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Energy decay and blow-up of solutions for a class of system of generalized nonlinear Klein-Gordon equations with source and damping terms

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Abstract: In this work, we investigate generalized coupled nonlinear Klein-Gordon equations with nonlinear damping and source terms and initial-boundary value conditions, in a bounded domain. We obtain decay of solutions by use of Nakao inequality. The blow up of solutions with negative initial energy is also established.

Key words: Decay,blow up, generalized Klein-Gordon equation

1. Introduction

In this paper, we study the initial-boundary value problem for the following coupled nonlinear generalized Klein–Gordon equations with nonlinear damping terms and source terms

$$u_{tt} - div(|\nabla u|^{\alpha-1}\nabla u) + m_1^2 u + |u_t|^{p-1}u_t = g_1(u,v), \qquad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

$$v_{tt} - div(|\nabla v|^{\alpha-1}\nabla v) + m_2^2 v + |v_t|^{q-1}v_t = g_2(u,v), \qquad (x,t) \in \Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \qquad x \in \Omega, \tag{1.3}$$

$$v(x,0) = v_0(x), v_t(x,0) = v_1(x), \qquad x \in \Omega, \tag{1.4}$$

$$u(x,t) = v(x,t) = 0, \qquad x \in \partial\Omega, \tag{1.5}$$

where Ω is a bounded domain of $R^n (n = 1,2,3)$, with smooth boundary $\partial\Omega$, $p,q \geq 1$, $\alpha \geq 1$ and $m_1, m_2 > 0$ are real numbers.

There are many results on the Cauchy problem for a class of the system Klein-Gordon equations [10, 11, 13, 17]. For instance, Segal[14] first proposed the following nonlinear system of Klein-Gordon equations

$$\begin{cases} u_{tt} - \Delta u + m_1^2 u + g_1 u^2 v = 0, \\ v_{tt} - \Delta v + m_2^2 v + g_2 u v^2 = 0, \end{cases} \tag{1.6}$$

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where m_1 and m_2 are nonzero constants, which define the movement of charged mesons in an electromagnetic field. I. Segal discussed the problem (1.6) of the global existence of the Cauchy problem with $g_1 > 0, g_2 > 0$. Blow up of solutions of (1.6) with $g_1 < 0, g_2 < 0$ was first established in [6, 7].

In the case of $\alpha = 1$, the problem (1.1)-(1.5) becomes to the following form

$$\begin{cases} u_{tt} - \Delta u + m_1^2 u + |u_t|^{p-1} u_t = g_1(u,v), \\ v_{tt} - \Delta v + m_2^2 v + |v_t|^{q-1} v_t = g_2(u,v). \end{cases} \tag{1.7}$$

Pişkin [13] proved the uniform decay of solutions by using Nakao's inequality and blow-up solutions in finite time with negative initial energy of the system (1.7). In addition, Ye [17] proved the global existence by using the potential well method and asymptotic stability by use of Komornik's lemma [5] of the system (1.7) with $p = q$. Wu [15] also discussed the blow-up of global solutions under some conditions for a system of (1.7).

When $p = q = 1$, Wu[16] studied the global existence, nonexistence, and asymptotic behavior of solutions for the system (1.7). When $m_1 = m_2 = 0$, Agre and Rammaha [2] proved the global existence and the nonexistence of solutions for the system (1.7) by applying the same techniques as in [3].

In this paper, the global existence of solution of the problem (1.1)-(1.5) was proved, and decay rates of energy which decays exponentially for $p = q = 1$ and polynomially for $p, q > 1$, were established by the use of Nakao's inequality [9]. The blow-up result for solutions with negative initial energy was established for $r > \max\{p, q\}$ by applying the technique of [3].

2. Preliminaries

In this section, we present some assumptions and lemmas, in the proof of our main result. We shall write $\|.\|$ and $\|.\|_p$ to define the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively. There exists a function $G(u,v)$ such that $\frac{\partial G}{\partial u} = g_1(u,v)$, $\frac{\partial G}{\partial v} = g_2(u,v)$.

Concerning the functions $g_1(u,v)$ and $g_2(u,v)$, we take

$$g_1(u,v) = (r+1)[a|u+v|^{r-1}(u+v) + b|u|^{\frac{r-3}{2}} u |v|^{\frac{r+1}{2}}],$$

$$g_2(u,v) = (r+1)[a|u+v|^{r-1}(u+v) + b|u|^{\frac{r+1}{2}} |v|^{\frac{r-3}{2}} v],$$

where $a, b > 0$ real numbers and r satisfies

$$\begin{cases} 1 < r, & n \le 2, \\ 1 < r \le \dfrac{(n+2}{(n-2}, & n > 2. \end{cases} \tag{2.1}$$

In accordance with the above equalities, it can easily verify that

$$u g_1(u,v) + v g_2(u,v) = (r+1) G(u,v), \qquad \forall (u,v) \in R^2, \tag{2.2}$$

$$G(u,v) = [a|u+v|^{r+1} + 2b|uv|^{\frac{r+1}{2}}]. \tag{2.3}$$

Lemma 2.1 *[8] There exist two positive constants c_0 and c_1 such that*

$$c_0(|u|^{r+1}+|v|^{r+1}) \leq G(u,v) \leq c_1(|u|^{r+1}+|v|^{r+1}) \tag{2.4}$$

is satisfied.

We consider the following functionals

$$J(t)=\frac{1}{2}\left(\frac{2}{\alpha+1}\|\nabla u\|_{\alpha+1}^{\alpha+1}+\frac{2}{\alpha+1}\|\nabla v\|_{\alpha+1}^{\alpha+1}+m_1^2\|u\|^2+m_2^2\|v\|^2\right)-\int_\Omega G(u,v)dx \tag{2.5}$$

and

$$I(t)=\frac{2}{\alpha+1}\|\nabla u\|_{\alpha+1}^{\alpha+1}+\frac{2}{\alpha+1}\|\nabla v\|_{\alpha+1}^{\alpha+1}+m_1^2\|u\|^2+m_2^2\|v\|^2-(r+1)\int_\Omega G(u,v)dx. \tag{2.6}$$

We define the total energy functional associated with (1.1)-(1.5) as follows:

$$E(t)=\frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2+\frac{2}{\alpha+1}\|\nabla u\|_{\alpha+1}^{\alpha+1}+\frac{2}{\alpha+1}\|\nabla v\|_{\alpha+1}^{\alpha+1}+m_1^2\|u\|^2+m_2^2\|v\|^2\right)-\int_\Omega G(u,v)dx. \tag{2.7}$$

We also denote

$$W=\left\{(u,v):(u,v)\in W_0^{1,\alpha+1}(\Omega)\times W_0^{1,\alpha+1}(\Omega), I(u,v)>0\right\}\cup\{0,0\}. \tag{2.8}$$

Lemma 2.2 *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$E'(t)=-\left(\|u_t\|_{p+1}^{p+1}+\|v_t\|_{q+1}^{q+1}\right)\leq 0. \tag{2.9}$$

Proof Multiplying equation (1.1) by u_t and equation (1.2) by v_t, and integrating over Ω, using integrating by parts and summing up the product results, we obtain

$$E(t)-E(0)=-\int_0^t\left(\|u_\tau\|_{p+1}^{p+1}+\|v_\tau\|_{q+1}^{q+1}\right)d\tau \qquad for \qquad t\geq 0. \tag{2.10}$$

□

Lemma 2.3 *(Sobolev-Poincare Inequality) [1] Let p be a real number with $2 \leq p < \infty (n=1,2)$ and $2\leq p\leq \frac{2n}{n-2}(n\geq 3)$, thus there is a constant $C_*=C_*(\Omega,p)$ such that*

$$\|u\|_p \leq C_*\|\nabla u\|, \qquad \forall u\in H_0^1(\Omega).$$

Lemma 2.4 *(Nakao Inequality) [9] Let $\varphi(t)$ be nonnegative and nonincreasing function defined on $[0,T], T>1$ and suppose that there are constants $w_0>0$ and $m\geq 0$ such that*

$$\varphi^{1+m}(t)\leq w_0(\varphi(t)-\varphi(t+1)), \qquad t\in[0,T].$$

Thus we obtain for all $t \in [0,T]$,

$$\begin{cases} \varphi(t) \leq \varphi(0) e^{-w_1 [t-1]^+}, & m=0, \\ \varphi(t) \leq \left(\varphi(0)^{-m} + w_0^{-1} m [t-1]^+\right)^{\frac{-1}{m}}, & m>0, \end{cases} \tag{2.11}$$

where $[t-1]^+ = max\{t-1,0\}$ *and* $w_1 = \ln\left(\frac{w_0}{w_0-1}\right)$.

Now, we specify the local existence theorem that can be established by combination arguments of [2, 3, 12].

Theorem 2.5 *(Local Existence) Assume that (2.1) holds. Thus, there exist* p,q *satisfying*

$$\begin{cases} 1 \leq p,q, & n \leq 2, \\ 1 \leq p,q \leq \dfrac{n+2}{n-2}, & n>2 \end{cases}$$

and further $(u_0,v_0) \in W_0^{1,\alpha+1}(\Omega) \cap L^{r+1}(\Omega)$, $(u_1,v_1) \in L^2(\Omega) \cap L^2(\Omega)$. *Thus, problem (1.1)-(1.5) has a unique local solution*

$$u,v \in \left(C[0,T); W_0^{1,\alpha+1}(\Omega) \cap L^{r+1}(\Omega)\right),$$

$u_t \in C\left([0,T); L^2(\Omega)\right) \cap L^{p+1}(\Omega \times [0,T))$ *and* $v_t \in C\left([0,T); L^2(\Omega)\right) \cap L^{q+1}(\Omega \times [0,T))$.
Moreover, at least one of the following statements holds true:
(i) $T=\infty$,
(ii) $\|u_t\|^2 + \|v_t\|^2 + \frac{2}{\alpha+1}\|\nabla u\|_{\alpha+1}^{\alpha+1} + \frac{2}{\alpha+1}\|\nabla v\|_{\alpha+1}^{\alpha+1} + m_1^2\|u\|^2 + m_2^2\|v\|^2 \to \infty$ *as* $t \to T^-$.

3. Global existence and decay of solutions

Lemma 3.1 *Assume that (2.1) holds and* $\alpha > 1$ *and* $r > \alpha$ *satisfy*

$$r+1 \leq \frac{n(\alpha+1)}{n-(\alpha+1)}, \qquad \alpha+1 < n. \tag{3.1}$$

Let $(u_0,v_0) \in W$ *and* $(u_1,v_1) \in L^2(\Omega) \times L^2(\Omega)$ *such that*

$$\beta = \frac{c_1 C_*^{r+1}(r+1)(\alpha+1)}{2}\left[\frac{(r+1)(\alpha+1)}{r-1}E(0)\right]^{\frac{r-\alpha}{\alpha+1}} < 1, \tag{3.2}$$

then $(u,v) \in W$, *for all* $t \geq 0$.

Proof Suppose not. Then for some $T_m > 0$, $(u(T_m), v(T_m)) \notin W$. Since $(u(0),v(0)) \in W$ and $I(0)>0$, then by continuity of $u(t)$ and $v(t)$ that

$$I(t) > 0, \tag{3.3}$$

for some interval near $t=0$. Let $T_m > 0$ be a maximal time, when (3.3) holds on $[0,T_m]$. So, for $\forall t \in [0,T_m]$,

$$I(T_m) = 0$$

and

$$I(t) > 0, \qquad \forall 0 \le t \le T_m.$$

According to (2.5) and (2.6), we obtain

$$\begin{aligned} J(t) &= \frac{1}{r+1} I(t) + \frac{r-1}{2(r+1)} \left(\frac{2}{\alpha+1} \|\nabla u\|_{\alpha+1}^{\alpha+1} + \frac{2}{\alpha+1} \|\nabla v\|_{\alpha+1}^{\alpha+1} + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 \right) \\ &\ge \frac{r-1}{2(r+1)} \left(\frac{2}{\alpha+1} \|\nabla u\|_{\alpha+1}^{\alpha+1} + \frac{2}{\alpha+1} \|\nabla v\|_{\alpha+1}^{\alpha+1} + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 \right). \end{aligned} \tag{3.4}$$

By using (3.4), (2.9) and definition of $E(t)$, we have

$$\frac{2}{\alpha+1} \|\nabla u\|_{\alpha+1}^{\alpha+1} + \frac{2}{\alpha+1} \|\nabla v\|_{\alpha+1}^{\alpha+1} \le \frac{2(r+1)}{r-1} J(t) \le \frac{2(r+1)}{r-1} E(t) \le \frac{2(r+1)}{r-1} E(0). \tag{3.5}$$

Hence,

$$\|\nabla u\|_{\alpha+1} + \|\nabla v\|_{\alpha+1} \le \left(\frac{(r+1)(\alpha+1)}{r-1} E(0) \right)^{\frac{1}{\alpha+1}}. \tag{3.6}$$

According to Sobolev embedding inequality, we have

$$\|u\|_{r+1}^{r+1} \le C_*^{r+1} \|\nabla u\|_{\alpha+1}^{r+1} = C_*^{r+1} \|\nabla u\|_{\alpha+1}^{r-\alpha} \|\nabla u\|_{\alpha+1}^{\alpha+1} \tag{3.7}$$

and

$$\|v\|_{r+1}^{r+1} \le C_*^{r+1} \|\nabla v\|_{\alpha+1}^{r+1} = C_*^{r+1} \|\nabla v\|_{\alpha+1}^{r-\alpha} \|\nabla v\|_{\alpha+1}^{\alpha+1}. \tag{3.8}$$

Combining (3.7) and (3.8) with (3.6) implies $\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1} \le C_*^{r+1} \left(\frac{(r+1)(\alpha+1)}{r-1} E(0) \right)^{\frac{r-\alpha}{\alpha+1}} \left(\|\nabla u\|_{\alpha+1}^{\alpha+1} + \|\nabla v\|_{\alpha+1}^{\alpha+1} \right)$.

Applying (3.2) to above inequality with (2.4), we get $I(T_m) > 0$

$$\begin{aligned} (r+1) \int_\Omega G(u,v)dx &\le c_1 (r+1) \left(\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1} \right) \\ &\le \beta \frac{2}{\alpha+1} \left(\|\nabla u\|_{\alpha+1}^{\alpha+1} + \|\nabla v\|_{\alpha+1}^{\alpha+1} \right) \\ &< \frac{2}{\alpha+1} \left(\|\nabla u\|_{\alpha+1}^{\alpha+1} + \|\nabla v\|_{\alpha+1}^{\alpha+1} \right). \end{aligned} \tag{3.9}$$

Consequently, by using (2.6), we deduce that $I(t) > 0$ for all $t \in [0, T_m]$, which contradicts $I(t) = 0$. The lemma's proof is complete. □

Lemma 3.2 *Let the assumptions of Lemma 3.1 hold. Thus, there exists* $\eta_1 = 1 - \beta$ *so that*

$$(r+1) \int_\Omega G(u,v)dx \le (1-\eta_1) \left(\frac{2}{\alpha+1} \|\nabla u\|_{\alpha+1}^{\alpha+1} + \frac{2}{\alpha+1} \|\nabla v\|_{\alpha+1}^{\alpha+1} + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 \right).$$

Proof From (3.9), we obtain

$$
\begin{aligned}
(r+1)\int_{\Omega} G(u,v)dx &\leq \beta\left(\frac{2}{\alpha+1}\|\nabla u\|_{\alpha+1}^{\alpha+1}+\frac{2}{\alpha+1}\|\nabla v\|_{\alpha+1}^{\alpha+1}\right)\\
&\leq \beta\left(\frac{2}{\alpha+1}\|\nabla u\|_{\alpha+1}^{\alpha+1}+\frac{2}{\alpha+1}\|\nabla v\|_{\alpha+1}^{\alpha+1}+m_1^2\|u\|^2+m_2^2\|v\|^2\right).
\end{aligned}
$$

Let $\beta = 1-\eta_1$, then we have the result. □

Remark 3.3 *Hence, we can deduce from Lemma 3.2*

$$
\frac{2}{\alpha+1}\|\nabla u\|_{\alpha+1}^{\alpha+1}+\frac{2}{\alpha+1}\|\nabla v\|_{\alpha+1}^{\alpha+1}+m_1^2\|u\|^2+m_2^2\|v\|^2 \leq \frac{1}{\eta_1}I(t). \tag{3.10}
$$

Theorem 3.4 *Assume that (2.1) holds. Let $(u_0,v_0)\in W$ satisfying (2.8). Thus, the solution of problem (1.1)-(1.5) is global.*

Proof It suffices to show that $\|u_t\|^2+\|v_t\|^2+\frac{2}{\alpha+1}\|\nabla u\|_{\alpha+1}^{\alpha+1}+\frac{2}{\alpha+1}\|\nabla v\|_{\alpha+1}^{\alpha+1}+m_1^2\|u\|^2+m_2^2\|v\|^2$ is bounded independently of t. To indicate this, using (2.6) and (2.7) we have

$$
\begin{aligned}
E(0) &\geq E(t)=\frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{1}{2}\left(\frac{2}{\alpha+1}\|\nabla u\|_{\alpha+1}^{\alpha+1}\right.\\
&\quad \left.+\frac{2}{\alpha+1}\|\nabla v\|_{\alpha+1}^{\alpha+1}+m_1^2\|u\|^2+m_2^2\|v\|^2\right)-\int_{\Omega} G(u,v)dx\\
&= \frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+J(t)\\
&= \frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)+\frac{1}{r+1}I(t)\\
&\quad +\frac{r-1}{2\,(r+1)}\left(\frac{2}{\alpha+1}\|\nabla u\|_{\alpha+1}^{\alpha+1}+\frac{2}{\alpha+1}\|\nabla v\|_{\alpha+1}^{\alpha+1}+m_1^2\|u\|^2+m_2^2\|v\|^2\right)\\
&\geq \frac{1}{2}\left(\|u_t\|^2+\|v_t\|^2\right)\\
&\quad +\frac{r-1}{2\,(r+1)}\left(\frac{2}{\alpha+1}\left(\|\nabla u\|_{\alpha+1}^{\alpha+1}+\|\nabla v\|_{\alpha+1}^{\alpha+1}\right)+m_1^2\|u\|^2+m_2^2\|v\|^2\right)
\end{aligned}
$$

because $I(t)\geq 0$. Therefore,

$$
\|u_t\|^2+\|v_t\|^2+\frac{2}{\alpha+1}\|\nabla u\|_{\alpha+1}^{\alpha+1}+\frac{2}{\alpha+1}\|\nabla v\|_{\alpha+1}^{\alpha+1}+m_1^2\|u\|^2+m_2^2\|v\|^2 \leq CE(0)
$$

where $C=\frac{2(r+1)}{r-1}$. Thus by Theorem 2.5, we get the result of global existence. □

Theorem 3.5 *Assume that (2.1) and (2.8) hold, and further* $(u_0, v_0) \in W$. *Then, we obtain the following decay estimates:*

$$E(t) \leq \begin{cases} E(0)e^{-w_1[t-1]^+}, & p = q = 1 \\ \left(E(0)^{-m} + C_9^{-1}m[t-1]^+\right)^{\frac{-1}{m}}, & p, q > 1 \end{cases}$$

where w_1, m, *and* C_9 *are positive constants.*

Now, we shall derive the decay estimate of the solution in Theorem 3.5 by using Nakao inequality.

Proof By integration of (2.9) over $[t, t+1]$, $t > 0$, we obtain

$$E(t) - E(t+1) = \int_t^{t+1} \left(\|u_\tau(\tau)\|_{p+1}^{p+1} + \|v_\tau(\tau)\|_{q+1}^{q+1}\right) d\tau = D_1^{p+1}(t) + D_2^{q+1}(t) \tag{3.11}$$

where

$$D_1^{p+1}(t) = \int_t^{t+1} \left(\|u_\tau(\tau)\|_{p+1}^{p+1}\right) d\tau \tag{3.12}$$

and

$$D_2^{q+1}(t) = \int_t^{t+1} \left(\|v_\tau(\tau)\|_{q+1}^{q+1}\right) d\tau. \tag{3.13}$$

Hölder inequality and by virtue of (3.12), we observe that

$$\int_t^{t+1}\int_\Omega |u_t|^2 dxdt \leq \int_t^{t+1} |\Omega|^{\frac{p-1}{p+1}} \|u_t\|_{p+1}^2 dt = |\Omega|^{\frac{p-1}{p+1}} D_1^2(t) = CD_1^2(t). \tag{3.14}$$

Similarly, Hölder inequality and due to (3.13), we obtain

$$\int_t^{t+1}\int_\Omega |v_t|^2 dxdt \leq |\Omega|^{\frac{q-1}{q+1}} D_2^2(t) = CD_2^2(t). \tag{3.15}$$

Hence, from (3.14) and (3.15), there exist $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$\|u_t(t_i)\| \leq CD_1(t), \qquad i = 1, 2 \tag{3.16}$$

and

$$\|v_t(t_i)\| \leq CD_2(t), \qquad i = 1, 2. \tag{3.17}$$

By multiplying (1.1) and (1.2) by u and v, respectively, and integrating it over $\Omega \times [t_1, t_2]$, we have

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt \quad &\leq \quad -\int_{t_1}^{t_2}\int_{\Omega}[uu_{tt} + vv_{tt}]dxdt \\ &\quad -\int_{t_1}^{t_2}\int_{\Omega}[|u_t|^{p-1}u_t u]dxdt - \int_{t_1}^{t_2}\int_{\Omega}[|v_t|^{q-1}v_t v]dxdt. \end{aligned} \tag{3.18}$$

To estimate of the first term of the right-hand side of (3.18), by using (1.1)-(1.5), integrating by parts and Cauchy–Schwarz inequality, we get

$$\begin{aligned}\int_{t_1}^{t_2} I(t)dt \quad &\leq \quad \|u_t(t_1)\| \|u(t_1)\| + \|u_t(t_2)\| \|u(t_2)\| \\ &\quad + \|v_t(t_1)\| \|v(t_1)\| + \|v_t(t_2)\| \|v(t_2)\| \\ &\quad + \int_{t_1}^{t_2} \|u_t\|^2 dt + \int_{t_1}^{t_2} \|v_t\|^2 dt \\ &\quad -\int_{t_1}^{t_2}\int_{\Omega}[|u_t|^{p-1}u_t u]dxdt - \int_{t_1}^{t_2}\int_{\Omega}[|v_t|^{q-1}v_t v]dxdt. \end{aligned} \tag{3.19}$$

Now, our purpose is to estimate the right hand side of the inequality. First, we will estimate the last two terms in the right-hand side of inequality (3.19). By applying Hölder inequality, we get

$$\int_{t_1}^{t_2}\int_{\Omega}[|u_t|^{p-1}u_t u]dxdt \leq \int_{t_1}^{t_2}\left[\|u_t(t)\|_{p+1}^{p} \|u(t)\|_{p+1}\right]dt \tag{3.20}$$

and

$$\int_{t_1}^{t_2}\int_{\Omega}[|v_t|^{q-1}v_t v]dxdt \leq \int_{t_1}^{t_2}\left[\|v_t(t)\|_{q+1}^{q} \|v(t)\|_{q+1}\right]dt. \tag{3.21}$$

According to (3.5) and Sobolev–Poincare inequality, we obtain for $p \geq 1$

$$\begin{aligned}\int_{t_1}^{t_2}\left[\|u_t(t)\|_{p+1}^p\|u(t)\|_{p+1}\right]dt &\le C_*\int_{t_1}^{t_2}\left[\|u_t(t)\|_{p+1}^p\|\nabla u\|\right]dt\\ &\le C_*\left(\frac{2(r+1)}{r-1}\right)^{\frac12}\int_{t_1}^{t_2}\left[\|u_t(t)\|_{p+1}^p E^{\frac12}(s)\right]dt\\ &\le C_*\left(\frac{2(r+1)}{r-1}\right)^{\frac12}\sup_{t_1\le s\le t_2}E^{\frac12}(s)\int_{t_1}^{t_2}\left[\|u_t\|_{p+1}^p\right]dt\\ &\le C_*\sqrt{\frac{2(r+1)}{r-1}}\sup_{t_1\le s\le t_2}E^{\frac12}(s)D_1^p(t). \end{aligned}\tag{3.22}$$

Similarly, we obtain for $q\ge1$

$$\int_{t_1}^{t_2}\left[\|v_t(t)\|_{q+1}^q\|v(t)\|_{q+1}\right]dt\le C_*\sqrt{\frac{2(r+1)}{r-1}}\sup_{t_1\le s\le t_2}E^{\frac12}(s)D_2^q(t).\tag{3.23}$$

Now, from (3.5), (3.16), and Sobolev–Poincare inequality, we get

$$\|u_t(t_i)\|\,\|u(t_i)\|\le C_1D_1(t)\sup_{t_1\le s\le t_2}E^{\frac12}(s),\tag{3.24}$$

where $C_1=2C_*\sqrt{\frac{2(r+1)}{r-1}}C$. Similary, from (3.5), (3.17), and Sobolev–Poincare inequality, we obtain

$$\|v_t(t_i)\|\,\|v(t_i)\|\le C_2D_2(t)\sup_{t_1\le s\le t_2}E^{\frac12}(s),\tag{3.25}$$

where $C_2=2C_*\sqrt{\frac{2(r+1)}{r-1}}C$. Substitute (3.20)-(3.25) into (3.19) by (3.14) and (3.15), we obtain

$$\begin{aligned}\int_{t_1}^{t_2}I(t)dt \le\; & C_3\Bigg\{\sup_{t_1\le s\le t_2}E^{\frac12}(s)\left(D_1(t)+D_2(t)\right)+D_1^2(t)+D_2^2(t)\\ &+C_*\sqrt{\frac{2(r+1)}{r-1)}}\sup_{t_1\le s\le t_2}E^{\frac12}(s)\left(D_1^p(t)+D_2^q(t)\right)\Bigg\},\end{aligned}\tag{3.26}$$

where $C_3=\max\{C_1,C_2,C,1\}$. Morever, from definition of $E(t)$, $I(t)$ and Remark 3.3, we get

$$E(t)\le\frac12\left(\|u_t\|^2+\|v_t\|^2\right)+C_4I(t),\tag{3.27}$$

where $C_4=\frac{1}{\eta_1}\frac{r-1}{2(r+1)}+\frac{1}{r+1}$. By integrating (3.27) over $[t_1,t_2]$, we get

$$\int_{t_1}^{t_2}E(t)dt\le\frac12\int_{t_1}^{t_2}\left(\|u_t\|^2+\|v_t\|^2\right)dt+C_4\int_{t_1}^{t_2}I(t)dt.$$

Hence, by (3.14), (3.15), and (3.26), we have

$$\begin{aligned}\int_{t_1}^{t_2} E(t)dt &\le \frac{1}{2}C\left(D_1^2(t)+D_2^2(t)\right)\\ &+C_4C_3\left\{\sup_{t_1\le s\le t_2} E^{\frac{1}{2}}(s)\left(D_1(t)+D_2(t)\right)+D_1^2(t)+D_2^2(t)\right.\\ &\left.+C_*\sqrt{\frac{2(r+1)}{r-1}}\sup_{t_1\le s\le t_2} E^{\frac{1}{2}}(s)\left(D_1^p(t)+D_2^q(t)\right)\right\}. \end{aligned} \tag{3.28}$$

Now, by integrating $\frac{d}{dt}E(t)$ over $[t,t_2]$, we have

$$E(t)=E(t_2)+\int_t^{t_2}\left(\|u_\tau(\tau)\|_{p+1}^{p+1}+\|v_\tau(\tau)\|_{q+1}^{q+1}\right)d\tau. \tag{3.29}$$

Therefore, since $t_2-t_1\ge\frac{1}{2}$, we deduce that

$$\int_{t_1}^{t_2} E(t)dt \ge (t_2-t_1)E(t_2)\ge \frac{1}{2}E(t_2).$$

That is,

$$E(t_2)\le 2\int_{t_1}^{t_2} E(t)dt. \tag{3.30}$$

Therefore, exploiting (3.11), (3.29), (3.30) and because $t_1,t_2\in[t,t+1]$, we obtain

$$\begin{aligned}E(t) &\le 2\int_{t_1}^{t_2} E(t)dt+\int_t^{t+1}\left(\|u_\tau(\tau)\|_{p+1}^{p+1}+\|v_\tau(\tau)\|_{q+1}^{q+1}\right)d\tau\\ &= 2\int_{t_1}^{t_2} E(t)dt+D_1^{p+1}(t)+D_2^{q+1}(t).\end{aligned} \tag{3.31}$$

Then, from (3.28), we obtain

$$\begin{aligned}E(t)&\le (C+2C_4C_3)\left(D_1^2(t)+D_2^2(t)\right)+D_1^{p+1}(t)+D_2^{q+1}(t)\\ &+C_5E^{\frac{1}{2}}(t)\left(D_1(t)+D_2(t)+D_1^p(t)+D_2^q(t)\right),\end{aligned} \tag{3.32}$$

where $C_5=2C_4C_3\max\left(1,C_*\sqrt{\frac{2(r+1)}{r-1}}\right)$.

Hence, by arithmetic-geometric mean inequality, we deduce that

$$E(t)\le C_6\left[D_1^2(t)+D_2^2(t)+D_1^{p+1}(t)+D_2^{q+1}(t)+D_1^{2p}(t)+D_2^{2q}(t)\right], \tag{3.33}$$

where $C_6 = max(2C + 4C_4C_3 + C_5^2, 2, C_5^2)$. Now we distinguish two cases.
Case 1: When $p = q = 1$, we get from (3.33)

$$E(t) \leq 3C_6 \left[D_1^2(t) + D_2^2(t)\right] = 3C_6 \left[E(t) - E(t+1)\right]. \tag{3.34}$$

By Lemma 2.4, we have

$$E(t) \leq E(0)e^{-w_1[t-1]^+}, \tag{3.35}$$

where $[t-1]^+ = max\{t-1, 0\}$ and $w_1 = \ln\left(\frac{3C_6}{3C_6-1}\right)$.
Case 2: When $p, q > 1$, we get from (3.33)

$$\begin{aligned} E(t) &\leq C_6 D_1^2(t)\left[1 + D_1^{p-1}(t) + D_1^{2(p-1)}(t)\right] + C_6 D_2^2(t)\left[1 + D_2^{q-1}(t) + D_2^{2(q-1)}(t)\right] \\ &\leq C_6\left[1 + D_1^{p-1}(t) + D_1^{2(p-1)}(t) + D_2^{q-1}(t) + D_2^{2(q-1)}(t)\right]\left(D_1^2(t) + D_2^2(t)\right). \end{aligned} \tag{3.36}$$

Thus since $E(t) \leq E(0)$ for $\forall t \geq 0$, we obtain from (3.11)

$$\begin{aligned} E(t) &\leq C_6\left[1 + D_1^{p-1}(t) + D_1^{2(p-1)}(t) + D_2^{q-1}(t) + D_2^{2(q-1)}(t)\right]\left(D_1^2(t) + D_2^2(t)\right) \\ &\leq C_6\left[1 + E^{\frac{p-1}{p+1}}(0) + E^{\frac{q-1}{q+1}}(0) + E^{\frac{2(p-1)}{p+1}}(0) + E^{\frac{2(q-1)}{q+1}}(0)\right]\left(D_1^2(t) + D_2^2(t)\right) \\ &\leq C_7\left(D_1^2(t) + D_2^2(t)\right), \qquad t \geq 0, \end{aligned} \tag{3.37}$$

where $C_7 = C_6\left[1 + E^{\frac{p-1}{p+1}}(0) + E^{\frac{q-1}{q+1}}(0) + E^{\frac{2(p-1)}{p+1}}(0) + E^{\frac{2(q-1)}{q+1}}(0)\right]$. When we take $m = \max\left\{\frac{p-1}{2}, \frac{q-1}{2}\right\}$; then we get

$$\begin{aligned} E(t)^{1+m} &\leq \left[C_7\left(D_1^2(t) + D_2^2(t)\right)\right]^{1+m} \\ &= C_7^{1+m}\left(D_1^{2+2m}(t) + D_2^{2+2m}(t)\right) \\ &= C_8\left(D_1^{2+2m}(t) + D_2^{2+2m}(t)\right), \end{aligned} \tag{3.38}$$

where $C_8 = C_7^{1+m}$. Consequently, (3.38) is equal to

$$\begin{aligned} E(t)^{1+m} &\leq C_8\left(D_1^{p+1}(t)D_1^{2m-p+1}(t) + D_2^{q+1}(t)D_2^{2m-q+1}(t)\right) \\ &\leq C_8\left(D_1^{p+1}(t)E^{\frac{2m-p+1}{p+1}}(0) + D_2^{q+1}(t)E^{\frac{2m-q+1}{q+1}}(0)\right) \\ &\leq C_9\left(D_1^{p+1}(t) + D_2^{q+1}(t)\right) \\ &= C_9\left(E(t) - E(t+1)\right), \end{aligned} \tag{3.39}$$

where $C_9 = C_8 \max\left\{E^{\frac{2m-p+1}{p+1}}(0), E^{\frac{2m-q+1}{q+1}}(0)\right\}$.

Thus, from Lemma 2.4 ve (3.39), we have for $t \in [0, T]$ and $m > 0$

$$E(t) \leq \left(E(0)^{-m} + C_9^{-1}m[t-1]^+\right)^{\frac{-1}{m}}.$$

This completes the proof of Theorem 12. □

4. Blow up of solutions

Theorem 4.1 *Suppose that* $r+1 > \max\{p+1, q+1\}$*,the initial energy* $E(0) < 0$ *and* $\alpha < r$*. If so, the solution for this system blows up in finite time* T^* *where* $T^* \leq \frac{1-\sigma}{\xi\sigma\psi^{\frac{\sigma}{1-\sigma}}(0)}$*.* $\psi(t)$ *and* σ *are given (4.1) and (4.2), respectively.*

Proof We assume that the solution exists for all the time, we arrive at a contradiction. Define $H(t) = -E(t)$, $E(0) < 0$ and (2.9) gives $0 < H(0) \leq H(t)$. Denote

$$\psi(t) = H^{1-\sigma}(t) + \varepsilon\left(\int_\Omega uu_t dx + \int_\Omega vv_t dx\right), \tag{4.1}$$

where ε is a positive and small constant to be determined, and

$$0 < \sigma \leq \min\left\{\frac{r-p}{(r+1)p}, \frac{r-q}{(r+1)q}, \frac{r-1}{2(r+1)}\right\}. \tag{4.2}$$

Our aim is to show that $\psi(t)$ satisfies a differential inequality of the following form

$$\psi'(t) \geq \xi\psi^\zeta(t), \qquad \zeta > 1.$$

This will result in a blow up in finite time. By differentiation of (4.1), we have

$$\psi'(t) = (1-\sigma)H^{-\sigma}(t)H'(t)$$

$$+\varepsilon\left(\int_\Omega u_t u_t dx + \int_\Omega v_t v_t dx\right) + \varepsilon\left(\int_\Omega uu_{tt}dx + \int_\Omega vv_{tt}dx\right). \tag{4.3}$$

By multiplying (1.1) by u and (1.2) by v, respectively, and integrating it over $\Omega \times [t_1, t_2]$, by (2.2) and (4.3), we obtain

$$\begin{aligned}\psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\left(\|u_t\|^2 + \|v_t\|^2\right) - \varepsilon\left(\|\nabla u\|_{\alpha+1}^{\alpha+1} + \|\nabla v\|_{\alpha+1}^{\alpha+1}\right) \\ &\quad -\varepsilon\left(m_1^2\|u\|^2 + m_2^2\|v\|^2\right) - \varepsilon\left(\int_\Omega uu_t|u_t|^{p-1}dx + \int_\Omega vv_t|v_t|^{q-1}dx\right) \\ &\quad +\varepsilon(r+1)\int_\Omega G(u,v)dx. \end{aligned} \tag{4.4}$$

From definition of $H(t)$, we obtain

$$\begin{aligned}-\varepsilon\left(\|\nabla u\|_{\alpha+1}^{\alpha+1} + \|\nabla v\|_{\alpha+1}^{\alpha+1}\right) &= \varepsilon(\alpha+1)H(t) - \varepsilon(\alpha+1)\int_\Omega G(u,v)dx \\ &\quad +\varepsilon\left(\frac{\alpha+1}{2}\right)\left(\|u_t\|^2 + \|v_t\|^2\right) + \varepsilon\left(\frac{\alpha+1}{2}\right)\left(m_1^2\|u\|^2 + m_2^2\|v\|^2\right). \end{aligned} \tag{4.5}$$

Substitute (4.5) into (4.4) to get

$$\begin{aligned}\psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\left(\frac{\alpha+3}{2}\right)\left(\|u_t\|^2+\|v_t\|^2\right)+\varepsilon\left(\alpha+1\right)H(t)\\ &+\varepsilon(r-\alpha)\int_\Omega G(u,v)dx+\varepsilon\left(\frac{\alpha-1}{2}\right)(m_1^2\|u\|^2+m_2^2\|v\|^2)\\ &-\varepsilon\left(\int_\Omega uu_t|u_t|^{p-1}dx+\int_\Omega vv_t|v_t|^{q-1}dx\right).\end{aligned} \tag{4.6}$$

Now, we use of the following Young's inequality to estimate the last term in (4.6)

$$xy\leq\frac{\delta^j x^j}{j}+\frac{\delta^{-k}y^k}{k}$$

where $x,y\geq 0,\ \delta>0,\ j,k\in R^+$ such that $\frac{1}{j}+\frac{1}{k}=1$. Therefore, applying the previous inequality and from $H'(t)=\|u_t\|_{p+1}^{p+1}+\|v_t\|_{q+1}^{q+1}$, we have

$$\begin{aligned}\int_\Omega uu_t|u_t|^{p-1}dx &\leq \frac{\delta_1^{p+1}}{p+1}\|u\|_{p+1}^{p+1}+\frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}\|u_t\|_{p+1}^{p+1}\\ &\leq \frac{\delta_1^{p+1}}{p+1}\|u\|_{p+1}^{p+1}+\frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}H'(t)\end{aligned}$$

and

$$\begin{aligned}\int_\Omega vv_t|v_t|^{q-1}dx &\leq \frac{\delta_2^{q+1}}{q+1}\|v\|_{q+1}^{q+1}+\frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\|v_t\|_{q+1}^{q+1}\\ &\leq \frac{\delta_2^{q+1}}{q+1}\|v\|_{q+1}^{q+1}+\frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}H'(t),\end{aligned}$$

where δ_1 and δ_2 are real numbers depending on the time t. Consequently, we obtain from (4.6)

$$\begin{aligned}\psi'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\left(\frac{\alpha+3}{2}\right)\left(\|u_t\|^2+\|v_t\|^2\right)+\varepsilon\left(\alpha+1\right)H(t)\\ &+\varepsilon(r-\alpha)\int_\Omega G(u,v)dx+\varepsilon\left(\frac{\alpha-1}{2}\right)(m_1^2\|u\|^2+m_2^2\|v\|^2)\\ &-\varepsilon\left(\frac{\delta_1^{p+1}}{p+1}\|u\|_{p+1}^{p+1}+\frac{\delta_2^{q+1}}{q+1}\|v\|_{q+1}^{q+1}\right)-\varepsilon\left(\frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}+\frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\right)H'(t).\end{aligned} \tag{4.7}$$

Therefore, by taking δ_1 and δ_2 so that $\delta_1^{-\frac{p+1}{p}} = n_1 H^{-\sigma}(t), \delta_2^{-\frac{q+1}{q}} = n_2 H^{-\sigma}(t)$, where $n_1, n_2 > 0$ are specified later, we have

$$\delta_1^{p+1} = n_1{}^{-p} H^{\sigma p}(t) \leq n_1{}^{-p} c_1{}^{\sigma p} (\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1})^{\sigma p} \tag{4.8}$$

and

$$\delta_2^{q+1} = n_2{}^{-q} H^{\sigma q}(t) \leq n_2{}^{-q} c_1{}^{\sigma q} (\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1})^{\sigma q}, \tag{4.9}$$

because $H(t) = -E(t) \leq \int_\Omega G(u,v)dx \leq c_1 \left(\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1} \right)$. Substituting (4.8) and (4.9) into (4.7), we get

$$\begin{aligned}
\psi'(t) \quad \geq \quad & \left(1 - \sigma - \frac{\varepsilon p n_1}{p+1} - \frac{\varepsilon q n_2}{q+1}\right) H^{-\sigma}(t) H'(t) + \varepsilon \left(\frac{\alpha+3}{2}\right) \left(\|u_t\|^2 + \|v_t\|^2\right) \\
& + \varepsilon\,(\alpha+1)\, H(t) + \varepsilon (r-\alpha) \int_\Omega G(u,v) dx + \varepsilon \left(\frac{\alpha-1}{2}\right) (m_1^2 \|u\|^2 + m_2^2 \|v\|^2) \\
& - \varepsilon \left(\frac{n_1{}^{-p} c_1{}^{\sigma p}}{p+1}\right) (\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1})^{\sigma p} \|u\|_{p+1}^{p+1} \\
& - \varepsilon \left(\frac{n_2{}^{-q} c_1{}^{\sigma q}}{q+1}\right) (\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1})^{\sigma q} \|v\|_{q+1}^{q+1}.
\end{aligned} \tag{4.10}$$

Since $L^{r+1}(\Omega) \hookrightarrow L^{p+1}(\Omega), L^{r+1}(\Omega) \hookrightarrow L^{q+1}(\Omega)$, we have

$$\|u\|_{p+1}^{p+1} \leq C \|u\|_{r+1}^{p+1}, \qquad \|v\|_{q+1}^{q+1} \leq C \|v\|_{r+1}^{q+1}.$$

Thus

$$(\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1})^{\sigma p} \|u\|_{p+1}^{p+1} \leq C_{10} (\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1})^{\sigma p + \frac{p+1}{r+1}} \tag{4.11}$$

and

$$(\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1})^{\sigma q} \|v\|_{q+1}^{q+1} \leq C_{11} (\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1})^{\sigma q + \frac{q+1}{r+1}}. \tag{4.12}$$

Using (4.2) and the following inequality[4]:
$z^v \leq z + 1 \leq \left(1 + \frac{1}{\omega}\right)(z + \omega),\ \forall z \geq 0,\ 0 < v \leq 1,\ \omega > 0$, we obtain, for $t \geq 0$,

$$\begin{aligned}
\left(\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1}\right)^{\sigma p + \frac{p+1}{r+1}} \quad & \leq \quad d\left(\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1} + H(0)\right) \\
& \leq \quad d\left(\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1} + H(t)\right)
\end{aligned} \tag{4.13}$$

and

$$\left(\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1}\right)^{\sigma q + \frac{q+1}{r+1}} \leq d\left(\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1} + H(t)\right) \tag{4.14}$$

for $\omega = H(0)$ and $d = 1 + \frac{1}{H(0)}$. Substituting (4.11)-(4.14) into (4.10), by (2.4) we have

$$\begin{aligned}\psi'(t) \geq\ & \left(1-\sigma-\frac{\varepsilon p n_1}{p+1}-\frac{\varepsilon q n_2}{q+1}\right)H^{-\sigma}(t)H'(t)+\varepsilon\left(\frac{\alpha+3}{2}\right)\left(\|u_t\|^2+\|v_t\|^2\right)\\ &+\varepsilon\left(\alpha+1-\frac{{n_1}^{-p}{c_1}^{\sigma p}C_{10}d}{p+1}-\frac{{n_2}^{-q}{c_1}^{\sigma q}C_{11}d}{q+1}\right)H(t)+\varepsilon\left(\frac{\alpha-1}{2}\right)(m_1^2\|u\|^2+m_2^2\|v\|^2)\\ &+\varepsilon\left(c_0(r-\alpha)-\frac{{n_1}^{-p}{c_1}^{\sigma p}C_{10}d}{p+1}-\frac{{n_2}^{-q}{c_1}^{\sigma q}C_{11}d}{q+1}\right)\left(\|u\|_{r+1}^{r+1}+\|v\|_{r+1}^{r+1}\right).\end{aligned} \tag{4.15}$$

We choose n_1, n_2 large enough so that

$$c_0(r-\alpha)-\frac{{n_1}^{-p}{c_1}^{\sigma p}C_{10}d}{p+1}-\frac{{n_2}^{-q}{c_1}^{\sigma q}C_{11}d}{q+1} \geq \frac{c_0(r-\alpha)}{2}$$

and

$$\alpha+1-\frac{{n_1}^{-p}{c_1}^{\sigma p}C_{10}d}{p+1}-\frac{{n_2}^{-q}{c_1}^{\sigma q}C_{11}d}{q+1} \geq \frac{\alpha+1}{2}.$$

Choose ε small enough so that $1-\sigma-\frac{\varepsilon p n_1}{p+1}-\frac{\varepsilon q n_2}{q+1} \geq 0$. Then we get

$$\begin{aligned}\psi'(t) \geq\ & \varepsilon\left(\frac{\alpha+3}{2}\right)\left(\|u_t\|^2+\|v_t\|^2\right)+\varepsilon\left(\frac{\alpha+1}{2}\right)H(t)\\ &+\varepsilon\left(\frac{\alpha-1}{2}\right)(m_1^2\|u\|^2+m_2^2\|v\|^2)+\varepsilon\left(\frac{c_0(r-\alpha)}{2}\right)(\|u\|_{r+1}^{r+1}+\|v\|_{r+1}^{r+1})\\ \geq\ & \eta\left(\|u_t\|^2+\|v_t\|^2+H(t)+m_1^2\|u\|^2+m_2^2\|v\|^2+\|u\|_{r+1}^{r+1}+\|v\|_{r+1}^{r+1}\right),\end{aligned} \tag{4.16}$$

where $\eta = \min\left\{\varepsilon\frac{(\alpha+3)}{2}, \varepsilon\frac{(\alpha+1)}{2}, \varepsilon\frac{(\alpha-1)}{2}, \varepsilon\frac{c_0(r-\alpha)}{2}\right\}$. Consequently, we have

$$\psi(t) \geq \psi(0) = H^{1-\sigma}(0)+\varepsilon\left(\int_\Omega u_0u_1dx+\int_\Omega v_0v_1dx\right) > 0, \qquad \forall t \geq 0. \tag{4.17}$$

Next we estimate $\psi^{\frac{1}{1-\sigma}}(t)$. We have

$$\begin{aligned}\psi^{\frac{1}{1-\sigma}}(t) &= \left[H^{1-\sigma}(t)+\varepsilon\left(\int_\Omega uu_tdx+\int_\Omega vv_tdx\right)\right]^{\frac{1}{1-\sigma}}\\ &\leq 2^{\frac{\sigma}{1-\sigma}}\left[H(t)+\varepsilon^{\frac{1}{1-\sigma}}\left(\int_\Omega uu_tdx+\int_\Omega vv_tdx\right)^{\frac{1}{1-\sigma}}\right].\end{aligned} \tag{4.18}$$

By Hölder's inequality, the Sobolev embedding theorem $L^{r+1}(\Omega) \hookrightarrow L^2(\Omega)$, and Young's inequality, we have

$$
\begin{aligned}
\left| \int_\Omega u u_t dx + \int_\Omega v v_t dx \right|^{\frac{1}{1-\sigma}} &\leq C\left(\|u\|^{\frac{1}{1-\sigma}} \|u_t\|^{\frac{1}{1-\sigma}} + \|v\|^{\frac{1}{1-\sigma}} \|v_t\|^{\frac{1}{1-\sigma}} \right) \\
&\leq C\left(\|u\|_{r+1}^{\frac{1}{1-\sigma}} \|u_t\|^{\frac{1}{1-\sigma}} + \|v\|_{(r+1)}^{\frac{1}{1-\sigma}} \|v_t\|^{\frac{1}{1-\sigma}} \right) \\
&\leq C\left(\|u\|_{r+1}^{\frac{\mu}{1-\sigma}} \|u_t\|^{\frac{\lambda}{1-\sigma}} + \|v\|_{r+1}^{\frac{\mu}{1-\sigma}} \|v_t\|^{\frac{\lambda}{1-\sigma}} \right), \qquad (4.19)
\end{aligned}
$$

where $\frac{1}{\mu} + \frac{1}{\lambda} = 1$. We get $\lambda = 2(1-\sigma)$, to obtain $\mu = \frac{2(1-\sigma)}{1-2\sigma} \leq r+1$ by (4.2). Hence, (4.19) comes

$$
\left| \int_\Omega u u_t dx + \int_\Omega v v_t dx \right|^{\frac{1}{1-\sigma}} \leq C\left(\|u_t\|^2 + \|v_t\|^2 + \|u\|_{r+1}^{\frac{2}{1-2\sigma}} + \|v\|_{r+1}^{\frac{2}{1-2\sigma}} \right). \qquad (4.20)
$$

From (4.2), since $\frac{2}{1-2\sigma} \leq r+1$, furthermore, we have

$$
\|u\|_{r+1}^{\frac{2}{1-2\sigma}} = \left(\|u\|_{r+1}^{r+1} \right)^{\frac{2}{(1-2\sigma)(r+1)}} \leq d\left(\|u\|_{r+1}^{r+1} + H(t) \right),
$$

$$
\|v\|_{r+1}^{\frac{2}{1-2\sigma}} = \left(\|v\|_{r+1}^{r+1} \right)^{\frac{2}{(1-2\sigma)(r+1)}} \leq d\left(\|v\|_{r+1}^{r+1} + H(t) \right)
$$

and

$$
\begin{aligned}
\left| \int_\Omega u u_t dx + \int_\Omega v v_t dx \right|^{\frac{1}{1-\sigma}} &\leq C\left(\|u_t\|^2 + \|v_t\|^2 + \|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1} + H(t) \right) \\
&\leq C\left(\|u_t\|^2 + \|v_t\|^2 + H(t) + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 + \|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1} \right). \qquad (4.21)
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\psi^{\frac{1}{1-\sigma}}(t) &\leq 2^{\frac{\sigma}{1-\sigma}} \Big[H(t) + \varepsilon^{\frac{1}{1-\sigma}} C\left(\|u_t\|^2 + \|v_t\|^2 + H(t) \right) \\
&\quad + \varepsilon^{\frac{1}{1-\sigma}} C\left(m_1^2 \|u\|^2 + m_2^2 \|v\|^2 + \|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1} \right) \Big] \\
&\leq C_* \left(\|u_t\|^2 + \|v_t\|^2 + H(t) + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 + \|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1} \right), \qquad (4.22)
\end{aligned}
$$

where $C_* = 2^{\frac{\sigma}{1-\sigma}}(1 + \varepsilon^{\frac{1}{1-\sigma}} C)$.

A combination of (4.16) and (4.22), we conclude that

$$
\psi'(t) \geq \xi \psi^{\frac{1}{1-\sigma}}(t), \qquad \frac{1}{1-\sigma} > 1, \qquad (4.23)
$$

where ξ is some positive constant. A simple integration of (4.23) yields

$$\psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\xi\sigma t}{1-\sigma}}.$$

Thus the solution of $H(t)$ blows up in a finite time T^*, with

$$T^* \leq \frac{1-\sigma}{\xi\sigma\psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

□

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