TÜBİTAK

Turkish Journal of Mathematics

Volume 47 | Number 4

Article 9

1-1-2023

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Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article
Turk J Math
(2023) 47: 1169 – 1181
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doi:10.55730/1300-0098.3419

Szabó's algorithm and applications

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Received: 22.09.2022 • **Accepted/Published Online:** 27.02.2023 • **Final Version:** 16.05.2023

Abstract: In this paper, Szabó's algorithm is used as the main tool to find locally symmetric left invariant Riemannian metrics on some 4-dimensional Lie groups. Locally symmetric left invariant Riemannian Lie groups constitute an important subclass of Riemannian Lie groups with zero-divergence Weyl-tensor the so-called C-manifolds. Some properties of the curvature operator of these 4-dimensional C-manifolds are studied.

Key words: Lie algebra, Lie group, locally symmetric metric, Z-decomposition, Weyl tensor

1. Introduction

A left invariant Riemannian metric on a connected Riemannian manifold is locally symmetric if the covariant derivative of its Riemannian curvature tensor vanishes. It is well known that the class of locally symmetric Riemannian metrics contains Riemannian metrics with constant sectional curvature, product of locally symmetric Riemannian metrics, the left invariant Riemannian metrics induced by the opposite of the Killing form on a connected compacts semisimple Lie groups and so on. A manifold endowed with a locally symmetric Riemannian metric is a locally symmetric Riemannian space. Cartan in [3, 4] observes that irreducible globally symmetric spaces are homogeneous spaces G/H where G is a connected simple Lie group, H a compact subgroup of G and the Riemannian symmetric metric is the G-invariant Riemannian metric induced by the killing form of the Lie algebra $\mathfrak{g}$ of G. He used this insight to classify Riemannian symmetric spaces in connection with the theory of Lie groups [9].

A Lie group G together with a left invariant Riemannian metric g is called a Riemannian Lie group. A left invariant Riemannian metric g on G induces an inner product $g(e) = \langle\, , \,\rangle$ on the Lie algebra $\mathfrak{g}$ of G and conversely, any inner product on $\mathfrak{g}$ gives rise to a unique left invariant Riemannian metric on G.

Let (G, g) be a connected n-dimensional Riemannian Lie group, endowed with Levi-Civita connection ∇ , with Lie algebra $\mathfrak{g}$. A $\mathbb{R}$-bilinear function L is called Levi-civita product if

$$\forall u, v \in \mathfrak{g} \quad L_u v := (\nabla_{u^l} v^l)(e),$$

where u^l and v^l are left invariant vector fields associated to u and v. We defined by K the Riemannian curvature tensor at the identity element e by

$$\forall u, v, w \in \mathfrak{g} \quad K(u, v)w = L_{[u,v]}w - L_u L_v w + L_v L_u w.$$

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2010 *AMS Mathematics Subject Classification:* 53C20, 53C35, 53C30

The algebraic condition of local symmetricity on (G, g) is therefore equivalent to

$$[L_{e_m}, K(e_i, e_j)] = K(L_{e_m}e_i, e_j) + K(e_i, L_{e_m}e_j), \tag{1.1}$$

where $(e_1, e_2, \cdots, e_n)$ is an $g(e)$-orthonormal basis of the Lie algebra $\mathfrak{g}$ with respect to the inner product and [ ] is the commutator.

The above equations induce a system of $\dfrac{n^3(n-1)^2}{4}$ polynomial equations on the structure constants of the Lie algebra $\mathfrak{g}$.

For connected Lie groups of dimension 1 or 2, it follows from a direct computation that every left invariant Riemannian metric is locally symmetric.

For connected Lie groups of dimension 3, locally symmetric left invariant Riemannian metrics are well known, see [14].

For 4-dimensional Riemannian Lie groups, we have from Equation (1.1) a system of 144 polynomial equations for each of the 12 nonisomorphic associated Euclidean Lie algebra. Those systems are not easy to handle. In order to avoid the algebraic condition of local symmetricity for these Lie groups, we use the Szabó's algorithm as the main tool to split any left Riemannian metric of $\mathcal{C}$-spaces [2] as a direct product of left invariant metrics on Lie groups of lower dimension. In this paper, we are interested in finding locally symmetric left invariant Riemannian metrics on some 4-dimensional connected Lie groups.

The outline of this paper is as follows: Section 2 is devoted to some basic knowledge on the curvature tensor. In Section 3, we describe Szabó's algorithm and recall some main results on locally symmetric Riemannian Lie groups of dimension 3. The Szabó's algorithm is applied to some 4-dimensional Euclidean Lie algebra. In Section 4, we prove Theorem 4.1.

2. Preliminaries

Let (G, g) be an n-dimensional connected Riemannian Lie group and denote by $\mathcal{H}^{inf}$ and $\mathcal{H}$ the infinitesimal holonomy group and the primitive holonomy group at the identity element e of G, respectively. Their Lie algebras are denoted by $\mathfrak{h}^{inf}$ and $\mathfrak{h}$, respectively. For more details about the holonomy group and his subgroups, see [12].

2.1. The linear curvature operator

Let $x, y, v, w \in \mathfrak{g}$. We recall that on the set $\wedge^2\mathfrak{g}$ of bivectors of $\mathfrak{g}$, the inner product denoted $\langle\, ,\, \rangle_{\wedge^2\mathfrak{g}}$ is defined by

$$\langle x \wedge y, v \wedge w\rangle_{\wedge^2\mathfrak{g}} \quad = \quad \langle x, v\rangle\langle y, w\rangle - \langle x, w\rangle\langle y, v\rangle. \tag{2.1}$$

Let $\overline{K}$ be the self-adjoint operator called the curvature operator, associated to the symmetric bilinear form K (Riemannian curvature) on the euclidian space $(\wedge^2\mathfrak{g}, \langle\, ,\, \rangle_{\wedge^2\mathfrak{g}})$:

$$\langle\overline{K}(x \wedge y), v \wedge w\rangle_{\wedge^2\mathfrak{g}} = K(x \wedge y, v \wedge w) := \langle K(x, y)v, w\rangle = K(x, y, v, w). \tag{2.2}$$

The relation (2.1) induces a linear isomorphism from the set $\wedge^2\mathfrak{g}$ of bivectors to the set $\mathfrak{o}(\mathfrak{g})$ of skew-symmetric endomorphisms, such that for $v \wedge w \in \wedge^2\mathfrak{g}$ and $x \in \mathfrak{g}$,

$$\widetilde{v \wedge w}(x) = \langle v, x\rangle w - \langle w, x\rangle v. \tag{2.3}$$

In the rest of this paper, the skew-symmetric endomorphism associated to a bivector h will be denoted $\widetilde{h}$.

Definition 2.1 *[16] Let h be any arbitrary bivector of $\mathfrak{g}$.*

1. *The decomposition*
$$\mathfrak{g} = U_0 \oplus U_1 \oplus \cdots \oplus U_q, \tag{2.4}$$
where U_0 is the kernel of $\widetilde{h}$ and U_k ,$1 \leq k \leq q$ are 2-dimensional invariant real planes of the real operator $\widetilde{h}$, is the Jordan decomposition of the Lie algebra $\mathfrak{g}$ with respect to h.

2. *The number q in the Jordan decomposition is the* rank *of h.*

Remark 2.2 $q = \sum m_k$ *where m_k is the multiplicity of the nonzero eigenvalue $\lambda_k = i\nu_k$, $\nu_k > 0$ of $\widetilde{h}$.*

Definition 2.3 *[16] Let h be any arbitrary bivector. The decomposition*
$$h = \sum_{k=1}^{q} v_k \wedge w_k, \tag{2.5}$$
where $v_k, w_k \in U_k$ and U_k are the 2-dimensional subspaces of the Jordan decomposition is the Darboux decomposition or the Darboux normal form of the bivector h.

Remark 2.4 *[16] If the multiplicity m_k of nonzero eigenvalue $\lambda_k = i\nu_k$ of $\widetilde{h}$ is greater than one, the Darboux decomposition is not unique.*

Definition 2.5 *[16] The eigenvector $h \in \wedge^2\mathfrak{g}$ of the curvature operator $\overline{K}$ is said to be irreducible if any Darboux normal form of h does not split into two nontrivial summands such that they are also eigenvectors of $\overline{K}$.*

Let V_0 be the subspace of $\mathfrak{g}$ such that the action of $\mathcal{H}$ on V_0 is trivial. From [2, 16] the subspace ${V_0}^{\perp}$ decomposes into orthogonal, invariant and irreducible subspaces V_i, $i > 1$ under the action of $\mathcal{H}$ such that
$$\mathfrak{g} \, = \, V_0 \, \oplus \, V_1 \, \oplus \, V_2 \oplus \cdots \oplus \, V_k. \tag{2.6}$$

Here, irreducible means that there are no nontrivial invariant subspace of V_i.

Definition 2.6 *[16] The V-decomposition of the Lie algebra $\mathfrak{g}$ of a connected Riemannian Lie group (G, g) is an $g(e)$-orthogonal and irreducible decomposition of $\mathfrak{g}$ with respect to the primitive holonomy group at $e \in G$.*

2.2. The $\mathcal{C}$-spaces Riemannian Lie groups

Definition 2.7 *[7, 15, 17] A Riemannian Lie group (G, g) of dimension ≥ 4 is called a $\mathcal{C}$-space if its Weyl tensor W satisfies $divW = 0$.*

In the literature, there are some classifications of the metrics Lie algebras of 4-dimensional $\mathcal{C}$-spaces (see [7, 15, 17] for more details). Table 1 below gives the Lie algebras and the structure constants in an orthonormal basis (e_1, e_2, e_3, e_4), for 4-dimensional Riemannian Lie groups with respect to their structure constants and the spectrum of the curvature operator.

Table 1. Lie Algebras of 4-dimensional $\mathcal{C}-$spaces riemannian Lie groups.

Lie algebras	Structure constants	$\mathrm{Spec}(\overline{K}) = \{\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{23}, \sigma_{24}, \sigma_{34}\}$
$4\mathbb{A}_1$	$C^k_{i,j} = 0,\ \forall i, j, k,$	$\{0, 0, 0, 0, 0, 0\}$
$\mathbb{A}_{3,6} \oplus \mathbb{A}_1 c > 0$	$C^1_{2,3} = c,\ C^2_{1,3} = -c$	$\{0, 0, 0, 0, 0, 0\}$
$\mathbb{A}_{3,9} \oplus \mathbb{A}_1$ $a > 0$	$C^2_{1,3} = a\sqrt{1+m^2}$ $C^3_{1,2} = C^1_{2,3} = -a\sqrt{1+m^2}$ $C^1_{2,4} = -C^2_{1,4} = a\,m\sqrt{1+m^2}$	$\left\{\frac{a^2(1+m^2)}{4}, \frac{a^2(1+m^2)}{4}, 0, \frac{a^2(1+m^2)}{4}, 0, 0\right\}$
$\mathbb{A}_{3,3} \oplus \mathbb{A}_1$ $a > 0$	$C^1_{1,3} = C^2_{2,3} = a$	$\{-a^2, -a^2, 0, -a^2, 0, 0\}$
$\mathbb{A}^{\alpha}_{3,7} \oplus \mathbb{A}_1$ $a > 0,\ \alpha > 0$	$C^1_{1,3} = C^2_{2,3} = \alpha a$ $C^1_{2,3} = -C^2_{1,3} = a$	$\{-\alpha^2 a^2, -\alpha^2 a^2, 0, -\alpha^2 a^2, 0, 0\}$
$\mathbb{A}^{\alpha,\beta}_{4,5}$ $a > 0$ $\alpha = \beta = 1$	$C^1_{1,4} = C^2_{2,4} = C^3_{3,4} = a$	$\{-a^2, -a^2, -a^2, -a^2, -a^2, -a^2\}$
$\mathbb{A}^{\alpha,\beta}_{4,6}$ $\alpha = \beta,$ $a > 0, \alpha > 0$,W=0	$C^1_{1,4} = C^2_{2,4} = C^3_{3,4} = \alpha a$ $C^2_{3,4} = -C^3_{2,4} = a$	$\{-\alpha^2 a^2, -\alpha^2 a^2, -\alpha^2 a^2, -\alpha^2 a^2, -\alpha^2 a^2, -\alpha^2 a^2\}$
$\mathbb{A}_{4,12}$ $a > 0, d > 0$	$C^1_{1,3} = C^2_{2,3} = \sqrt{a^2+b^2}$ $C^1_{2,4} = -C^2_{1,4} = \frac{a\,d}{\sqrt{a^2+b^2}}$ $C^1_{2,3} = -C^2_{1,3} = \frac{b\,d}{\sqrt{a^2+b^2}}$	$\{-(a^2+b^2), -(a^2+b^2), 0, -(a^2+b^2), 0, 0\}$
$\mathbb{A}_2 \oplus 2\mathbb{A}_1,\ a > 0$	$C^2_{1,2} = a$	$\{-a^2, 0, 0, 0, 0, 0\}$
$2\mathbb{A}_2,\ a > 0,\ b > 0$	$C^2_{1,2} = a,\ C^4_{3,4} = b$	$\{-a^2, 0, 0, 0, 0, -b^2\}$
$\mathbb{A}^{\alpha,\beta}_{4,5}, \alpha\beta \neq 0, -1 \leq \alpha \leq \beta \leq 1,\ a > 0$	$C^1_{1,4} = C^2_{2,4} = C^3_{3,4} = a$	$\{-a^2, -a^2, -a^2, -a^2, -a^2, -a^2\}$
$\mathbb{A}^{\alpha,\beta}_{4,6}, \alpha \neq 0, \beta \geq 0,\ \alpha \neq 1,\ a > 0, W \neq 0, divW = 0$	$C^1_{1,4} = \alpha a,\ C^3_{2,4} = -C^2_{3,4} = -a$	$\{0, 0, -(\alpha a)^2, 0, 0, 0\}$
$\mathbb{A}^{\alpha,\beta}_{4,6}, \alpha \neq 0,\ \beta > 0,\ a > 0, W \neq 0, divW = 0$	$C^1_{1,4} = C^2_{2,4} = C^3_{3,4} = \beta a,\ C^3_{2,4} = -C^2_{3,4} = -a$	$\{-(\beta a)^2, -(\beta a)^2, -(\beta a)^2, -(\beta a)^2, -(\beta a)^2, -(\beta a)^2\}$
$\mathbb{A}^{\beta}_{4,9}, -1 < \beta \leq 1,\ a > 0$	$C^1_{1,4} = C^1_{2,3} = 2a,\ C^2_{2,4} = C^3_{3,4} = a$	$\{0, 0, -6a^2, -2a^2, -2a^2, -2a^2\}$
$\mathbb{A}^{\alpha}_{4,11},\ \alpha > 0,\ a > 0$	$C^1_{1,4} = C^1_{2,3} = 2a\alpha,$ $C^2_{2,4} = C^3_{3,4} = a\alpha,$ $C^3_{2,4} = -C^2_{3,4} = -a$	$\{0, 0, -6a^2\alpha^2, -2a^2\alpha^2, -2a^2\alpha^2, -2a^2\alpha^2\}$

2.3. Some main results on Riemannian Lie groups of dimension $n \leq 3$

An Euclidian 4-dimensional Lie algebra on which Szabò's algorithm is effective, splits into a direct sum (as linear spaces), of nontrivial Lie algebras both of dimension 2 or dimension 1 and 3. Riemannian Lie groups of dimension 1 and 2 are locally symmetric Riemannian manifolds. The structure constants of the Lie algebra of a 3-dimensional locally symmetric Riemannian Lie group satisfy the following conditions:

Proposition 2.8 *[14] Let G be a connected 3-dimensional real unimodular Lie group with left-invariant Riemannian metric. (G, g) is a locally symmetric Riemannian Lie group if and only if in the Lie algebra $\mathfrak{g}$ of G, there exists an $g(e) = \langle , \rangle$-orthonormal basis $(e_1,\ e_2,\ e_3)$ in which the brackets of the Lie algebra are presented in Table 2.*

Table 2. 3-dimensional unimodular case.

Lie algebra	Structure constants	Restrictions
$\mathbb{R}^3$	$[e_1, e_2] = [e_2, e_3] = [e_1, e_3] = 0$	
$\mathbb{R}^2 \rtimes \mathfrak{so}(2)$	$[e_1, e_2] = ae_3,\ \ [e_3, e_1] = ae_2$	$a > 0$
$\mathfrak{su}(2)$	$[e_1, e_2] = ae_3,\ \ [e_2, e_3] = ae_1,\ \ [e_3, e_1] = ae_2$	$a > 0$

Proposition 2.9 *[14] Let G be a connected 3-dimensional real nonunimodular Lie group with left-invariant Riemannian metric. (G, g) is a locally symmetric Riemannian Lie group if and only if in the Lie algebra $\mathfrak{g}$ of G, there exist an $g(e) = \langle , \rangle$-orthonormal basis $(e_1,\ e_2,\ e_3)$ in which the brackets of the Lie algebra are presented in Table 3.*

Table 3. 3-dimensional nonunimodular case.

Lie algebra	Structure constants	Restrictions
$\mathfrak{g}_I$	$[e_1, e_2] = ae_2,\ \ [e_1, e_3] = ae_3$	$a > 0$
$\mathfrak{g}_D$	$[e_1, e_2] = ae_2 + be_3,\ \ [e_1, e_3] = -be_2 + ae_3$	$a > 0, b > 0,\ D \neq 0$
$\mathfrak{g}_0$	$[e_1, e_2] = ae_2$	$a > 0$

For more details on locally symmetric left invariant 3-dimensional riemannian Lie groups, see [14].

3. Szabó's algorithm

3.1. Szabó's algorithm proof

Let h be an irreducible eigenvector of $\overline{K}$ with nonnull eigenvalue. $\widetilde{h}$ leaves the invariant and irreducible subspaces of the orthogonal V-decomposition invariant. Thus, we have the following proposition:

Proposition 3.1 *Let h be an irreducible eigenvector of a nonzero eigenvalue of $\overline{K}$, the nontrivial invariant subspace*

$$H^1 = U_1 \oplus U_2 \oplus \cdots \oplus U_q$$

of $\widetilde{h}$ is contained in a unique invariant and irreducible subspace of the V-decomposition.

Proof Let $\mathfrak{g} = U_0 \oplus U_1 \oplus \cdots\cdots \oplus U_q$ be the Jordan decomposition of $\mathfrak{g}$ with respect to $\widetilde{h}$ and $\mathfrak{g} = V_0 \oplus V_1 \oplus \cdots\cdots \oplus V_k$ a $V-$decomposition of $\mathfrak{g}$ such that $U_1 \subset V_1$. Suppose that for $j > 1, U_j$ is a subset of one of the subspaces V_i, for some $i > 1$. The equation $\lambda h = \overline{K}(h)$ is equivalent to

$$\lambda v_1 \wedge w_1 - \overline{K}(v_1 \wedge w_1) = -\lambda \sum_{j=2}^{q} v_j \wedge w_j + \sum_{j=2}^{q} \overline{K}(v_j \wedge w_j). \tag{3.1}$$

The skew-symmetric linear operator associated to the left hand side and the right hand side of Equation (3.1) satisfies the relation

$$\lambda \widetilde{v_1 \wedge w_1} - K(v_1, w_1) = -\lambda \sum_{j=2}^{q} \widetilde{v_j \wedge w_j} + \sum_{j=2}^{q} K(v_j, w_j). \tag{3.2}$$

Let $u = \sum_{s=0}^{k} u_s$ be the decomposition of u with respect to the V-decomposition. Since $K(u_i, w_i)$ and $\widetilde{u_i \wedge w_i}$ are linear operators on V_i and $K(u_i, w_i)(v_j) = 0$ for all $u_i, w_i \in Vi$ and $v_j \in V_j$ with $i \neq j$, we have

$$K(v_1, w_1)(u) = K(v_1, w_1)(u_1) \in V_1, \quad \lambda \widetilde{v_1 \wedge w_1}(u) \in V_1,$$

$$\lambda \sum_{j=2}^{q} \widetilde{v_j \wedge w_j}(u) = \lambda \sum_{j=2}^{q} (\langle v_j, u\rangle w_j - \langle w_j, u\rangle v_j) \in \bigoplus_{j=2}^{q} V_j$$

$$\text{and} \quad \sum_{j=2}^{q} K(v_j, w_j)(u) = \sum_{j=2}^{q} K(v_j, w_j)(u_j) \in \bigoplus_{j=2}^{q} V_j.$$

For all $u \in \mathfrak{g}$,

$$\begin{cases} K(v_1, w_1)(u) - \lambda \widetilde{v_1 \wedge w_1}(u) \in V_1 \cap \bigoplus_{j=2}^{q} V_j = \{0\} \\ \sum_{j=2}^{q} K(v_j, w_j)(u) - \lambda \sum_{j=2}^{q} \widetilde{v_j \wedge w_j}(u) \in V_1 \cap \bigoplus_{j=2}^{q} V_j = \{0\} \end{cases},$$

thus

$$\begin{cases} \overline{K}(v_1 \wedge w_1) & = & \lambda\ v_1 \wedge w_1 \\ \overline{K}(\sum_{j=2}^{q} v_j \wedge w_j) & = & \lambda\ \sum_{j=2}^{q} v_j \wedge w_j \end{cases}. \tag{3.3}$$

It follows that the eigenvector h splits into two nontrivial sum such that they are also eigenvectors of $\overline{K}$. This contradicts the irreducibility of h. □

Let $\{h_1, h_2, \cdots, h_\rho, h_{\rho+1}, \cdots, h_{\frac{n(n-1)}{2}}\}$ be a system of linearly independent irreducible eigenvectors of $\overline{K}$ which form a basis of $\wedge^2\mathfrak{g}$ and assume that just the first ρ vectors are corresponding to nonzero eigenvalues. Therefore, $\mathfrak{h}$ is the free Lie algebra on $\{\widetilde{h}_1, \widetilde{h}_2, \cdots, \widetilde{h}_\rho\}$.

For irreducible bivectors h_k, $1 \leq k \leq \rho$, let us consider the Jordan decomposition

$$\mathfrak{g} = U_{k0} \oplus U_{k1} \oplus U_{k2} \oplus \cdots \oplus U_{kN_k},$$

where U_{k0} is the kernel of $\widetilde{h}_k$, $U_{kl}, l \in \{1, 2, \cdots, N_k\} \subset \mathbb{N}$ the real $\widetilde{h}_k$ invariant 2-plan in the Jordan decomposition and N_k the rank of h_k. Let

$$H_k^0 = U_{k0} \quad \text{and} \quad H_k^1 = U_{k1} \oplus U_{k2} \oplus \cdots \oplus U_{kN_k}, \tag{3.4}$$

We have

$$\widetilde{h}_k H_k^0 = 0, \qquad \text{and} \qquad \widetilde{h}_k H_k^1 \subset H_k^1.$$

Let us choose an arbitrary vector h_{k_1}, $1 \leq k_1 \leq \rho$, and consider its subspaces $H_{k_1}^0$ and $H_{k_1}^1$ constructed above. H_k^1 is a nontrivial linear subspace of a unique V_i, $i > 0$ of the V-decomposition.

Proposition 3.2 *If for any* h_i , $i \neq k_1$ *the relation* $H_{k_1}^1 \subseteq H_i^0$ *holds, then* $H_{k_1}^1 = V_i$.

Proof For all real number t, $exp\left(t \sum\limits_{i=1}^{\rho} \alpha_i \widetilde{h}_i\right) = Id + \sum\limits_{i=1}^{\infty} \frac{t^k}{k!} \left(\sum\limits_{i=1}^{\rho} \alpha_i \widetilde{h}_i\right)^k \in \mathcal{H}_e$.

By the above, the $(\cdot)$-action of $\mathcal{H}$ on H_k^1 is completely determined by the action of the generators $\widetilde{h}_1, \widetilde{h}_2, \cdots, \widetilde{h}_\rho$ of the free Lie algebra $\mathfrak{h}_e$ on H_k^1. The image of H_k^1 by $\sum\limits_{i=1}^{\rho} \alpha_i \widetilde{h}_i$ is a linear subspace of H_k^1, written

$$\alpha_1 \widetilde{h}_1 \cdot H_k^1 + \alpha_2 \widetilde{h}_2 \cdot H_k^1 + \cdots + \alpha_k \widetilde{h}_k \cdot H_k^1 + \cdots + \alpha_\rho \widetilde{h}_\rho \cdot H_k^1 \subset H_k^1.$$

Suppose that $\alpha_j \widetilde{h}_j \cdot H_k^1$ is trivial with $j \neq k$, *i.e.* $H_k^1 \subset H_j^0$ for $j \in \{1, 2, \cdots, \rho\}$ and $j \neq k$. Then H_k^1 is one of the subspace V_i, $i > 0$ of the V-decomposition. □

Proposition 3.3 *Let* h_1 *and* h_2 *be two irreducible eigenvectors with nonzero eigenvalue of* $\overline{K}$. *If* $H_1^1 \nsubseteq H_2^0$, *then* $H_1^1 + H_2^1$ *is contained in a unique invariant and irreducible subspace of the* $V-$*decomposition.*

Proof Let $h_1 = \sum\limits_{i=1}^{N_1} v_i \wedge w_i$ and $h_2 = \sum\limits_{s=1}^{N_2} v'_s \wedge w'_s$ be the Darboux normal form of h_1 and h_2, respectively. We suppose that

$$H_1^1 \subset V_k \quad \text{and} \quad H_2^1 \subset V_l \quad \text{with} \quad k \neq l.$$

Let $x = \sum\limits_{i=1}^{N_1} (\alpha_i v_i + \beta_i w_i) \in H_1^1$, x is an element of V_k and we have:

$$\begin{aligned}
\widetilde{h}_2(x) &= \sum_{s=1}^{N_2} \sum_{i=1}^{N_1} \alpha_i v'_s \wedge w'_s(v_i) + \sum_{s=1}^{N_2} \sum_{i=1}^{N_1} \beta_i v'_s \wedge w'_s(w_i) \\
&= \sum_{s=1}^{N_2} \sum_{i=1}^{N_1} \alpha_i (\langle v'_s, v_i \rangle w'_s - \langle w'_s, v_i \rangle v'_s) + \sum_{s=1}^{N_2} \sum_{i=1}^{N_1} \beta_i (\langle v'_s, w_i \rangle w'_s - \langle w'_s, w_i \rangle v'_s) \\
&= 0.
\end{aligned}$$

In fact, $\langle u, v \rangle = 0$ for all $u \in V_k$, $v \in V_l$, since the V-decomposition is an orthogonal decomposition. Thus, $x \in \ker \widetilde{h}_2$ and $H_1^1 \subset \ker \widetilde{h}_2 = H_2^0$. This contradicts the fact that $H_1^1 \nsubseteq H_2^0$. Therefore $V_k = V_l$. □

Remark 3.4 *The distribution* (V_i) *of* $\mathfrak{g}$ *is not always involutive.*

Let V_i, $i > 0$ be a linear subspace given by the V- decomposition . Let us consider for $i > 0$, the subspaces

$$Z_i = \text{Span}\{u_1, L_{u_1}u_2, L_{u_1}L_{u_2}u_3, \cdots, L_{u_1}L_{u_2}\cdots L_{u_l}u_{l+1}, u_i \in V_i, l \in \mathbb{N} \setminus \{0\}\}$$

of $\mathfrak{g}$ and Z_0 the complete subspace in $\mathfrak{g}$ which is totally orthogonal to the space $Z_1 + Z_2 + \cdots + Z_k$.

Remark 3.5 $Z_0 \subset V_0$, $V_i \subset Z_i$ *and* $\mathcal{H}^{inf} Z_i \subset Z_i$.

We have the orthogonal splitting

$$\mathfrak{g} = Z_0 \oplus Z_1 \oplus Z_2 \oplus \cdots \oplus Z_k. \tag{3.5}$$

See [16]. It is a decomposition of the Lie algebra $\mathfrak{g}$ into invariant and irreducible subspaces with respect to the infinitesimal holonomy group at e.

Definition 3.6 *[16]*

The splitting (3.5) is called a Z*-decomposition of the Lie algebra* $\mathfrak{g}$ *of a connected Riemannian Lie group* (G, g).

Proposition 3.7 *[16] The subspaces* Z_i *induce on* G *a totally parallel distribution. Thus they are involutive, and the integral manifolds are totally geodesic.*

Remark 3.8

1. *If* (G, g) *is locally symmetric Riemannian Lie group, then* $V_i = Z_i$, *since* $\mathcal{H}^{inf} = \mathcal{H}$ *and* V_i *is a nonnull* $\mathcal{H}$*-invariant subspace of* $\mathcal{H}$*-irreducible subspace* Z_i;
2. Z_j, $j \geq 0$ *is a Lie subalgebra of* $\mathfrak{g}$.

Proposition 3.9 *[1, 5, 10, 12] If an Euclidean Lie algebra* $\mathfrak{g}$ *admits a* Z*-decomposition, then the associated connected Riemannian Lie group* (G, g) *splits into a direct product of Riemannian immersed Lie subgroups of* G.

3.2. Steps of Szabó's algorithm

Step 1: V-decomposition of the Euclidean Lie algebra $\mathfrak{g}$

1. i) $\mathcal{H}H^1_{k_1} \subseteq H^1_{k_1}$ so that $H^1_{k_1}$ is a nontrivial $\mathcal{H}$-invariant subspace of V_k;

   ii) $\mathcal{H}$ acts irreducibly on V_k.

   Therefore $H^1_{k_1} = V_k$, $k > 0$ is one of the subspaces of the V-decomposition.

2. If there exist vectors $h_{k_2}, h_{k_3}, \cdots, h_{k_l}$ with $1 \leq k_i \leq \rho$, such that $H^1_{k_1} \not\subseteq H^0_{k_i}$ holds for each k_i,

   (a) either, if for any h_i, $i \notin \{k_2, k_3, \cdots, k_l\}$, the relation $H^1_{k_1} + H^1_{k_2} + \cdots + H^1_{k_l} \subseteq H^0_i$ holds. Then $H^1_{k_1} + H^1_{k_2} + \cdots + H^1_{k_l}$ is the subspace V_k of the V-decomposition,

(b) or, there exist h_i, $i \notin \{k_2, k_3, \cdots, k_l\}$ such that $H^1_{k_1} + H^1_{k_2} + \cdots + H^1_{k_l} \nsubseteq H^0_i$. Therefore, we extend the system $H^1_{k_1} + H^1_{k_2} + \cdots + H^1_{k_l}$ with the elements H^1_i.

This process can be repeated for $H^1_{k_1} + H^1_{k_2} + \cdots + H^1_{k_l} + H^1_i$ until we obtain the *maximal system* $h_{k_1}, h_{k_2}, \cdots, h_{k_j}$ such that:

- For every index i, $1 \leq i \leq j$, there is another index i', $1 \leq i' < i$ with $H^1_{k_{i'}} \nsubseteq H^0_{k_i}$.
- For every index $i \notin \{k_1, k_2, \cdots, k_j\}$, we get the relation $H^1_{k_s} \subseteq H^0_i \quad s \in \{1, 2, \cdots, j\}$.

Then $V_k = H^1_{k_1} + H^1_{k_2} + \cdots + H^1_{k_j}, \ k > 0$.

3. By continuing the procedure we can construct all other invariant subspaces V_k, $k > 0$.

Step 2: Z-decomposition of the Euclidean Lie algebra $\mathfrak{g}$

Construct all the invariant subspaces Z_k, $k \geq 0$.

4. Applications of Szabó's algorithm

We apply Szabó's algorithm on the euclidian Lie algebras of 4-dimensional $\mathcal{C}$- spaces Riemannian Lie groups. Precisely we have the following.

Theorem 4.1 *Let (G, g) be a 4-dimensional connected Riemannian Lie group. If the metric Lie algebra $(\mathfrak{g}, g(e) = \langle\, , \rangle)$ belongs to Table 4 below, then g is locally symmetric.*

Proof According to the classification table of 4-dimensional Riemannian Lie groups with nonnull harmonic Weyl tensor, see Table 1, we compute the matrix of the curvature tensor in the basis $(e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4)$, where (e_1, e_2, e_3, e_4) is an orthonormal basis of $\mathfrak{g}$.

1. **The Lie algebra $\mathbb{A}_2 \oplus 2\mathbb{A}_1$.**

For this Lie algebra, the nonnull components of the curvature tensors are:

$$\begin{array}{lll} K(e_1,e_2)e_1 = -(a^2 + \frac{3}{4}b^2)e_2, & K(e_1,e_4)e_1 = \frac{1}{4}b^2 e_4, & K(e_2,e_4)e_2 = \frac{1}{4}b^2 e_4, \\ K(e_1,e_2)e_2 = (a^2 + \frac{3}{4}b^2)e_1, & K(e_1,e_4)e_4 = -\frac{1}{4}b^2 e_1, & K(e_2,e_4)e_4 = -\frac{1}{4}b^2 e_2 \end{array}.$$

Linear curvature operator K is $\mathtt{diag}\left(-(a^2 + \frac{3}{4}b^2), 0, \frac{1}{4}b^2, 0, \frac{1}{4}b^2, 0\right)$.

<u>Case I</u> : $b \neq 0$

The eigenvectors $h_1 = e_1 \wedge e_2$, $h_2 = e_1 \wedge e_4$, $h_3 = e_2 \wedge e_4$ associated respectively to nonnull eigenvalues $-(a^2 + \frac{3}{4}b^2)$, $\frac{1}{4}b^2$, $\frac{1}{4}b^2$ are irreducible. $H^0_1 = \mathrm{Span}\{e_3, e_4\}$, $H^1_1 = \mathrm{Span}\{e_1, e_2\}$, $H^0_2 = \mathrm{Span}\{e_2, e_3\}$, $H^1_2 = \mathrm{Span}\{e_1, e_4\}$, $H^0_3 = \mathrm{Span}\{e_1, e_3\}$ and $H^1_3 = \mathrm{Span}\{e_2, e_4\}$.

$H^1_1 \nsubseteq H^0_2$ and $H^1_1 \nsubseteq H^0_3$. The subspace $V_1 = H^1_1 + H^1_2 + H^1_3 = \mathtt{Span}\{e_1, e_2, e_4\}$ is one of subspaces $V_j, j > 0$ and the V-decomposition of $\mathbb{A}_2 \oplus 2\mathbb{A}_1$ is $\mathbb{A}_2 \oplus 2\mathbb{A}_1 = V_0 \oplus V_1$, where $V_0 = \mathtt{Span}\{e_3\}$. The Z-decomposition of $\mathbb{A}_2 \oplus 2\mathbb{A}_1$ with $b \neq 0$ is

$$\mathbb{A}_2 \oplus 2\mathbb{A}_1 = Z_0 \oplus Z_1,$$

Table 4. Szabó's decomposition Lie Algebra.

Lie algebras	Structure constants	Results obtained
$\mathbb{A}_2 \oplus 2\mathbb{A}_1$	$C_{1,2}^2 = a, \quad a > 0$	Szabó's algorithm effective locally symmetric
$2\mathbb{A}_2$	$C_{1,2}^2 = a,\ C_{3,4}^4 = b$ $a > 0,\ b > 0$	Szabó's algorithm effective locally symmetric
$\mathbb{A}_{4,6}^{\alpha,\beta}$ Weyl tensor $W \neq 0$ $\alpha \neq 0, \beta \geq 0, \alpha \neq 1,\ a > 0$	$C_{1,4}^1 = \alpha a,\ C_{2,4}^3 = -C_{3,4}^2 = -a$	Szabó's algorithm effective locally symmetric
$\mathbb{A}_{3,9} \oplus \mathbb{A}_1$ $a > 0$	$C_{1,3}^2 = a\sqrt{1+m^2}$ $C_{1,2}^3 = C_{2,3}^1 = -a\sqrt{1+m^2}$ $C_{2,4}^1 = -C_{1,4}^2 = a\,m\sqrt{1+m^2}$	Szabó's algorithm effective locally symmetric
$\mathbb{A}_{3,3} \oplus \mathbb{A}_1$ $a > 0$	$C_{1,3}^1 = C_{2,3}^2 = a$	Szabó's algorithm effective locally symmetric
$\mathbb{A}_{3,7}^{\alpha} \oplus \mathbb{A}_1$ $a > 0,\ \alpha > 0$	$C_{1,3}^1 = C_{2,3}^2 = \alpha a$ $C_{2,3}^1 = -C_{1,3}^2 = a$	Szabó's algorithm effective locally symmetric
$\mathbb{A}_{4,12}$ $a > 0, d > 0$	$C_{1,3}^1 = C_{2,3}^2 = \sqrt{a^2+b^2}$ $C_{2,4}^1 = -C_{1,4}^2 = \frac{a\,d}{\sqrt{a^2+b^2}}$ $C_{2,3}^1 = -C_{1,3}^2 = \frac{b\,d}{\sqrt{a^2+b^2}}$	Szabó's algorithm effective locally symmetric

where $Z_1 = \mathtt{Span}\{e_1, e_2, e_4\}$ and $Z_0 = \text{Span}\{e_3\}$.

For the Lie algebra Z_1, the Lie brackets are

$$[e_2, e_1] = -ae_1 - be_4, \quad [e_2, e_4] = 0, \quad [e_1, e_4] = 0.$$

Moreover $[Z_0, Z_1] = \{0\}$.

For this decomposition, left invariant Riemannian metric induced by the restriction of the inner product $\langle\, ,\, \rangle$ on the 3-dimensional Lie algebra Z_1 is not locally symmetric by proposition 2.8 or 2.9.

Therefore if $b \neq 0$, then $\mathbb{A}_2 \oplus 2\mathbb{A}_1$ does not admit locally symmetric metric.

Case II : $b = 0$.

The eigenvector $h_1 = e_1 \wedge e_2$ associated to the nonnull eigenvalue $-a^2$ is irreducible. $H_1^0 = \mathtt{Span}\{e_3, e_4\}$ and $H_1^1 = \mathtt{Span}\{e_1, e_2\}$. The subspace $V_1 = \text{Span}\{e_1, e_2\}$ is one of the subspaces V_j, $j > 0$. The V-decomposition is $\mathbb{A}_2 \oplus 2\mathbb{A}_1 = V_0 \oplus V_1$, where $V_0 = \mathtt{Span}\{e_3, e_4\}$. The Z-decomposition of $\mathbb{A}_2 \oplus 2\mathbb{A}_1$ with $b = 0$ is

$$\mathbb{A}_2 \oplus 2\mathbb{A}_1 = Z_0 \oplus Z_1,$$

where $Z_1 = \mathtt{Span}\{e_1, e_2\}$ and $Z_0 = \mathtt{Span}\{e_3, e_4\}$.

In addition, Z_0 is a 2-dimensional abelian Lie algebra and $[Z_0, Z_1] = \{0\}$.

For this decomposition, left invariant Riemannian metrics induced by the restriction of the inner product $\langle\, ,\, \rangle$ on the 2-dimensional Lie algebras Z_0 and Z_1 are locally.

Therefore if $b = 0$, then $\mathbb{A}_2 \oplus 2\mathbb{A}_1$ admit locally symmetric metric.

2. **The Lie algebra** $2\mathbb{A}_2$**.**

The matrix of the curvature operator is $[\overline{K}] = \mathrm{diag}(-a^2, 0, 0, 0, 0, -b^2)$. For the nonnull eigenvalue $-a^2$ and $-b^2$ of $[\overline{K}]$, the associated and irreducible eigenvectors are $h_1 = e_1 \wedge e_2$ and $h_2 = e_3 \wedge e_4$. Therefore $H_1^0 = \mathrm{Span}\{e_3, e_4\}$, $H_1^1 = \mathrm{Span}\{e_1, e_2\}$, $H_2^0 = \mathrm{Span}\{e_1, e_2\}$, and $H_2^1 = \mathrm{Span}\{e_3, e_3\}$.

i) $H_1^1 \subset H_2^0$; therefore, H_1^1 is one of the subspaces V_j, $j > 0$ in the V-decomposition.

ii) $H_2^1 \subset H_1^0$; hence, H_2^1 is one of the subspaces V_i, $i > 0$ in the V-decomposition.

Setting $V_1 = H_1^1$ and $V_2 = H_2^1$, the V-decomposition of $2\mathbb{A}_2$ is $2\mathbb{A}_2 = V_1 \oplus V_2$ and the Z-decomposition is

$$2\mathbb{A}_2 = Z_1 \oplus Z_2,$$

where $Z_1 = \mathtt{Span}\{e_1, e_2\}$ and $Z_2 = \mathtt{Span}\{e_3, e_4\}$.

The components Z_1 and Z_2 of the Z-decomposition satisfy $[Z_1, Z_2] = \{0\}$. Moreover, the left invariant Riemannian metrics induced by the restriction on $Z_i, i \in \{1, 2\}$ of the inner product $\langle\, ,\rangle$ are locally symmetric since $\dim Z_1 = \dim Z_2 = 2$. Thus, the metric g is locally symmetric.

3. **The Lie algebra** $\mathbb{A}_{4,6}^{\alpha,\beta}$, $\alpha \neq 1$**.**

The nonnull structure constants are $C_{1,4}^1 = \alpha a$, $C_{2,4}^3 = -a$, $C_{3,4}^2 = a$, $a > 0$ and the matrix of the curvature tensor linear operator is $[\overline{K}] = \mathrm{diag}(0, 0, -(\alpha a)^2, 0, 0, 0)$. For the nonnull eigenvalue $-(\alpha a)^2$ of $[\overline{K}]$, the associated irreducible eigenvector is $h_1 = e_1 \wedge e_4$. The Z-decomposition is

$$\mathfrak{g} = Z_0 \oplus Z_1,$$

where $Z_1 = \mathrm{Span}\{e_1, e_4\}$ and $Z_0 = \mathrm{Span}\{e_2, e_3\}$. On the other hand, the left invariant Riemannian metrics induces by the restriction on $Z_i, i \in \{0, 1\}$ of the inner product $\langle\, ,\rangle$ are locally symmetric, since $\dim Z_0 = \dim Z_1 = 2$. Therefore, the metric g is locally symmetric. Also $[Z_0, Z_1] = Z_0$ and Z_0 is an abelian Lie algebra.

4. **The Lie algebra** $\mathbb{A}_{3,3} \oplus \mathbb{A}_1$**:**

The matrix of the linear curvature operator $\overline{K}$ is $\mathtt{diag}(-a^2,\, -a^2,\, 0,\, -a^2,\, 0,\, 0,)$. The eigenvectors $h_1 = e_1 \wedge e_2$, $h_2 = e_1 \wedge e_3$, $h_3 = e_2 \wedge e_3$ associated respectively to nonzero eigenvalues $-a^2$, $-a^2$, $-a^2$ are irreducible. By direct computation, the Z-decomposition of $\mathbb{A}_{3,3} \oplus \mathbb{A}_1$ is

$$\mathbb{A}_{3,3} \oplus \mathbb{A}_1 = Z_0 \oplus Z_1,$$

where $Z_1 = \mathtt{Span}\{e_1, e_2, e_3\}$ $Z_0 = \mathtt{Span}\{e_4\}$.

For the Lie algebra Z_1, the Lie brackets are

$$[e_3, e_1] = -ae_1 + 0e_2, \quad [e_3, e_2] = 0e_1 - ae_2, \quad \text{and} \quad [e_1, e_2] = 0.$$

Furthermore, $[Z_0, Z_1] = \{0\}$.

The restriction of the inner product $\langle\,,\,\rangle$ on Z_1 induces a locally symmetric left invariant Riemannian metric by proposition 2.8. Therefore the inner product $\langle\,,\,\rangle$ on $\mathbb{A}_{3,3}\oplus\mathbb{A}_1$ is locally symmetric as product of locally symmetric left invariant metrics.

For other Lie algebras, $\mathbb{A}_{3,9}\oplus\mathbb{A}_1$, $\mathbb{A}^{\alpha}_{3,7}\oplus\mathbb{A}_1$, and $\mathbb{A}_{4,12}$, the irreducible eigenvectors associated to nonnull eigenvalues are $h_1 = e_1\wedge e_2$, $h_2 = e_1\wedge e_3$ and $h_3 = e_2\wedge e_3$. By direct computation, the Z-decomposition is

$$\mathfrak{g} = Z_0\oplus Z_1,$$

where $Z_0 = \{e_4\}$ and $Z_1 = \mathrm{Span}\{e_1, e_2, e_3\}$. The Lie brackets on the 3-dimensional components Z_1 of the above Lie algebras are as follows:

(a) For $\mathbb{A}_{3,9}\oplus\mathbb{A}_1$,

$$[e_1, e_2] = -a\sqrt{1+m^2}\,e_3, \quad [e_2, e_3] = -a\sqrt{1+m^2}\,e_1, \quad [e_3, e_1] = -a\sqrt{1+m^2}\,e_2.$$

Moreover, $[Z_0, Z_1]\subset Z_1$.

(b) For $\mathbb{A}^{\alpha}_{3,7}\oplus\mathbb{A}_1$,

$$[e_3, e_1] = -\alpha a\, e_1 + a\, e_2, \quad [e_3, e_2] = -ae_1 - \alpha\, ae_2, \quad [e_1, e_2] = 0.$$

and it holds that $[Z_0, Z_1] = \{0\}$.

(c) For $\mathbb{A}_{4,12}$,

$$[e_3, e_1] = -\sqrt{a^2+b^2}\,e_1 + \frac{bd}{\sqrt{a^2+b^2}}\,e_2, \quad [e_3, e_2] = -\frac{bd}{\sqrt{a^2+b^2}}\,e_1 - \sqrt{a^2+b^2}e_2,$$

$$[e_1, e_2] = 0.$$

We also have $[Z_0, Z_1]\subset Z_1$.

For each of these Lie algebras, the left invariant Riemannian metrics induced by the restriction of the inner product $\langle\,,\,\rangle$ on Z_1 are locally symmetric by Propositions 2.8 and 2.9. Therefore, the metrics g are locally symmetric.

□

5. Conclusion

We point out a rich and complete important class of examples of 4-dimensionnal locally symmetric Riemannian manifolds: the 4-dimensional locally symmetric Riemannian Lie groups which are decomposable with respect to the infinitesimal holonomy group into a product of locally symmetric Riemannian Lie groups of lower dimension. For 4-dimensionnal metric Lie algebras $\mathbb{A}^{\alpha,\beta}_{4,5}\,(\alpha\beta\neq 0,\ -1\leq\alpha\leq\beta\leq 1), \mathbb{A}^{\alpha,\beta}_{4,6}\,(\alpha\neq 0,\ \beta>0$ and $W\neq 0), \mathbb{A}_{3,6}\oplus\mathbb{A}_1$ and $4\mathbb{A}_1$ of C-spaces Riemannian Lie groups, the Szabó's algorithm is not effective. But, the associated left invariant Riemannian metrics are locally symmetric since their sectional curvatures are constants. For the remaining metric Lie algebras $\mathbb{A}^{\beta}_{4,9}$, $\mathbb{A}^{\alpha,}_{4,11}$ $(\alpha>0)$ and $\mathbb{A}_{3,3}\oplus\mathbb{A}_1(b\neq 0)$ of C-spaces Riemannian Lie groups, only the direct computation of algebraic condition of locally symmetricity seems afordable. But this method is very tedious.

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