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Spin structures on generalized real Bott manifolds

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Abstract: In this paper, we give a necessary and sufficient condition for a generalized real Bott manifold to have a spin structure in terms of column vectors of the associated matrix. We also give an interpretation of this result to the associated acyclic ω-weighted digraphs. Using this, we obtain a family of real Bott manifolds that does not admit spin structure.

Key words: Generalized Bott manifold, small cover, acyclic digraph

1. Introduction

A generalized real Bott tower of height k is a sequence of real projective bundles

$$B_k \longrightarrow B_{k-1} \longrightarrow \cdots \longrightarrow B_1 \longrightarrow \{pt\} \tag{1.1}$$

where B_i is the projectivization of the Whitney sum of $n_i + 1$ real line bundles over B_{i-1}. This notion is introduced by Choi et al. [3] as a generalization of the notion of a Bott tower given in [8]. The manifold B_k is called a real Bott manifold when $n_i = 1$ for each i and a generalized real Bott manifold, otherwise. The manifold B_k can be realized as a small cover over $\prod_{i=1}^{k} \Delta^{n_i}$ where Δ^{n_i} is the n_i-simplex [10, Corollary 4.6]. It is also known that every small cover over a product of simplices is a generalized real Bott manifold [3, Remark 6.5].

Let P be a simple convex polytope of dimension n with the facet set $\mathcal{F}(P) = \{F_1, \cdots, F_m\}$. For every small cover M over P, there is an associated $(n \times (m-n))$ matrix $A = [a_{ij}]$ with entries in $\mathbb{Z}_2$ which can be used to reconstruct M (see Section 2). Moreover, the mod 2 cohomology ring structure of M depends only on the face poset of P and the matrix A. More precisely, let $\mathbb{Z}_2[P]$ be the Stanley-Reisner ring of P, that is, the quotient of the polynomial ring $\mathbb{Z}_2[x_1, \cdots, x_m]$ with the ideal I generated by the square free monomials $x_{i_1} \cdots x_{i_r}$ for which $F_{i_1} \cap \cdots \cap F_{i_r}$ is empty. There is a graded ring isomorphism between $H^*(M, \mathbb{Z}_2)$ and $\mathbb{Z}_2[P]/J$ where J is the homogeneous ideal generated by the monomials $x_i + \sum_{j=1}^{m-n} a_{ij} x_{n+j}$, [4, Theorem 4.14]. Here the degree of x_i is 1. In [4, Corollary 6.8], Davis and Januskiewicz show that the total Stiefel-Whitney

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class of P is given by

$$w(M)=\Big(\prod_{i=1}^{m-n}(1+x_{n+i})\Big)\cdot\Big(\prod_{i=1}^{n}\big(1+\sum_{j=1}^{m-n}a_{ij}x_{n+j}\big)\Big) \mod I. \tag{1.2}$$

Therefore, the coefficient of x_i in the first Stiefel-Whitney class of M is one more than the sum of the entries of the $(i-n)$-th column of A when $i>n$ and zero, otherwise. Hence, the small cover M is orientable if and only if the sum of the entries of the i-th column of the matrix A is congruent to 1 modulo 2 for each $i\geq 1$ [12, Theorem, 1.7]. Since A is a matrix over $\mathbb{Z}_2$, the sum of the entries of the j-th column of A is equivalent to the dot product of the column vector with itself. Therefore, the small cover M is orientable if and only if $A_j\cdot A_j\equiv 1$ modulo 2, where A_j denote the j-th column vector of A.

In Section 2, we observe that a small cover M has a spin structure when $A_i\cdot A_i\equiv 3 \pmod 4$ and $A_i\cdot A_j\equiv 0 \pmod 2$ for all $1\leq i<j\leq m-n$ (Corollary 2.4). It turns out that when P is a product of simplices of dimensions greater than 1, the converse is also true (Corollay 3.2). In other words, when each B_i is a projectivization of the Whitney sum of 3 or more line bundles, the generalized Bott manifold B_k has a spin structure if and only if $A_i\cdot A_i\equiv 3 \pmod 4$ and $A_i\cdot A_j\equiv 0 \pmod 2$ for all $1\leq i<j\leq m-n$. In Theorem 3.1, we give a criterion for an arbitrary generalized Bott manifold B_k to have a spin structure. It is equivalent to the criterion given in [6].

In [7, Lemma 2.1], Gasior gives a formula for the second Stiefel-Whitney class of $M(A)$ in terms of the second Stiefel-Whitney classes of $M(A_{ij})$, where A_{ij} is an $n\times n$ matrix whose k-th column is A_k if $k=i,j$ and 0 otherwise, called an elementary component. After reducing the problem to elementary components, the author gives a necessary and sufficient condition on existence of a spin structure on them in [7, Theorem 1.2] which can also be obtained as a corollary of Theorem 3.1. Moreover, Proposition 3.7 is a generalization of this result to the generalized real Bott manifolds.

It is well-known that real Bott manifolds can be classified by acyclic digraphs [3]. In [5, Theorem 4.5], Dsouza gives a necessary and sufficient condition on the associated digraph for a given real Bott manifold to have a spin structure. In [9], Güçlükan İlhan and Gürbüzer show that for every generalized Bott manifold B_k, there is an associated acyclic digraph D_{B_k} on labeled vertices $\{v_1,\cdots,v_k\}$ where each edge from a vertex v_i has a vector weight in $\mathbb{Z}_2^{n_i}$. In Section 4, we generalize the condition given by Dsouza and Uma to a condition on D_{B_k} for the associated generalized Bott tower B_k to have a spin structure (Theorem 4.2).

The Wu formula implies that $w_3(M)=0$ whenever $w_1(M)$ and $w_2(M)$ are zero. Therefore, the result of Section 3 gives us sufficient conditions for $w_3(M)$ to be zero. In Section 5, we obtain a formula for $w_3(M)$ when M is a small cover over a product of simplices of dimensions greater than or equal to 3. As a corollary, we give necessary conditions for the vanishing of the third Stiefel-Whitney class of M. We obtain similar results for w_4 and we classify small covers over a product of simplices of dimensions greater than or equal to 4 whose first four Stiefel-Whitney classes are zero.

2. Small covers

Let P be an n-dimensional simple convex polytope and $\mathcal{F}(P)=\{F_1,F_2,\ldots,F_m\}$ be the set of facets of P. A small cover over P is an n-dimensional smooth closed manifold M with a locally standard $\mathbb{Z}_2^n$-action whose orbit space is P. Two small covers M_1 and M_2 over P are said to be Davis-Jansukiewicz equivalent if there is a weakly $\mathbb{Z}_2^n$-equivariant homeomorphism between M_1 and M_2 covering the identity on P. The Davis-Januskiewicz classes of small covers over P are given by the characteristic functions.

A characteristic function $\lambda : \mathcal{F}(P) \to \mathbb{Z}_2^n$ over P is a $\mathbb{Z}_2^n$-coloring function satisfying the following nonsingularity condition:

$$F_{i_1} \cap \cdots \cap F_{i_n} \neq \emptyset \quad \Rightarrow \langle \lambda(F_{i_1}), \ldots, \lambda(F_{i_n}) \rangle = \mathbb{Z}_2^n.$$

In [3], Davis and Janueskiewicz construct a small cover $M(\lambda)$ associated to a given characteristic function λ as the quotient space of the space $(P \times \mathbb{Z}_2^n)$ and the equivalence relation defined by

$$(p, g) \sim (q, h) \text{ if } p = q \text{ and } g^{-1}h \in \langle \lambda(F_{i_1}), \ldots, \lambda(F_{i_k}) \rangle$$

where the intersection $\bigcap_{j=1}^{k} F_{i_j}$ is the minimal face containing p in its relative interior.

Theorem 2.1 *[3, Proposition 1.8] For every small cover M over P, there is a characteristic function λ with $\mathbb{Z}_2^n$-homeomorphism $M(\lambda) \to M$ covering the identity on P.*

The group $GL(n, \mathbb{Z}_2)$ acts freely on the set of characteristic functions over P by composition. Moreover, the orbit space of this action is in one-to-one correspondence with the Davis-Januskiewicz equivalence classes of small covers over P. Fix a basis $e_1, \cdots, e_n$ for $\mathbb{Z}_2^n$ and reorder facets of P in such a way that $\bigcap_{i=1}^{n} F_i \neq \emptyset$. By the above theorem, for a given small cover M over P, there is an $(n \times (m-n))$-matrix $A = [a_{ij}]$ such that M and $M(\lambda)$ are Davis-Januszkiewicz equivalent where

$$\lambda(F_i) = \begin{cases} e_i, & i \leq n \\ \sum_j a_{ji} e_j & i > n. \end{cases}$$

Theorem 2.2 (Theorem 4.14, [3]) *The mod 2 cohomology ring of M is $\mathbb{Z}[P]/J$, where J is the homogeneous ideal generated by the monomials $x_i + \sum_{j=1}^{m-n} a_{ij} x_{n+j}$.*

Let $w_i(M)$ and $w(M)$ denote the i-th and the total Stiefel-Whitney classes of M, respectively. By Corollary 6.8 in [3], the total-Stiefel Whitney class of a small cover over M is given by the equation (1.2). Let A_j denote the j-th column vector of A. Then the first Stiefel-Whitney class of M is given by the following formula

$$w_1(M) = \sum_{i=1}^{m-n} (1 + \sum_j a_{ji}) \cdot x_{i+n} = \sum_{i=1}^{m-n} (1 + A_i \cdot A_i) \cdot x_{i+n}$$

since $a_{ji}^2 = a_{ji}$. Hence, M is orientable if and only if $A_i \cdot A_i \equiv 1 \pmod 2$ for all $1 \leq i \leq m-n$. By comparing the degree 2-terms in each side of the equation (1.2), one obtains a similar formula for the second Stiefel-Whitney class of M.

Proposition 2.3 *The second Stiefel-Whitney class of M is*

$$w_2(M) = \sum_{i=1}^{m-n} \alpha_i \cdot x_{i+n}^2 + \sum_{1 \le i < j \le m-n} \beta_{ij} \cdot x_{i+n} \cdot x_{j+n} \qquad \pmod I \tag{2.1}$$

where $\alpha_i = \binom{1 + A_i \cdot A_i}{2}$ *and* $\beta_{ij} = (1 + A_i \cdot A_i)(1 + A_j \cdot A_j) + A_i \cdot A_j$.

Proof The coefficient of x_{i+n}^2 in the equation (1.2) equals the coefficient of y^2 in $(1+y)\Big(\prod_j (1 + a_{ji}y)\Big)$, which is the $(k_i + 1)$-th power of $1 + y$, where k_i is the number of 1s in A_i. Since the entries of A_i are either 0 or 1, the number of 1's in A_i is equal to $A_i \cdot A_i$. Hence, the coefficient of x_{i+n}^2 in (1.2) is $\binom{1+A_i \cdot A_i}{2}$.

To find β_{ij}, first note that $|\{t| \; a_{ti} = a_{tj} = 1\}| = A_i \cdot A_j$. Therefore, β_{ij} is equal to the coefficient of $y_i y_j$ in the product

$$(1 + y_i)^{(A_i \cdot A_i - A_i \cdot A_j + 1)}(1 + y_j)^{(A_j \cdot A_j - A_i \cdot A_j + 1)}(1 + y_i + y_j)^{A_{ij}}.$$

Hence, we have

$$\begin{aligned} \beta_{ij} &= (A_i \cdot A_i - A_i \cdot A_j + 1)(A_j \cdot A_j + 1) + A_{ij}(A_j \cdot A_j) \\ &= (1 + A_i \cdot A_i)(1 + A_j \cdot A_j) - A_i \cdot A_j. \end{aligned}$$

Since we work with $\mathbb{F}_2$ coefficients, the result follows. □

Corollary 2.4 *Let M be a small cover over P with an associated reduced matrix A. If $A_i \cdot A_i \equiv 3 \pmod 4$ and $A_i \cdot A_j \equiv 0 \pmod 2$ for all possible $i < j$ then M has a spin structure.*

3. Existence of spin structure

In this section, we give a necessary and sufficient condition for the existence of spin structure for generalized Bott manifolds. Let B_k be a generalized real Bott manifold given in (1.1). One can realize B_k as a small cover over $P = \prod_{i=1}^{k} \Delta^{n_i}$, where $\sum_{i=1}^{k} n_i = n$. The facets of P is given by the following set

$$\mathcal{F} = \{F_j^i = \Delta^{n_1} \times \cdots \times \Delta^{n_{i-1}} \times f_j^i \times \Delta^{n_{i+1}} \times \cdots \times \Delta^{n_k} | \; 1 \le i \le k, \; 0 \le j \le n_i\},$$

where $\{f_0^i, \ldots, f_{n_i}^i\}$ is the set of facets of the simplex Δ^{n_i}. Note that P has $(n+k)$-facets and the intersection $\bigcap_{j \neq 0} F_j^i$ is nonempty. Hence, B_k can be represented by a $(n \times k)$ matrix $A = [a_{ij}]$ by choosing $F_l = F_j^i$ for $l = n_1 + \cdots + n_{i-1} + j$ and $1 \le j \le n_i$ and $F_l = F_0^i$ for $l = n + i$. Following [2, 3], one can see A as a $(k \times k)$ vector matrix $A = [\mathbf{v}_{ij}]$ where $\mathbf{v}_{ij} \in \mathbb{Z}_2^{n_i}$. Here $\mathbf{v}_{ij}$ is the column vector whose l-th entry is $a_{n_1 + \cdots + n_{i-1} + l, j}$.

Note that facets in $\mathcal{F} \backslash \{F_{j_1}^1, \cdots, F_{j_k}^k\}$ intersect at a vertex for every $0 \le j_i \le n_i$ and $1 \le i \le k$. Moreover, a family of facets containing the set $\{F_0^i, \cdots, F_{n_i}^i\}$ has an empty intersection for any $1 \le i \le k$. Let $A_{l_1 \cdots l_k}$ be a $(k \times k)$ matrix whose j-th row is the l_j-th row of A for $1 \le l_i \le n_i$ and $1 \le i \le k$. In [3], using these facts, it is shown that the characteristic function corresponding to A satisfies the nonsingularity condition if and only if every principal minor of $A_{l_1\cdots l_k}$ is 1 for all $1 \le l_i \le n_i$ and $1 \le i \le k$. This forces $(\mathbf{v}_{ii})_t = 1$ for all $1 \le i \le k$ and $1 \le t \le n_i$.

Note that the Stanley-Reisner ring of P is

$$\mathbb{Z}_2[x_{10}, \cdots, x_{1n_1}, \cdots, x_{k0}, \cdots, x_{kn_k}]/I$$

where I is the homogeneous ideal generated by monomial products $x_{i0}\cdots x_{in_i}$, $1 \le i \le k$. In this notation, x_{ij} corresponds to $x_{n_1+\cdots+n_{i-1}+j}$ when $1 \le j \le n_i$ and to x_{n+i} when $j = 0$ in the equation (1.2). Therefore, the second Stiefel-Whitney class of B_k is equal to

$$w_2(M) = \sum_{i=1}^{k} \alpha_i \cdot x_{i0}^2 + \sum_{1\le i<j\le k} \beta_{ij} \cdot x_{i0} \cdot x_{j0}$$

modulo I where α_i and β_{ij} are as given in Proposition 2.3. From now on, we assume that $n_i = 1$ for $1 \le i \le l$ and $n_i > 1$, otherwise. This means that the only relations involving the monomials of degree 2 are

$$x_{i0}^2 = \sum_{j\ne i} \mathbf{v}_{ij} \cdot x_{i0} \cdot x_{j0}$$

for $1 \le i \le l$ (here, the vector $\mathbf{v}_{ij} \in \mathbb{Z}_2$ is considered a scalar). Therefore, we have

$$\begin{aligned} w_2(M) \quad = \quad & \sum_{i=l+1}^{k} \alpha_i \cdot x_{i0}^2 + \sum_{l\le i<j\le k} \beta_{ij} \cdot x_{i0} \cdot x_{j0} \\ & + \sum_{i<j\le l} (\beta_{ij} + \mathbf{v}_{ij} \cdot \alpha_i + \mathbf{v}_{ji} \cdot \alpha_j) \cdot x_{i0} \cdot x_{j0} \\ & + \sum_{i<l+1\le j\le k} (\beta_{ij} + \mathbf{v}_{ij} \cdot \alpha_i) \cdot x_{i0} \cdot x_{j0} \end{aligned} \tag{3.1}$$

Theorem 3.1 *The generalized real Bott manifold B_k has a spin structure if and only if the following conditions are satisfied:*

i) $A_i \cdot A_i \equiv 1 \pmod 2$ *when* $i \le l$ *and* $A_i \cdot A_i \equiv 3 \pmod 4$*; otherwise,*

ii) $A_i \cdot A_j \equiv 0 \pmod 2$ *for all* $l \le i < j \le k$*,*

iii) $A_i \cdot A_j$ *and* $\dfrac{\mathbf{v}_{ij} \cdot (A_i \cdot A_i + 1) + \mathbf{v}_{ji} \cdot (A_j \cdot A_j + 1)}{2}$ *have the same parity when* $1 \le i < j \le l$.

iv) $A_i \cdot A_j$ *and* $\dfrac{\mathbf{v}_{ij} \cdot (A_i \cdot A_i + 1)}{2}$ *have the same parity when* $1 \le i < l + 1 \le j \le k$.

Proof The manifold B_k has a spin structure if and only if it is orientable and w_2 vanishes. Recall that the manifold B_k is orientable if and only if $A_i \cdot A_i$ is congruent to 1 modulo 2. In this case, $\beta_{ij} \equiv A_i \cdot A_j$ modulo 2. Then the theorem follows from the equation (3.1) and the fact that $\binom{1+A_j\cdot A_j}{2}$ have the same parity with $\frac{A_i \cdot A_j + 1}{2}$. □

It is well-known that the vector matrix A is equivalent to an upper triangular one in which the entries of the diagonal vectors are all 1 via conjugation by a permutation matrix [3, Lemma 5.1]. Under this assumption, the above theorem is equivalent to the [6, Theorem 4.7]. In [6, Theorem 4.7], the ordering in the product is chosen so that the last $k-l$ of the simplices have dimension 1. Moreover, T_s and T_{rs} in [6, Theorem 4.7] are equivalent to $\binom{A_s \cdot A_s}{2}$ and $A_r \cdot A_s$, respectively and the orientability condition is equivalent to $A_s \cdot A_s \equiv 1 \pmod 2$.

By Theorem 3.1, it follows that the converse of Corollary 2.4 is also true when $l = 0$.

Corollary 3.2 *The generalized real Bott manifold with $l = 0$ has a spin structure if and only if $A_i \cdot A_i \equiv 3 \pmod 4$ and $A_i \cdot A_j \equiv 0 \pmod 2$ for all $1 \leq i < j \leq k$, where A is the reduced matrix.*

Example 3.3 *Let $P = \Delta^2 \times \Delta^3 \times \Delta^5$ and B be a 3-step generalized Bott manifold corresponding to the reduced matrix*

$$A = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right].$$

Then B has a spin structure by the above Corollary.

The following corollary also follows from the Proposition 5.1 of [13].

Corollary 3.4 *If a generalized real Bott manifold over $P = \prod_{t=1}^{k} \Delta^{n_t}$ with $l = 0$ admits a spin structure, then $n_j \equiv 3$ (mod 4) for some j.*

Proof Let $P = \prod_{t=1}^{k} \Delta^{n_t}$ and M be a small cover over P with an associated vector matrix A. If B is a vector matrix obtained by conjugating A via permutation matrix P_σ then $A_i \cdot A_i = B_{\sigma(i)} \cdot B_{\sigma(i)}$ and $A_i \cdot A_j = B_{\sigma(i)} \cdot B_{\sigma(j)}$. Therefore, we can assume that A is an upper triangular vector matrix in which the entries of the diagonal vectors are all 1. Then we have $A_1 \cdot A_1 = n_1$. So if M has a spin structure, $n_1 \equiv 3$ (mod 4). □

When $l = k$, we have the following result.

Corollary 3.5 *The real Bott manifold B_k has a spin structure if and only if*

i) $A_i \cdot A_i \equiv 1 \pmod 2$, $1 \leq i \leq k$,

ii) $A_i \cdot A_j$ *and* $\dfrac{\mathbf{v}_{ij} \cdot (A_i \cdot A_i + 1) + \mathbf{v}_{ji} \cdot (A_j \cdot A_j + 1)}{2}$ *have the same parity when* $1 \leq i < j \leq l$.

The above corollary is equivalent to Theorem 3.2 in [5] where A is assumed to be upper-triangular. In particular, Theorem 1.2 in [7] directly follows from the corollary.

Example 3.6 *Let $P = I \times \Delta^2 \times \Delta^2$. Then a small cover B_3 over P corresponds to a vector matrix*

$$A = \left[\begin{array}{c|c|c} 1 & a_{12} & a_{13} \\ \hline a_{21} & 1 & a_{23} \\ a_{31} & 1 & a_{33} \\ \hline a_{41} & a_{42} & 1 \\ a_{51} & a_{52} & 1 \end{array}\right].$$

If B_3 has a spin structure, then $a_{12} + a_{42} + a_{52} = 1$ and $a_{13} + a_{23} + a_{33} = 1$ by part i of Theorem 3.1 and $a_{12}a_{13} + a_{23} + a_{33} + a_{42} + a_{52} \equiv 0 \pmod 2$ by part ii of Theorem 3.1. By substituting the first two equations to the last one, we get $a_{12} = a_{13} = 0$ and hence $a_{42} + a_{52} = a_{23} + a_{33} = 1$. On the other hand, at least one of the vectors $\begin{pmatrix} a_{42} \\ a_{52} \end{pmatrix}$ and $\begin{pmatrix} a_{23} \\ a_{33} \end{pmatrix}$ must be zero by the nonsingularity condition. Hence, there is no small cover over $I \times \Delta^2 \times \Delta^2$ with a spin structure when $n \geq 2$.

It is well-known that when n_i's are all even, there is no orientable small cover over P [1]. Hence, small covers over P have no spin structures when all the n_i's are even. In the next section, we generalize the above example to have a nonexistence result for every small cover over $P = I \times \Delta^{2n_1} \times \cdots \times \Delta^{2n_k}$ for $k \geq 2$. When $k = 1$, a small cover over $I \times \Delta^{4t}$ does not have a spin structure since $A_2 \cdot A_2$ is either $4t$ or $4t + 1$. However, the small cover over $P = I \times \Delta^{4t+2}$ corresponding to a characteristic function λ which sends F_0^1 to e_1 and F_0^2 to $e_1 + e_2 + \cdots + e_{4t+3}$ has a spin structure.

Given a dimension function $\omega : \{1, 2, \ldots, n\} \to \mathbb{N}$, let I_ω be the identity vector matrix associated to ω, i.e. the (i, j)-entry of I_ω is 1 when $\omega(1) + \cdots + \omega(j - 1) + 1 \leq i \leq \omega(1) + \cdots + \omega(j)$, and 0, otherwise. To generalize Theorem 1.2 in [7] to our case, we denote the matrix $A - I_\omega$, where $\omega(i) = n_i$ by B.

Proposition 3.7 *The generalized real Bott manifold with an associated matrix B has a spin structure if and only if for all $1 \leq i < j \leq k$, the generalized Bott manifold corresponding to B_{ij} has a spin structure, where B_{ij} is the vector matrix whose l-th column is B_l if $l = i, j$ and 0, otherwise.*

4. ω-weighted digraph interpretation

In [2], Choi shows that there is a bijection between the set of real Bott manifolds and acyclic digraphs with n-labeled vertices which sends B_k to a graph whose adjacency matrix is $A - I_k$. In [5, Theorem 4.5], Dsouza and Uma give an interpretation of existence of a spin structure for real Bott manifolds in terms of associated digraphs. In this section, we generalize [5, Theorem 4.5] to small covers over a product of simplices.

Definition 4.1 *Given a dimension function $\omega : V \to \mathbb{N}$, a digraph with vertex set V is called ω-vector weighted if every edge (u, v) is assigned a nonzero vector $\mathbf{w}(\mathbf{u}, \mathbf{v})$ in $\mathbb{Z}_2^{\omega(u)}$.*

Let G be a ω-vector weighted digraph. For convenience, we take the weight of (u, v) to be the zero vector in $\mathbb{Z}_2^{\omega(u)}$ when there is no edge from u to v. If (u, v) is an edge of G, then u is called an in-neighbor of

v and v is called an out-neighbor of u. Let $N_G^-(v)$ and $N_G^+(v)$ denote the set of in-neighbors and out-neighbors of v in G. We define in-degree $\deg^-(v)$ and out degree $\deg^+(v)$ of v as follows:

$$\deg^-(v) = \sum_{u \in N_G^-(v)} \mathbf{w}(\mathbf{u},\mathbf{v}) \cdot \mathbf{w}(\mathbf{u},\mathbf{v})$$

$$\deg^+(v) = \sum_{z \in N_G^+(v)} \mathbf{w}(\mathbf{v},\mathbf{z}) \cdot \mathbf{w}(\mathbf{v},\mathbf{z}).$$

We can consider a digraph as an ω-weighted digraph with $\omega(i) = 1$ for each i. In this case, the notion of in-degree and out-degree of a vertex of a ω-weighted digraph agrees with those of digraphs. An adjacency matrix $A_\omega(G)$ of an ω-weighted digraph G with labeled vertices $v_1, \cdots, v_n$ is defined to be an $(n \times n)$ ω-vector matrix whose (i,j)-th entry is $\mathbf{w}(\mathbf{v_i},\mathbf{v_j})$. An ω-vector weighted digraph is called acyclic if it does not contain any directed cycle.

As shown in [9], there is a one-to-one correspondence between the set of small covers over the product $P = \Delta^{n_1} \times \cdots \Delta^{n_k}$ and the set of acylic ω-weighted digraphs where $\omega : \{v_1, \cdots, v_k\} \to \mathbb{N}$ is defined by $\omega(v_i) = n_i$. The correspondence is obtained by sending a small cover with an associated matrix A to a ω-weighted digraph whose adjacency matrix is $A - I_\omega$. For a given small cover B over P, we denote the associated acyclic ω-weighted digraph by D_B. Recall that the dot product of a vector $\mathbf{v}$ over $\mathbb{Z}_2$ with itself is equal to the number of nonzero coordinates of $\mathbf{v}$. Therefore, $A_i \cdot A_j$ is equal to $A_\omega(D_B)_i \cdot A_\omega(D_B)_i + \omega(i)$ when $i = j$ and $A_\omega(D_B)_i \cdot (A_\omega D_B)_j + \mathbf{w}(\mathbf{v_i},\mathbf{v_j}) \cdot \mathbf{w}(\mathbf{v_i},\mathbf{v_j}) + \mathbf{w}(\mathbf{v_j},\mathbf{v_i}) \cdot \mathbf{w}(\mathbf{v_j},\mathbf{v_i})$, otherwise. Moreover, $A_\omega(D_B)_i \cdot A_\omega(D_B)_i$ is equal to $\deg^-(v_i)$. Let M_{ij} be the sum of $\omega(u, v_i) \cdot \omega(u, v_j)$ where u runs in the set of in-neighbor of both v_i and v_j. Then $A_\omega(D_B)_i \cdot A_\omega(D_B)_j = M_{ij}$.

Theorem 4.2 *The generalized real Bott manifold B with associated w-weighted digraph D_B has a spin structure if and only if the following conditions are satisfied:*

i) Indegree of a vertex v of D_B is even if $\omega(v) = 1$ and is congruent to $-\omega(v) + 3$ modulo 4, otherwise,

ii) M_{ij} is even if v_i is neither in-neighbor nor out-neighbor v_j with $i \neq j$,

iii) M_{ij} and $\dfrac{\mathbf{w}(\mathbf{v_i},\mathbf{v_j}) \cdot \deg^-(v_i)}{2}$ have the same parity when v_i is an in-neighbor of v_j with $\omega(v_i) = 1$,

iv) M_{ij} and $\mathbf{w}(\mathbf{v_i},\mathbf{v_j}) \cdot \mathbf{w}(\mathbf{v_i},\mathbf{v_j})$ have the same parity when v_i is an in-neighbor of v_j with $\omega(v_i) > 1$.

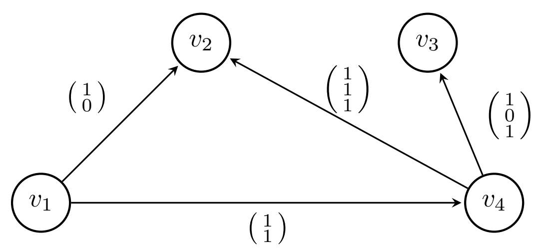
Proof If v_i is neither in-neighbor nor out-neighbor v_j, conditions iii and iv of Theorem 3.1 is equivalent to the statement that $A_i \cdot A_j$ is even for $i \neq j$. In this case, we also have $M_{ij} = A_i \cdot A_j$. Otherwise, either v_i or v_j is an in-neighbor of the other one. Since $M_{ij} = M_{ji}$, without loss of generality, we can assume that v_i is. Then $A_i \cdot A_j = M_{ij} + \mathbf{w}(\mathbf{v_i},\mathbf{v_j}) \cdot \mathbf{w}(\mathbf{v_i},\mathbf{v_j})$. Therefore, when $\omega(v_i) = 1$, combining conditions iii and iv of Theorem 3.1, one obtains condition iii above. When $\omega(v_i) > 1$, iv can be obtained by combining parts ii and iv of Theorem 3.1.

□

Example 4.3 *Let $P = \Delta^2 \times \Delta^3 \times \Delta^3 \times \Delta^3$, and B be a 4-step generalized Bott manifold corresponding to the reduced matrix*

$$A = \left[\begin{array}{c|c|c|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right].$$

Then $\omega : \{1,2,3,4\} \to \mathbb{N}$ with $\omega(1) = 2$, $\omega(2) = \omega(3) = \omega(4) = 3$ and an ω-weighted digraph corresponding to B is as given below. Since $\deg^-(v_3) = 2$, B has no spin structure by part i of the above theorem.



G

Corollary 4.4 *A small cover over $P = I \times \Delta^{2n_1} \cdots \times \Delta^{2n_k}$ does not have a spin structure when $k \geq 2$.*

Proof Let M be a small cover over P, and G be the associated acyclic ω-weighted digraph. Assume for a contradiction that M has a spin structure. The underlying digraph of G has a source, say v_i. Since indegree of v_i is zero, the weight of v_i must be 1. Let v_j be a source of the digraph obtained by removing v_i from the underlying digraph and, v_k be a source of the digraph obtained by removing v_i and v_j. Then the in-degrees of vertices v_j and v_k are $\mathbf{w}(\mathbf{v_i}, \mathbf{v_j})$ and $\mathbf{w}(\mathbf{v_i}, \mathbf{v_k}) + \mathbf{w}(\mathbf{v_j}, \mathbf{v_k}) \cdot \mathbf{w}(\mathbf{v_j}, \mathbf{v_k})$, respectively. By part i of the above theorem, both of them must be odd. In particular $\mathbf{w}(\mathbf{v_i}, \mathbf{v_j}) = 1$ and, $\mathbf{w}(\mathbf{v_i}, \mathbf{v_k})$ and $\mathbf{w}(\mathbf{v_j}, \mathbf{v_k}) \cdot \mathbf{w}(\mathbf{v_j}, \mathbf{v_k})$ have different parities. On the other hand, $M_{jk} = \mathbf{w}(\mathbf{v_i}, \mathbf{v_k})$ as a dot product of j-th and k-th column of the adjacency matrix. Since M_{jk} and $\mathbf{w}(\mathbf{v_j}, \mathbf{v_k}) \cdot \mathbf{w}(\mathbf{v_j}, \mathbf{v_k})$ have different parities, v_j cannot be an in-neighbor of v_k, by part iv of the above theorem. This means that $\mathbf{w}(\mathbf{v_j}, \mathbf{v_k})$ is the zero vector. Hence, by part ii, M_{jk} must be even and hence $\mathbf{w}(\mathbf{v_i}, \mathbf{v_k}) = 0$. Contradiction. □

5. Higher Stiefel-Whitney classes

It is well-known that the Stiefel-Whitney classes w_i of a smooth manifold satisfy the Wu formula [11]

$$Sq^i(w_j) = \sum_{t=0}^{i} \binom{j+t-i-1}{t} w_{i-t} w_{j+t}$$

where Sq^i denotes the Steenrod squares. Therefore, for any $i \leq j$ with $i+j=m$, one has

$$\binom{j-1}{i} w_m = Sq^i(w_m) + \sum_{t=0}^{i-1} \binom{j+t-i-1}{t} w_{i-t} w_{j+t}.$$

Substituting $m=3$ and $i=1$ gives $w_3 = Sq^1(w_2) + w_1 w_2$. This means that whenever w_1 and w_2 are both zero, so is w_3. Therefore, the following result directly follows from Corollary 3.2.

Proposition 5.1 *The first three Stiefel-Whitney classes of a small cover over a product of simplices of dimensions greater than or equal to 2 are zero if and only if $A_i \cdot A_i \equiv 3 \pmod 4$ and $A_i \cdot A_j \equiv 0 \pmod 2$ for all $i \neq j$ where A is the associated reduced matrix.*

Now we show that the conditions of the above proposition are not necessary for $w_3(M)$ to be zero. For this, let $k_S(A)$ denote the size of the set $\{t|\ a_{ts} = 1 \text{ for all } s \in S\}$ for any $S \subseteq \{1, 2, \cdots, k\}$. We write k_S instead of $k_S(A)$ when it is clear from the context. Note that $k_{\{i\}} = A_i \cdot A_i$ and $k_{\{i,j\}} = A_i \cdot A_j$.

Theorem 5.2 *The third Stiefel-Whitney class of a small cover M over $P = \prod_{i=1}^{k} \Delta^{n_i}$ modulo I is equal to*

$$w_3(M) = \sum_{1 \leq i \leq k} \binom{k_{\{i\}}+1}{3} x_{i0}^3 + \sum_{i \neq j} P(i,j) x_{i0}^2 x_{j0} + \sum_{i_1 < i_2 < i_3} Q(i_1, i_2, i_3) x_{i_1 0} x_{i_2 0} x_{i_3 0}$$

where

$$P(i,j) = \binom{k_{\{i\}}+1}{2} \cdot (k_{\{j\}}+1) - k_{\{i\}} \cdot k_{\{i,j\}}, \tag{5.1}$$

$$Q(i_2, i_2, i_3) = \left(\prod_{p=1}^{3} (k_{\{i_p\}}+1) \right) + \sum_{p=1}^{3} (k_{\{i_p\}}+1) \cdot k_{\{i_1,i_2,i_3\}-\{i_p\}}. \tag{5.2}$$

Proof One can easily find the coefficient of x_{i0}^3 as in the Stiefel-Whitney classes of smaller dimensions. The coefficient of $x_{i0}^2 x_{j0}$ is equal to the coefficient of $y_1^2 y_2$ in the polynomial

$$(1+y_1)^{k_{\{i\}}-k_{\{i,j\}}+1}(1+y_2)^{k_{\{j\}}-k_{\{i,j\}}+1}(1+y_1+y_2)^{k_{\{i,j\}}}$$

as before. We can pick y_2 either from the factor $(1+y_2)^{k_{\{j\}}-k_{\{i,j\}}+1}$ or from the factor $(1+y_1+y_2)^{k_{\{i,j\}}}$. If we chose it from the second one, we have to choose y_1^2 from $(1+y_1)^{k_{\{i\}}-k_{\{i,j\}}+1}(1+y_1+y_2)^{k_{\{i,j\}}-1}$. Therefore, we have

$$\begin{aligned} P(i,j) &= (k_{\{j\}} - k_{\{i,j\}} + 1) \binom{k_{\{i\}}+1}{2} + k_{\{i,j\}} \binom{k_{\{i\}}}{2} \\ &= \binom{k_{\{i\}}+1}{2} (k_{\{j\}}+1) - k_{\{i,j\}}. \end{aligned}$$

The coefficient of the monomial $x_{i_1 0} x_{i_2 0} x_{i_3 0}$ in $w_3(M)$ is equal to the coefficient of $y_1 y_2 y_3$ in the product

$$\left(\prod_{j=1}^{3} (1+y_j)^{k_{\{i_j\}} - \sum_{p \neq j} k_{\{i_p, i_j\}} + k_{\{i_1,i_2,i_3\}} + 1} \right) \cdot \left(\prod_{p \neq q} (1+y_p+y_q)^{k_{\{i_p,i_q\}} - k_{\{i_1,i_2,i_3\}}} \right) \cdot \left(1+y_1+y_2+y_3\right)^{k_{\{i_1,i_2,i_3\}}} \tag{5.3}$$

Now we can choose y_3 from either of the factors $(1+y_3)^{k_{\{i_3\}}-\sum_{p\neq 3} k_{\{i_p,i_3\}}+k_{\{i_1,i_2,i_3\}}+1}$, $(1+y_1+y_3)^{k_{\{i_1,i_3\}}-k_{\{i_1,i_2,i_3\}}}$, $(1+y_2+y_3)^{k_{\{i_2,i_3\}}-k_{\{i_1,i_2,i_3\}}}$ or $(1+y_1+y_2+y_3)^{k_{\{i_1,i_2,i_3\}}}$. Therefore, we have

$$\begin{aligned} Q(i_1,i_2,i_3) &= \Big(k_{\{i_3\}}+k_{\{i_1,i_2,i_3\}}+1-\sum_{p\neq 3} k_{\{i_p,i_3\}}\Big)\cdot\Big(\big(1+k_{\{i_1\}}\big)\big(1+k_{\{i_2\}}\big)+k_{\{i_1,i_2\}}\Big) \\ &\quad +\Big(\sum_{p=1}^{2}\big(k_{\{i_p,i_3\}}-k_{\{i_1,i_2,i_3\}}\big)\cdot\big(k_{\{i_p\}}\big(1+k_{\{i_1,i_2\}-\{i_p\}}\big)+k_{\{i_1,i_2\}}\big)\Big) \\ &\quad +k_{\{i_1,i_2,i_3\}}\cdot\big(k_{i_1}k_{i_2}+k_{\{i_1,i_2\}}-1\big). \end{aligned}$$

By algebraically manipulating terms, one can easily obtain the desired formula for $Q(i_1,i_2,i_3)$. □

Note that the above theorem is also true for small covers over an arbitrary simple convex polytope when the cohomology classes are represented appropriately. Moreover, one can easily find a formula for the third Stiefel-Whitney class of a small cover over a product of simplices as in the equation (3.1) by taking the relations coming from I into account. Here, we focus on the case where the dimension of simplies are all greater than equal to 3 in which I does not contain any relation of dimension 3 to obtain a simple formula.

Corollary 5.3 *Let M be a small cover over $P=\prod_{i=1}^{k}\Delta^{n_i}$ with $n_i\geq 3$. Then $w_3(M)=0$ if and only if the following conditions hold:*

i) $k_{\{i\}}\not\equiv 2 \pmod 4$,

ii) If $k_{\{i\}}$ or $k_{\{j\}}$ is odd then $k_{\{i,j\}}\equiv 1 \pmod 2$ if and only if either $k_{\{i\}}\equiv 0 \pmod 4$ and $k_{\{j\}}\equiv 1 \pmod 4$ or vice a versa,

iii) If $k_{\{i_1\}}\equiv k_{\{i_2\}}\equiv k_{\{i_3\}}\equiv 0 \pmod 4$ for $i_1<i_2<i_3$ then $k_{\{i_1,i_2\}}+k_{\{i_1,i_3\}}+k_{\{i_2,i_3\}}\equiv 1 \pmod 2$.

Proof Since I does not contain a monomial of degree less than or equal to 3 when $P=\prod_{i=1}^{k}\Delta^{n_i}$ with $n_i\geq 3$, $w_3(M)$ is zero if and only if $\binom{k_{\{i\}}+1}{3}\equiv 0 \pmod 2$ for all i, $P(i,j)\equiv 0 \pmod 2$ for all $i\neq j$ and $Q(i_1,i_2,i_3)\equiv 0 \pmod 2$ for all $i_1<i_2<i_3$. Here the first condition is equivalent to condition i. If neither $k_{\{i\}}$ nor $k_{\{j\}}$ is divisible by 4 then $P(i,j)\equiv P(j,i)\equiv 0 \pmod 2$ if and only if $k_{\{i,j\}}\equiv 0 \pmod 2$. Let $k_{\{i\}}\equiv 0 \pmod 4$. Then $P(i,j)\equiv 0 \pmod 2$ for all $j\neq i$. Moreover, $P(j,i)\equiv\binom{k_{\{j\}}+1}{2}-k_{\{j\}}k_{\{i,j\}}$ is even if and only if either $k_{\{j\}}\equiv 0 \pmod 4$ or $k_{\{j\}}\equiv 1 \pmod 4$ and $k_{\{i,j\}}\equiv 1 \pmod 2$, or $k_{\{j\}}\equiv 3 \pmod 4$ and $k_{\{i,j\}}\equiv 0 \pmod 2$. Therefore, when condition i holds, $P(i,j)\equiv P(j,i)\equiv 0 \pmod 2$ if and only if M satisfies condition ii.

Now suppose that conditions i and ii hold. If $k_{\{i_1\}},k_{\{i_2\}}$ and $k_{\{i_3\}}$ are all divisible by 4, then we have

$$Q(i_1,i_2,i_3)\equiv 1+k_{\{i_2,i_3\}}+k_{\{i_1,i_3\}}+k_{\{i_1,i_2\}} \pmod 2$$

and hence, $Q(i_1,i_2,i_3)\equiv 0 \pmod 2$ if and only if iii holds for the triple (i_1,i_2,i_3). Now suppose that at least one of them is not divisible by 4. WLOG, assume that $k_{\{i_1\}}\not\equiv 0 \pmod 4$. Then $(1+k_{\{p\}})k_{\{i_1,p\}}\equiv 1$ (mod

2) if and only if $k_{\{p\}} \equiv 0 \pmod 4$ and $k_{\{i_1\}} \equiv 1 \pmod 4$. Therefore, we have

$$Q(i_1,i_2,i_3) \equiv (1+k_{\{i_2\}})k_{\{i_1,i_3\}} + (1+k_{\{i_3\}})k_{\{i_1,i_2\}} \equiv 0 \pmod 2.$$

This proves the theorem. □

Since $k_i = \deg^-(v_i) + \omega(i)$ and $k_{ij} = M_{ij} + \mathbf{w}(\mathbf{v_i},\mathbf{v_j}) \cdot \mathbf{w}(\mathbf{v_i},\mathbf{v_j}) + \mathbf{w}(\mathbf{v_j},\mathbf{v_i}) \cdot \mathbf{w}(\mathbf{v_j},\mathbf{v_i})$, we have the following.

Corollary 5.4 *Let D_M be an ω-weighted acyclic digraph associated to a small cover M over $P = \prod_{i=1}^{k} \Delta^{n_i}$ with $n_i \geq 3$. Then $w_3(M) = 0$ if and only if the following conditions hold for vertices of D_M:*

i) $\deg^-(v_i) + \omega(i) \not\equiv 2$ *(mod 4),*

ii) If $\deg^-(v_i) + \omega(i)$ *or* $\deg^-(v_j) + \omega(j)$ *is odd then* $M_{ij} + \mathbf{w}(\mathbf{v_i},\mathbf{v_j}) \cdot \mathbf{w}(\mathbf{v_i},\mathbf{v_j}) + \mathbf{w}(\mathbf{v_j},\mathbf{v_i}) \cdot \mathbf{w}(\mathbf{v_j},\mathbf{v_i}) \equiv 1$ *(mod 2) if and only if either* $\deg^-(v_i) + \omega(i) \equiv 0$ *(mod 4) and* $\deg^-(v_j) + \omega(j) \equiv 1$ *(mod 4) or vice versa,*

iii) If $\deg^-(v_{i_1}) + \omega(i_1) \equiv \deg^-(v_{i_2}) + \omega(i_2) \equiv \deg^-(v_{i_3}) + \omega(i_3) \equiv 0$ *(mod 4), then*

$$\sum_{p \neq q} \Big(M_{i_p i_q} + \mathbf{w}(\mathbf{v_{i_p}},\mathbf{v_{i_q}}) \cdot \mathbf{w}(\mathbf{v_{i_p}},\mathbf{v_{i_q}}) + \mathbf{w}(\mathbf{v_{i_q}},\mathbf{v_{i_p}}) \cdot \mathbf{w}(\mathbf{v_{i_q}},\mathbf{v_{i_p}}) \Big) \equiv 1 \ (mod\ 2).$$

As shown above, when M is a generalized Bott manifold, the Stiefel-Whitney classes of M of dimensions less than or equal to 3 can be written in terms of the dot products of columns of the associated reduced vector matrix A. It is natural to ask whether this is true for all dimensions. The following theorem gives an affirmative answer to this question.

Theorem 5.5 *The fourth Stiefel-Whitney class of a small cover M over $P = \prod_{i=1}^{k} \Delta^{n_i}$ modulo I is equal to*

$$\begin{aligned} w_4(M) &= \sum \binom{k_{\{i\}}+1}{4} x_{i0}^4 + \sum P_1(i,j) x_{i0}^3 x_{j0} + \sum P_2(i,j) x_{i0}^2 x_{j0}^2 \\ &\quad + \sum Q(i_1,i_2,i_3)) x_{i_1 0}^2 x_{i_2 0} x_{i_3 0} + \sum R(i_1,i_2,i_3,i_4) x_{i_1 0} x_{i_2 0} x_{i_3 0} x_{i_4 0} \end{aligned}$$

where

$$
\begin{aligned}
P_1(i,j) &= \binom{k_{\{i\}}+1}{3}\cdot\left(k_{\{j\}}+1\right)-\binom{k_{\{i\}}}{2}\cdot k_{\{i,j\}},\\
P_2(i,j) &= \binom{k_{\{i\}}+1}{2}\cdot\binom{k_{\{j\}}+1}{2}-k_{\{i\}}k_{\{j\}}k_{\{i,j\}}+\binom{k_{\{i,j\}}}{2},\\
Q(i_1,i_2,i_3) &= \binom{k_{\{i_1\}}+1}{2}\Big(\left(k_{\{i_2\}}+1\right)\left(k_{\{i_3\}}+1\right)-k_{\{i_2,i_3\}}\Big)-k_{\{i_1\}}\Big(\sum_{p\neq 1}k_{\{i_1,i_p\}}\left(k_{\{i_2,i_3\}-\{i_p\}}+1\right)\Big)\\
&\quad +k_{\{i_1,i_2\}}k_{\{i_1,i_3\}}-k_{\{i_1,i_2,i_3\}},\\
R(i_1,i_2,i_3,i_4) &= \Big(\prod_{p=1}^{4}\left(k_{\{i_p\}}+1\right)\Big)-\sum_{p\neq q}\Big(\left(k_{\{i_p\}}+1\right)\left(k_{\{i_q\}}+1\right)-\frac{k_{\{i_p,i_q\}}}{2}\Big)\cdot k_{\{i_1,i_2,i_3,i_4\}-\{i_p,i_q\}}.
\end{aligned}
$$

Proof Since the rest can be found similarly, we only provide a proof for the formula for $Q(i_1,i_2,i_3)$. Here $Q(i_1,i_2,i_3)$ is equal to the coefficient of $y_1y_2y_3$ in (5.3). One can choose y_3 from either of the factors $(1+y_3)^{k_{\{i_3\}}-\sum_{p\neq 3}k_{\{i_p,i_3\}}+k_{\{i_1,i_2,i_3\}}+1}$, $(1+y_1+y_3)^{k_{\{i_1,i_3\}}-k_{\{i_1,i_2,i_3\}}}$, $(1+y_2+y_3)^{k_{\{i_2,i_3\}}-k_{\{i_1,i_2,i_3\}}}$, or $(1+y_1+y_2+y_3)^{k_{\{i_1,i_2,i_3\}}}$. Therefore, we have

$$
\begin{aligned}
Q(i_1,i_2,i_3) &= \Big(k_{\{i_3\}}+k_{\{i_1,i_2,i_3\}}+1-\sum_{p\neq 3}k_{\{i_p,i_3\}}\Big)\cdot\left[\binom{k_{\{i_1\}}+1}{2}\cdot\left(k_{\{i_2\}}+1\right)-k_{\{i_1\}}k_{\{i_1,i_2\}}\right]\\
&\quad +\left(k_{\{i_1,i_3\}}-k_{\{i_1,i_2,i_3\}}\right)\cdot\left[\binom{k_{\{i_1\}}}{2}\cdot\left(k_{\{i_2\}}+1\right)-\left(k_{\{i_1\}}-1\right)k_{\{i_1,i_2\}}\right]\\
&\quad +\left(k_{\{i_2,i_3\}}-k_{\{i_1,i_2,i_3\}}\right)\cdot\left[\binom{k_{\{i_1\}}+1}{2}\cdot k_{\{i_2\}}-k_{\{i_1\}}k_{\{i_1,i_2\}}\right]\\
&\quad +k_{\{i_1,i_2,i_3\}}\cdot\left[\binom{k_{\{i_1\}}}{2}\cdot k_{\{i_2\}}-\left(k_{\{i_1\}}-1\right)\left(k_{\{i_1,i_2\}}-1\right)\right].
\end{aligned}
$$

Since the sum of the first factors of each term in the RHS of the equation is $k_{\{i_3\}}+1$, the result easily follows. □

Corollary 5.6 *Let M be a small cover over $P=\prod_{i=1}^{k}\Delta^{n_i}$ with $n_i\geq 4$. Then $w_4(M)=0$ if and only if the following conditions hold:*

i) $k_{\{i\}}\equiv 0,1,2$ *or* 7 *(mod* 8*),*

ii) $k_{\{i,j\}}$ *must satisfy the following table:*

$k_{\{i\}}$ *(mod 8)*	$k_{\{j\}}$ *(mod 8)*	$k_{\{i,j\}}$ *(mod 4)*
0	0	1
0	*1*	0 *or* 1
0	2	1
1	1	2
1	2	2
2	2	3
-	7	0

iii) $k_{\{i,j,l\}}$ *must satisfy the following table*

$k_{\{i\}}$ *(mod 8)*	$k_{\{j\}}$ *(mod 8)*	$k_{\{l\}}$ *(mod 8)*	$k_{\{i,j,l\}}$ *(mod 2)*
0	0	0	1
0	0	*1*	$k_{\{0,1\}}$
0	0	2	1
0	1	1	$k_{\{0,1\}}$
0	1	2	$k_{\{0,1\}}$
0	2	2	1
1	1	1	0
1	1	2	0
1	2	2	0
2	2	2	1

Proof Note that $\binom{k_{\{i\}}+1}{4} \equiv 0 \pmod 2$ if and only if $k_{\{i\}}$ satisfies the condition i. Here $k_{\{i,j\}}$ and $k_{\{i,j,l\}}$ depend on the values of $k_{\{i\}}, k_{\{j\}}$ up to modulo 8, and $k_{\{i\}}, k_{\{j\}}$ and $k_{\{l\}}$ up to modulo 8, respectively. Let θ_i denote the integer between 0 and 7 that is congruent to $k_{\{i\}}$ modulo 8.

Suppose that $w_4(M) = 0$. Therefore, $P_1(i,j), P_2(i,j), Q(i_1,i_2,i_3)$ and $R(i_1,i_2,i_3,i_4)$ are zero modulo 2 for all possible combinations. When $\theta_i = 7$, $P_1(i,j) \equiv k_{\{i,j\}}$ and $P_2(i,j) \equiv k_{\{i,j\}} + \binom{k_{\{i,j\}}}{2}$. This gives that $k_{\{i,j\}} \equiv 0 \pmod 4$ when $\theta_i = 7$. When $\theta_i = 2$, $P_1(i,j) \equiv k_{\{i\}} + 1 + k_{\{i,j\}} \pmod 2$ and hence we have $k_{\{i,j\}} \equiv 0 \pmod 2$ when $\theta_j = 1, 7$ and $k_{\{i,j\}} \equiv 1 \pmod 2$ when $\theta_j = 0, 2$. Since when $(\theta_i, \theta_j) = (2,2)$, $P_2(i,j) \equiv 1 + \binom{k_{\{i,j\}}}{2} \pmod 2$, $k_{\{i,j\}} \equiv 3 \pmod 4$. When $\theta_i = 0$, $P_2(i,j) \equiv 0 \pmod 2$ yields $k_{\{i,j\}} \equiv 0$ or 1 (mod 4). In particular, we have $k_{\{i,j\}} \equiv 1 \pmod 4$ when $(\theta_i, \theta_j) = (0,2)$. Similarly, when $\theta_i = 1$, $P_2(i,j) \equiv 0 \pmod 2$ gives $k_{\{i,j\}} \equiv 1$ or $2 \pmod 4$ and hence we have $k_{\{i,j\}} \equiv 2 \pmod 4$ when $(\theta_i, \theta_j) = (1,2)$.

When $\theta_i = 0$ for all $i \in \{i_1, i_2, i_3, i_4\}$, $R(i_1,i_2,i_3,i_4) \equiv 1 + k_{\{i_1,i_2\}} \pmod 2$ and hence it is zero modulo 2 if and only if $k_{\{i_1,i_2\}} \equiv 1 \pmod 2$. Since $k_{\{i_1,i_2\}} \equiv 0$ or $1 \pmod 4$ whenever $\theta_i = 0$, we have $k_{\{i_1,i_2\}} \equiv 1$ (mod 4) in this case. Similarly, when $\theta_i = 1$ for all $i \in \{i_1, i_2, i_3, i_4\}$, $R(i_1,i_2,i_3,i_4) \equiv 0 \pmod 2$ yields $k_{\{i_1,i_2\}} \equiv 2$ (mod 4) since it is either 1 or 2 modulo 4.

Under these assumptions, when $\theta_i = 7$ for one of the i_1, i_2 or i_3, $Q(i_1,i_2,i_3) \equiv k_{\{i_1,i_2,i_3\}} \equiv 0 \pmod 2$. When $(\theta_{i_1}, \theta_{i_2}, \theta_{i_3}) = (2,0,0)$, $Q(i_1,i_2,i_3) \equiv 1 + k_{\{i_1,i_2,i_3\}} \equiv 0 \pmod 2$. Similarly, $Q(i_1,i_2,i_3) \equiv 0 \pmod 2$ for $(\theta_{i_1}, \theta_{i_2}, \theta_{i_3}) = (2, p_1, p_2)$ and $(\theta_{i_1}, \theta_{i_2}, \theta_{i_3}) = (0, q_1, q_2)$ where $0 \leq p_t \leq 2$ and $0 \leq q_t \leq 1$ give the all the remaining restrictions on $k_{\{i_1,i_2,i_3\}}$ and proves the only if part of the theorem. One can easily check that under these restrictions, $w_4(M) = 0$. □

Whenever m is not a power of 2, the Wu formula can be used to express w_m in terms of lower Stiefel-

Whitney classes and their Steenrod squares. Hence, one can conclude that whenever the lower dimensional Stiefel-Whitney classes are zero then so is w_m for $m \neq 2^p$ for any p. Hence, we have the following result.

Corollary 5.7 *Let M be a small cover over $P = \prod_{i=1}^{k} \Delta^{n_i}$ with $n_i \geq 4$ with an associated matrix A. Then the first seven Stiefel-Whitney classes of M are zero if and only if $A_i \cdot A_i \equiv 7 \pmod 8$, $A_i \cdot A_j \equiv 0 \pmod 4$ and $k_{\{i,j,l\}} = |\{t | a_{it} = a_{jt} = a_{lt} = 1\}| \equiv 0 \pmod 2$ for all $i < j < l$.*

Proof By Proposition 5.1 and the above argument, it suffices to show that if $A_i \cdot A_i \equiv 7 \pmod 8$, $A_i \cdot A_j \equiv 0 \pmod 4$ and $k_{\{i,j,l\}} = |\{t | a_{it} = a_{jt} = a_{lt} = 1\}| \equiv 0 \pmod 2$ for all $i < j < l$ then $w_4(M) = 0$. This directly follows from Theorem 5.5. □

When m is a power of 2, for all $i + j = m$, $\binom{j-1}{i}$ is always even and hence one can not use the Wu formula to find w_m. Considering the results of the paper, we believe that for each $m = 2^t$, $k_{\{S\}}$'s where S is a subset of size t of $\{1, 2, \cdots, k\}$ will appear as a coefficient of $w_m(M)$ and we conjecture the following.

Conjecture 5.8 *Let M be a small cover over $P = \prod_{i=1}^{k} \Delta^{n_i}$ with $n_i \geq 2^t$ with an associated matrix A. Then the first $2^{t+1} - 1$ Stiefel-Whitney classes of M are zero if and only if for any $S \subseteq \{1, 2, \cdots, k\}$ of size less than or equal to $t + 1$, $k_S = |\{i| \ a_{si} = 1 \text{ for any } s \in S\}|$ is congruent to -1 modulo 2^{t+1} when $|S| = 1$ and is congruent to 0 modulo $2^{t+1-|S|}$, otherwise.*

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