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Spectral and topological properties of linear operators on a Hilbert space

On (M, k)-quasi-*-paranormal operators

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Abstract: We introduce the class of (M, k)-quasi-$*$-paranormal operators on a Hilbert space H. This class extends the classes of $*$-paranormal and k-quasi-$*$-paranormal operators. An operator T on a complex Hilbert space is called (M, k)-quasi-$*$-paranormal if there exists $M > 0$ such that

$$\sqrt{M}\left\|T^{k+2}x\right\|\left\|T^{k}x\right\| \geq \left\|T^{*}T^{k}x\right\|^{2}$$

for all $x \in H$. In the present article, we give operator matrix representation of a (M, k)-quasi-$*$-paranormal operator. The compactness, the invariant subspace, and some topological properties of this class of operators are studied. Several properties of this class of operators are also presented.

Key words: M-*-paranormal operator, (M, k)-quasi-*-paranormal operator, SVEP, invariant subspacce

1. Introduction

In what follows H will be an infinite dimensional separable complex Hilbert space. By an operator on H, we mean a bounded linear transformation from H to H. Let $B(H)$ be the Banach algebra of operators on H. Denote by $N(T)$ and $R(T)$ respectively, for the null space and the range of an operator T in $B(H)$. As an extension of normal operators, P. Halmos introduced the class of hyponormal operators (defined by $TT^* \leq T^*T$)[8]. Although there are still many interesting problems for hyponormal operators yet to solve (e.g., the invariant subspace problem), one of the recent hot topics in operator theory is to study of natural extensions of hyponormal operators. Below are some of these nonhyponormal operators. Recall that an operator $T \in B(H)$ is said to be quasi-hyponormal if $T^{*2}T^2 \geq (T^*T)^2$; paranormal if $||T^2x|| ||x|| \geq ||Tx||^2$ for all unit vector $x \in H$; k-paranormal if $||T^kx|| ||x|| \geq ||Tx||^k$ for all $x \in H$. An operator T is called $*-$ paranormal if $||T^*x||^2 \leq ||T^2x|| ||x||$ and T is called $k*$-paranormal if $||T * x||^k \leq ||T^kx||$ for all unit vector x in H where k is a natural number with $k \geq 2$. The class of $*$-paranormal operators and more generally the class of k^*-paranormal operators was originally introduced in [15] and [16] with different names as k-hyponormal or operators of class $(H;\ k)$. For more results for such operators, one can refer [2], [3], [5], and [20]. As a

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generalization of $*$-class A operators and $*$-paranormal operators, the author in [10, 11] introduced the class of k-quasi-$*$-paranormal operators ($||T^*T^kx||^2 \leq ||T^{k+2}x|||T^kx||$) for all unit vector $x \in H$ where k is a natural number and the class of k-quasi-$*$-class A operators ($T^k(|T^2| - |T^*|^2)T^k \geq 0$; where k is a natural number). For more details for such operators, one can refer [12], [2], [3], [5], and [20]. We introduce the class of (M,k)-quasi-*-paranormal operators generalizing the class of k-quasi-*-paranormal operators.

Definition 1.1 *An operator $T \in B(H)$ is said to be M -*-paranormal if there exists $M > 0$ such that*

$$MT^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geq 0$$

for all $\lambda > 0$.

Definition 1.2 *An operator $T \in B(H)$ is said to be (M,k)-quasi-*-paranormal if there exists $M > 0$ and a positive integer k such that*

$$T^{*k}(MT^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0$$

for all $\lambda > 0$.

This definition is equivalent to

$$\sqrt{M}\left\|T^{k+2}x\right\|\left\|T^kx\right\| \geq \left\|T^*T^kx\right\|^2$$

for all $x \in H$. An $(M,1)$-quasi-*-paranormal is M-quasi-*-paranormal. Obviously,

$$M\text{-*-paranormal} \subset M\text{-quasi-*-paranormal} \subset (M,k)\text{-quasi-*-paranormal}$$

and

$$(M,k)\text{-quasi-*-paranormal} \subset (M,k+1)\text{-quasi-*-paranormal}$$

By a direct calculation, the operator

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

acting on the Hilbert space $\mathbb{C}^2$ is a $(M,2)$-quasi-*-paranormal operator that is not M-quasi-*-paranormal. This shows that the classes above do not coincide.

Example On the usual Hilbert space ℓ_2 equipped with its standard basis $(e_n)_n$, let S_r be the right weighted shift defined by

$$S_re_n = \alpha_ne_{n+1}$$

where $(\alpha_n)_n$ is a decreasing complex sequence. Then, S_r is (M,k)-quasi-*-paranormal if and only if

$$|\alpha_{n-1}|^2 < \sqrt{M}\,|\alpha_n|\,|\alpha_{n+1}|$$

for all n. Indeed, we have

$$(S_r^{*2}S_r^2 - 2\lambda S_rS_r^* + \lambda^2)e_n = (|\alpha_n|^2\,|\alpha_{n+1}|^2 - 2\lambda\,|\alpha_{n-1}|^2 + \lambda^2)e_n$$

$$S_r^ke_n = \alpha_n\alpha_{n+1}...\alpha_{n+k-1}e_{n+k}$$

$$S_r^{*k}e_n = \overline{\alpha_{n-1}\alpha_{n-2}...\alpha_{n-k}}e_{n-k}$$

Thus, for all $\lambda > 0$ and all n,

$$\begin{aligned}
&\left\langle S_r^{*k}(MS_r^{*2}S_r^2 - 2\lambda S_r S_r^* + \lambda^2)S_r^k e_n, e_n\right\rangle \geq 0\\
&\Leftrightarrow (M\left|\alpha_n\right|^2\left|\alpha_{n+1}\right|^2 - 2\lambda\left|\alpha_{n-1}\right|^2 + \lambda^2)\alpha_n\alpha_{n+1}...\alpha_{n+k-1}\overline{\alpha_{n-1}\alpha_{n-2}...\alpha_{n-k}} \geq 0\\
&\Leftrightarrow (M\left|\alpha_n\right|^2\left|\alpha_{n+1}\right|^2 - 2\lambda\left|\alpha_{n-1}\right|^2 + \lambda^2)\left|\alpha_n\right|^2\left|\alpha_{n+1}\right|^2 ... \left|\alpha_{n+k-1}\right|^2 \geq 0\\
&\Leftrightarrow M\left|\alpha_n\right|^2\left|\alpha_{n+1}\right|^2 - 2\lambda\left|\alpha_{n-1}\right|^2 + \lambda^2 \geq 0
\end{aligned}$$

Since $\lambda > 0$ is arbitrary,

$$\left|\alpha_{n-1}\right|^2 - \sqrt{M}\left|\alpha_n\right|\left|\alpha_{n+1}\right| < 0.$$

In the present article, we give operator matrix representation of a (M,k)-quasi-$*$-paranormal operator. The compactness, the invariant subspace, and some topological properties of this class of operators are studied. Several properties of this class of operators are also presented.

2. Main results

We start with the following useful theorem

Theorem 2.1 *Let $T \in B(H)$ be a (M,k)-quasi-*-paranormal operator. If $R(T^k)$ is dense in H, then T is M-*-paranormal.*

Proof Let $x \in H$. Since $R(T^k)$ is dense in H, there exists a sequence $(x_n)_n$ in H such that $x = \lim\limits_{n\to\infty} T^k x_n$. Since T is (M,k)-quasi-*-paranormal,

$$\sqrt{M}\left\|T^{k+2}x\right\|\left\|T^k x\right\| \geq \left\|T^*T^k x\right\|^2$$

Hence, by the continuity of the inner product,

$$\begin{aligned}
\sqrt{M}\left\|T^2x\right\|\left\|x\right\| &= \sqrt{M}\left\|\lim_{n\to\infty} T^{k+2}x_n\right\|\left\|\lim_{n\to\infty} T^k x_n\right\|\\
&= \sqrt{M}\lim_{n\to\infty}\left\|T^{k+2}x_n\right\|\left\|T^k x_n\right\|\\
&\geq \lim_{n\to\infty}\left\|T^*T^k x_n\right\|^2 = \left\|\lim_{n\to\infty} T^*T^k x_n\right\|^2\\
&= \left\|T^*x\right\|^2
\end{aligned}$$

Thus, T is M-*-paranormal. □

Corollary 2.2 *Let T be a nonzero (M,k)-quasi-*-paranormal operator but not M-*-paranormal. Then, T has a nontrivial closed invariant subspace.*

Proof Suppose that T has no nontrivial closed invariant subspace. Since $T \neq 0$, $N(T) \neq H$ and $\overline{R(T)} \neq \{0\}$ are nontrivial closed invariant subspaces for T. Thus, we must have $N(T) = \{0\}$ and $\overline{R(T)} = H$. By Theorem 2.1, T is M-*-paranormal, which contradicts the hypothesis. □

Theorem 2.3 *Let $T \in B(H)$ be a (M,k)-quasi-*-paranormal operator. If $\overline{R(T^k)} \neq H$, then T admits the matrix representation*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $H = \overline{R(T^k)} \oplus N(T^{*k})$. Furthermore, T_1 is M--paranormal, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Proof Since T is (M,k)-quasi-*-paranormal,

$$\langle T^{*k}(MT^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k y, y\rangle \geq 0$$

for all $y \in H$. Hence,

$$\langle (MT^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k y, T^k y\rangle \geq 0$$

Thus, for all $x \in \overline{R(T^k)}$,

$$\langle (MT^{*2}T^2 - 2\lambda TT^* + \lambda^2)x, x\rangle = \langle (MT_1^{*2}T_1^2 - 2\lambda T_1T_1^* + \lambda^2)x, x\rangle \geq 0$$

Consequently, T_1 is M-*-paranormal. Let now P be the orthogonal projection on $\overline{R(T^k)}$. For all $x = x_1 + x_2, y = y_1 + y_2 \in H$, we have

$$\langle T_3^k x_2, y_2\rangle = \langle T^k(I-P)x, (I-P)y\rangle = \langle (I-P)x, T^{*k}(I-P)y\rangle = 0$$

Thus, $T_3^k = 0$. Furthermore, $\sigma(T_1) \cup \sigma(T_3) = \sigma(T) \cup \Omega$, where Ω is the union of holes in $\sigma(T)$ which happen to be a subset of $\sigma(T_1) \cap \sigma(T_3)$ by [6, Corollary 7], with the interior of $\sigma(T_1) \cap \sigma(T_3) = \emptyset$, and T_3 is nilpotent. Thus, $\sigma(T) = \sigma(T_1) \cup \{0\}$. □

Corollary 2.4 *Let $T \in B(H)$ be (M,k)-quasi-*-paranormal. If the restriction $T_1 = T\Big|\overline{R(T^k)}$ is invertible, then T is similar to the sum of an M-*-paranormal operator and a nilpotent operator.*

Proof Let

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } H = \overline{R(T^k)} \oplus N(T^{*k})$$

Then, A_1 is M-*-paranormal by Theorem 2.3. Since T_1 is invertible, $0 \notin \sigma(T)$. Hence, $\sigma(T_1) \cap \sigma(T_3) = \emptyset$. By Rosenblum's Corollary [17], [18], there exists $X \in B(H)$ for which $T_1X - XT_3 = T_2$. Thus,

$$T = \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$$

□

In the following, we give a similar result of [7, Proposition 2.6] for our class of operators.

Theorem 2.5 *Let $T \in B(H)$ be a (M,k)-quasi-*-paranormal operator. If there exists an integer $n, n \geq k+2$ for which T^n is compact, then T^k is compact for $k \geq 2$, and T is compact for $k = 0$ or $k = 1$.*

Proof It sufficies to show that T^{n-1} is compact. For $n \geq k+2$, and since T is (M,k)-quasi-*-paranormal,

$$\sqrt{M} \left\| T^{k+2} \frac{T^{n-k-2}x}{\|T^{n-k-2}x\|} \right\| \left\| T^k \frac{T^{n-k-2}x}{\|T^{n-k-2}x\|} \right\| \geq \left\| T^* T^k \frac{T^{n-k-2}x}{\|T^{n-k-2}x\|} \right\|^2$$

for all x in H. Hence,

$$\sqrt{M} \|T^n x\| \left\|T^{n-2}x\right\| \geq \left\|T^* T^{n-2} x\right\|^2 \tag{2.1}$$

Let $(x_p)_p$ be a bounded sequence which converges weakly to 0 as $p \to \infty$. Thus, by (1) and the compactness of T^n,

$$\left\|T^* T^{n-2} x_p\right\| \to 0 \tag{2.2}$$

as $p \to \infty$. If $n = 2$, then T^* is compact by (2). Therefore, T so is. For $n \geq 3$, we have by (2),

$$T^{*(n-1)} T^{(n-1)} = T^{*(n-1)} T^* T^{(n-1)} T$$

is compact. Thus, T^{n-1} is compact. □

Theorem 2.6 *Let $T \in B(H)$ be a (M,k)-quasi-*-paranormal operator. If $M \subset H$ is a closed invariant subspace of T, then the restriction $T|M$ is (M,k)-quasi-*-paranormal.*

Proof With respect to the decomposition $H = M \oplus M^{\perp}$, T can be written

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

Hence, for all integer k, $k \geq 2$, we get

$$T^k = \begin{pmatrix} A^k & \sum\limits_{p=0}^{k-1} A^{k-1-p} B C^p \\ 0 & C^k \end{pmatrix}$$

Since T is (M,k)-quasi-*-paranormal, there exists $M > 0$ such that for all $\lambda \in \mathbb{C}$

$$T^{*k}(M T^{*2} T^2 - 2\lambda T T^* + \lambda^2) T^k \geq 0$$

Hence, we obtain

$$T^{*k}(M T^{*2} T^2 - 2\lambda T T^* + \lambda^2) T^k = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$$

where

$$X = A^{*k}(MA^{*2}A^2 - 2\lambda AA^* - 2\lambda BB^* + \lambda^2)A^k$$

$$Y = A^{*k}(MA^{*2}A^2 - 2\lambda AA^* - 2\lambda BB^* + \lambda^2)\sum_{p=0}^{k-1} A^{k-1-p}BC^p +$$

$$+A^{*k}(MA^{*2}(AB + BC) - 2\lambda BC^*)C^k$$

and some operator $Z \in B(H)$.

By [4, Theorem 6], $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0$ if and only if $X, Z \geq 0$ and $Y = X^{\frac{1}{2}}WZ^{\frac{1}{2}}$ for some contraction W. Thus,

$$A^{*k}(MA^{*2}A^2 - 2\lambda AA^* - 2\lambda BB^* + \lambda^2)A^k \geq 0$$

Since $\lambda BB^* \geq 0$,

$$A^{*k}(MA^{*2}A^2 - 2\lambda AA^* + \lambda^2)A^k \geq 0$$

Consequently, the restriction $A = T\,|M$ is (M,k)-quasi-*-paranormal. □

Theorem 2.7 *If $B \in B(H)$ is unitarily equivalent to an (M,k)-quasi-*-paranormal operator T on H, then B is also (M,k)-quasi-*-paranormal.*

Proof There exists a unitary operator U on H for which $B = U^*TU$. Since A is (M,k)-quasi-*-paranormal,

$$\begin{aligned} & B^{*k}(MB^{*2}B^2 - 2\lambda BB^* + \lambda^2)B^k = \\ = \;& (U^*TU)^{*k}\left[M\,(U^*TU)^{*2}\,(U^*TU)^2 - 2\lambda U^*TU\,(U^*TU)^* + \lambda^2\right](U^*TU)^k \\ = \;& U^*T^{*k}U\left[MU^*T^{*2}UU^*A^2U - 2\lambda U^*T^2U + \lambda^2\right]U^*T^kU \\ = \;& U^*T^{*k}(MT^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^kU \geq 0 \end{aligned}$$

Thus, B is (M,k)-quasi-*-paranormal. □

Theorem 2.8 *Let $T \in B(H)$ be an (M,k)-quasi-*-paranormal operator, and let $S \in B(H)$ be an isometric operator. Then TS is (M,k)-quasi-*-paranormal whenever T commutes with S.*

Proof Since T is (M,k)-quasi-*-paranormal,

$$\begin{aligned} & (TS)^{*k}(M(TS)^{*2}(TS)^2 - 2\lambda TS(TS)^* + \lambda^2)(TS)^k = \\ & = S^{*k}T^{*k}\left[MS^*T^*S^*T^*TSTS - 2\lambda TSS^*T^* + \lambda^2\right]S^kT^k \\ & = T^{*k}S^{*k}\left[MT^{*2}T^2 - 2\lambda TSS^*T^* + \lambda^2\right]S^kT^k \\ & = T^{*k}S^{*k-1}\left[MS^*T^{*2}T^2S - 2\lambda S^*TSS^*T^*S + \lambda^2S^*S\right]S^{k-1}T^k \\ & = T^{*k}S^{*k-1}\left[MT^{*2}T^2 - 2\lambda TT^* + \lambda^2\right]S^{k-1}T^k \\ & = S^{*k-1}T^{*k}\left[MT^{*2}T^2 - 2\lambda TT^* + \lambda^2\right]T^kS^{k-1} \geq 0 \end{aligned}$$

which achieves the proof. □

Definition 2.9 *[1] An operator T in $B(H)$ is said to have the Single Valued Extension Property, briefly SVEP, at a complex number α, if for each open neighborhood V of α, the unique analytic function $f : V \to H$ satisfying*

$$(T - \lambda)f(\lambda) = 0$$

for all $\lambda \in V$ is $f \equiv 0$.
Furthermore, T is said to have SVEP [13, 14] if T has SVEP at every complex number.

Definition 2.10 *[1] For $T \in B(H)$, the smallest integer m such that $N(T^m) = N(T^{m+1})$ is said to be the ascent of T, and is denoted by $\alpha(T)$. If no such integer exists, we shall write $\alpha(T) = \infty$.*

Definition 2.11 *[1] The smallest integer m such that $R(T^m) = R(T^{m+1})$ is said to be the descent of T, and is denoted by $\delta(T)$. If no such integer exists, we set $\delta(T) = \infty$.*

According to [1], $\alpha(T) = \delta(T)$ whenever $\alpha(T)$ and $\delta(T)$ are both finite. Now, we give the value of the ascent for an (M, k)-quasi-*-paranormal operator.

Theorem 2.12 *$\alpha(T) = k$ for an (M, k)-quasi-*-paranormal operator $T \in B(H)$, i.e. $N(T^k) = N(T^{k+1})$.*

Proof Let $x \in N(T^{k+1})$. Hence, $T^{k+1}x = 0 = T^{k+2}x$. Since T is (M, k)-quasi-*-paranormal operator, there exists $M > 0$ such that

$$0 = \left\| \sqrt{M} T^{k+2} x \right\| \left\| T^k x \right\| \geq \left\| T^* T^k x \right\|^2$$

Hence, $T^*T^kx = 0$. Thus, for all $z \in H$

$$\left\langle T^* T^k x, z \right\rangle = 0$$

i.e.

$$\left\langle T^k x, Tz \right\rangle = 0$$

for all $z \in H$. Therefore, $T^kx \in R(T)^{\perp}$. Since $R\left(T^k\right) \subset R(T)$,

$$T^k x \in R\left(T^k\right)^{\perp} \cap R\left(T^k\right) = \{0\}$$

which implies that $x \in N(T^k)$. This achieves the proof because clearly $N(T^k) \subset N(T^{k+1})$. □

Corollary 2.13 *An operator (M, k) -quasi-*-paranormal operator has SVEP.*

Proof Immediately follows from Theorem 2.12 and [1, Theorem 3.8]. □

Theorem 2.14 *The class of (M, k)-quasi-*-paranormal operators is arcwise connected.*

Proof It suffices to show that the class is closed for the multiplication by scalars, i.e. if T is (M,k)-quasi-*-paranormal, then λT so is. Let then T be an (M,k)-quasi-*-paranormal operator, and let α be any complex scalar. For all $x \in H$ we have

$$\begin{aligned} \|(\alpha T)^\star(\alpha T)^k x\|^2 &= |\alpha|^{2k+2}\|T^\star T^k x\|^2 \leq \sqrt{M}|\alpha|^{2k+2}\|T^{k+2}x\|\|T^k x\| \\ &= \sqrt{M}\|(\alpha T)^{k+2}x\|\|(\alpha T)^k x\| \end{aligned}$$

□

Remarks 1. It is clear that the class of (M,k)-quasi-*-paranormal operators is nested with respect to M, i.e.

$$(M,k)\text{-quasi-*-paranormal} \subset (M',k)\text{-quasi-*-paranormal}$$

whenever $M \leq M'$.

The class of (M,k)-quasi-*-paranormal operators is not convex. In fact, operators $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ are $(4,k)$-quasi-*-paranormal. However, the operator $\frac{1}{2}(T+S)$ is not $(4,k)$-quasi-*-paranormal.

Also, the operator $T - I$ is not $(4,k)$-quasi-*-paranormal. This shows that the above class is not translation invariant.

2. Theorem 2.7 is in general false if the operator U is invertible and not unitary. Indeed, the bilateral weighted shift S defined on the Hilbert space $\ell_2(\mathbb{Z})$ by

$$Se_n = \begin{cases} e_{n+1}, & n \leq 1 \text{ or } n \geq 3 \\ \sqrt{2}e_3 & n = 2 \end{cases}$$

is in particular $(3,k)$-quasi-*-paranormal, and the operator

$$Ue_n = \begin{cases} e_{n+1}, & n \leq 1 \text{ or } n \geq 3 \\ \frac{1}{3}e_3 & n = 2 \end{cases}$$

is invertible. But the operator $U^{-1}SU$ is not $(3,k)$-quasi-*-paranormal.

3. The adjoint of an (M,k)-quasi-*-paranormal operator may not be (M,k)-quasi-*-paranormal. As an example, the operator

$$Te_n = \begin{cases} e_{n+1}, & n \leq 1 \text{ or } n \geq 3 \\ \sqrt{2}e_3 & n = 2 \end{cases}$$

is $(\sqrt{2},k)$-quasi-*-paranormal. Nonetheless, its adjoint is not $(\sqrt{2},k)$-quasi-*-paranormal.

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