)(a^2a^-)^D\right)^*(a^2a^-)(a^2a^-)^Da^{m+2}xa^-. \end{aligned}$$

Then it follows from Lemma 2.4 and Theorem 3.1 that

$$
\begin{aligned}
&(a^2a^-)^D + \left((a^2a^-)^D\right)^{m+1}\left((a^2a^-)(a^2a^-)^D a^{m+2}xa^-\right)\left(1-(a^2a^-)(a^2a^-)^D\right)\\
&= a^D aa^- + (a^D)^{m+1}aa^-(a^{m+2}xa^-)(1-a^Da^2a^-)\\
&= a^D aa^- + (a^D)^{m+1}a^{m+2}xa^-(1-a^Da^2a^-)\\
&= a^D aa^- + a^{m+2}x^{m+2}a^-(1-a^Da^2a^-)\\
&= a^D aa^- + axa^-(1-a^Da^2a^-).
\end{aligned}
$$

Therefore, $a^Daa^- + axa^-(1-a^Da^2a^-)$ is an $m$-weak group inverse of $a^2a^-$.

Next, we give an $(m+1)$-weak group inverse of $a$. Let $y$ be an $m$-weak group inverse of $a^2a^-$. Since $\mathrm{ind}(a^2a^-) \le k+1$ and Lemma 2.4, we get that

$$
\begin{aligned}
(aa^D)^*a^{m+1} &= \left((a^2a^-)^{k+1}(a^D)^{k+1}\right)^*(a^2a^-)^m a\\
&= \left((a^D)^{k+1}\right)^*\left((a^2a^-)^{k+1}\right)^*(a^2a^-)^m a\\
&= \left((a^D)^{k+1}\right)^*\left((a^2a^-)^{k+1}\right)^*(a^2a^-)^{m+1} ya\\
&= (aa^D)^*(a^2a^-)^{m+1}ya\\
&= (aa^D)^*(a^2a^-)^D(a^2a^-)^{m+2}ya\\
&= (aa^D)^*aa^D(a^2a^-)^{m+1}ya.
\end{aligned}
$$

Then it also follows from Lemma 2.4 and Theorem 3.1 that

$$
\begin{aligned}
&a^D + (a^D)^{m+2}aa^D(a^2a^-)^{m+1}ya(1-aa^D)\\
&= a^D + (a^D)^{m+2}a^{m+2}a^-ya(1-aa^D)\\
&= a^D + a^Daa^-(a^2a^-)y^2a(1-aa^D)\\
&= a^D + a^D(a^2a^-)y^2a(1-aa^D)\\
&= a^D + a^Dya(1-aa^D).
\end{aligned}
$$

Hence, $a^D + a^Dya(1-aa^D)$ is an $(m+1)$-weak group inverse of $a$. □

**Corollary 3.5** *Let $m \in \mathbb{N}^+$ and $a \in R$ be regular with an inner inverse $a^-$. If $a \in R^{ⓦ_{m+1}}$ and each idempotent element in $R$ is left $*$-cancellable, then $(a^2a^-)^{ⓦ_m} = aa^{ⓦ_{m+1}}a^-$ and $a^{ⓦ_{m+1}} = \left((a^2a^-)^{ⓦ_m}\right)^2 a$.*

**Proof** According to the proof of Theorem 3.4, we get

$$
\left((a^2a^-)(a^2a^-)^D\right)^*(a^2a^-)^m = \left((a^2a^-)(a^2a^-)^D\right)^*(a^2a^-)(a^2a^-)^D a^{m+2}a^{ⓦ_{m+1}}a^-.
$$

Then from Lemma 2.4 and Proposition 3.3, it follows that

$$\begin{aligned}
(a^2a^-)^{\textcircled{w}_m} &= \left((a^2a^-)^D\right)^{m+2}(a^2a^-)a^{m+2}a^{\textcircled{w}_{m+1}}a^- \\
&= \left((a^2a^-)^D\right)^{m+1}(a^Daa^-)(a^2a^-)a^{m+2}a^{\textcircled{w}_{m+1}}a^- \\
&= \left((a^2a^-)^D\right)^{m+1}a^Da^{m+3}a^{\textcircled{w}_{m+1}}a^- \\
&= \left((a^2a^-)^D\right)^{m+1}a^{m+2}a^{\textcircled{w}_{m+1}}a^- \\
&= (a^D)^{m+1}aa^-a^{m+2}a^{\textcircled{w}_{m+1}}a^- \\
&= (a^D)^{m+1}a^{m+2}a^{\textcircled{w}_{m+1}}a^- \\
&= a^{m+2}\left(a^{\textcircled{w}_{m+1}}\right)^{m+2}a^- \\
&= aa^{\textcircled{w}_{m+1}}a^-.
\end{aligned}$$

A similar argument gives that

$$\begin{aligned}
a^{\textcircled{w}_{m+1}} &= (a^D)^{m+2}(a^2a^-)^{m+1}(a^2a^-)^{\textcircled{w}_m}a = (a^D)^{m+2}a^{m+2}a^-(a^2a^-)^{\textcircled{w}_m}a \\
&= (a^2a^-)^D(a^2a^-)^{\textcircled{w}_m}a = \left((a^2a^-)^{\textcircled{w}_m}\right)^2 a.
\end{aligned}$$

□

In [23], Shi et al. found that Cline's formula for pseudocore inverses does not hold. Here, for a regular element $a \in R$ and $b = aa^-$, we prove that $ab \in R^{\textcircled{D}}$ is equivalent to $ba \in R^{\textcircled{D}}$.

**Proposition 3.6** *Let $a \in R$ be regular with an inner inverse $a^-$. Then $a \in R^{\textcircled{D}}$ if and only if $a^2a^- \in R^{\textcircled{D}}$. In this case, $a^{\textcircled{D}} = (a^2a^-)^{\textcircled{D}}$.*

**Proof** From $aa^DR = (a^2a^-)(a^2a^-)^DR$ and Lemma 2.2, the equivalence of $a \in R^{\textcircled{D}}$ and $a^2a^- \in R^{\textcircled{D}}$ follows directly. Also, it is easy to check that $aa^{\textcircled{D}} \in \left((a^2a^-)(a^2a^-)^D\right)\{1,3\}$. Combining Lemma 2.4, we get that

$$(a^2a^-)^{\textcircled{D}} = (a^2a^-)^D\left((a^2a^-)(a^2a^-)^D\right)^{(1,3)} = a^Daa^-aa^{\textcircled{D}} = a^{\textcircled{D}}.$$

□

Particularly, when $a \in R^{\{1,3\}}$, we can have the relevant result for weak core inverses.

**Theorem 3.7** *Let $a \in R^{\{1,3\}}$ with a $\{1,3\}$-inverse $a^{(1,3)}$. Then $a \in R^{wC}$ if and only if $a^2a^{(1,3)} \in R^{\textcircled{D}}$. In this case, $a^{wC} = (a^2a^{(1,3)})^{\textcircled{w}}$.*

**Proof** It is obvious by Lemma 2.6 and Proposition 3.6. For the expression of the weak core inverse, since $a^{\textcircled{D}} = (a^2a^{(1,3)})^{\textcircled{D}}$, we obtain that

$$a^{wC} = (a^{\textcircled{D}})^2a^2a^{(1,3)} = \left((a^2a^{(1,3)})^{\textcircled{D}}\right)^2a^2a^{(1,3)} = (a^2a^{(1,3)})^{\textcircled{w}}.$$

□

**Remark 3.8** *It is worth noting that $a^2a^{(1,3)} \in R^{ⓦ}$ does not imply $a \in R^{wC}$ when $a \in R^{\{1,3\}}$ with a $\{1,3\}$-inverse $a^{(1,3)}$. Taking the example as in [28, Example 3.4], we know $a = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in R^{ⓦ_2}$ but $a \notin R^{Ⓓ}$. From Theorem 3.4, it was shown that $a \in R^{ⓦ_2}$ if and only if $a^2a^{(1,3)} \in R^{ⓦ}$. However, $a \notin R^{wC}$ by Lemma 2.6.*

In [6], Ferreyra et al. considered the equivalent condition for $A^{wC} = A^{ⓦ}$ when $A \in \mathbb{C}^{n\times n}$. Later, Zhou and Chen [28] gave several equivalent conditions for $a^{wC} = a^{ⓦ_m}$ for $m \in \mathbb{N}\backslash\{1\}$, when $a \in R^{wC}$. Here we investigate the case of $a^{wC} = a^{ⓦ}$ when $a \in R^{wC}$, which generalizes the result of Ferreyra et al. [6].

**Theorem 3.9** *Let $a \in R^{wC}$. Then $a^{wC} = a^{ⓦ}$ if and only if*

$$(aa^{ⓦ} - aa^{Ⓓ})(1 - a_2a_2^{(1,3)}) = 0\,.$$

**Proof** According to the proof of [28, Corollary 3.2], we get that $aa^{(1,3)} = p_1 + p_2$ with $p_1p_2 = 0$, where $p_1 = a_1a_1^{(1,3)}$ and $p_2 = a_2a_2^{(1,3)}$. Then

$$a^{ⓦ}p_1 = (a^{Ⓓ})^2ap_1 = a_1^{\#}a_1a_1^{(1,3)} = a_1^{\text{\textcircled{\#}}} = a^{Ⓓ}\,.$$

It follows that

$$\begin{aligned} a^{wC} = a^{ⓦ} &\Leftrightarrow a^{ⓦ}aa^{(1,3)} = a^{ⓦ} \\ &\Leftrightarrow a^{ⓦ}(1 - aa^{(1,3)}) = 0 \\ &\Leftrightarrow a^{ⓦ}(1 - p_1 - p_2) = 0 \\ &\Leftrightarrow a^{ⓦ}(1 - p_2) - a^{ⓦ}p_1(1 - p_2) = 0 \\ &\Leftrightarrow (a^{ⓦ} - a^{Ⓓ})(1 - p_2) = 0 \\ &\Leftrightarrow (aa^{ⓦ} - aa^{Ⓓ})(1 - a_2a_2^{(1,3)}) = 0. \end{aligned}$$

□

**Example 3.10** *$a^{wC} = a^{Ⓓ}$ does not imply that $a^{wC} = a^{ⓦ}$. For example, let $R = \mathbb{C}^{4\times 4}$ with the conjugate transpose as the involution. Take $a = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in R$. Then we have $aa^{(1,3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Following [6, Theorem 3.12], [24, Theorem 3.2] and [25, Theorem 3.1], we get that $a^{ⓦ} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $a^{Ⓓ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = a^{wC}$. However, $a^{wC} \neq a^{ⓦ}$.*

Following [28, Definition 4.7], an element $a \in R$ is called weak core element if $a \in R^{wC}$ with $a^{wC} = a^D aa^{(1,3)}$. The forthcoming example shows that $a$ is not weak core element when $a^{wC} = a^{ⓦ}$.

**Example 3.11** *Let $R = \mathbb{C}^{6\times 6}$ with the conjugate transpose as the involution. Take*

$$a = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

*Then according to [6, Theorem 5.3], we know that $a$ is not weak core element. By [6, Theorem 3.12], we get* $a^{ⓦ} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. *It is easy to verify that* $aa^{(1,3)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. *Then it follows* $a^{wC} = a^{ⓦ} aa^{(1,3)} = a^{ⓦ}$.

## 4. Characterizations of weak core inverses

In [27], some characterizations of weak group inverses are given. In this section, we will present some new characterizations of weak core inverses analogously.

**Theorem 4.1** *Let $a \in R$ be $\{1,3\}$-invertible with a $\{1,3\}$-inverse $a^{(1,3)}$ and $x \in R$. Then the following conditions are equivalent.*

(i) $a \in R^{wC}$ *and* $x = a^{wC}$.

(ii) $xax = x$, $xR = a^m R = a^{m+1}R$, $R(a^m)^* a^2 a^{(1,3)} = Rx$ *and* $(a^m)^* aR = (a^m)^* a^m R$ *for some* $m \in \mathbb{N}^+$.

(iii) $xax = x$, $xR = a^m R \subseteq a^{m+1}R$, $R(a^m)^* a^2 a^{(1,3)} \subseteq Rx$ *and* $(a^m)^* aR \subseteq (a^m)^* a^m R$ *for some* $m \in \mathbb{N}^+$.

(iv) $xax = x$, ${}^\circ(a^{m+1}) \subseteq {}^\circ(a^m) = {}^\circ x$, $x^\circ \subseteq \big((a^m)^* a^2 a^{(1,3)}\big)^\circ$ *and* $(a^m)^* aR \subseteq (a^m)^* a^m R$ *for some* $m \in \mathbb{N}^+$.

**Proof** (i)$\Rightarrow$(ii). By the hypothesis, we can suppose $\mathrm{ind}(a) = m$. Following [28, Proposition 3.14], we also get

$$xax = x,\ ax^2 = x \text{ and } ax = aa^{ⓦ} aa^{(1,3)}.$$

Since

$$x = ax^2 = a^m x^{m+1} = a^{m+1} x^{m+2},$$

we get $xR \subseteq a^{m+1}R \subseteq a^m R$. Since $xa^{m+1} = a^{ⓦ} a^{m+1} = a^m$, we get $a^m R \subseteq xR$. Hence, $xR = a^m R = a^{m+1}R$.

Since

$$(a^m)^* a^2 a^{(1,3)} = (a^m)^* a^2 a^{ⓦ} aa^{(1,3)} = (a^m)^* a^2 x,$$

we have $R(a^m)^*a^2a^{(1,3)} \subseteq Rx$. From

$$x = (a^{\textcircled{\tiny D}})^2a^2a^{(1,3)} = (a^{\textcircled{\tiny D}})^2a^m(a^{\textcircled{\tiny D}})^ma^2a^{(1,3)}$$
$$= (a^{\textcircled{\tiny D}})^2\left((a^{\textcircled{\tiny D}})^m\right)^*(a^m)^*a^2a^{(1,3)} \in R(a^m)^*a^2a^{(1,3)},$$

we get $Rx \subseteq R(a^m)^*a^2a^{(1,3)}$. Then $Rx = R(a^m)^*a^2a^{(1,3)}$.

In addition, $(a^m)^*aR \subseteq (a^m)^*a^mR$ follows directly by Theorem 3.1. Obviously $(a^m)^*a^mR \subseteq (a^m)^*aR$. Then $(a^m)^*aR = (a^m)^*a^mR$.

(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(iv). It follows by Lemma 2.1.

(iv)$\Rightarrow$(i). From $xa - 1 \in {}^\circ x$, we get $(xa-1)a^m = 0$. Then $xa^{m+1} = a^m$. Since $axa^{m+1} = a^{m+1}$, it follows that $ax - 1 \in {}^\circ(a^{m+1})$. Then $(ax-1)x = 0$ implies $ax^2 = x$. Hence, $a \in R^D$ with $\mathrm{ind}(a) \leq m$ by Lemma 2.3.

From $(a^m)^*aR \subseteq (a^m)^*a^mR$, we get $a \in R^{\textcircled{\tiny W}}$ by Theorem 3.1. Hence, $a \in R^{wC}$. Since

$$\left((a^m)^*a^2a^{(1,3)}\right)^\circ \subseteq \left((a^{\textcircled{\tiny D}})^2\left((a^{\textcircled{\tiny D}})^m\right)^*(a^m)^*a^2a^{(1,3)}\right)^\circ = \left((a^{\textcircled{\tiny D}})^2a^2a^{(1,3)}\right)^\circ = (a^{wC})^\circ$$

and $x^\circ \subseteq \left((a^m)^*a^2a^{(1,3)}\right)^\circ$, we can derive from $x(ax-1) = 0$ that $a^{wC}(ax-1) = 0$. Then $a^{wC} = a^{wC}ax = ax^2 = x$ by Lemma 2.4. $\square$

**Remark 4.2** *From the example of Remark 3.8, we know that the condition* $(a^m)^*aR \subseteq (a^m)^*a^mR$ *of Theorem 4.1* (iii) *is not superfluous. In fact, we take* $x = a^D$. *By computation,* $(a^2)^*a = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $(a^2)^*a^2 = (a^2)^*a^2a^{(1,3)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Then* $xax = x$, $xR = a^2R \subseteq a^3R$ *and* $R(a^2)^*a^2a^{(1,3)} \subseteq Rx$. *However,* $(a^2)^*aR \subsetneq (a^2)^*a^2R$. *Hence,* $a \notin R^{wC}$.

**Theorem 4.3** *Let* $a \in R$ *be* $\{1,3\}$*-invertible with a* $\{1,3\}$*-inverse* $a^{(1,3)}$. *Then the following conditions are equivalent.*

(i) $a \in R^{wC}$.

(ii) *There exists an idempotent* $q \in R$ *such that* $qR = a^mR = a^{m+1}R$, $Ra^m = Ra^{m+1}$, $R(a^m)^*a^2a^{(1,3)} = Rq$ *and* $(a^m)^*aR = (a^m)^*a^mR$ *for some* $m \in \mathbb{N}^+$.

(iii) *There exists an idempotent* $q \in R$ *such that* $qR = a^mR \subseteq a^{m+1}R$, $Ra^m \subseteq Ra^{m+1}$, $R(a^m)^*a^2a^{(1,3)} \subseteq Rq$ *and* $(a^m)^*aR \subseteq (a^m)^*a^mR$ *for some* $m \in \mathbb{N}^+$.

(iv) $a^{m+1}$ *is regular and there exists an idempotent* $q \in R$ *such that* ${}^\circ(a^{m+1}) \subseteq {}^\circ(a^m) = {}^\circ q$, $(a^{m+1})^\circ \subseteq (a^m)^\circ$, $q^\circ \subseteq \left((a^m)^*a^2a^{(1,3)}\right)^\circ$ *and* $(a^m)^*aR \subseteq (a^m)^*a^mR$ *for some* $m \in \mathbb{N}^+$.

*In this case,* $a^{wC} = a^m(a^{m+1})^-q$ *for any* $(a^{m+1})^- \in a^{m+1}\{1\}$.

**Proof** (i)$\Rightarrow$(ii). Following the assumption, we suppose $\mathrm{ind}(a)=m$. Let $q=aa^{wC}$. Then

$$q=aa^{\textcircled{w}}aa^{(1,3)}=a^m(a^{\textcircled{w}})^m aa^{(1,3)}=a^{m+1}(a^{\textcircled{w}})^{m+1}aa^{(1,3)},$$

which implies that $qR\subseteq a^{m+1}R\subseteq a^mR$. Also, from

$$a^m=a^{\textcircled{w}}a^{m+1}=aa^{\textcircled{w}}a^{\textcircled{w}}a^{m+1}=aa^{\textcircled{w}}a^m=aa^{\textcircled{w}}aa^{(1,3)}a^m=aa^{wC}a^m=qa^m,$$

we obtain $a^mR\subseteq qR$ and $Ra^m=Ra^{m+1}$. Hence, $qR=a^mR=a^{m+1}R$.

Since

$$(a^m)^*a^2a^{(1,3)}=(a^m)^*a^2a^{\textcircled{w}}aa^{(1,3)}=(a^m)^*aq,$$

we get $R(a^m)^*a^2a^{(1,3)}\subseteq Rq$. From

$$\begin{aligned} q&=a^{\textcircled{D}}a^2a^{(1,3)}=a^{\textcircled{D}}a^m(a^{\textcircled{D}})^m a^2a^{(1,3)}\\ &=a^{\textcircled{D}}\left((a^{\textcircled{D}})^m\right)^*(a^m)^*a^2a^{(1,3)}\in R(a^m)^*a^2a^{(1,3)}, \end{aligned}$$

we get $Rq\subseteq R(a^m)^*a^2a^{(1,3)}$. Then $R(a^m)^*a^2a^{(1,3)}=Rq$.

Also, $(a^m)^*aR\subseteq(a^m)^*a^mR$ follows directly by Theorem 3.1. Obviously, $(a^m)^*a^mR\subseteq(a^m)^*aR$. Hence, $(a^m)^*aR=(a^m)^*a^mR$.

(ii)$\Rightarrow$(iii) is clear.

(iii)$\Rightarrow$(iv). It follows directly by Lemma 2.1.

(iv)$\Rightarrow$(i). Let $(a^{m+1})^-$ be an inner inverse of $a^{m+1}$. Since

$$(1-(a^{m+1})^-a^{m+1})\subseteq(a^{m+1})^\circ\subseteq(a^m)^\circ,$$

it follows that $a^m=a^m(a^{m+1})^-a^{m+1}$. From $1-q\in{}^\circ q={}^\circ(a^m)$, we can get $a^m=qa^m$. Since

$$(1-a^{m+1}(a^{m+1})^-)\in{}^\circ(a^{m+1})\subseteq{}^\circ q,$$

we can get that $q=a^{m+1}(a^{m+1})^-q$.

Let $x=a^m(a^{m+1})^-q$. Then $ax=q$,

$$\begin{aligned} xa^{m+1}&=a^m(a^{m+1})^-qa^{m+1}=a^m(a^{m+1})^-a^{m+1}=a^m\\ \text{and } ax^2&=a^{m+1}(a^{m+1})^-qa^m(a^{m+1})^-q=qa^m(a^{m+1})^-q=x. \end{aligned}$$

Hence, $a\in R^D$ with $\mathrm{ind}(a)\le m$ by Lemma 2.3.

From $(a^m)^*aR\subseteq(a^m)^*a^mR$, we can get $a\in R^{\textcircled{w}}$ by Theorem 3.1. Therefore, $a\in R^{wC}$. Since

$$\left((a^m)^*a^2a^{(1,3)}\right)^\circ\subseteq\left((a^{\textcircled{D}})^2a^2a^{(1,3)}\right)^\circ=(a^{wC})^\circ$$

and $q^\circ\subseteq\left((a^m)^*a^2a^{(1,3)}\right)^\circ$, it follows that $ax-1\in q^\circ\subseteq(a^{wC})^\circ$. Then by Lemma 2.4, we have

$$a^{wC}=a^{wC}ax=ax^2=x=a^m(a^{m+1})^-q.$$

□

Let $p, q \in R$ be idempotent. If $x \in R$, then $x$ can be represented as a sum

$$x = pxq + px(1-q) + (1-p)xq + (1-p)x(1-q)$$

or as a formal matrix

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}_{p\times q}, \tag{4.1}$$

where $x_{11} = pxq$, $x_{12} = px(1-q)$, $x_{21} = (1-p)xq$ and $x_{22} = (1-p)x(1-q)$, which is well-known as Peirce decomposition.

Finally, we investigate maximal classes of elements concerning the weak core inverse by Peirce decomposition.

**Theorem 4.4** *Let $a \in R^{wC}$ with $\mathrm{ind}(a) = k$ and $y \in a\{1\}$. If there exists $x \in R$ satisfying $xa^{k+1} = a^k$, then the following conditions are equivalent.*

(i) $a^{wC} = xay$.

(ii) $xa = a^{ⓦ}a$ *and* $a^{(1,3)} - y \in (1 - a^{ⓦ}a)R$.

(iii) $x = \begin{pmatrix} a^{Ⓓ} & x_2 \\ 0 & x_4 \end{pmatrix}_{p\times p}$ *with* $x_2a = a^{ⓦ}a - a^{Ⓓ}a$ *and* $x_4a = 0$, $y = \begin{pmatrix} a^{wC} & 0 \\ y_3 & y_4 \end{pmatrix}_{q\times\gamma}$, *where* $p = aa^{Ⓓ}$, $q = a^{ⓦ}a$ *and* $\gamma = aa^{(1,3)}$.

*In this case,* $(a^*)^\circ \subseteq \left((a^k)^*a^2y\right)^\circ$.

**Proof** (i)⇒(ii). From $a^{wC} = xay$ and $y \in a\{1\}$, we get

$$a^{ⓦ}a = a^{ⓦ}aa^{(1,3)}a = a^{wC}a = xaya = xa.$$

Also, since $a^{ⓦ}aa^{(1,3)} = xay = a^{ⓦ}ay$, we can obtain $a^{ⓦ}a(a^{(1,3)} - y) = 0$, which implies that

$$a^{(1,3)} - y = (1 - a^{ⓦ}a)(a^{(1,3)} - y) \in (1 - a^{ⓦ}a)R.$$

(ii)⇒(iii). Let $p = aa^{Ⓓ}$, $q = a^{ⓦ}a$ and $\gamma = aa^{(1,3)}$. Then following Peirce decomposition, we get that

$$a = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times p} = \begin{pmatrix} aa^{ⓦ}a & a - aa^{ⓦ}a \\ 0 & 0 \end{pmatrix}_{\gamma\times q},$$

where $a_{11} = aa^{Ⓓ}aaa^{Ⓓ} = a^2a^{Ⓓ}$, $a_{12} = a_1 - a_{11}$ and $a_2$ is nilpotent of index $k$. Then $a^k = \begin{pmatrix} a_{11}^k & \widetilde{a_{12}} \\ 0 & 0 \end{pmatrix}_{p\times p}$, where $\widetilde{a_{12}} = \sum_{j=0}^{k-1} a_{11}^j a_{12} a_2^{k-1-j}$.

Suppose that $x=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}_{p\times p}$ satisfies $xa^{k+1}=a^k$. Hence, we conclude that

$$\begin{cases} x_1 a_{11}^{k+1} = a_{11}^k \\ x_1 \sum\limits_{j=0}^{k} a_{11}^j a_{12} a_2^{k-j} = \widetilde{a_{12}} \\ x_3 a_{11}^{k+1} = 0 \\ x_3 \sum\limits_{j=0}^{k} a_{11}^j a_{12} a_2^{k-j} = 0 \end{cases} \Rightarrow \begin{cases} x_1 a a^{\textcircled{D}} = a^{\textcircled{D}} \\ x_1 a_{12} a_2^k + x_1 a_{11} \widetilde{a_{12}} = \widetilde{a_{12}} \\ x_3 a a^{\textcircled{D}} = 0 \\ x_3 \sum\limits_{j=0}^{k} a_{11}^j a_{12} a_2^{k-j} = 0. \end{cases}$$

Then $x_3 = x_3 a a^{\textcircled{D}} = 0$ and $x_1 = x_1 a a^{\textcircled{D}} = a^{\textcircled{D}}$. Therefore, $x=\begin{pmatrix} a^{\textcircled{D}} & x_2 \\ 0 & x_4 \end{pmatrix}_{p\times p}$.

Following Peirce decomposition and Lemma 2.4, we can obtain $a^{\textcircled{w}}=\begin{pmatrix} a^{\textcircled{D}} & a^{\textcircled{w}}-a^{\textcircled{D}} \\ 0 & 0 \end{pmatrix}_{p\times p}$, then $a^{\textcircled{w}}a = \begin{pmatrix} a^{\textcircled{D}}a_{11} & a^{\textcircled{D}}a_{12}+(a^{\textcircled{w}}-a^{\textcircled{D}})a_2 \\ 0 & 0 \end{pmatrix}_{p\times p}$. In addition, we can calculate $xa = \begin{pmatrix} a^{\textcircled{D}}a_{11} & a^{\textcircled{D}}a_{12}+x_2a_2 \\ 0 & x_4a_2 \end{pmatrix}_{p\times p}$. Hence, it follows from $xa=a^{\textcircled{w}}a$ and Lemma 2.4 that

$$\begin{pmatrix} aa^{\textcircled{D}} & a^{\textcircled{D}}a - aa^{\textcircled{D}} + x_2a_2 \\ 0 & x_4a_2 \end{pmatrix}_{p\times p} = \begin{pmatrix} aa^{\textcircled{D}} & a^{\textcircled{w}}a - aa^{\textcircled{D}} \\ 0 & 0 \end{pmatrix}_{p\times p}.$$

Then $x_2a_2 = a^{\textcircled{w}}a - a^{\textcircled{D}}a$ and $x_4a_2=0$, it follows that $x_2a = a^{\textcircled{w}}a - a^{\textcircled{D}}a$ and $x_4a=0$.

From $a^{(1,3)}-y\in(1-a^{\textcircled{w}}a)R$, we get $a^{\textcircled{w}}ay = a^{\textcircled{w}}aa^{(1,3)}$. Let $y=\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}_{q\times\gamma}$. Then

$$y_1 = qy\gamma = a^{\textcircled{w}}ayaa^{(1,3)} = a^{\textcircled{w}}aa^{(1,3)} = a^{wC}$$

and

$$y_2 = qy(1-\gamma) = a^{\textcircled{w}}ay(1-aa^{(1,3)}) = a^{\textcircled{w}}aa^{(1,3)}(1-aa^{(1,3)}) = 0.$$

Hence, $y=\begin{pmatrix} a^{wC} & 0 \\ y_3 & y_4 \end{pmatrix}_{q\times\gamma}$.

(iii)$\Rightarrow$(i). Following Peirce decomposition and Lemma 2.4, we can also get $a=\begin{pmatrix} aa^{\textcircled{w}}a & aa^{\textcircled{D}}a - aa^{\textcircled{w}}a \\ 0 & a-aa^{\textcircled{D}}a \end{pmatrix}_{p\times q}$. Then by computation, we get

$$xay = \begin{pmatrix} a^{wC}+(a^{\textcircled{D}}a-a^{\textcircled{w}}a+x_2a)y_3 & (a^{\textcircled{D}}a-a^{\textcircled{w}}a+x_2a)y_4 \\ x_4ay_3 & x_4ay_4 \end{pmatrix}_{p\times\gamma}.$$

Since $x_2a = a^{\textcircled{w}}a - a^{\textcircled{D}}a$ and $x_4a=0$, it follows that

$$xay = \begin{pmatrix} a^{wC} & 0 \\ 0 & 0 \end{pmatrix}_{p\times\gamma} = pa^{wC}\gamma = aa^{\textcircled{D}}a^{wC}aa^{(1,3)} = a^{wC}.$$

In this case, for arbitrary $u \in (a^*)^\circ$, we have $a^*u = 0$. Hence, $aa^{(1,3)}u = 0$. Following $a^{(1,3)} - y \in (1 - a^{ⓦ}a)R$, we can obtain that $a^{ⓦ}ay = a^{ⓦ}aa^{(1,3)}$. Hence,

$$(a^k)^*a^2yu = (a^k)^*a^2a^{ⓦ}ayu = (a^k)^*a^2a^{ⓦ}aa^{(1,3)}u = 0.$$

Then $u \in \left((a^k)^*a^2y\right)^\circ$. Therefore, $(a^*)^\circ \subseteq \left((a^k)^*a^2y\right)^\circ$. $\square$


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