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Energy decay and blow-up of solutions for a class of system of generalized nonlinear Klein-Gordon equations with source and damping terms

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Abstract: In this work, we investigate generalized coupled nonlinear Klein-Gordon equations with nonlinear damping and source terms and initial-boundary value conditions, in a bounded domain. We obtain decay of solutions by use of Nakao inequality. The blow up of solutions with negative initial energy is also established.

Key words: Decay, blow up, generalized Klein-Gordon equation

1. Introduction

In this paper, we study the initial-boundary value problem for the following coupled nonlinear generalized Klein–Gordon equations with nonlinear damping terms and source terms

\[
\begin{align*}
    u_{tt} - \text{div}(|\nabla u|^{\alpha-1}\nabla u) + m_1^2 u + |u_t|^{p-1} u_t &= g_1(u,v), \quad (x,t) \in \Omega \times (0,T), \\
    v_{tt} - \text{div}(|\nabla v|^{\alpha-1}\nabla v) + m_2^2 v + |v_t|^{q-1} v_t &= g_2(u,v), \quad (x,t) \in \Omega \times (0,T), \\
    u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\
    v(x,0) &= v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega, \\
    u(x,t) = v(x,t) &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^n (n = 1, 2, 3) \), with smooth boundary \( \partial \Omega \), \( p, q \geq 1, \alpha \geq 1 \) and \( m_1, m_2 > 0 \) are real numbers.

There are many results on the Cauchy problem for a class of the system Klein-Gordon equations \([10, 11, 13, 17]\). For instance, Segal\([14]\) first proposed the following nonlinear system of Klein-Gordon equations

\[
\begin{align*}
    u_{tt} - \Delta u + m_1^2 u + g_1 u^2 v &= 0, \\
    v_{tt} - \Delta v + m_2^2 v + g_2 u v^2 &= 0, \quad (x,t) \in \Omega \times (0,T), \\
    u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x), \\
    v(x,0) &= v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega, \\
    u(x,t) = v(x,t) &= 0, \quad x \in \partial \Omega.
\end{align*}
\]

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where \( m_1 \) and \( m_2 \) are nonzero constants, which define the movement of charged mesons in an electromagnetic field. I. Segal discussed the problem (1.6) of the global existence of the Cauchy problem with \( g_1 > 0, g_2 > 0 \). Blow up of solutions of (1.6) with \( g_1 < 0, g_2 < 0 \) was first established in [6, 7].

In the case of \( \alpha = 1 \), the problem (1.1)-(1.5) becomes to the following form

\[
\begin{cases}
  u_{tt} - \Delta u + m_1^2 u + |u_t|^{p-1}u_t = g_1(u, v), \\
  v_{tt} - \Delta v + m_2^2 v + |v_t|^{q-1}v_t = g_2(u, v).
\end{cases}
\]  

Pişkin [13] proved the uniform decay of solutions by using Nakao’s inequality and blow-up solutions in finite time with negative initial energy of the system (1.7). In addition, Ye [17] proved the global existence by using the potential well method and asymptotic stability by use of Komornik’s lemma [5] of the system (1.7) with \( p = q \). Wu [15] also discussed the blow-up of global solutions under some conditions for a system of (1.7).

When \( p = q = 1 \), Wu [16] studied the global existence, nonexistence, and asymptotic behavior of solutions for the system (1.7). When \( m_1 = m_2 = 0 \), Agre and Rammaha [2] proved the global existence and the nonexistence of solutions for the system (1.7) by applying the same techniques as in [3].

In this paper, the global existence of solution of the problem (1.1)-(1.5) was proved, and decay rates of energy which decays exponentially for \( p = q = 1 \) and polynomially for \( p, q > 1 \), were established by the use of Nakao’s inequality [9]. The blow-up result for solutions with negative initial energy was established for \( r > \max \{p, q\} \) by applying the technique of [3].

2. Preliminaries

In this section, we present some assumptions and lemmas, in the proof of our main result. We shall write \( \| \cdot \| \) and \( \| \cdot \|_p \) to define the usual \( L^2(\Omega) \) norm and \( L^p(\Omega) \) norm, respectively. There exists a function \( G(u, v) \) such that \( \frac{\partial G}{\partial u} = g_1(u, v), \frac{\partial G}{\partial v} = g_2(u, v) \).

Concerning the functions \( g_1(u, v) \) and \( g_2(u, v) \), we take

\[
g_1(u, v) = (r + 1)[a|u + v|^{r-1}(u + v) + b|u|^{\frac{r+1}{r}}|v|^{\frac{r+1}{r}}],
\]

\[
g_2(u, v) = (r + 1)[a|u + v|^{r-1}(u + v) + b|u|^{\frac{r+1}{r}}|v|^{\frac{r+1}{r}}],
\]

where \( a, b > 0 \) real numbers and \( r \) satisfies

\[
\begin{cases}
  1 < r , & n \leq 2, \\
  1 < r \leq \frac{(n + 2)}{(n - 2)}, & n > 2.
\end{cases}
\]  

In accordance with the above equalities, it can easily verify that

\[
u g_1(u, v) + v g_2(u, v) = (r + 1)G(u, v), \quad \forall (u, v) \in R^2,
\]

\[
G(u, v) = [a|u + v|^{r+1} + 2b|uv|^{\frac{r+1}{r}}].
\]
Lemma 2.1 [8] There exist two positive constants $c_0$ and $c_1$ such that
\[
c_0(|u|^{r+1} + |v|^{r+1}) \leq G(u, v) \leq c_1(|u|^{r+1} + |v|^{r+1})
\] (2.4)
is satisfied.

We consider the following functionals
\[
J(t) = \frac{1}{2} \left( \frac{2}{\alpha + 1} \|\nabla u\|_{\alpha+1}^{\alpha+1} + \frac{2}{\alpha + 1} \|\nabla v\|_{\alpha+1}^{\alpha+1} + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 \right) - \int_\Omega G(u, v) \, dx
\]
(2.5)
and
\[
I(t) = \frac{2}{\alpha + 1} \|\nabla u\|_{\alpha+1}^{\alpha+1} + \frac{2}{\alpha + 1} \|\nabla v\|_{\alpha+1}^{\alpha+1} + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 - (r + 1) \int_\Omega G(u, v) \, dx.
\]
(2.6)

We define the total energy functional associated with (1.1)-(1.5) as follows:
\[
E(t) = \frac{1}{2} \left( \|u_t\|^2 + \|v_t\|^2 + \frac{2}{\alpha + 1} \|\nabla u\|_{\alpha+1}^{\alpha+1} + \frac{2}{\alpha + 1} \|\nabla v\|_{\alpha+1}^{\alpha+1} + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 \right) - \int_\Omega G(u, v) \, dx
\]
(2.7)

We also denote
\[
W = \{(u, v) : (u, v) \in W_0^{1,\alpha+1}(\Omega) \times W_0^{1,\alpha+1}(\Omega), I(u, v) > 0 \} \cup \{0, 0\}.
\]
(2.8)

Lemma 2.2 $E(t)$ is a nonincreasing function for $t \geq 0$ and
\[
E'(t) = - \left( \|u_t\|_{p+1}^{p+1} + \|v_t\|_{q+1}^{q+1} \right) \leq 0.
\]
(2.9)

Proof Multiplying equation (1.1) by $u_t$ and equation (1.2) by $v_t$, and integrating over $\Omega$, using integrating by parts and summing up the product results, we obtain
\[
E(t) - E(0) = - \int_0^t \left( \|u_\tau\|_{p+1}^{p+1} + \|v_\tau\|_{q+1}^{q+1} \right) \, d\tau \quad for \quad t \geq 0.
\]
(2.10)

Lemma 2.3 (Sobolev-Poincare Inequality) [1] Let $p$ be a real number with $2 \leq p < \infty (n = 1, 2)$ and $2 \leq p \leq \frac{2n}{n-2} (n \geq 3)$, thus there is a constant $C_* = C_*(\Omega, p)$ such that
\[
\|u\|_p \leq C_* \|\nabla u\|, \quad \forall u \in H_0^1(\Omega).
\]

Lemma 2.4 (Nakao Inequality) [9] Let $\varphi(t)$ be nonnegative and nonincreasing function defined on $[0, T], T > 1$ and suppose that there are constants $w_0 > 0$ and $m \geq 0$ such that
\[
\varphi^{1+m}(t) \leq w_0 (\varphi(t) - \varphi(t+1)), \quad t \in [0, T].
\]
Thus we obtain for all $t \in [0,T]$,

$$
\begin{align*}
\varphi(t) &\leq \varphi(0)e^{-w_1(0)[t-1]^+}, & m &= 0, \\
\varphi(t) &\leq (\varphi(0)^{-m} + w_0^{-1}m[t-1]^+)^{\frac{1}{m}}, & m &> 0,
\end{align*}
$$

(2.11)

where $|t-1|^+ = \max\{t-1,0\}$ and $w_1 = \ln \left(\frac{w_0}{w_0-1}\right)$.

Now, we specify the local existence theorem that can be established by combination arguments of [2, 3, 12].

**Theorem 2.5 (Local Existence)** Assume that (2.1) holds. Thus, there exist $p,q$ satisfying

$$
\begin{align*}
1 \leq p,q, & n \leq 2, \\
1 \leq p,q \leq \frac{n+2}{n-2}, & n > 2
\end{align*}
$$

and further $(u_0, v_0) \in W^{0,\alpha+1}_0(\Omega) \cap L^{r+1}(\Omega)$, $(u_1, v_1) \in L^2(\Omega) \cap L^2(\Omega)$. Thus, problem (1.1)-(1.5) has a unique local solution

$$
(u,v) \in \left(C[0,T); W^{0,\alpha+1}_0(\Omega) \cap L^{r+1}(\Omega)\right),
$$

$u_t \in C \left([0,T); L^2(\Omega) \cap L^{r+1}(\Omega \times [0,T])\right)$ and $v_t \in C \left([0,T); L^2(\Omega) \cap L^{r+1}(\Omega \times [0,T])\right)$.

Moreover, at least one of the following statements holds true:

(i) $T = \infty$,

(ii) $\|u_t\|^2 + \|v_t\|^2 + \frac{2}{\alpha+1} \|\nabla u\|_{\alpha+1}^2 + \frac{2}{\alpha+1} \|\nabla v\|_{\alpha+1}^2 + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 \to \infty$ as $t \to T^-$.

3. Global existence and decay of solutions

**Lemma 3.1** Assume that (2.1) holds and $\alpha > 1$ and $r > \alpha$ satisfy

$$
\frac{r+1}{\alpha+1} \leq \frac{n(\alpha+1)}{n-(\alpha+1)}, \quad \alpha+1 < n.
$$

(3.1)

Let $(u_0, v_0) \in W$ and $(u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$ such that

$$
\beta = \frac{c_1C^r_{\alpha+1}(r+1)(\alpha+1)}{2} \left[\frac{(r+1)(\alpha+1)}{r-1}E(0)\right]^{\frac{r-n}{\alpha+1}} < 1,
$$

(3.2)

then $(u,v) \in W$, for all $t \geq 0$.

**Proof** Suppose not. Then for some $T_m > 0$, $(u(T_m), v(T_m)) \notin W$. Since $(u(0), v(0)) \in W$ and $I(0) > 0$, then by continuity of $u(t)$ and $v(t)$ that

$$
I(t) > 0,
$$

(3.3)

for some interval near $t = 0$. Let $T_m > 0$ be a maximal time, when (3.3) holds on $[0,T_m]$. So, for $\forall t \in [0,T_m]$, $I(T_m) = 0$.
and

$$I(t) > 0, \quad 0 \leq t \leq T_m.$$ 

According to (2.5) and (2.6), we obtain

$$J(t) = \frac{1}{r+1} I(t) + \frac{r-1}{2(r+1)} \left( \frac{2}{\alpha + 1} \| \nabla u \|_{\alpha+1}^{\alpha+1} + \frac{2}{\alpha + 1} \| \nabla v \|_{\alpha+1}^{\alpha+1} + m^2 \| u \|^2 + m^2 \| v \|^2 \right)$$

$$\geq \frac{r-1}{2(r+1)} \left( \frac{2}{\alpha + 1} \| \nabla u \|_{\alpha+1}^{\alpha+1} + \frac{2}{\alpha + 1} \| \nabla v \|_{\alpha+1}^{\alpha+1} + m^2 \| u \|^2 + m^2 \| v \|^2 \right). \quad (3.4)$$

By using (3.4), (2.9) and definition of $E(t)$, we have

$$\frac{2}{\alpha + 1} \| \nabla u \|_{\alpha+1}^{\alpha+1} + \frac{2}{\alpha + 1} \| \nabla v \|_{\alpha+1}^{\alpha+1} \leq \frac{2(r+1)}{r-1} J(t) \leq \frac{2(r+1)}{r-1} E(t) \leq \frac{2(r+1)}{r-1} E(0). \quad (3.5)$$

Hence,

$$\| \nabla u \|_{\alpha+1} + \| \nabla v \|_{\alpha+1} \leq \left( \frac{(r+1)(\alpha + 1)}{r-1} E(0) \right)^{\frac{1}{r+1}}. \quad (3.6)$$

According to Sobolev embedding inequality, we have

$$\| u \|_{r+1}^{r+1} \leq C_s \| \nabla u \|_{\alpha+1}^{r+1} = C_s \| \nabla u \|_{\alpha+1}^{r-\alpha} \| \nabla u \|_{\alpha+1}^{\alpha+1} \quad (3.7)$$

and

$$\| v \|_{r+1}^{r+1} \leq C_s \| \nabla v \|_{\alpha+1}^{r+1} = C_s \| \nabla v \|_{\alpha+1}^{r-\alpha} \| \nabla v \|_{\alpha+1}^{\alpha+1}. \quad (3.8)$$

Combining (3.7) and (3.8) with (3.6) implies $\| u \|_{r+1}^{r+1} + \| v \|_{r+1}^{r+1} \leq C_s \left( \frac{(r+1)(\alpha + 1)}{r-1} E(0) \right)^{\frac{r-\alpha}{r+1}} \left( \| \nabla u \|_{\alpha+1}^{\alpha+1} + \| \nabla v \|_{\alpha+1}^{\alpha+1} \right)$.

Applying (3.2) to above inequality with (2.4), we get $I(T_m) > 0$

$$(r+1) \int_{\Omega} G(u,v) \, dx \leq c_1 (r+1) \left( \| u \|_{r+1}^{r+1} + \| v \|_{r+1}^{r+1} \right)$$

$$\leq \beta \frac{2}{\alpha + 1} \left( \| \nabla u \|_{\alpha+1}^{\alpha+1} + \| \nabla v \|_{\alpha+1}^{\alpha+1} \right)$$

$$< \frac{2}{\alpha + 1} \left( \| \nabla u \|_{\alpha+1}^{\alpha+1} + \| \nabla v \|_{\alpha+1}^{\alpha+1} \right). \quad (3.9)$$

Consequently, by using (2.6), we deduce that $I(t) > 0$ for all $t \in [0,T_m]$, which contradicts $I(t) = 0$.

The lemma’s proof is complete. \(\square\)

**Lemma 3.2** Let the assumptions of Lemma 3.1 hold. Thus, there exists $\eta_1 = 1 - \beta$ so that

$$(r+1) \int_{\Omega} G(u,v) \, dx \leq (1 - \eta_1) \left( \frac{2}{\alpha + 1} \| \nabla u \|_{\alpha+1}^{\alpha+1} + \frac{2}{\alpha + 1} \| \nabla v \|_{\alpha+1}^{\alpha+1} + m^2 \| u \|^2 + m^2 \| v \|^2 \right).$$
Proof From (3.9), we obtain

$$(r + 1) \int_\Omega G(u,v)dx \leq \beta \left( \frac{2}{\alpha + 1} \|\nabla u\|^{\alpha+1}_{\alpha+1} + \frac{2}{\alpha + 1} \|\nabla v\|^{\alpha+1}_{\alpha+1} \right)$$

$$\leq \beta \left( \frac{2}{\alpha + 1} \|\nabla u\|^{\alpha+1}_{\alpha+1} + \frac{2}{\alpha + 1} \|\nabla v\|^{\alpha+1}_{\alpha+1} + m_1^2 u^2 + m_2^2 v^2 \right).$$

Let $\beta = 1 - \eta_1$, then we have the result. \qed

Remark 3.3 Hence, we can deduce from Lemma 3.2

$$\frac{2}{\alpha + 1} \|\nabla u\|^{\alpha+1}_{\alpha+1} + \frac{2}{\alpha + 1} \|\nabla v\|^{\alpha+1}_{\alpha+1} + m_1^2 u^2 + m_2^2 v^2 \leq \frac{1}{\eta_1} I(t).$$ (3.10)

Theorem 3.4 Assume that (2.1) holds. Let $(u_0, v_0) \in W$ satisfying (2.8). Thus, the solution of problem (1.1)-(1.5) is global.

Proof It suffices to show that $\|u_t\|^2 + \|v_t\|^2 + \frac{2}{\alpha + 1} \|\nabla u\|^{\alpha+1}_{\alpha+1} + \frac{2}{\alpha + 1} \|\nabla v\|^{\alpha+1}_{\alpha+1} + m_1^2 u^2 + m_2^2 v^2$ is bounded independently of $t$. To indicate this, using (2.6) and (2.7) we have

$$E(0) \geq E(t) = \frac{1}{2} \left( \|u_t\|^2 + \|v_t\|^2 \right) + \frac{1}{2} \left( \frac{2}{\alpha + 1} \|\nabla u\|^{\alpha+1}_{\alpha+1} \right) + \frac{r - 1}{2(r + 1)} \left( \frac{2}{\alpha + 1} \|\nabla u\|^{\alpha+1}_{\alpha+1} + \frac{2}{\alpha + 1} \|\nabla v\|^{\alpha+1}_{\alpha+1} + m_1^2 u^2 + m_2^2 v^2 \right)$$

$$\geq \frac{1}{2} \left( \|u_t\|^2 + \|v_t\|^2 \right) + \frac{r - 1}{2(r + 1)} \left( \frac{2}{\alpha + 1} \|\nabla u\|^{\alpha+1}_{\alpha+1} + \|\nabla v\|^{\alpha+1}_{\alpha+1} + m_1^2 u^2 + m_2^2 v^2 \right)$$

because $I(t) \geq 0$. Therefore,

$$\|u_t\|^2 + \|v_t\|^2 + \frac{2}{\alpha + 1} \|\nabla u\|^{\alpha+1}_{\alpha+1} + \frac{2}{\alpha + 1} \|\nabla v\|^{\alpha+1}_{\alpha+1} + m_1^2 u^2 + m_2^2 v^2 \leq CE(0)$$

where $C = \frac{2(r+1)}{r-1}$. Thus by Theorem 2.5, we get the result of global existence. \qed

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Theorem 3.5 Assume that (2.1) and (2.8) hold, and further \((u_0, v_0) \in W\). Then, we obtain the following decay estimates:

\[
E(t) \leq \begin{cases} 
E(0)e^{-w_1|t-1|^+}, & p = q = 1 \\
\left(E(0)^{-m} + C_0^{-1}m|t-1|^+\right)^{\frac{1}{m}}, & p, q > 1
\end{cases}
\]

where \(w_1, m,\) and \(C_0\) are positive constants.

Now, we shall derive the decay estimate of the solution in Theorem 3.5 by using Nakao inequality.

**Proof** By integration of (2.9) over \([t, t+1], t > 0\), we obtain

\[
E(t) - E(t+1) = \int_t^{t+1} \left(\|u_\tau(\tau)\|_{p+1}^{p+1} + \|v_\tau(\tau)\|_{q+1}^{q+1}\right) d\tau = D_1^{p+1}(t) + D_2^{q+1}(t) \tag{3.11}
\]

where

\[
D_1^{p+1}(t) = \int_t^{t+1} \left(\|u_\tau(\tau)\|_{p+1}^{p+1}\right) d\tau \tag{3.12}
\]

and

\[
D_2^{q+1}(t) = \int_t^{t+1} \left(\|v_\tau(\tau)\|_{q+1}^{q+1}\right) d\tau. \tag{3.13}
\]

Hölder inequality and by virtue of (3.12), we observe that

\[
\int_t^{t+1} \int_\Omega |u_\tau|^2 dx dt \leq \int_t^{t+1} |\Omega|^{\frac{2}{p+1}} \|u_\tau\|_{p+1}^2 dt = |\Omega|^{\frac{2}{p+1}} D_1^2(t) = CD_1^2(t). \tag{3.14}
\]

Similarly, Hölder inequality and due to (3.13), we obtain

\[
\int_t^{t+1} \int_\Omega |v_\tau|^2 dx dt \leq |\Omega|^{\frac{2}{q+1}} D_2^2(t) = CD_2^2(t). \tag{3.15}
\]

Hence, from (3.14) and (3.15), there exist \(t_1 \in \left[t, t + \frac{1}{4}\right]\) and \(t_2 \in \left[t + \frac{3}{4}, t + 1\right]\) such that

\[
\|u_i(t_i)\| \leq CD_1(t), \quad i = 1, 2 \tag{3.16}
\]

and

\[
\|v_i(t_i)\| \leq CD_2(t), \quad i = 1, 2. \tag{3.17}
\]
By multiplying (1.1) and (1.2) by $u$ and $v$, respectively, and integrating it over $\Omega \times [t_1, t_2]$, we have

$$\int_{t_1}^{t_2} I(t) dt \leq - \int_{t_1}^{t_2} \int_{\Omega} [u \partial_t u + v \partial_t v] dx dt$$

$$+ \int_{t_1}^{t_2} \int_{\Omega} [u|^{p-1} u v] dx dt - \int_{t_1}^{t_2} \int_{\Omega} [v|^{q-1} v v] dx dt.$$

(3.18)

To estimate of the first term of the right-hand side of (3.18), by using (1.1)-(1.5), integrating by parts and Cauchy–Schwarz inequality, we get

$$\int_{t_1}^{t_2} I(t) dt \leq \|u(t_1)\| \|u(t_1)\| + \|u(t_2)\| \|u(t_2)\|$$

$$+ \|v(t_1)\| \|v(t_1)\| + \|v(t_2)\| \|v(t_2)\|$$

$$+ \int_{t_1}^{t_2} \|u\|^2 dt + \int_{t_1}^{t_2} \|v\|^2 dt$$

$$- \int_{t_1}^{t_2} \int_{\Omega} [u|^{p-1} u u] dx dt - \int_{t_1}^{t_2} \int_{\Omega} [v|^{q-1} v v] dx dt.$$

(3.19)

Now, our purpose is to estimate the right hand side of the inequality. First, we will estimate the last two terms in the right-hand side of inequality (3.19). By applying Hölder inequality, we get

$$\int_{t_1}^{t_2} \int_{\Omega} [u|^{p-1} u u] dx dt \leq \int_{t_1}^{t_2} \left[ \|u_t(t)\|_{p+1} \|u(t)\|_{p+1} \right] dt$$

(3.20)

and

$$\int_{t_1}^{t_2} \int_{\Omega} [v|^{q-1} v v] dx dt \leq \int_{t_1}^{t_2} \left[ \|v_t(t)\|_{q+1} \|v(t)\|_{q+1} \right] dt.$$

(3.21)

According to (3.5) and Sobolev–Poincare inequality, we obtain for $p \geq 1$
\[
\int_{t_1}^{t_2} \left[ \|u(t)\|_{p+1}^p \|u(t)\|_{p+1} \right] dt \leq C_* \int_{t_1}^{t_2} \left[ \|u_t(t)\|_{p+1}^p \|\nabla u\| \right] dt \\
\leq C_* \left( \frac{2(r+1)}{r-1} \right)^{\frac{1}{2}} \int_{t_1}^{t_2} \left[ \|u_t(t)\|_{p+1} E^\frac{1}{2}(s) \right] dt \\
\leq C_* \left( \frac{2(r+1)}{r-1} \right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s) \int_{t_1}^{t_2} \left[ \|u_t\|_{p+1} \right] dt \\
\leq C_* \sqrt{\frac{2(r+1)}{r-1}} \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s)D_1^2(t). \tag{3.22}
\]

Similarly, we obtain for \( q \geq 1 \)
\[
\int_{t_1}^{t_2} \left[ \|v(t)\|_{q+1}^q \|v(t)\|_{q+1} \right] dt \leq C_* \sqrt{\frac{2(r+1)}{r-1}} \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s)D_2^2(t). \tag{3.23}
\]

Now, from (3.5), (3.16), and Sobolev–Poincare inequality, we get
\[
\|u_i(t_i)\| \|u(t_i)\| \leq C_1 D_1(t) \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s), \tag{3.24}
\]
where \( C_1 \) = \( 2C_* \sqrt{\frac{2(r+1)}{r-1}} C \). Similarly, from (3.5), (3.17), and Sobolev–Poincare inequality, we obtain
\[
\|v_i(t_i)\| \|v(t_i)\| \leq C_2 D_2(t) \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s), \tag{3.25}
\]
where \( C_2 \) = \( 2C_* \sqrt{\frac{2(r+1)}{r-1}} C \). Substitute (3.20)-(3.25) into (3.19) by (3.14) and (3.15), we obtain
\[
\int_{t_1}^{t_2} I(t) dt \leq C_3 \left\{ \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s) \left( D_1(t) + D_2(t) \right) + D_1^2(t) + D_2^2(t) \right\} + C_* \sqrt{\frac{2(r+1)}{r-1}} \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s) \left( D_1^2(t) + D_2^2(t) \right), \tag{3.26}
\]
where \( C_3 = \max \{ C_1, C_2, C_1 \} \). Moreover, from definition of \( E(t) \), \( I(t) \) and Remark 3.3, we get
\[
E(t) \leq \frac{1}{2} \left( \|u_t\|^2 + \|v_t\|^2 \right) + C_4 I(t), \tag{3.27}
\]
where \( C_4 = \frac{1}{m} \frac{r-1}{2(r+1)} + \frac{1}{r+1} \). By integrating (3.27) over \([t_1, t_2] \), we get
\[
\int_{t_1}^{t_2} E(t) dt \leq \frac{1}{2} \int_{t_1}^{t_2} \left( \|u_t\|^2 + \|v_t\|^2 \right) dt + C_4 \int_{t_1}^{t_2} I(t) dt.
\]
Hence, by (3.14), (3.15), and (3.26), we have
\[
\int_{t_1}^{t_2} E(t) dt \leq \frac{1}{2} C \left( D_1^2(t) + D_2^2(t) \right) + C_4 C_3 \left\{ \sup_{t_1 \leq s \leq t_2} E^2(s) (D_1(t) + D_2(t)) + D_1^2(t) + D_2^2(t) \right\} + C_4 C_3 \left( \frac{2}{r+1} \sup_{t_1 \leq s \leq t_2} E^2(s) (D_1^r(t) + D_2^r(t)) \right). \tag{3.28}
\]

Now, by integrating \( \frac{d}{dt} E(t) \) over \([t, t_2]\), we have
\[
E(t) = E(t_2) + \int_{t}^{t_2} \left( \| u_{\tau}(\tau) \|^u_{p+1} + \| v_{\tau}(\tau) \|^w_{q+1} \right) d\tau. \tag{3.29}
\]

Therefore, since \( t_2 - t_1 \geq \frac{1}{2} \), we deduce that
\[
\int_{t_1}^{t_2} E(t) dt \geq (t_2 - t_1) E(t_2) \geq \frac{1}{2} E(t_2).
\]
That is,
\[
E(t_2) \leq 2 \int_{t_1}^{t_2} E(t) dt. \tag{3.30}
\]

Therefore, exploiting (3.11), (3.29), (3.30) and because \( t_1, t_2 \in [t, t+1] \), we obtain
\[
E(t) \leq 2 \int_{t_1}^{t_2} E(t) dt + \int_{t}^{t+1} \left( \| u_{\tau}(\tau) \|^u_{p+1} + \| v_{\tau}(\tau) \|^w_{q+1} \right) d\tau
\]
\[
= 2 \int_{t_1}^{t_2} E(t) dt + D_1^{p+1}(t) + D_2^{q+1}(t). \tag{3.31}
\]
Then, from (3.28), we obtain
\[
E(t) \leq (C + 2C_4 C_3) \left( D_1^2(t) + D_2^2(t) \right) + D_1^{p+1}(t) + D_2^{q+1}(t)
\]
\[
+ C_5 E^2(t) (D_1(t) + D_2(t)) + D_1^r(t) + D_2^r(t), \tag{3.32}
\]
where \( C_5 = 2C_4 C_3 \max \left( 1, \sqrt{\frac{2(r+1)}{r-1}} \right) \).

Hence, by arithmetic-geometric mean inequality, we deduce that
\[
E(t) \leq C_6 \left[ D_1^2(t) + D_2^2(t) + D_1^{p+1}(t) + D_2^{q+1}(t) + D_1^{2p}(t) + D_2^{2q}(t) \right]. \tag{3.33}
\]
where $C_6 = \max(2C + 4C_4C_3 + C_5^2, 2, C_2^2)$. Now we distinguish two cases.

Case 1: When $p = q = 1$, we get from (3.33)

$$E(t) \leq 3C_6 \left[ D_1^2(t) + D_2^2(t) \right] = 3C_6 [E(t) - E(t + 1)].$$

(3.34)

By Lemma 2.4, we have

$$E(t) \leq E(0)e^{-w_1[t-1]^+},$$

(3.35)

where $[t-1]^+ = \max\{t-1, 0\}$ and $w_1 = \ln \left( \frac{3C_6}{\pi C_7} \right)$.

Case 2: When $p, q > 1$, we get from (3.33)

$$E(t) \leq C_6D_1^2(t) \left[ 1 + D_1^{q-1}(t) + D_1^{2(p-1)}(t) \right] + C_6D_2^2(t) \left[ 1 + D_2^{q-1}(t) + D_2^{2(q-1)}(t) \right]
\leq C_6 \left[ 1 + D_1^{q-1}(t) + D_1^{2(p-1)}(t) + D_2^{q-1}(t) + D_2^{2(q-1)}(t) \right] \left( D_1^2(t) + D_2^2(t) \right).$$

(3.36)

Thus since $E(t) \leq E(0)$ for all $t \geq 0$, we obtain from (3.11)

$$E(t) \leq C_6 \left[ 1 + D_1^{q-1}(t) + D_1^{2(p-1)}(t) + D_2^{q-1}(t) + D_2^{2(q-1)}(t) \right] \left( D_1^2(t) + D_2^2(t) \right)
\leq C_7 \left( D_1^2(t) + D_2^2(t) \right), \quad t \geq 0,$$

(3.37)

where $C_7 = C_6 \left[ 1 + E^{\frac{p-1}{p+1}}(0) + E^{\frac{q-1}{q+1}}(0) + E^{\frac{2(p-1)}{p+1}}(0) + E^{\frac{2(q-1)}{q+1}}(0) \right]$. When we take $m = \max \left\{ \frac{p-1}{2}, \frac{q-1}{2} \right\}$, then we get

$$E(t)^{1+m} \leq \left[ C_7 \left( D_1^2(t) + D_2^2(t) \right) \right]^{1+m}
= C_7^{1+m} \left( D_1^{2+2m}(t) + D_2^{2+2m}(t) \right)
= C_8 \left( D_1^{2+2m}(t) + D_2^{2+2m}(t) \right),$$

(3.38)

where $C_8 = C_7^{1+m}$. Consequently, (3.38) is equal to

$$E(t)^{1+m} \leq C_8 \left( D_1^{p+1}(t)D_1^{2m-p+1}(t) + D_2^{q+1}(t)D_2^{2m-q+1}(t) \right)
\leq C_8 \left( D_1^{p+1}(t)E^{\frac{2m-p+1}{p+1}}(0) + D_2^{q+1}(t)E^{\frac{2m-q+1}{q+1}}(0) \right)
\leq C_9 \left( E^{p+1}(t) + D_2^{q+1}(t) \right)
= C_9 \left( E(t) - E(t + 1) \right),$$

(3.39)

where $C_9 = C_8 \max \left\{ E^{\frac{2m-p+1}{p+1}}(0), E^{\frac{2m-q+1}{q+1}}(0) \right\}$.

Thus, from Lemma 2.4 and (3.39), we have for $t \in [0, T]$ and $m > 0$

$$E(t) \leq (E(0)^{1-m} + C_9^{-1}m[t-1]^+) \frac{1}{m}.$$  

This completes the proof of Theorem 12.  

□
4. Blow up of solutions

Theorem 4.1 Suppose that \( r + 1 > \max \{ p + 1, q + 1 \} \), the initial energy \( E(0) < 0 \) and \( \alpha < r \). If so, the solution for this system blows up in finite time \( T^* \) where \( T^* \leq \frac{1 - \alpha}{\xi \psi \alpha + 1 (0)} \). \( \psi(t) \) and \( \sigma \) are given (4.1) and (4.2), respectively.

**Proof** We assume that the solution exists for all the time, we arrive at a contradiction. Define \( H(t) = -E(t) \), \( E(0) < 0 \) and (2.9) gives \( 0 < H(0) \leq H(t) \). Denote

\[
\psi(t) = H^{1 - \sigma}(t) + \varepsilon \left( \int u u_t dx + \int v v_t dx \right), \tag{4.1}
\]

where \( \varepsilon \) is a positive and small constant to be determined, and

\[
0 < \sigma \leq \min \left\{ \frac{r - p}{(r + 1)p}, \frac{r - q}{(r + 1)q}, \frac{r - 1}{2(r + 1)} \right\}. \tag{4.2}
\]

Our aim is to show that \( \psi(t) \) satisfies a differential inequality of the following form

\[
\psi'(t) \geq \xi \psi^{\zeta}(t), \quad \zeta > 1.
\]

This will result in a blow up in finite time. By differentiation of (4.1), we have

\[
\psi'(t) = (1 - \sigma) H^{-\sigma}(t) H'(t)
+ \varepsilon \left( \int u u_t dx + \int v v_t dx \right) + \varepsilon \left( \int u u_t dx + \int v v_t dx \right). \tag{4.3}
\]

By multiplying (1.1) by \( u \) and (1.2) by \( v \), respectively, and integrating it over \( \Omega \times [t_1, t_2] \), by (2.2) and (4.3), we obtain

\[
\psi'(t) = (1 - \sigma) H^{-\sigma}(t) H'(t)
+ \varepsilon \left( \int u u_t dx + \int v v_t dx \right) + \varepsilon \left( \int u u_t dx + \int v v_t dx \right)
- \varepsilon \left( m_1^2 ||u||^2 + m_2^2 ||v||^2 \right) - \varepsilon \left( \int u u_t dx + \int v v_t dx \right)
+ \varepsilon (r + 1) \int G(u, v) dx. \tag{4.4}
\]

From definition of \( H(t) \), we obtain

\[
- \varepsilon \left( ||\nabla u||^{\alpha + 1}_{\alpha + 1} + ||\nabla v||^{\alpha + 1}_{\alpha + 1} \right) = \varepsilon (\alpha + 1) H(t) - \varepsilon (\alpha + 1) \int G(u, v) dx
+ \varepsilon \left( \frac{\alpha + 1}{2} \right) \left( ||u||^2 + ||v||^2 \right) + \varepsilon \left( \frac{\alpha + 1}{2} \right) \left( m_1^2 ||u||^2 + m_2^2 ||v||^2 \right). \tag{4.5}
\]

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Substitute (4.5) into (4.4) to get
\[
\psi'(t) = (1 - \sigma)H^{-\sigma}(t)H'(t) + \varepsilon \left( \frac{\alpha + 3}{2} \right) \left( \|u_t\|^2 + \|v_t\|^2 \right) + \varepsilon (\alpha + 1) H(t) \\
+ \varepsilon (r - \alpha) \int_\Omega G(u, v) dx + \varepsilon \left( \frac{\alpha - 1}{2} \right) (m_1^2 \|u\|^2 + m_2^2 \|v\|^2) \\
- \varepsilon \left( \int_\Omega uu_t |u_t|^{p-1} dx + \int_\Omega vv_t |v_t|^{q-1} dx \right).
\]
(4.6)

Now, we use the following Young’s inequality to estimate the last term in (4.6)
\[
xy \leq \frac{\delta j x^j}{j} + \frac{\delta^{-k} y^k}{k}
\]
where \(x, y \geq 0, \delta > 0, j, k \in \mathbb{R}^+\) such that \(\frac{1}{j} + \frac{1}{k} = 1\). Therefore, applying the previous inequality and from \(H'(t) = \|u_t\|_{p+1}^{p+1} + \|v_t\|_{q+1}^{q+1}\), we have
\[
\int_\Omega uu_t |u_t|^{p-1} dx \leq \frac{\delta_1^{p+1}}{p+1} \|u_t\|_{p+1}^{p+1} + \frac{p \delta_1^{p+1}}{p+1} \|u_t\|_{p+1}^{p+1} \\
\leq \frac{\delta_1^{p+1}}{p+1} \|u_t\|_{p+1}^{p+1} + \frac{p \delta_1^{p+1}}{p+1} H'(t)
\]
and
\[
\int_\Omega vv_t |v_t|^{q-1} dx \leq \frac{\delta_2^{q+1}}{q+1} \|v_t\|_{q+1}^{q+1} + \frac{q \delta_2^{q+1}}{q+1} \|v_t\|_{q+1}^{q+1} \\
\leq \frac{\delta_2^{q+1}}{q+1} \|v_t\|_{q+1}^{q+1} + \frac{q \delta_2^{q+1}}{q+1} H'(t),
\]
where \(\delta_1\) and \(\delta_2\) are real numbers depending on the time \(t\). Consequently, we obtain from (4.6)
\[
\psi'(t) \geq (1 - \sigma)H^{-\sigma}(t)H'(t) + \varepsilon \left( \frac{\alpha + 3}{2} \right) \left( \|u_t\|^2 + \|v_t\|^2 \right) + \varepsilon (\alpha + 1) H(t) \\
+ \varepsilon (r - \alpha) \int_\Omega G(u, v) dx + \varepsilon \left( \frac{\alpha - 1}{2} \right) (m_1^2 \|u\|^2 + m_2^2 \|v\|^2) \\
- \varepsilon \left( \frac{\delta_1^{p+1}}{p+1} \|u_t\|_{p+1}^{p+1} + \frac{\delta_2^{q+1}}{q+1} \|v_t\|_{q+1}^{q+1} \right) - \varepsilon \left( \frac{p \delta_1^{p+1}}{p+1} + \frac{q \delta_2^{q+1}}{q+1} \right) H'(t).
\]
(4.7)
Therefore, by taking $\delta_1$ and $\delta_2$ so that $\delta_1 \frac{p+1}{p} = n_1 H^{-\sigma}(t), \delta_2 \frac{q+1}{q} = n_2 H^{-\sigma}(t)$, where $n_1, n_2 > 0$ are specified later, we have

$$\delta_1^{p+1} = n_1 \frac{1}{-p} H^{\sigma p}(t) \leq n_1 \frac{-p c_1}{1} \sigma p (\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1})^{\sigma p}$$

and

$$\delta_2^{q+1} = n_2 \frac{-q}{-q} H^{\sigma q}(t) \leq n_2 \frac{q c_1}{1} \sigma q (\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1})^{\sigma q},$$

because $H(t) = -E(t) \leq \int_{\Omega} G(u,v)dx \leq c_1 (\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1})$. Substituting (4.8) and (4.9) into (4.7), we get

$$\psi'(t) \geq \left(1 - \sigma - \frac{\varepsilon p n_1}{p+1} - \frac{\varepsilon q n_2}{q+1}\right) H^{-\sigma}(t) H'(t) + \varepsilon \left(\frac{\alpha + 3}{2}\right) \left(\|u\|^2 + \|v\|^2\right)$$

$$+ \varepsilon (\alpha + 1) H(t) + \varepsilon (r - \alpha) \int_{\Omega} G(u,v)dx + \varepsilon \left(\frac{\alpha - 1}{2}\right) \left(m_1^2 \|u\|^2 + m_2^2 \|v\|^2\right)$$

$$- \varepsilon \left(n_1 \frac{-p c_1}{p+1}\right) (\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1})^{\sigma p} \|u\|^{p+1}_{p+1}$$

$$- \varepsilon \left(n_2 \frac{q c_1}{q+1}\right) (\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1})^{\sigma q} \|v\|^{q+1}_{q+1}.$$  

Since $L^{r+1}(\Omega) \hookrightarrow L^{p+1}(\Omega), L^{r+1}(\Omega) \hookrightarrow L^{q+1}(\Omega)$, we have

$$\|u\|^{p+1}_{p+1} \leq C \|u\|^{r+1}_{r+1}, \quad \|v\|^{q+1}_{q+1} \leq C \|v\|^{r+1}_{r+1}.$$  

Thus

$$(\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1})^{\sigma p} \|u\|^{p+1}_{p+1} \leq C_{10}(\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1})^{\sigma p + \frac{r+1}{r}}$$

and

$$(\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1})^{\sigma q} \|v\|^{q+1}_{q+1} \leq C_{11}(\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1})^{\sigma q + \frac{r+1}{r}}.$$  

Using (4.2) and the following inequality[4]:

$z^v \leq z + 1 \leq (1 + \frac{1}{\omega}) (z + \omega), \forall z \geq 0, 0 < v \leq 1, \omega > 0$, we obtain, for $t \geq 0$,

$$\left(\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1}\right)^{\sigma p + \frac{r+1}{r}} \leq d \left(\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1} + H(0)\right)$$

$$\leq d \left(\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1} + H(t)\right)$$

and

$$\left(\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1}\right)^{\sigma q + \frac{r+1}{r}} \leq d \left(\|u\|^{r+1}_{r+1} + \|v\|^{r+1}_{r+1} + H(t)\right)$$
for $\omega = H(0)$ and $d = 1 + \frac{1}{p(0)}$. Substituting (4.11)-(4.14) into (4.10), by (2.4) we have

\[
\psi'(t) \geq \left(1 - \sigma - \frac{\varepsilon m_1}{p + 1} - \frac{\varepsilon q n_2}{q + 1}\right) H^{-\sigma}(t) H'(t) + \varepsilon \left(\frac{\alpha + 3}{2}\right) \left(\|u_t\|^2 + \|v_t\|^2\right) + \varepsilon \left(\frac{\alpha - 1}{2}\right) \left(m_1^2\|u\|^2 + m_2^2\|v\|^2\right) + \varepsilon \left(c_0(r - \alpha)\right) \left(\|u\|_{r+1}^2 + \|v\|_{r+1}^2\right).
\]

We choose $n_1, n_2$ large enough so that

\[
c_0(r - \alpha) - \frac{n_1^{-p} c_1^{-p} C_{10}^d}{p + 1} - \frac{n_2^{-q} c_1^{-q} C_{11}^d}{q + 1} \geq \frac{c_0(r - \alpha)}{2}
\]

and

\[
\alpha + 1 - \frac{n_1^{-p} c_1^{-p} C_{10}^d}{p + 1} - \frac{n_2^{-q} c_1^{-q} C_{11}^d}{q + 1} \geq \frac{\alpha + 1}{2}.
\]

Choose $\varepsilon$ small enough so that $1 - \sigma - \frac{\varepsilon m_1}{p+1} - \frac{\varepsilon q n_2}{q+1} \geq 0$. Then we get

\[
\psi'(t) \geq \varepsilon \left(\frac{\alpha + 3}{2}\right) \left(\|u_t\|^2 + \|v_t\|^2\right) + \varepsilon \left(\frac{\alpha + 1}{2}\right) H(t) + \varepsilon \left(\frac{\alpha - 1}{2}\right) \left(m_1^2\|u\|^2 + m_2^2\|v\|^2\right) + \varepsilon \left(c_0(r - \alpha)\right) \left(\|u\|_{r+1}^2 + \|v\|_{r+1}^2\right) \geq \eta \left(\|u_t\|^2 + \|v_t\|^2 + H(t) + m_1^2\|u\|^2 + m_2^2\|v\|^2 + \|u\|_{r+1}^2 + \|v\|_{r+1}^2\right),
\]

where $\eta = \min \left\{\varepsilon \left(\frac{\alpha + 3}{2}\right), \varepsilon \left(\frac{\alpha + 1}{2}\right), \varepsilon \left(\frac{\alpha - 1}{2}\right), \varepsilon \left(c_0(r - \alpha)\right)\right\}$. Consequently, we have

\[
\psi(t) \geq \psi(0) = H^{1-\sigma}(0) + \varepsilon \left(\int_\Omega u_0 u_1 dx + \int_\Omega v_0 v_1 dx\right) > 0, \quad \forall t \geq 0.
\]

Next we estimate $\psi^{\frac{1}{1-\sigma}}(t)$. We have

\[
\psi^{\frac{1}{1-\sigma}}(t) = \left[H^{1-\sigma}(t) + \varepsilon \left(\int_\Omega u_t dx + \int_\Omega v_t dx\right)\right]^{\frac{1}{1-\sigma}} \leq 2^{\frac{\gamma}{1-\sigma}} \left[H(t) + \varepsilon \left(\int_\Omega u_t dx + \int_\Omega v_t dx\right)^{\frac{1}{1-\sigma}}\right].
\]
By Hölder’s inequality, the Sobolev embedding theorem \( L^{r+1}(\Omega) \hookrightarrow L^2(\Omega) \), and Young’s inequality, we have

\[
\left| \int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx \right|^{\frac{1}{\sigma}} \leq C \left( ||u||^{\frac{1}{r+1}} ||u_t||^{\frac{1}{r+1}} + ||v||^{\frac{1}{r+1}} ||v_t||^{\frac{1}{r+1}} \right) \\
\leq C \left( ||u||^{\frac{1}{r+1}} ||u_t||^{\frac{1}{r+1}} + ||v||^{\frac{1}{(r+1)}} ||v_t||^{\frac{1}{r+1}} \right) \\
\leq C \left( ||u||^{\frac{\mu}{r+1}} ||u_t||^{\frac{\mu}{r+1}} + ||v||^{\frac{\mu}{r+1}} ||v_t||^{\frac{\mu}{r+1}} \right),
\]

(4.19)

where \( \frac{1}{\mu} + \frac{1}{\lambda} = 1 \). We get \( \lambda = 2(1 - \sigma) \), to obtain \( \mu = \frac{2(1+\sigma)}{1-2\sigma} \leq r + 1 \) by (4.2). Hence, (4.19) comes

\[
\left| \int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx \right|^{\frac{1}{\sigma}} \leq C \left( ||u||^2 + ||v||^2 + ||u||^{\frac{2}{r+1}} + ||v||^{\frac{2}{r+1}} \right).
\]

(4.20)

From (4.2), since \( \frac{2}{1-2\sigma} \leq r + 1 \), furthermore, we have

\[
||u||^{\frac{2}{r+1}} = \left( ||u||^{\frac{r+1}{r+1}} \right)^{\frac{2}{r+1}} \leq d \left( ||u||^{\frac{r+1}{r+1}} + H(t) \right),
\]

\[
||v||^{\frac{2}{r+1}} = \left( ||v||^{\frac{r+1}{r+1}} \right)^{\frac{2}{r+1}} \leq d \left( ||v||^{\frac{r+1}{r+1}} + H(t) \right)
\]

and

\[
\left| \int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx \right|^{\frac{1}{\sigma}} \leq C \left( ||u||^2 + ||v||^2 + ||u||^{\frac{2}{r+1}} + ||v||^{\frac{2}{r+1}} + H(t) \right)
\]

\[
\leq C \left( ||u||^2 + ||v||^2 + H(t) + m_1^2 ||u||^2 + m_2^2 ||v||^2 + ||u||^{\frac{r+1}{r+1}} + ||v||^{\frac{r+1}{r+1}} \right).
\]

(4.21)

Thus we obtain

\[
\psi^{1-\sigma}(t) \leq 2^{\frac{1}{r+1}} \left[ H(t) + \varepsilon^{\frac{1}{r+1}} C \left( ||u||^2 + ||v||^2 + H(t) \right) + \varepsilon^{\frac{1}{r+1}} C \left( m_1^2 ||u||^2 + m_2^2 ||v||^2 + ||u||^{\frac{r+1}{r+1}} + ||v||^{\frac{r+1}{r+1}} \right) \right]
\]

\[
\leq C_* \left( ||u||^2 + ||v||^2 + H(t) + m_1^2 ||u||^2 + m_2^2 ||v||^2 + ||u||^{\frac{r+1}{r+1}} + ||v||^{\frac{r+1}{r+1}} \right),
\]

(4.22)

where \( C_* = 2^{\frac{1}{r+1}} (1 + \varepsilon^{\frac{1}{r+1}} C) \).

A combination of (4.16) and (4.22), we conclude that

\[
\psi'(t) \geq \xi \psi^{1-\sigma}(t), \quad \frac{1}{1-\sigma} > 1,
\]

(4.23)
where $\xi$ is some positive constant. A simple integration of (4.23) yields

$$\psi^{1-\sigma} (t) \geq \frac{1}{\psi^{1-\sigma} (0) - \xi t^\alpha}. $$

Thus the solution of $H(t)$ blows up in a finite time $T^*$, with

$$T^* \leq \frac{1 - \sigma}{\xi \sigma \psi^{1-\sigma} (0)}. \quad \Box$$

References


