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# On the rank of generalized order-preserving transformation semigroups

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**Abstract:** For any two non-empty (disjoint) chains $X$ and $Y$, and for a fixed order-preserving transformation $\theta : Y \to X$, let $\mathcal{GO}(X,Y;\theta)$ be the generalized order-preserving transformation semigroup. Let $\mathcal{O}(Z)$ be the order-preserving transformation semigroup on the set $Z = X \cup Y$ with a defined order. In this paper, we show that $\mathcal{GO}(X,Y;\theta)$ can be embedded in $O(Z,Y) = \{\alpha \in \mathcal{O}(Z) : Z\alpha \subseteq Y\}$, the semigroup of order-preserving transformations with restricted range. If $\theta \in \mathcal{GO}(Y,X)$ is one-to-one, then we show that $\mathcal{GO}(X,Y;\theta)$ and $O(X, im(\theta))$ are isomorphic semigroups. If we suppose that $|X| = m$, $|Y| = n$, and $|im(\theta)| = r$ where $m,n,r \in \mathbb{N}$, then we find the rank of $\mathcal{GO}(X,Y;\theta)$ when $\theta$ is one-to-one but not onto. Moreover, we find lower bounds for $rank(\mathcal{GO}(X,Y;\theta))$ when $\theta$ is neither one-to-one nor onto and when $\theta$ is onto but not one-to-one.

**Key words:** Generalized order-preserving transformation semigroup, the semigroup of order-preserving transformations with restricted range, generating set, rank

## 1. Introduction

A full transformation on a non-empty set $X$ is a self-mapping on $X$. The set of all transformations on $X$ forms a semigroup $\mathcal{T}(X)$ under the composition $\circ$ of transformations, which is called the (full) transformations semigroup on $X$. For a non-empty chain $X$, a transformation $\alpha \in \mathcal{T}(X)$ is called order-preserving if $x_1 \leq x_2$ implies $x_1\alpha \leq x_2\alpha$ for all $x_1, x_2 \in X$. If we denote the set of all order-preserving transformations on $X$ by $\mathcal{O}(X)$, then $\mathcal{O}(X)$ is a subsemigroup of $\mathcal{T}(X)$, which is called the order-preserving transformations semigroup on $X$. For non-empty (disjoint) sets $X$ and $Y$, let $\mathcal{T}(Z)$ be the full transformation semigroup on the set $Z = X \cup Y$. Then it is clear that the set $T(Z,Y) = \{\alpha \in \mathcal{T}(Z) : im(\alpha) \subseteq Y\}$ is a subsemigroup of $\mathcal{T}(Z)$, which is called a semigroup of transformations with restricted range. $T(Z,Y)$ was introduced by Symons in [20]. Since then, there have been many kinds of research on transformation semigroups with restricted range (see, for examples [5, 14, 18]). Note that $T(Z,Y)$ is not regular in general. Sanwong and Sommanee proved in [15] that the set $F(Z,Y) = \{\alpha \in \mathcal{T}(Z) : im(\alpha) = Y\alpha\}$, is the largest regular subsemigroup of $T(Z,Y)$, where $Y\alpha = \{y\alpha : y \in Y\}$. For any non-empty (disjoint) sets $X$ and $Y$, let $\mathcal{GT}(X,Y)$ denote the set of all (full) transformations from $X$ to $Y$. For a fixed transformation $\theta : Y \to X$, Magill defined a so-called sandwich operation $*$ on $\mathcal{GT}(X,Y)$ as follows:

$$\alpha * \beta = \alpha \circ \theta \circ \beta \text{ for } \alpha, \beta \in \mathcal{GT}(X,Y)$$

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in [13]. With this operation, $\mathcal{GT}(X,Y)$ is also a semigroup which is called a generalized transformation semigroup, and denoted by $\mathcal{GT}(X,Y;\theta)$. It is well known the analogue of Cayley's theorem for finite groups, that is every finite semigroup is isomorphic to a subsemigroup of a suitable finite transformation semigroup. Hence the transformation semigroups and their generalizations have an important role in Semigroup Theory. As on $\mathcal{T}(X)$, there have been many kinds of research on generalized transformation semigroups in the literature (see, for examples [1, 11, 16, 17, 19]). For a fixed element $a$ of a semigroup $S$, if we define a sandwich operation $\star_a$ by $x \star_a y = x \cdot a \cdot y$ for all $x, y \in S$, then $(S, \star_a)$ is a semigroup, and this semigroup is called the variant of $S$ with respect to $a$ and denoted by $S^a$. In [1, Lemma 2.1], it is shown that $\mathcal{GT}(X,Y;\theta)$ can be embedded in $T(X\cup Y, Y) = \{\alpha \in \mathcal{T}(X \cup Y) : (X\cup Y)\alpha \subseteq Y\}$ (see, for a different embedding [16, Theorem 2.3 ]). Moreover, if $\theta : Y \to X$ is one-to-one, it is shown that $\mathcal{GT}(X,Y;\theta)$ and $T(X, im(\theta))$ are isomorphic semigroups in [1, Theorem 2.2].

For any non-empty (disjoint) chains $(X, \leq_1)$ and $(Y, \leq_2)$, a transformation $\alpha \in \mathcal{GT}(X,Y)$ is called order-preserving if $x_1 \leq_1 x_2$ implies $x_1\alpha \leq_2 x_2\alpha$ for all $x_1, x_2 \in X$. Let $\mathcal{GO}(X,Y)$ denote the set of all order-preserving transformations from $X$ to $Y$. As above, for a fixed order-preserving transformation $\theta : Y \to X$, we define a sandwich operation $*$ on $\mathcal{GO}(X,Y)$ by $\alpha * \beta = \alpha \circ \theta \circ \beta$ for all $\alpha, \beta \in \mathcal{GO}(X,Y)$. With this operation, it is clear that $\mathcal{GO}(X,Y)$ is also a semigroup which is called a generalized order-preserving transformations semigroup, and denoted by $\mathcal{GO}(X,Y;\theta)$. If $X = Y$, then $\mathcal{GO}(X,Y;\theta)$ is denoted by $\mathcal{O}(X;\theta)$, and also if $\theta \in \mathcal{O}(X)$ is the identity transformation on $X$, then it is clear that $\mathcal{O}(X;\theta) = \mathcal{O}(X)$. For a fixed order-preserving transformation $\theta : Y \to X$, we define an order on the set $Z = X \cup Y$ as follows: For all $z_1, z_2 \in Z$,

$$z_1 \leq z_2 \quad \text{if and only if} \quad \begin{cases} z_1 \leq_1 z_2 & \text{and } z_1, z_2 \in X, \\ z_1 \leq_2 z_2 & \text{and } z_1, z_2 \in Y, \\ z_1 \leq_1 z_2\theta & \text{and } z_1 \in X,\ z_2 \in Y, \\ z_1\theta \leq_1 z_2,\ \ z_1\theta \neq z_2 & \text{and } z_1 \in Y,\ z_2 \in X. \end{cases} \tag{1.1}$$

Now it is clear that $Z$ is a chain with this order, and denoted by $(Z, \leq)$. For example, if $X = \{1 < 2 < 3 < 4 < 5\}$, $Y = \{6 < 7 < 8 < 9\}$ and $\theta = \begin{pmatrix} 6 & 7 & 8 & 9 \\ 1 & 3 & 3 & 4 \end{pmatrix}$, then

$$Z = \{1 < 6 < 2 < 3 < 7 < 8 < 4 < 9 < 5\}.$$

Let $\mathcal{O}(Z)$ denote the full order-preserving transformations semigroup on the chain $(Z, \leq)$. In this paper, we are interested in generalized order-preserving transformations semigroup $\mathcal{GO}(X,Y;\theta)$ and semigroups of order-preserving transformations with restricted range. In [12, Theorem 3.1], the regularity of the semigroup $\mathcal{GO}(X,Y;\theta)$ is characterized. Further, they provided necessary and sufficient conditions for $\mathcal{GO}(X,Y;\theta)$ to be isomorphic to $\mathcal{O}(X)$ and $\mathcal{O}(Y)$, respectively. For any non-empty (disjoint) chains $(X, \leq_1)$ and $(Y, \leq_2)$, let $\mathcal{O}(Z)$ be the order-preserving transformations semigroup on the chain $Z = X \cup Y$ with the order defined in (1.1). Then the subsemigroup

$$O(Z,Y) = \{\alpha \in \mathcal{O}(Z) : Z\alpha \subseteq Y\}$$

is a semigroup of order-preserving transformations with restricted range. $O(Z,Y)$ is not regular in general. Fernandes et al. proved in [6] that the set

$$FO(Z,Y) = \{\alpha \in \mathcal{O}(Z) : Z\alpha = Y\alpha\},$$

consisting of all regular elements in $O(Z,Y)$, is the largest regular subsemigroup of $O(Z,Y)$.

For any non-empty subset $A$ of a semigroup $S$, the subsemigroup generated by $A$, that is the smallest subsemigroup of $S$ containing $A$, is denoted by $\langle A\rangle$. If there exists a finite subset $A$ of $S$ such that $\langle A\rangle = S$, then $S$ is called a finitely generated semigroup. The rank of a finitely generated semigroup $S$ is defined by

$$rank(S) = \min\{\,|A| : \langle A\rangle = S\,\}.$$

One of the most important research areas in computational algebra is to find a (minimal) generating set for some important algebraic structures such as regular semigroups and some transformation semigroups (see, for examples [2–4, 7, 9, 15]). Let $Y = \{y_1 < y_2 < \cdots < y_n\}$ be a finite chain, and let $V$ be any subset of $Y$. Then an element $v$ of $V$ is called *captive* if either $v \in \{y_1, y_n\}$ or $v = y_i$ for $2 \le i \le n-1$ and $y_{i-1}, y_{i+1} \in V$. The set of all captive elements of $V$ is denoted by $V^\sharp$. For example, if $Y = \{1 < 2 < 3 < 4 < 5 < 6\}$, and if $U = \{2 < 3 < 5\}$, $V = \{1 < 3 < 4 < 5\}$ and $W = \{1 < 2 < 3 < 6\}$, then $U^\sharp = \emptyset$, $V^\sharp = \{1,4\}$ and $W^\sharp = \{1,2,6\}$. In [6, Theorem 4.3], the rank of $O(Z,Y)$ was computed. In particular, if $|Z| = m$ and $|Y| = n$, then the authors proved that

$$rank(O(Z,Y)) = \left(\begin{array}{c} m-1 \\ n-1 \end{array}\right) + |Y^\sharp|,$$

where $Y^\sharp$ denotes the set of all captive elements of $Y$ (For unexplained terms in semigroup theory, see [8, 10].)

In the following section, for non-empty (disjoint) chains $X$ and $Y$, we show that $\mathcal{GO}(X,Y;\theta)$ can be embedded in $O(X\cup Y, Y)$. If $\theta : Y \to X$ is one-to-one, we show that $\mathcal{GO}(X,Y;\theta)$ and $O(X, im(\theta))$ are isomorphic semigroups. If $|X| \ge |Y|$, then $\mathcal{GO}(X,Y;\theta)$ is isomorphic to $O(X, im(\rho))^\Theta$, the variant of $O(X, im(\rho))$ with respect to $\Theta$, defined in Equation (2.1).

For $m,n,r \in \mathbb{N}$, let $|X| = m$, $|Y| = n$, and $|im(\theta)| = r$. In the last section, if $\theta$ is neither onto nor one-to-one, i.e. $r < m$ and $r < n$, first we show that

$$rank(\mathcal{GO}(X,Y;\theta)) \ge \sum_{s=r+1}^{\min\{m,n\}} \left(\begin{array}{c} m-1 \\ s-1 \end{array}\right)\left(\begin{array}{c} n \\ s \end{array}\right)$$

Moreover, if $\theta$ is onto but not one-to-one, i.e. $r = m < n$, we show that

$$rank(\mathcal{GO}(X,Y;\theta)) \ge \left(\begin{array}{c} n \\ m \end{array}\right).$$

Finally, let $\theta : Y \to X$ be one-to-one but not onto, i.e. $r = n < m$. Since we prove in Theorem 2.2 that $\mathcal{GO}(X,Y;\theta)$ and $O(X, im(\theta))$ are isomorphic, it follows from [6, Theorem 4.3] that

$$rank(\mathcal{GO}(X,Y;\theta)) = \left(\begin{array}{c} m-1 \\ r-1 \end{array}\right) + (im\theta)^\sharp$$

where $(im\theta)^\sharp$ is the set of all captive elements of $im\theta$. Moreover, if $\theta : Y \to X$ is both one-to-one and onto, then since $\mathcal{GO}(X,Y;\theta))$ and $\mathcal{O}(X)$ are isomorphic, we conclude that

$$rank(\mathcal{GO}(X,Y;\theta)) = rank(\mathcal{O}(X)) = |X| + 1 = m+1$$

for $m \ge 2$.

## 2. Some connections

For non-empty (disjoint) chains $(X, \leq_1)$ and $(Y, \leq_2)$, let $\theta$ be any fixed order-preserving transformation from $Y$ to $X$. In this section, we investigate some connections between the generalized order-preserving transformations semigroup $\mathcal{GO}(X, Y; \theta)$ and the semigroup of order-preserving transformations with restricted range $O(Z, Y)$ where $Z = X \cup Y$ is the chain with the order defined in (1.1). With these notations, since the proof is slightly different we state and prove the following lemma which is similar to Lemma 2.1 in [1].

**Lemma 2.1** *$\mathcal{GO}(X, Y; \theta)$ can be embedded in $O(X \cup Y, Y)$.*

**Proof** Let $Z = X \cup Y$. For each $\alpha \in \mathcal{GO}(X, Y; \theta)$, consider the transformation $\widehat{\alpha} \in O(Z, Y)$ defined by

$$x\widehat{\alpha} = \begin{cases} x\alpha & \text{if } x \in X \\ (x\theta)\alpha & \text{if } x \in Y. \end{cases}$$

For any $x, y \in Z$ with $x \leq y$, we show that $\widehat{\alpha}$ is an order-preserving transformation in the following four cases.
Case 1: Suppose that both $x, y \in X$. Then, since $x \leq_1 y$ and $\alpha \in \mathcal{GO}(X, Y; \theta)$, we have $x\widehat{\alpha} = x\alpha \leq_2 y\alpha = y\widehat{\alpha}$, and so from (1.1), $x\widehat{\alpha} \leq y\widehat{\alpha}$.
Case 2: Suppose that both $x, y \in Y$. Then, since $x \leq_2 y$ and $\theta \in \mathcal{GO}(Y, X)$, we have $x\theta \leq_1 y\theta$, and similarly, $x\widehat{\alpha} = (x\theta)\alpha \leq_2 (y\theta)\alpha = y\widehat{\alpha}$, and so from (1.1), $x\widehat{\alpha} \leq y\widehat{\alpha}$.
Case 3: Suppose that $x \in X$ and $y \in Y$. Similarly, we have $x \leq_1 y\theta$, and then, $x\widehat{\alpha} = x\alpha \leq_2 (y\theta)\alpha = y\widehat{\alpha}$, and so $x\widehat{\alpha} \leq y\widehat{\alpha}$.
Case 4: Suppose that $y \in X$ and $x \in Y$. This time, we have $x\theta \leq_1 y$, $x\theta \neq y$, and then, $x\widehat{\alpha} = (x\theta)\alpha \leq_2 y\alpha = y\widehat{\alpha}$, and so $x\widehat{\alpha} \leq y\widehat{\alpha}$, as required.

As shown in the proof of Lemma 2.1 in [1], it is similar to show that $\widehat{\alpha * \beta} = \widehat{\alpha} \circ \widehat{\beta}$ for all $\alpha, \beta \in \mathcal{GO}(X, Y; \theta)$. Moreover, it is similar to show that the map $\psi : \mathcal{GO}(X, Y; \theta) \to O(Z, Y)$ defined by $\alpha\psi = \widehat{\alpha}$ for all $\alpha \in \mathcal{GO}(X, Y; \theta)$ is a one-to-one homomorphism.$\square$

Since the cardinalities of the semigroups $\mathcal{GO}(X, Y; \theta)$ and $O(Z, Y)$ are different in general, $\psi$ is not onto. We suppose that $|X| \geq |Y|$ for the rest of this section.

**Theorem 2.2** *If $\theta \in \mathcal{GO}(Y, X)$ is one-to-one, then $\mathcal{GO}(X, Y; \theta)$ is isomorphic to $O(X, im(\theta))$. In particular, if $\theta$ is both one-to-one and onto, then $\mathcal{GO}(X, Y; \theta)$ is isomorphic to $\mathcal{O}(X)$.*

**Proof** Let the map $\varphi : \mathcal{GO}(X, Y; \theta) \to O(X, im(\theta))$ be defined by $\alpha\varphi = \alpha \circ \theta$ for each $\alpha \in \mathcal{GO}(X, Y; \theta)$. Suppose that for $x, y \in X$, $x \leq_1 y$. Since $\alpha \in \mathcal{GO}(X, Y; \theta)$, we have $x\alpha \leq_2 y\alpha$, and since $\theta \in \mathcal{GO}(Y, X)$, we have $(x\alpha)\theta \leq_1 (y\alpha)\theta$, and so $\alpha\varphi = \alpha \circ \theta \in O(X, im(\theta))$. Then, as shown in the proof of Theorem 2.2 in [1], it is similar to show that $\varphi$ is a one-to-one homomorphism.

Now consider the map $\check{\theta} : Y \to im(\theta)$ defined by $y\check{\theta} = y\theta$ for all $y \in Y$. Then it is clear that $\check{\theta}$ is one-to-one, onto and order-preserving, and that the inverse $\check{\theta}^{-1}$ of $\check{\theta}$ is also one-to-one, onto and order-preserving. For each $\beta \in O(X, im(\theta))$ and for all $x, y \in X$, if $x \leq_1 y$, then we have $x\beta \leq_1 y\beta$, and so $(x\beta)\check{\theta}^{-1} \leq_2 (y\beta)\check{\theta}^{-1}$. Thus $\beta \circ \check{\theta}^{-1} \in \mathcal{GO}(X, Y)$, and similarly, $(\beta \circ \check{\theta}^{-1})\varphi = \beta$. Therefore, $\varphi$ is onto, and so an isomorphism. Since $O(X, X) = \mathcal{O}(X)$, the proof is completed. $\square$

For any mapping $\theta : Y \to X$, it is shown in [1] that there exists a one-to-one mapping $\rho : Y \to X$ such that $im(\theta) \subseteq im(\rho)$. If both $X$ and $Y$ are infinite, this is not true for order-preserving cases in general. However, if $Y$ is finite, there exists a one-to-one and order-preserving mapping $\rho : Y \to X$ such that $im(\theta) \subseteq im(\rho)$. Indeed, suppose that $Y = \{y_1 < \cdots < y_n\}$ and that $|im(\theta)| = r$. Since $r \le n \le |X|$, there exist $x_1, \ldots, x_n \in X$ such that $x_1 < \cdots < x_n$ and $im(\theta) \subseteq \{x_1, \ldots, x_n\}$. Now we fix $\{x_1 < \cdots < x_n\}$ which contains $im(\theta)$ and define $\rho : Y \to X$ by $y_i\rho = x_i$ for every $1 \le i \le n$. Then it is clear that $\rho$ is one-to-one and order-preserving. For example, let $X = \{1 < 2 < 3 < 4 < 5 < 6\}$ (or the set of all positive integers with the usual order) and $Y = \{7 < 8 < 9\}$. If $\theta_1 = \begin{pmatrix} 7 & 8 & 9 \\ 1 & 1 & 4 \end{pmatrix}$, then we consider $\rho_1 = \begin{pmatrix} 7 & 8 & 9 \\ 1 & 4 & 5 \end{pmatrix}$, and if $\theta_2 = \begin{pmatrix} 7 & 8 & 9 \\ 2 & 5 & 5 \end{pmatrix}$, then we consider $\rho_2 = \begin{pmatrix} 7 & 8 & 9 \\ 2 & 3 & 5 \end{pmatrix}$. Suppose that $Y = \{y_1 < \cdots < y_n\}$ is a finite chain. For any $\theta \in \mathcal{GO}(Y, X)$, we consider the one-to-one and order-preserving mapping $\rho : Y \to X$ such that $im(\theta) \subseteq im(\rho) = \{x_1 < \cdots < x_n\}$, which is defined above. Then we consider the transformation $\Theta \in \mathcal{T}(X)$ defined by

$$x\Theta = \begin{cases} (x_1\rho^{-1})\theta & \text{if } x < x_2 \\ (x_i\rho^{-1})\theta & \text{if } x_i \leqq x < x_{i+1} \text{ for } i = 2, \ldots, n-1 \\ (x_n\rho^{-1})\theta & \text{if } x_n \leqq x \end{cases} \tag{2.1}$$

for each $x \in X$. Notice that the restriction of $\Theta$ to $im(\rho)$ is equal to $\rho^{-1}\theta$, that is $\Theta_{|_{im(\rho)}} = \rho^{-1}\theta$. Then we show that $\Theta$ is order-preserving. Since $\rho : Y \to X$ is one-to-one and order-preserving mapping, $\rho^{-1} : im(\rho) \to Y$ exists and is order-preserving. Moreover, since $\theta : Y \to X$ is order-preserving, it follows that $\Theta$ is order-preserving. Now it is clear that $\Theta \in O(X, im(\rho))$. For example, according to the above examples, we have

$$\Theta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 4 & 4 \end{pmatrix} \text{ and } \Theta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 2 & 5 & 5 & 5 & 5 \end{pmatrix}.$$

A non-empty subset $A$ of a chain $X$ is called *convex* if for all $a, b \in A$ and $x \in X$, $a \le x \le b$ implies $x \in A$. A (finite) partition $P = \{A_1, \ldots, A_n\}$ of a chain $X$ is called an *ordered partition* if $a < b$ for all $a \in A_i$ and $b \in A_{i+1}$ $(1 \le i \le n-1)$. In addition, if for every $1 \le i \le n$, $A_i$ is convex, $P$ is called an *ordered convex partition* of $X$. Recall that for a finite chain $Y$, the set of all kernel classes of an order-preserving transformation $\beta : X \to Y$ is a finite ordered convex partition of $X$ (see, for example, [8]).

With these notations we have the following result:

**Theorem 2.3** *If $|X| \ge |Y|$ and $Y$ is finite, then $\mathcal{GO}(X, Y; \theta)$ is isomorphic to $O(X, im(\rho))^{\Theta}$, the variant of $O(X, im(\rho))$ with respect to $\Theta$.*

**Proof** For each $\alpha \in \mathcal{GO}(X, Y; \theta)$, define the following transformation:

$$\widetilde{\alpha} = \alpha \circ \rho. \tag{2.2}$$

Since both $\alpha$ and $\rho$ are order-preserving, it is clear that $\widetilde{\alpha} \in O(X, im(\rho))$. For any $\alpha, \beta \in \mathcal{GO}(X, Y; \theta)$, since all of $\alpha$, $\beta$, $\theta$ and $\rho$ are order-preserving, it follows from the multiplication defined on $\mathcal{GO}(X, Y; \theta)$ and Equation (2.2) that

$$\widetilde{\alpha * \beta} = \widetilde{\alpha \circ \theta \circ \beta} = \alpha \circ \theta \circ \beta \circ \rho$$

is order-preserving. In addition, since $im(\alpha \circ \rho) \subseteq im(\rho)$ and $\Theta_{|_{im(\rho)}} = \rho^{-1}\theta$ from Equation (2.1) it follows that

$$\widetilde{\alpha * \beta} = \alpha \circ (\rho \circ \rho^{-1}) \circ \theta \circ \beta \circ \rho \equiv (\alpha \circ \rho) \circ (\rho^{-1} \circ \theta) \circ (\beta \circ \rho) = \widetilde{\alpha} \circ \Theta \circ \widetilde{\beta} = \widetilde{\alpha} \star_{\Theta} \widetilde{\beta}.$$

Now it is clear that the mapping $\varphi : \mathcal{GO}(X,Y;\theta) \to O(X, im(\rho))^{\Theta}$ defined by $\alpha\varphi = \widetilde{\alpha}$ for all $\alpha \in \mathcal{GO}(X,Y;\theta)$ is a homomorphism. Since $\rho$ is one-to-one, it is also clear that $\varphi$ is one-to-one.

For each $\beta \in O(X, im(\rho))$, notice that $\{\, x\beta^{-1} : x \in im(\beta)\,\}$ is a finite ordered convex partition of $X$. Now it is clear that the mapping $\breve{\beta} : X \to Y$ defined by $(x\beta^{-1})\breve{\beta} = x\rho^{-1}$ for each $x \in im(\beta)$ is order-preserving, and so $\breve{\beta} \in \mathcal{GO}(X,Y;\theta)$. Moreover, we have the identities $\breve{\beta}\varphi = \beta$ and $im(\varphi) = \mathcal{O}(X, im(\rho))$, as required. □

## 3. Generating sets and ranks

For any non-empty (disjoint) chains $X$ and $Y$, let $a, b \in X$. Then we define the following convex subsets of $X$:

$$\begin{aligned} (-,a) &= \{x \in X : x < a\}, & (-,a] &= \{x \in X : x \leqq a\}, \\ (b,-) &= \{x \in X : b < x\}, & [b,-) &= \{x \in X : b \leqq x\}, \\ (a,b) &= \{x \in X : a < x < b\}, & (a,b] &= \{x \in X : a < x \leqq b\}, \\ [a,b) &= \{x \in X : a \leqq x < b\} \quad \text{and} & [a,b] &= \{x \in X : a \leqq x \leqq b\}. \end{aligned} \tag{3.1}$$

(We suppose $a < b$ in the last four subsets above). Similarly, we define the convex subsets for $Y$. In this section we suppose that $\theta : Y \to X$ has a finite height, that is the cardinality of $im(\theta)$ is finite. Let $P = \{A_1, \ldots, A_r\}$ be a finite partition of $X$, if $P$ is ordered, say $A_1 < \cdots < A_r$, then we write $P = (A_1 < \cdots < A_r)$. For any $\alpha \in \mathcal{GO}(X,Y;\theta)$ with a finite height $s$, there exists a subchain $\{y_1 < \cdots < y_s\}$ of $Y$ such that $im(\alpha) = \{y_1, \ldots, y_s\}$. In this case, we write as $im(\alpha) = \{y_1 < \cdots < y_s\}$. Moreover, if we let $A_i = y_i\alpha^{-1}$ for each $1 \leq i \leq s$, then the set of kernel classes $\{A_1, \ldots, A_s\}$ of $\alpha$ is an ordered convex partition of $X$. In this case, $\mathrm{Ker}\,(\alpha) = (A_1 < \cdots < A_s)$, and moreover, $\alpha$ can be represented by the following tabular form:

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ y_1 & y_2 & \cdots & y_s \end{pmatrix}.$$

For any semigroup $S$, let $S^2 = \{\, st \,:\, s, t \in S\,\}$. With the above notations, we have the following lemma:

**Lemma 3.1** *For $S = \mathcal{GO}(X,Y;\theta)$, if $|im(\theta)| = r$ is finite, then*

$$S^2 = \{\, \alpha \in S \,:\, |im(\alpha)| \leq r\,\}.$$

**Proof** Suppose that $im(\theta) = \{x_1 < \cdots < x_r\}$ and $T = \{\, \alpha \in S \,:\, |im(\alpha)| \leq r\,\}$. Thus $\mathrm{Ker}\,(\theta) = (x_1\theta^{-1} < \cdots < x_r\theta^{-1})$ is an ordered convex partition of $Y$. Since, for all $\alpha, \beta \in S$, $|im(\alpha * \beta)| = |im(\alpha \circ \theta \circ \beta)| \leq |im(\theta)| \leq r$, it follows that $S^2 \subseteq T$.

For any $\alpha \in T$, let $im(\alpha) = \{y_1 < \cdots < y_s\}$ $(1 \leq s \leq r)$, and let $A_i = y_i\alpha^{-1}$ for every $1 \leq i \leq s$. Then we choose a unique element $z_i \in x_i\theta^{-1}$ for every $1 \leq i \leq s$, and so we have a subchain $\{z_1 < \cdots < z_s\}$ of $Y$. Moreover, as defined in Equations (3.1), we define the following convex subsets of $X$:

$$B_1 = (-, x_1], \quad B_i = (x_{i-1}, x_i] \quad (2 \leq i \leq s-1) \text{ and } B_s = (x_{s-1}, -).$$

Then it is clear that $(B_1 < \cdots < B_s)$ is an ordered convex partition of $X$. Finally, we define the following transformations

$$\beta = \begin{pmatrix} A_1 & \cdots & A_s \\ z_1 & \cdots & z_s \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} B_1 & \cdots & B_s \\ y_1 & \cdots & y_s \end{pmatrix}.$$

Then it is clear that $\beta, \gamma \in S$. Moreover, since $\alpha = \beta \circ \theta \circ \gamma = \beta * \gamma \in S^2$, it follows that $T \subseteq S^2$. Therefore, we have the identity $T = S^2$. □

In the rest of this section, for finite positive integers $m$ and $n$, we suppose that $|X| = m$ and $|Y| = n$. Moreover, without loss of generality, we take $X = X_m = \{1 < 2 < \cdots < m\}$ and $Y = \{y_1 < y_2 < \cdots < y_n\}$. Thus, for every $\alpha \in \mathcal{GO}(X, Y; \theta)$, we notice that $|im(\alpha)| \leq \min\{m, n\}$, and so for every $1 \leq s \leq \min\{m, n\}$, we have the following sets:

$$D_s = \{\, \alpha \in \mathcal{GO}(X, Y; \theta) \, : \, |im(\alpha)| = s \,\}.$$

**Proposition 3.2** *With above notations, for every* $1 \leq s \leq \min\{m, n\}$,

$$|D_s| = \binom{m-1}{s-1}\binom{n}{s}.$$

**Proof** For every $\alpha \in D_s$, recall that $\operatorname{Ker}(\alpha) = (A_1 < A_2 < \cdots < A_s)$ is an ordered convex partition of $X$ with $s$ terms, and that $im(\alpha) = \{y_1 < y_2 < \cdots < y_s\}$ is a subchain of $Y$ with $s$ terms. Conversely, if $P = (A_1 < A_2 < \cdots < A_s)$ is an ordered convex partition of $X$ with $s$ terms, and if $V = \{y_1 < y_2 < \cdots < y_s\}$ is a subchain of $Y$ with $s$ terms, there exists a unique $\alpha \in D_s$ such that $\operatorname{Ker}(\alpha) = P$ and $im(\alpha) = V$, namely $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ y_1 & y_2 & \cdots & y_s \end{pmatrix}$. Since there exist $\binom{m-1}{s-1}$ many ordered convex partitions of $X$ with $s$ terms, and $\binom{n}{s}$ many subchains of $Y$ with $s$ terms, it follows that $|D_s| = \binom{m-1}{s-1}\binom{n}{s}$, as required. □

For any semigroup $S$, if $A$ is any generating set of $S$, then it is clear that $A$ must contain $S \setminus S^2$, and so $|A| \geq |S \setminus S^2|$. Therefore, we have the following immediate corollary:

**Corollary 3.3** *For* $m, n, r \in \mathbb{Z}^+$, *let* $|X| = m$, $|Y| = n$, *and* $|im(\theta)| = r$. *If* $\theta$ *is neither onto nor one-to-one, or equivalently, if* $\min\{m, n\} > r$, *then*

$$rank(\mathcal{GO}(X, Y; \theta)) \geq \sum_{s=r+1}^{\min\{m,n\}} \binom{m-1}{s-1}\binom{n}{s}.$$

**Proof** Suppose that $\theta$ is neither onto nor one-to-one, or that $\min\{m, n\} > r$. Then it follows from Proposition 3.2 that

$$\begin{aligned} rank(\mathcal{GO}(X, Y; \theta)) \quad &\geq \quad |S \setminus S^2| = \left| \bigcup_{s=r+1}^{\min\{m,n\}} D_s \right| = \sum_{s=r+1}^{\min\{m,n\}} |D_s| \\ &= \quad \sum_{s=r+1}^{\min\{m,n\}} \binom{m-1}{s-1}\binom{n}{s}, \end{aligned}$$

as required. □

Suppose that $X = X_m = \{1 < 2 < \cdots < m\}$ and $Y = \{y_1 < y_2 < \cdots < y_n\}$ are finite chains, and that $\theta : Y \to X$ is onto but not one-to-one. Thus $|Y| = n \geq m + 1$, and from Proposition 3.2, the cardinality of

$$D_m = \left\{ \begin{pmatrix} 1 & 2 & \cdots & m \\ v_1 & v_2 & \cdots & v_m \end{pmatrix} : \{v_1 < v_2 < \cdots < v_m\} \text{ is a subchain of } Y \right\}$$

is $\begin{pmatrix} n \\ m \end{pmatrix}$. Moreover, every element of $D_m$ is one-to-one.

**Lemma 3.4** *For $m, n \in \mathbb{Z}^+$ with $m < n$, let $|X| = m$, $|Y| = n$ and $\theta \in \mathcal{GO}(Y, X)$. If $\theta$ is onto, but not one-to-one, then*

$$rank(\mathcal{GO}(X, Y; \theta)) \geq \begin{pmatrix} n \\ m \end{pmatrix}.$$

**Proof** Let $A$ be a generating set of $\mathcal{GO}(X, Y; \theta)$. Then, for any $\alpha \in D_m$, there exist $\alpha_1, \ldots, \alpha_q \in A$ such that $\alpha_1 \cdots \alpha_q = \alpha$. Since $|im(\alpha_q)| \leq m = |im(\alpha)|$, and $im(\alpha) \subseteq im(\alpha_q)$, we have $im(\alpha_q) = im(\alpha)$. Moreover, since both $\alpha$ and $\alpha_q$ are one-to-one and order-preserving, it follows that $\alpha_q = \alpha$, and so $\alpha \in A$. Therefore, $D_m \subseteq A$, and so $|A| \geq |D_m| = \begin{pmatrix} n \\ m \end{pmatrix}$, as required. $\square$

Finally, if $\theta : Y \to X$ is one-to-one order-preserving transformation, then it follows from Theorem 2.2 that $\mathcal{GO}(X, Y; \theta)$ and $O(X, im(\theta))$ are isomorphic. In addition, if $\theta$ is also onto, then $\mathcal{GO}(X, Y; \theta)$ and $\mathcal{O}(X)$ are isomorphic. For $|X| \geq 2$, it is proved in [9, Theorem 2.7] that the rank of $\mathcal{O}(X) \setminus \{1_X\}$, where $1_X$ is the identity element of $\mathcal{O}(X)$, is $|X|$. Moreover, since $\mathcal{O}(X) \setminus \{1_X\}$ is an ideal of $\mathcal{O}(X)$, every generating set of $\mathcal{O}(X)$ must contain $1_X$ and a generating set of $\mathcal{O}(X) \setminus \{1_X\}$, and so $rank(\mathcal{O}(X)) = |X| + 1$. Thus, if $\theta$ is one-to-one and onto, we have

$$rank(\mathcal{GO}(X, Y; \theta)) = rank(\mathcal{O}(X)) = |X| + 1 = m + 1$$

for $m \geq 2$.

For $|X| = m$ and for a subset $Y$ of $X$ with $r$ elements, it is proved in [6, Theorem 4.3] that

$$rank(O(X, Y)) = \begin{pmatrix} m-1 \\ r-1 \end{pmatrix} + |Y^\sharp|.$$

When $\theta$ is one-to-one but not onto, since $im(\theta)$ is a subset of $X$ with $r$ elements, from Theorem 2.2 we have the following result:

**Corollary 3.5** *Let $|X| = m$ and $|im(\theta)| = r$ where $1 \leq r \leq m - 1$. If $\theta$ is one-to-one but not onto, then*

$$rank(\mathcal{GO}(X, Y; \theta)) = \begin{pmatrix} m-1 \\ r-1 \end{pmatrix} + |(im(\theta))^\sharp|$$

*where $(im(\theta))^\sharp$ is the set of all captive elements of $im(\theta)$* .$\square$ $\square$

According to the experience, we have obtained during this work, the set $S \setminus S^2$ is not a (minimum) generating a set of $\mathcal{GO}(X, Y; \theta)$, as in [1, Lemma 3.3, Theorem 3.6]. In addition, the method used in [6] looks impossible to apply for finding a generating set of $\mathcal{GO}(X, Y; \theta)$. That is why we have the following open problem:

**Open Problem:** For $m, n, r \in \mathbb{Z}^+$, let $|X| = m$, $|Y| = n$, and $|im(\theta)| = r$. If $\theta$ is not one-to-one, then what is the rank of $\mathcal{GO}(X, Y; \theta)$?

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