TÜBİTAK

Turkish Journal of Mathematics

Volume 47 | Number 4 Article 5

1-1-2023

On Bell based Appell polynomials

ZEYNEP ÖZAT

MEHMET ALİ ÖZARSLAN

BAYRAM ÇEKİM

Recommended Citation

ÖZAT, ZEYNEP; ÖZARSLAN, MEHMET ALİ; and ÇEKİM, BAYRAM (2023) "On Bell based Appell polynomials," *Turkish Journal of Mathematics*: Vol. 47: No. 4, Article 5. https://doi.org/10.55730/1300-0098.3415

Available at: https://journals.tubitak.gov.tr/math/vol47/iss4/5

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 1099 – 1128
© TÜBİTAK
doi:10.55730/1300-0098.3415

On Bell based Appell polynomials

Zeynep ÖZAT[1,3,*], **Mehmet Ali ÖZARSLAN**[2], **Bayram ÇEKİM**[3]
[1]Graduate School of Natural and Applied Sciences, Gazi University, Ankara, Turkey
[2]Department of Mathematics, Faculty of Arts and Sciences, Eastern Mediterranean University, via Mersin 10, Gazimağusa, North Cyprus, Turkey
[3]Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey

Received: 19.08.2022 • **Accepted/Published Online:** 01.02.2023 • **Final Version:** 16.05.2023

Abstract: Recently, several Bell based polynomials such as Bernoulli, Euler, Genocchi and Apostol versions were defined and investigated. The main aim of this paper is to introduce the general family of Bell based Appell polynomials, which includes many new members in addition to the existing ones, and to investigate their properties including determinantal representation, recurrence relation, derivative formula, addition formula, shift operators and differential equation. Furthermore, we introduce 2-iterated Bell-Appell polynomials and investigate their similar properties. With the help of this 2-iterated family, we also obtain the closed form summation formulae between the usual and the generalized versions of the Bell based Appell polynomials. Finally, we introduce the Bell based Bernoulli-Euler, Bernoulli-Genocchi, Euler-Genocchi and Stirling-Appell polynomials of the second kind as special cases of 2-iterated Bell based Appell polynomials and state the corresponding results.

Key words: Appell polynomials, Bell polynomials, Bell based Bernoulli polynomials, Bell based Euler polynomials, Bell based Genocchi polynomials, Bell based Stirling polynomials of the second kind

1. Introduction

Special polynomials are one of the important fields of mathematics. It is because of this reason that, especially in recent years, general cases of special polynomials have been studied and new properties of these polynomials have been discovered. Later, the hybrid polynomial families have emerged. The Appell polynomials are also among the most important polynomials in the study of hybrid polynomials. The identity $\frac{d}{dx}\mathcal{A}_j(x) = j\mathcal{A}_{j-1}(x)$, $j = 1, 2, ...$ introduces Appell polynomials. Appell polynomials have a wide range of applications, including in number theory, approximation theory, analytic functions, and so on (see for details [2, 3, 8, 9, 12, 17, 19, 25, 29, 31, 33, 34, 36, 37, 39, 41, 42]). Because of these applications, various properties of them have been studied. Appell polynomials can be defined in a variety of equivalent definitions. Appell polynomials can also be defined via the generating relation

$$A(t)\,e^{xt} = \sum_{j=0}^{\infty} \mathcal{A}_j(x)\,\frac{t^j}{j!}, \tag{1.1}$$

*Correspondence: zeynep.ozat1@gazi.edu.tr

2010 *AMS Mathematics Subject Classification:* Primary 33C65, 11B73; Seconary, 11B68 where $A(t)$ which is the determining function denoted by a formal power series

$$A(t) = \sum_{j=0}^{\infty} A_j \frac{t^j}{j!}, \quad \mathcal{A}_0 \neq 0 \tag{1.2}$$

[2, 36]. Appell polynomials contain many polynomial families such as Bernoulli, Euler, Genocchi (Genocchi polynomials are an analogous of Bernoulli and Euler polynomials) and Hermite polynomials [5, 9, 27, 35, 37] which have many important application areas. Due to their usefulness in applied fields, different generalizations of Appell polynomials have been studied, in recent years.

Bell polynomials, which are families of important fields of study such as special functions, analytic number theory and combinatorics, have been the interest of many mathematicians. The Bell polynomials $Bel_j(x)$ which have a significant impact on number theory are defined by means of the generating function [6]

$$e^{x\left(e^t-1\right)} = \sum_{j=0}^{\infty} Bel_j(x) \frac{t^j}{j!}, \tag{1.3}$$

where $Bel_j(x)$ are called the single-variable Bell polynomials. For $x=1$, the Bell numbers Bel_j [6] are given by the following serie

$$e^{e^t-1} = \sum_{j=0}^{\infty} Bel_j \frac{t^j}{j!}. \tag{1.4}$$

The relationship between Stirling numbers of the second kind [26] and Bell polynomials is given as follows

$$\begin{aligned} Bel_j(x) &= e^{-x} \sum_{l=0}^{\infty} \frac{l^j}{j!} x^l \\ &= \sum_{l=0}^{j} x^l S_2(j,l), \end{aligned}$$

where $S_2(j,l)$ denote Stirling numbers of the second kind, defined by the relations [7, 37]

$$z^j = \sum_{l=0}^{j} S_2(j,l)\, z(z-1)\cdots(z-l+1), \tag{1.5}$$

and

$$\left(e^z-1\right)^l = l! \sum_{j=l}^{\infty} S_2(j,l) \frac{z^j}{j!}. \tag{1.6}$$

Bivariate Bell polynomials $Bel_j(x,y)$ were defined by means of the generating function [11]

$$e^{xt} e^{y\left(e^t-1\right)} = \sum_{j=0}^{\infty} Bel_j(x,y) \frac{t^j}{j!}. \tag{1.7}$$

The first five bivariate Bell polynomials $Bel_j(x,y)$ are as follows:

$$\begin{aligned}
Bel_0(x,y) &= 1,\\
Bel_1(x,y) &= x+y,\\
Bel_2(x,y) &= (x+y)^2+y,\\
Bel_3(x,y) &= (x+y)^3+3y^2+3xy+y,\\
Bel_4(x,y) &= (x+y)^3+6x^2y+12xy^2+4xy+6y^3+y^2+y.
\end{aligned}$$

Bivariate Appell polynomials play an important role in applied sciences because of their potentially useful properties. Many extensions of Appell polynomials such as Laguerre based, Hermite based, truncated exponential based, Δ_h-Gould-Hopper-Appell polynonials were investigated in detail (see [4, 18, 20, 22, 24, 28, 32]). In this paper, motivated by these special polynomials, we introduce the Bell based Appell polynomials

$$A(t)\,e^{xt}e^{y\left(e^t-1\right)}=\sum_{j=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_j(x,y)\,\frac{t^j}{j!}, \tag{1.8}$$

where $A(t)$ has a formal power series

$$A(t)=\sum_{l=0}^{\infty}\alpha_l\frac{t^l}{l!},\quad \alpha_0\neq 0. \tag{1.9}$$

We can obtain the following polynomial families for certain choices of $A(t)$.

- In the case of $A(t)=1$ and $x=0$, the polynomials in (1.8) reduce to the Bell polynomials [13, 35].
- In the case $A(t)=1$, the polynomials in (1.8) reduce to the bivariate Bell polynomials [11].
- In the case of $A(t)=\left(\frac{t}{e^t-1}\right)^{\alpha}$, we get the Bell based Bernoulli polynomials of order α [11].
- In the case of $A(t)=\left(\frac{2}{e^t+1}\right)^{\alpha}$, we have the Bell based Euler polynomials of order α [15].
- In the case of $A(t)=\left(\frac{2t}{e^t+1}\right)^{\alpha}$, we get the Bell based Genocchi polynomials of order α [10].
- In the case of $A(t)=\frac{\left(e^t-1\right)^k}{k!}$, the polynomials in (1.8) reduce to the Bell based Stirling polynomials [11].
- In the case of $A(t)=\left(\frac{2^{\sigma}t^{\mu}}{\lambda e^t+1}\right)^{\alpha}$, we get the Bell based Apostol type polynomials of order α for $\lambda,\sigma,\mu\in\mathbb{C}$ and $\alpha\in\mathbb{N}_0$ [16].
- In the case of $A(t)=\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}$, we have the Bell based Apostol-Bernoulli polynomials of order α [14].
- In the case of $A(t)=\left(\frac{2}{\lambda e^t+1}\right)^{\alpha}$, we obtain the Bell based Apostol-Euler polynomials of order α [14].

- In the case of $A(t) = \left(\frac{2t}{\lambda e^t+1}\right)^{\alpha}$, the polynomials in (1.8) reduce to the Bell based Apostol-Genocchi polynomials of order α [14].

- In the case of $A(t) = \left(\frac{1-u}{\lambda e^t-u}\right)^{\alpha}$, we have the Bell based Apostol-type Frobenius-Euler polynomials polynomials of order α [1].

- In the case of $A(t) = \left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha}$, the polynomials in (1.8) reduce to the Bell based unification of the Apostol-Bernoulli, Euler and Genocchi polynomials for $\lambda, \alpha, \beta \in \mathbb{C}$, $k \in \mathbb{N}_0$ and $a, b \in \mathbb{N} - \{0\}$.

This paper is organized as follows: In Section 2, the determinantal form, recurrence relation, derivative formula, shift operators, differential equation and addition formula are obtained for Bell based Appell polynomials. In Section 3, new properties of Bell based Bernoulli, Bell based Euler and Bell based Genocchi polynomials are given as special cases of the main results obtained in Section 2. In Section 4, we define the 2-iterated Bell based Appell polynomials ${}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)$ and derive their determinantal representation, recurrence relation, shift operators, differential equation and closed form summation formulae. In Section 5, we introduce some new 2-iterated Bell based Appell polynomials such as Bell based Bernoulli-Euler, Bernoulli-Genocchi, Euler-Genocchi and Stirling-Appell polynomials and state corresponding main results to those obtained in Section 4.

2. Bell based Appell polynomials

In this section, we give the determinantal form, recurrence relation, derivative formula, lowering and raising operators, differential equation and addition formula for the Bell based Appell polynomials.

From (1.7), (1.8), and (1.9), the explicit representation of the Bell based Appell polynomials is given as follows:

$$ {}_{\mathcal{B}}\mathcal{A}_j(x,y) = \sum_{l=0}^{j} \binom{j}{l} \alpha_{j-l} Bel_l(x,y). $$

Theorem 2.1 *The Bell based Appell polynomials ${}_{\mathcal{B}}\mathcal{A}_j(x,y)$ have the following determinantal representation:*

$$ {}_{\mathcal{B}}\mathcal{A}_j(x,y) = \frac{(-1)^j}{(\gamma_0)^{j+1}} \begin{vmatrix} Bel_0(x,y) & Bel_1(x,y) & \cdots & Bel_{j-1}(x,y) & Bel_j(x,y) \\ \gamma_0 & \gamma_1 & \cdots & \gamma_{j-1} & \gamma_j \\ 0 & \gamma_0 & \cdots & \binom{j-1}{1}\gamma_{j-2} & \binom{j}{1}\gamma_{j-1} \\ 0 & 0 & \cdots & \binom{j-1}{2}\gamma_{j-3} & \binom{j}{2}\gamma_{j-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_0 & \binom{j}{j-1}\gamma_1 \end{vmatrix}, \tag{2.1} $$

where $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n$ are the coefficients of the Maclaurin series of the function $\frac{1}{A(t)}$.

Proof Using the series representation

$$[A(t)]^{-1} = \sum_{j=0}^{\infty} \gamma_j \frac{t^j}{j!}, \tag{2.2}$$

and the generating functions (1.7) and (1.8), we get

$$e^{xt} e^{y\left(e^t-1\right)} = \sum_{l=0}^{\infty} \gamma_l \frac{t^l}{l!} \sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_j(x,y) \frac{t^j}{j!}. \tag{2.3}$$

Hence, we can write

$$\sum_{j=0}^{\infty} Bel_j(x,y) \frac{t^j}{j!} = \left(\sum_{l=0}^{\infty} \gamma_l \frac{t^l}{l!} \right) \left(\sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_j(x,y) \frac{t^j}{j!} \right). \tag{2.4}$$

Applying the Cauchy product rule, we have

$$\sum_{j=0}^{\infty} Bel_j(x,y) \frac{t^j}{j!} = \sum_{j=0}^{\infty} \sum_{l=0}^{j} \binom{j}{l} \gamma_l \, {}_{\mathcal{B}}\mathcal{A}_{j-l}(x,y) \frac{t^j}{j!}. \tag{2.5}$$

Comparing the coefficients of $\frac{t^j}{j!}$ in the equation given above, we have

$$Bel_j(x,y) = \sum_{l=0}^{j} \binom{j}{l} \gamma_l \, {}_{\mathcal{B}}\mathcal{A}_{j-l}(x,y). \tag{2.6}$$

Hence we get the following system of equations:

$$\begin{aligned}
&Bel_0(x,y) = \gamma_0 \, {}_{\mathcal{B}}\mathcal{A}_0(x,y), \\
&Bel_1(x,y) = \gamma_0 \, {}_{\mathcal{B}}\mathcal{A}_1(x,y) + \gamma_1 \, {}_{\mathcal{B}}\mathcal{A}_0(x,y), \\
&Bel_2(x,y) = \gamma_0 \, {}_{\mathcal{B}}\mathcal{A}_2(x,y) + \binom{2}{1} \gamma_1 \, {}_{\mathcal{B}}\mathcal{A}_1(x,y) + \gamma_2 \, {}_{\mathcal{B}}\mathcal{A}_0(x,y), \\
&\vdots \\
&Bel_{j-1}(x,y) = \gamma_0 \, {}_{\mathcal{B}}\mathcal{A}_{j-1}(x,y) + \binom{j-1}{1} \gamma_1 \, {}_{\mathcal{B}}\mathcal{A}_{j-2}(x,y) + \ldots \gamma_{j-1} \, {}_{\mathcal{B}}\mathcal{A}_0(x,y), \\
&Bel_j(x,y) = \gamma_0 \, {}_{\mathcal{B}}\mathcal{A}_j(x,y) + \binom{j}{1} \gamma_1 \, {}_{\mathcal{B}}\mathcal{A}_{j-1}(x,y) + \ldots + \gamma_j \, {}_{\mathcal{B}}\mathcal{A}_0(x,y).
\end{aligned}$$

Using Cramer's rule, we obtain

$$
{}_{\mathcal{B}}A_j(x,y)=\begin{vmatrix}
\gamma_0 & 0 & \cdots & 0 & Bel_0(x,y)\\
\gamma_1 & \gamma_0 & \cdots & 0 & Bel_1(x,y)\\
\gamma_2 & \binom{2}{1}\gamma_1 & \cdots & 0 & Bel_2(x,y)\\
\gamma_3 & \binom{3}{2}\gamma_2 & \cdots & 0 & Bel_3(x,y)\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
\gamma_{j-1} & \binom{j-1}{1}\gamma_{j-2} & \cdots & \gamma_0 & Bel_{j-1}(x,y)\\
\gamma_j & \binom{j}{1}\gamma_{j-1} & \cdots & \binom{j}{j-1}\gamma_1 & Bel_j(x,y)
\end{vmatrix}.\begin{vmatrix}
\gamma_0 & 0 & \cdots & 0 & 0\\
\gamma_1 & \gamma_0 & \cdots & 0 & 0\\
\gamma_2 & \binom{2}{1}\gamma_1 & \cdots & 0 & 0\\
\gamma_3 & \binom{3}{2}\gamma_2 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
\gamma_{j-1} & \binom{j-1}{1}\gamma_{j-2} & \cdots & \gamma_0 & 0\\
\gamma_j & \binom{j}{1}\gamma_{j-1} & \cdots & \binom{j}{j-1}\gamma_1 & \gamma_0
\end{vmatrix}^{-1}. \tag{2.7}
$$

Taking the determinant's transpose and applying the lower triangular matrix property, we get

$$
{}_{\mathcal{B}}A_j(x,y)=\frac{1}{(\gamma_0)^{j+1}}\begin{vmatrix}
\gamma_0 & \gamma_1 & \cdots & \gamma_{j-1} & \gamma_j\\
0 & \gamma_0 & \cdots & \binom{j-1}{1}\gamma_{j-2} & \binom{j}{1}\gamma_{j-1}\\
0 & 0 & \cdots & \binom{j-1}{2}\gamma_{j-3} & \binom{j}{2}\gamma_{j-2}\\
0 & 0 & \cdots & \binom{j-1}{3}\gamma_{j-4} & \binom{j}{3}\gamma_{j-3}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & \gamma_0 & \binom{j}{j-1}\gamma_1\\
Bel_0(x,y) & Bel_1(x,y) & \cdots & Bel_{j-1}(x,y) & Bel_j(x,y)
\end{vmatrix}.
$$

Lastly, using elementary row operations, the proof is completed. □

Theorem 2.2 *The recurrence relation satisfied by Bell based Appell polynomials* ${}_{\mathcal{B}}\mathcal{A}_j(x,y)$ *is given by*

$$
{}_{\mathcal{B}}\mathcal{A}_{j+1}(x,y)=x\ {}_{\mathcal{B}}\mathcal{A}_j(x,y)+\sum_{l=0}^{j}\binom{j}{l}\beta_l\ {}_{\mathcal{B}}\mathcal{A}_{j-l}(x,y)+y\sum_{l=0}^{j}\binom{j}{l}\ {}_{\mathcal{B}}\mathcal{A}_{j-l}(x,y)\ , \tag{2.8}
$$

where

$$
\frac{A'(t)}{A(t)}=\sum_{l=0}^{\infty}\beta_l\frac{t^l}{l!}. \tag{2.9}
$$

Proof Taking derivatives with respect to t on both sides of Equation (1.8), we obtain

$$
\sum_{j=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_{j+1}(x,y)\frac{t^j}{j!}=+xA(t)e^{xt}e^{y\left(e^t-1\right)}+\frac{A'(t)}{A(t)}A(t)e^{xt}e^{y\left(e^t-1\right)}+ye^tA(t)e^{xt}e^{y\left(e^t-1\right)}. \tag{2.10}
$$

Using Equations (1.8) and (2.9) and then applying Cauchy product rule, we have

$$\begin{aligned}\sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_{j+1}(x,y)\frac{t^j}{j!} &= \left(\sum_{l=0}^{\infty}\beta_l\frac{t^l}{l!}\right)\left(\sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_j(x,y)\frac{t^j}{j!}\right)+x\sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_j(x,y)\frac{t^j}{j!}+y\sum_{j=0}^{\infty}\frac{t^l}{l!}\sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_j(x,y)\frac{t^j}{j!}\\ &= \sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{j}{l}\beta_l\, {}_{\mathcal{B}}\mathcal{A}_{j-l}(x,y)\frac{t^j}{j!}+x\sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_j(x,y)\frac{t^j}{j!}+y\sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{j}{l} {}_{\mathcal{B}}\mathcal{A}_{j-l}(x,y)\frac{t^j}{j!}.\end{aligned} \tag{2.11}$$

By equating the coefficients of $\frac{t^j}{j!}$ on both sides of (2.11), the proof is completed. □

Theorem 2.3 *In view of the generating function of Bell based Appell polynomials, the following partial derivative formulae hold true:*

$$\frac{\partial}{\partial x} {}_{\mathcal{B}}\mathcal{A}_j(x,y) = j\, {}_{\mathcal{B}}\mathcal{A}_{j-1}(x,y), \tag{2.12}$$

$$\frac{\partial}{\partial y} {}_{\mathcal{B}}\mathcal{A}_j(x,y) = \Delta_x\left({}_{\mathcal{B}}\mathcal{A}_j(x,y)\right), \tag{2.13}$$

where

$$\Delta_x f(x,y) = f(x+1,y) - f(x,y).$$

Proof Taking derivatives with respect to x and y on both sides of (1.8), we can write

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\left\{ {}_{\mathcal{B}}\mathcal{A}_j(x,y)\right\}\frac{t^j}{j!} = t\left\{A(t)\,e^{xt}e^{y\left(e^t-1\right)}\right\}$$

and

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{\partial}{\partial y}\left\{ {}_{\mathcal{B}}\mathcal{A}_j(x,y)\right\}\frac{t^j}{j!} &= \left(e^t-1\right)\left\{A(t)\,e^{xt}e^{y\left(e^t-1\right)}\right\}\\ &= A(t)\,e^{(x+1)t}e^{y\left(e^t-1\right)} - A(t)\,e^{xt}e^{y\left(e^t-1\right)}\\ &= \sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_j(x+1,y)\frac{t^j}{j!} - \sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_j(x,y)\frac{t^j}{j!}.\end{aligned}$$

The desired result is obtained by arranging these equations and comparing the coefficients of $\frac{t^j}{j!}$.

□

Theorem 2.4 *For the Bell based Appell polynomials* ${}_{\mathcal{B}}\mathcal{A}_j(x,y)$*, we have the following lowering and raising operators:*

$${}_x\sigma_j^- := \frac{1}{j}D_x, \tag{2.14}$$

$${}_x\sigma_j^+ := x + y\sum_{l=0}^{j}\frac{{D_x}^l}{l!} + \sum_{l=0}^{j}\beta_l\frac{{D_x}^l}{l!}, \tag{2.15}$$

respectively, and the differential equation satisfied by the ${}_{\mathcal{B}}\mathcal{A}_j\left(x,y\right)$ *is given by*

$$\left[xD_x + y\sum_{l=0}^{j}\frac{{D_x}^{l+1}}{l!} + \sum_{l=0}^{j}\beta_l\frac{{D_x}^{l+1}}{l!} - j\right]\ {}_{\mathcal{B}}\mathcal{A}_j\left(x,y\right) = 0\ , \tag{2.16}$$

where $D_x := \frac{\partial}{\partial x}$ *and the* β_k*'s are the same as in* (2.9) *given by Theorem 2.2.*

Proof In light of the following derivative relation

$$D_x\ {}_{\mathcal{B}}\mathcal{A}_j\left(x,y\right) = j\ {}_{\mathcal{B}}\mathcal{A}_{j-1}\left(x,y\right), \tag{2.17}$$

we have

$$\frac{1}{j}D_x\ {}_{\mathcal{B}}\mathcal{A}_j\left(x,y\right) = \ {}_{\mathcal{B}}\mathcal{A}_{j-1}\left(x,y\right). \tag{2.18}$$

It demonstrates that provided the lowering operator

$${}_x\sigma_j^- := \frac{1}{j}D_x.$$

Now, we may express the term ${}_{\mathcal{B}}\mathcal{A}_{j-l}\left(x,y\right)$ in terms of the lowering operator in the recurrence relation (2.8) as follows:

$$\begin{aligned} {}_{\mathcal{B}}\mathcal{A}_{j-l}\left(x,y\right) \quad &= \left[\sigma_{j-l+1}^-\sigma_{j-l+2}^-\cdots\sigma_j^-\right]\ {}_{\mathcal{B}}\mathcal{A}_j\left(x,y\right) \\ &= \frac{(j-l)!}{j!}D_x^l\ {}_{\mathcal{B}}\mathcal{A}_j\left(x,y\right). \end{aligned} \tag{2.19}$$

By using Equations (2.8) and (2.19), we get

$$\left[x + y\sum_{l=0}^{j}\frac{{D_x}^l}{l!} + \sum_{l=0}^{j}{D_x}^l\frac{\beta_l}{l!}\right]\ {}_{\mathcal{B}}\mathcal{A}_j\left(x,y\right) = \ {}_{\mathcal{B}}\mathcal{A}_{j+1}\left(x,y\right); \tag{2.20}$$

therefore, the raising operator is obtained. In order to derive the differential equation, using the factorization method

$$\sigma_{j+1}^-\sigma_j^+\left(\ {}_{\mathcal{B}}\mathcal{A}_j\left(x,y\right)\right) = \ {}_{\mathcal{B}}\mathcal{A}_j\left(x,y\right), \tag{2.21}$$

we have

$$\left[xD_x + y\sum_{l=0}^{j}\frac{{D_x}^{l+1}}{l!} + \sum_{l=0}^{j}{D_x}^{l+1}\frac{\beta_l}{l!} - j\right]\ {}_{\mathcal{B}}\mathcal{A}_j\left(x,y\right) = 0\ ,$$

whence the result. □

Theorem 2.5 *Bell based Appell polynomials have the following addition formulae:*

$$ {}_{\mathcal{B}}\mathcal{A}_j\left(x+z,y\right) = \sum_{l=0}^{j}\binom{j}{l}\,{}_{\mathcal{B}}\mathcal{A}_{j-l}\left(z,y\right)x^l, \tag{2.22}$$

$$ {}_{\mathcal{B}}\mathcal{A}_j\left(x,y+z\right) = \sum_{l=0}^{j}\binom{j}{l}\,{}_{\mathcal{B}}\mathcal{A}_{j-l}\left(x,y\right)Bel_l\left(z\right), \tag{2.23}$$

$$ {}_{\mathcal{B}}\mathcal{A}_j\left(x+z,y+\omega\right) = \sum_{l=0}^{j}\binom{j}{l}\,{}_{\mathcal{B}}\mathcal{A}_{j-l}\left(x,y\right)Bel_l\left(z,\omega\right). \tag{2.24}$$

Proof Taking $x+z$ instead of x in Equation (1.8), we have

$$\begin{aligned}\sum_{j=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_j\left(x+z,y\right)\frac{t^j}{j!} &= A\left(t\right)e^{(x+z)t}e^{y\left(e^t-1\right)}\\ &= \left(\sum_{j=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_j\left(z,y\right)\frac{t^j}{j!}\right)\left(\sum_{l=0}^{\infty}x^l\frac{t^l}{l!}\right).\end{aligned}$$

Applying the Cauchy product rule, we obtain

$$\sum_{n=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_n\left(x+z,y\right)\frac{t^j}{j!} = \sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{j}{l}\,{}_{\mathcal{B}}\mathcal{A}_{j-l}\left(z,y\right)x^l\frac{t^j}{j!}. \tag{2.25}$$

Comparing the coeffcients of $\frac{t^j}{j!}$ on both sides of Equation (2.25) gives Equation (2.22). Similarly, Equation (2.23) is obtained by replacing y with $y+z$, and Equation (2.24) is proved by replacing x with $x+z$ and y with $y+\omega$. □

Corollary 2.6 *Bell based Appell polynomials have the following multiplication formula:*

$$ {}_{\mathcal{B}}\mathcal{A}_j\left(mx,ny\right) = \sum_{l=0}^{j}\binom{j}{l}\,{}_{\mathcal{B}}\mathcal{A}_{j-l}\left(x,y\right)Bel_l\left(\left(m-1\right)x,\left(n-1\right)y\right),\quad m,n\in\mathbb{N}. \tag{2.26}$$

This results from (2.24) *by taking* $z=\left(m-1\right)x$ *and* $\omega=\left(n-1\right)y$.

Theorem 2.7 *Bell based Appell polynomials satisfy the following relation*

$$\sum_{l=0}^{j}\binom{j}{l}\,{}_{\mathcal{B}}\mathcal{A}_l\left(x,y\right)\mathcal{A}_{j-l}\left(x,y\right) = \sum_{l=0}^{j}\binom{j}{l}\,{}_{\mathcal{B}}\mathcal{A}_j\left(2x,2y\right)\alpha_{j-l}.$$

Proof Consider the following equations:

$$A\left(t\right)e^{xt}e^{y\left(e^t-1\right)} = \sum_{j=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_j\left(x,y\right)\frac{t^j}{j!}, \tag{2.27}$$

$$A(t)\, e^{xt} e^{y\left(e^t-1\right)} = \sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_l\,(x,y)\,\frac{t^l}{l!}. \tag{2.28}$$

From (2.27) and (2.28), we obtain

$$A(t)\left[A(t)\, e^{2xt} e^{2y\left(e^t-1\right)}\right] = \left(\sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_j\,(x,y)\,\frac{t^j}{j!}\right)\left(\sum_{l=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_l\,(x,y)\,\frac{t^l}{l!}\right).$$

Using Equations (1.8), (1.9) and applying the Cauchy product rule, we have

$$\sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{j}{l}\ {}_{\mathcal{B}}\mathcal{A}_l\,(x,y)\,\mathcal{A}_{j-l}\,(x,y)\,\frac{t^j}{j!} = \sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{j}{l}\alpha_{j-l}\ {}_{\mathcal{B}}\mathcal{A}_j\,(2x,2y)\,\frac{t^j}{j!}.$$

Hence we get the result. □

3. Special cases of Bell based Appell polynomials

In this section, by choosing some special cases of the determining functions $A(t)$, we obtain the polynomials given in [10, 11, 15]. We present the recurrence relation, determinantal representation, shift operators and differential equation provided by Bell based Bernoulli polynomials, Bell based Euler polynomials and Bell based Genocchi polynomials. As far as we have known, all results are new for these existing families.

3.1. Bell based Bernoulli polynomials of order α

Bell based Bernoulli polynomials of order α are defined by the following generating function [11]:

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} e^{y\left(e^t-1\right)} = \sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{B}_j^{(\alpha)}\,(x,y)\,\frac{t^j}{j!}.$$

The first five Bell based Bernoulli polynomials ($\alpha = 1$) are as follows:

$$\begin{aligned}
{}_{\mathcal{B}}\mathcal{B}_0\,(x,y) &= 1,\\
{}_{\mathcal{B}}\mathcal{B}_1\,(x,y) &= x+y-\frac{1}{2},\\
{}_{\mathcal{B}}\mathcal{B}_2\,(x,y) &= (x+y)^2 - x + \frac{1}{6},\\
{}_{\mathcal{B}}\mathcal{B}_3\,(x,y) &= (x+y)^3 - \frac{3}{2}x^2 + \frac{1}{2}x + \frac{3}{2}y^2,\\
{}_{\mathcal{B}}\mathcal{B}_4\,(x,y) &= (x+y)^4 - 2x^3 + x^2 + 6xy^2 + 4y^3 + 2y^2 - \frac{1}{30}.
\end{aligned}$$

Corollary 3.1 *The Bell based Bernoulli polynomials* ${}_{\mathcal{B}}\mathcal{B}_j^{(\alpha)}\,(x,y)$ *satisfy the following recurrence relation*

$${}_{\mathcal{B}}\mathcal{B}_{j+1}^{(\alpha)}\,(x,y) = x\ {}_{\mathcal{B}}\mathcal{B}_j^{(\alpha)}\,(x,y) + y\sum_{l=0}^{j}\binom{j}{l}\ {}_{\mathcal{B}}\mathcal{B}_{j-l}^{(\alpha)}\,(x,y) - \alpha\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{j}{l}\binom{l+1}{m}\frac{\mathcal{B}_{l-m+1}}{(l+1)}\ {}_{\mathcal{B}}\mathcal{B}_{j-l}^{(\alpha)}\,(x,y)\ ,$$

where Bernoulli numbers $\mathcal{B}_l$ *are given by the following series* [5]

$$\frac{t}{e^t-1}=\sum_{l=0}^{\infty}\mathcal{B}_l\frac{t^l}{l!}.$$

Corollary 3.2 *For the Bell based Bernoulli polynomials* ${}_{\mathcal{B}}\mathcal{B}_j^{(\alpha)}(x,y)$*, we have the following lowering and raising operators*

$$\begin{aligned} {}_x\sigma_j^- : &= \frac{1}{j}D_x, \\ {}_x\sigma_j^+ : &= x+y\sum_{l=0}^{j}\frac{{D_x}^l}{l!}-\alpha\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{l+1}{m}\frac{\mathcal{B}_{l+1}}{(l+1)}\frac{{D_x}^l}{l!}\ , \end{aligned}$$

respectively. The polynomials ${}_{\mathcal{B}}\mathcal{B}_j^{(\alpha)}(x,y)$ *satisfy the following differential equation*

$$\left[xD_x+y\sum_{l=0}^{j}\frac{{D_x}^{l+1}}{l!}-\alpha\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{l+1}{m}\frac{\mathcal{B}_{l-m+1}}{(l+1)}\frac{{D_x}^{l+1}}{l!}-j\right]\ {}_{\mathcal{B}}\mathcal{B}_j^{(\alpha)}(x,y)=0.$$

Corollary 3.3 *The Bell based Bernoulli polynomials* ${}_{\mathcal{B}}\mathcal{B}_j(x,y)$ *have the following determinantal representation for* $\alpha=1$

$${}_{\mathcal{B}}\mathcal{B}_j(x,y)=(-1)^j\begin{vmatrix} Bel_0(x,y) & Bel_1(x,y) & \cdots & Bel_{j-1}(x,y) & Bel_j(x,y) \\ 1 & \frac{1}{2} & \cdots & \frac{1}{j} & \frac{1}{j+1} \\ 0 & 1 & \cdots & \binom{j-1}{1}\frac{1}{j-1} & \binom{j}{1}\frac{1}{j} \\ 0 & 0 & \cdots & \binom{j-1}{2}\frac{1}{j-2} & \binom{j}{2}\frac{1}{j-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \binom{j}{j-1}\frac{1}{2} \end{vmatrix}.$$

3.2. Bell based Euler polynomials of order α

Bell based Euler polynomials of order α are defined by the following generating function [15]:

$$\left(\frac{2}{e^t+1}\right)^{\alpha}e^{xt}e^{y\left(e^t-1\right)}=\sum_{j=0}^{\infty}{}_{\mathcal{B}}\mathcal{E}_j^{(\alpha)}(x,y)\frac{t^j}{j!}.$$

The first five Bell based Euler polynomials ($\alpha = 1$) are as follows:

$$\begin{aligned}
{}_{\mathcal{B}}\mathcal{E}_0(x,y) &= 1,\\
{}_{\mathcal{B}}\mathcal{E}_1(x,y) &= x+y-\frac{1}{2},\\
{}_{\mathcal{B}}\mathcal{E}_2(x,y) &= (x+y)^2-x,\\
{}_{\mathcal{B}}\mathcal{E}_3(x,y) &= (x+y)^3-\frac{3}{2}x^2+\frac{3}{2}y^2-\frac{1}{2}y+\frac{1}{4},\\
{}_{\mathcal{B}}\mathcal{E}_4(x,y) &= (x+y)^4-2x^3+x+6xy^2-2xy+4y^3+y^2.
\end{aligned}$$

Corollary 3.4 *The Bell based Euler polynomials ${}_{\mathcal{B}}\mathcal{E}_j^{(\alpha)}(x,y)$ satisfy the following recurrence relation*

$$ {}_{\mathcal{B}}\mathcal{E}_{j+1}^{(\alpha)}(x,y) = x\ {}_{\mathcal{B}}\mathcal{E}_j^{(\alpha)}(x,y) + y\sum_{l=0}^{j}\binom{j}{l}\ {}_{\mathcal{B}}\mathcal{E}_{j-l}^{(\alpha)}(x,y) - \frac{\alpha}{2}\sum_{l=0}^{j}\sum_{m=0}^{l}\binom{j}{l}\binom{l}{m}E_{l-m}\ {}_{\mathcal{B}}\mathcal{E}_{j-l}^{(\alpha)}(x,y)\ ,$$

where Euler numbers E_l [9] are given by the following series

$$\frac{2}{e^t+1} = \sum_{l=0}^{\infty} E_l \frac{t^l}{l!}.$$

Corollary 3.5 *For the Bell based Euler polynomials ${}_{\mathcal{B}}\mathcal{E}_j^{(\alpha)}(x,y)$, we have the lowering and raising operators*

$$\begin{aligned}
{}_x\sigma_j^- : &= \frac{1}{j}D_x,\\
{}_x\sigma_j^+ : &= x + y\sum_{l=0}^{j}\frac{{D_x}^l}{l!} - \frac{\alpha}{2}\sum_{l=0}^{j}\sum_{m=0}^{l}\binom{l}{m}E_{l-m}\frac{{D_x}^l}{l!},
\end{aligned}$$

respectively. The polynomials ${}_{\mathcal{B}}\mathcal{E}_j^{(\alpha)}(x,y)$ satisfy the following differential equation

$$\left[xD_x + y\sum_{l=0}^{j}\frac{{D_x}^{l+1}}{l!} - \frac{\alpha}{2}\sum_{l=0}^{j}\sum_{m=0}^{l}\binom{l}{m}E_{l-m}\frac{{D_x}^{l+1}}{l!} - j\right]\ {}_{\mathcal{B}}\mathcal{E}_j^{(\alpha)}(x,y) = 0. \tag{3.1}$$

Corollary 3.6 *The Bell based Euler polynomials ${}_{\mathcal{B}}\mathcal{E}_j(x,y)$ have the following determinantal representation for $\alpha = 1$*

$$ {}_{\mathcal{B}}\mathcal{E}_j(x,y) = (-1)^j \begin{vmatrix}
Bel_0(x,y) & Bel_1(x,y) & \cdots & Bel_{j-1}(x,y) & Bel_j(x,y)\\
1 & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2}\\
0 & 1 & \cdots & \binom{j-1}{1}\frac{1}{2} & \binom{j}{1}\frac{1}{2}\\
0 & 0 & \cdots & \binom{j-1}{2}\frac{1}{2} & \binom{j}{2}\frac{1}{2}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & \binom{j}{j-1}\frac{1}{2}
\end{vmatrix}.$$

3.3. Bell based Genocchi polynomials of order α

Bell based Genocchi polynomials of order α are defined by the following generating function [10]:

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} e^{y\left(e^t-1\right)} = \sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{G}_j^{(\alpha)}(x,y)\frac{t^j}{j!}.$$

The first five Bell based Genocchi polynomials ($\alpha = 1$) are as follows:

$$\begin{aligned}
{}_{\mathcal{B}}\mathcal{G}_0(x,y) &= 0,\\
{}_{\mathcal{B}}\mathcal{G}_1(x,y) &= 1,\\
{}_{\mathcal{B}}\mathcal{G}_2(x,y) &= 2(x+y)-1,\\
{}_{\mathcal{B}}\mathcal{G}_3(x,y) &= 3(x+y)^2-3x,\\
{}_{\mathcal{B}}\mathcal{G}_4(x,y) &= 4(x+y)^3-6x^2+6y^2-2y+1.
\end{aligned}$$

Corollary 3.7 *The Bell based Genocchi polynomials ${}_{\mathcal{B}}\mathcal{G}_j^{(\alpha)}(x,y)$ satisfy the following recurrence relation*

$$\begin{aligned}
{}_{\mathcal{B}}\mathcal{G}_{j+1}^{(\alpha)}(x,y) &= \left(\frac{j+1}{j+1-\alpha}\right)x\ {}_{\mathcal{B}}\mathcal{G}_j^{(\alpha)}(x,y) + \left(\frac{j+1}{j+1-\alpha}\right)y\sum_{l=0}^{j}\binom{j}{l}\ {}_{\mathcal{B}}\mathcal{G}_{j-l}^{(\alpha)}(x,y)\\
&\quad -\frac{\alpha}{2}\left(\frac{j+1}{j+1-\alpha}\right)\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{j}{l}\binom{l+1}{m}\frac{G_{l-m+1}}{(l+1)}\ {}_{\mathcal{B}}\mathcal{G}_{j-l}^{(\alpha)}(x,y)\ ,
\end{aligned}$$

where Genocchi numbers G_l [9] are given by the following series

$$\frac{2t}{e^t+1} = \sum_{l=0}^{\infty} G_l \frac{t^l}{l!}.$$

Corollary 3.8 *The Bell based Genocchi polynomials ${}_{\mathcal{B}}\mathcal{G}_j^{(\alpha)}(x,y)$ have the lowering and raising operators as*

$$\begin{aligned}
{}_x\sigma_j^- &:= \frac{1}{j}D_x,\\
{}_x\sigma_j^+ &:= \left(\frac{j+1}{j+1-\alpha}\right)x + \left(\frac{j+1}{j+1-\alpha}\right)y\sum_{l=0}^{j}\frac{D_x^l}{l!} - \frac{\alpha}{2}\left(\frac{j+1}{j+1-\alpha}\right)\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{l+1}{m}\frac{G_{l-m+1}}{(l+1)}\frac{D_x^l}{l!},
\end{aligned}$$

respectively. The polynomials ${}_{\mathcal{B}}\mathcal{G}_j^{(\alpha)}(x,y)$ satisfy the following differential equation

$$\begin{aligned}
&\Bigg[\left(\frac{j+1}{j+1-\alpha}\right)xD_x + \left(\frac{j+1}{j+1-\alpha}\right)y\sum_{l=0}^{j}\frac{{D_x}^{l+1}}{l!}\\
&-\frac{\alpha}{2}\left(\frac{j+1}{j+1-\alpha}\right)\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{l+1}{m}\mathcal{G}_{l-m+1}\frac{{D_x}^{l+1}}{(l+1)!} - \left(j+1-\left(\frac{j+1}{j+1-\alpha}\right)\right)\Bigg]\ {}_{\mathcal{B}}\mathcal{G}_j^{(\alpha)}(x,y) = 0.
\end{aligned}$$

Corollary 3.9 *The Bell based Genocchi polynomials* ${}_{\mathcal{B}}\mathcal{G}_j(x,y)$ *have the following determinantal representation for* $\alpha=1$

$$
{}_{\mathcal{B}}\mathcal{G}_j(x,y)=(-1)^j 2^{j+1}\begin{vmatrix}
Bel_0(x,y) & Bel_1(x,y) & \cdots & Bel_{j-1}(x,y) & Bel_j(x,y)\\
\frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2j} & \frac{1}{2(j+1)}\\
0 & \frac{1}{2} & \cdots & \binom{j-1}{1}\frac{1}{2(j-1)} & \binom{j}{1}\frac{1}{2j}\\
0 & 0 & \cdots & \binom{j-1}{2}\frac{1}{2(j-2)} & \binom{j}{2}\frac{1}{2(j-1)}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & \binom{j}{j-1}\frac{1}{4}
\end{vmatrix}.
$$

4. 2-iterated Bell based Appell polynomials

2-iterated polynomials are one of the important fields of study with potential applications in number theory, special function theory and combinatorics. Some of the principal polynomial families investigated in this area are the 2-iterated Appell polynomials, twice-iterated Δ_h Appell polynomials, twice-iterated $2D$ q-Appell polynomials, 2-iterated Bernoulli poynomials and 2-iterated Euler polynomials families in [21, 23, 38, 40].

In this section we introduce 2-iterated Bell based Appell polynomials and present equivalance theorem for the definition, recurrence relation, lowering and raising operators, differential equation and determinantal representation for them.

Let ${}_{\mathcal{B}}\mathcal{A}_j(x,y)$ be the Bell based Appell polynomials given by

$$
{}_{\mathcal{B}}\mathcal{A}_j(x,y)=\sum_{l=0}^{j}\binom{j}{l}\alpha_{j-l}Bel_l(x,y) \tag{4.1}
$$

with

$$
A(t)=\sum_{l=0}^{\infty}\alpha_l\frac{t^l}{l!},\quad \alpha_0\neq 0. \tag{4.2}
$$

Now, we introduce the 2-iterated Bell based Appell polynomials as follows:

Definition 4.1 *Let* ${}_{\mathcal{B}}\mathcal{A}_j(x,y)$ *be Bell based Appell polynomials given in (4.1) and* $P_j(x,y)$ *be any polynomial with explicit representation given as follows:*

$$
P_j(x,y)=\sum_{l=0}^{j}b(j,l)\,Bel_l(x,y). \tag{4.3}
$$

Let ${}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)$ *be the bivariate polynomial convolution defined by the Bell based Appell polynomials given as*

$$
{}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y):=\sum_{l=0}^{j}b(j,l)\,{}_{\mathcal{B}}\mathcal{A}_l(x,y),\quad j\in\mathbb{N}_0. \tag{4.4}
$$

Then the polynomials ${}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)$ *will be called 2-iterated Bell based Appell polynomials if they satisfy the following relation*

$$D_x\ {}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y) = j\ {}_{\mathcal{B}}\mathcal{A}_{j-1}^{[2]}(x,y), \quad j=1,2,\ldots.$$

Theorem 4.2 *The following statements are equivalent.*

(i) $\left\{{}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)\right\}_{j\in\mathbb{N}}$ *is a sequence of 2-iterated Bell based Appell polynomials.*

(ii) $\{P_j(x,y)\}_{j\in\mathbb{N}}$ *is a sequence of Bell based Appell polynomials with the determining function*

$$B(t) = \sum_{j=0}^{\infty} b_j \frac{t^j}{j!}, \quad b_0 \neq 0, \tag{4.5}$$

and $b(j,l)$ *are given by*

$$b(j,l) = \binom{j}{l} b_{j-l}. \tag{4.6}$$

(iii) The sequence of 2-iterated Bell based Appell polynomials $\left\{{}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)\right\}_{j\in\mathbb{N}}$ *have the following explicit represantation*

$${}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y) = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{j}{l}\binom{l}{m} b_{j-l}\alpha_m Bel_{l-m}(x,y). \tag{4.7}$$

(iv) The generating relation of the $\left\{{}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)\right\}_{j\in\mathbb{N}}$ *is given by*

$$A(t)\,B(t)\,e^{xt}e^{y\left(e^t-1\right)} = \sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)\frac{t^j}{j!}. \tag{4.8}$$

Proof $(i) \Leftrightarrow (ii)$ Let $\left\{{}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)\right\}_{j\in\mathbb{N}}$ be a sequence of 2-iterated Bell based Appell polynomials. Using the derivative operator $D_x := \frac{\partial}{\partial x}$ on both sides of (4.4), we obtain

$$j\,{}_{\mathcal{B}}\mathcal{A}_{j-1}^{[2]}(x,y) = \sum_{l=1}^{j} b(j,l)\,l\ {}_{\mathcal{B}}\mathcal{A}_{l-1}(x,y) = \sum_{l=0}^{j-1} b(j,l+1)(l+1)\ {}_{\mathcal{B}}\mathcal{A}_l(x,y). \tag{4.9}$$

Multiplying both sides by $\frac{1}{j}$, we have

$${}_{\mathcal{B}}\mathcal{A}_{j-1}^{[2]}(x,y) = \frac{1}{j}\sum_{l=0}^{j-1} b(j,l+1)(l+1)\ {}_{\mathcal{B}}\mathcal{A}_l(x,y),$$

or equivalently

$${}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y) = \frac{1}{j+1}\sum_{l=0}^{j} b(j+1,l+1)(l+1)\ {}_{\mathcal{B}}\mathcal{A}_l(x,y). \tag{4.10}$$

Comparing (4.4) and (4.10), we have

$$b(j,l)=\frac{l+1}{j+1}b(j+1,l+1).$$

Iterating this last equation l times, we obtain

$$\begin{aligned} b(j+1,l+1) &= \frac{(j+1)\,j\ldots(j-l+1)}{(l+1)\,l\ldots 1}b(j-l,0) \\ &= \binom{j+1}{l+1}b(j-l,0). \end{aligned}$$

Thus we get the following relation

$$b(j,l)=\binom{j}{l}b_{j-l}. \tag{4.11}$$

Writing (4.11) in (4.3), we have

$$\mathcal{P}_j(x,y)=\sum_{l=0}^{j}\binom{j}{l}b_{j-l}Bel_l(x,y). \tag{4.12}$$

Applying the derivative operator D_x on both sides of the last equation, we get

$$D_x(\mathcal{P}_j(x,y))=j\mathcal{P}_{j-1}(x,y). \tag{4.13}$$

Thus, $\{\mathcal{P}_j(x,y)\}_{j\in\mathbb{N}}$ is a sequence of Bell based Appell polynomials. The converse proposition $(ii)\Leftrightarrow(i)$ is obviously true.

Let $\{\mathcal{P}_j(x,y)\}_{j\in\mathbb{N}}$ be a sequence of Bell based Appell polynomials to obtain $(ii)\Leftrightarrow(iii)$. We have the following definition of the 2-iterated Bell based Appell polynomials

$${}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y):=\sum_{l=0}^{j}\binom{j}{l}b_{j-l}\ {}_{\mathcal{B}}\mathcal{A}_l(x,y), \quad j\in\mathbb{N}_0$$

where

$${}_{\mathcal{B}}\mathcal{A}_l(x,y)=\sum_{m=0}^{l}\binom{l}{m}\alpha_m Bel_{l-m}(x,y). \tag{4.14}$$

Thus, the 2-iterated Bell based Appell polynomials satisfy the explicit representation as follows:

$${}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)=\sum_{j=0}^{l}\binom{j}{l}b_{j-l}\sum_{l=0}^{m}\binom{l}{m}\alpha_m Bel_{l-m}(x,y)\ .$$

The converse proposition $(iii)\Leftrightarrow(ii)$ is is obviously true.

By writing the explicit form of ${}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)$ and using the Cauchy's product rule for the series, we can obtain $(iii)\Leftrightarrow(iv)$. □

4.1. Main properties of the 2-iterated Bell based Appell polynomials

Theorem 4.3 *The recurrence relation satisfied by the 2-iterated Bell based Appell polynomials* ${}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)$ *is given by*

$$ {}_{\mathcal{B}}\mathcal{A}_{j+1}^{[2]}(x,y)=x\ {}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)+y\sum_{l=0}^{j}\binom{j}{l}\ {}_{\mathcal{B}}\mathcal{A}_{j-l}^{[2]}(x,y)+\sum_{l=0}^{j}\binom{j}{l}\beta_l\ {}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)+\sum_{l=0}^{j}\binom{j}{l}\theta_l\ {}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)\ , \tag{4.15}$$

where

$$\frac{A'(t)}{A(t)}=\sum_{l=0}^{\infty}\beta_l\frac{t^l}{l!},\quad \frac{B'(t)}{B(t)}=\sum_{l=0}^{\infty}\theta_l\frac{t^l}{l!}. \tag{4.16}$$

Proof This theorem is proved in the same way of Theorem 2.2. □

Theorem 4.4 *For the 2-iterated* ${}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)$ *Bell based Appell polynomials, we have the lowering and raising operators as*

$$ {}_{x}\sigma_j^{-} := \frac{1}{j}D_x, \tag{4.17}$$

$$ {}_{x}\sigma_j^{+} := x+y\sum_{l=0}^{j}\frac{{D_x}^l}{l!}+\sum_{l=0}^{j}\beta_l\frac{{D_x}^l}{l!}+\sum_{l=0}^{j}\theta_l\frac{{D_x}^l}{l!}, \tag{4.18}$$

respectively, and the differential equation satisfied by the 2-iterated Bell based Appell polynomials ${}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)$ *is given by*

$$\left[xD_x+y\sum_{l=0}^{j}\frac{{D_x}^{l+1}}{l!}+\sum_{l=0}^{j}\beta_l\frac{{D_x}^{l+1}}{l!}+\sum_{l=0}^{j}\theta_l\frac{{D_x}^{l+1}}{l!}-j\right]\ {}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)=0\ , \tag{4.19}$$

where β_l *and* θ_l *are the same as in (4.16) given by Theorem 4.3.*

Proof This theorem is proved in the same way of Theorem 2.4. □

Theorem 4.5 *The 2-iterated Bell based Appell polynomials* ${}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)$ *have the following determinantal representation*

$$ {}_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)=\frac{(-1)^j}{(\lambda_0)^{j+1}}\begin{vmatrix} {}_{\mathcal{B}}\mathcal{A}_0(x,y) & {}_{\mathcal{B}}\mathcal{A}_1(x,y) & \cdots & {}_{\mathcal{B}}\mathcal{A}_{j-1}(x,y) & {}_{\mathcal{B}}\mathcal{A}_j(x,y) \\ \lambda_0 & \lambda_1 & \cdots & \lambda_{j-1} & \lambda_j \\ 0 & \lambda_0 & \cdots & \binom{j-1}{1}\lambda_{j-2} & \binom{j}{1}\lambda_{j-1} \\ 0 & 0 & \cdots & \binom{j-1}{2}\lambda_{j-3} & \binom{j}{2}\lambda_{j-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_0 & \binom{j}{j-1}\lambda_1 \end{vmatrix}, \tag{4.20}$$

where $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_j$ *are the coefficients of the Maclaurin series of the function* $\frac{1}{B(t)}$ *and*

$$_{\mathcal{B}}\mathcal{A}_j\left(x,y\right)=\sum_{l=0}^{j}\binom{j}{l}\lambda_l\ {}_{\mathcal{B}}\mathcal{A}_{j-l}^{[2]}.$$

Proof The theorem is proved in a same way as in Theorem 2.1. □

Theorem 4.6 *The 2-iterated Bell based Appell polynomials have the following summation formula*

$$_{\mathcal{B}}\mathcal{A}_j^{[2]}\left(x,y\right)=\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{l}\binom{j}{k}\binom{k}{l}\binom{l}{m}Bel_{j-k}\left(x,y\right)\delta_{k-l}\beta_{l-m}\beta_m\ , \tag{4.21}$$

where

$$\frac{A\left(t\right)}{B\left(t\right)}=\sum_{k=0}^{\infty}\delta_k\frac{t^k}{k!}\quad \textit{and}\quad B\left(t\right)=\sum_{l=0}^{\infty}\beta_l\frac{t^l}{l!}. \tag{4.22}$$

Proof Taking $\frac{A(t)}{B(t)}B\left(t\right)$ instead of $A\left(t\right)$ in (4.8) and using (4.22), we have

$$\left(\sum_{j=0}^{\infty}Bel_n(x,y)\frac{t^j}{j!}\right)\left(\sum_{k=0}^{\infty}\delta_k\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\beta_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\beta_m\frac{t^m}{m!}\right)=\sum_{j=0}^{\infty}\ {}_{\mathcal{B}}\mathcal{A}_j^{[2]}\left(x,y\right)\frac{t^j}{j!}.$$

Applying the Cauchy product rule, we obtain

$$\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{l}\binom{j}{k}\binom{k}{l}\binom{l}{m}Bel_{j-k}\left(x,y\right)\delta_{k-l}\beta_{l-m}\beta_m\frac{t^n}{n!}=\sum_{j=0}^{\infty}\ {}_{\mathcal{B}}\mathcal{A}_j^{[2]}\left(x,y\right)\frac{t^j}{j!}.$$

Comparing the coefficients of $\frac{t^j}{j!}$ on both sides of the result equation, the proof is completed. □

Corollary 4.7 *In the case* $A\left(t\right)=B\left(t\right)=\frac{t}{e^t-1}$, *we have the following summation formula*

$$_{\mathcal{B}}\mathcal{A}_j^{[2]}\left(x,y\right)=\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{j}{k}\binom{k}{l}\mathcal{B}_{k-l}\mathcal{B}_lBel_{j-k}\left(x,y\right)\ ,$$

where Bernoulli numbers $\mathcal{B}_k$ *[5] are given by the following series*

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}\mathcal{B}_k\frac{t^k}{k!}.$$

In the case $A\left(t\right)=\frac{2}{e^t+1}$ *and* $B\left(t\right)=\frac{2t}{e^t+1}$, *we have the following summation formula*

$$_{\mathcal{B}}\mathcal{A}_j^{[2]}\left(x,y\right)=\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{j}{k}\binom{k}{l}\frac{G_lG_{k-l+1}}{k+1}Bel_{j-k}\left(x,y\right)$$

where Genocchi numbers G_k *[9] are given by the following series*

$$\frac{2t}{e^t+1}=\sum_{k=0}^{\infty}G_k\frac{t^k}{k!}.$$

4.2. Closed form summation formulae for the 2-iterated Bell based Appell polynomials

In this section, we obtain closed form summation formulae between the usual and the generalized versions of the Bell based Appell polynomials. Then the corresponding closed form summation formulae are given for Bell based Bernoulli polynomials, Bell based Euler polynomials and Bell based Genocchi polynomials.

We start by defining the generalized $r-th$ order Bell based Appell polynomials via the following generating relation

$$[A(t)]^r e^{xt} e^{y\left(e^t-1\right)} = \sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_j^{[r]}(x,y)\frac{t^j}{j!}, \quad r\in\mathbb{N}.$$

We consider

$$\mathcal{F}^{(r)}(x,y;t) = [A(t)]^r e^{xt} e^{y\left(e^t-1\right)} = \sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_j^{[r]}(x,y)\frac{t^j}{j!}, \tag{4.23}$$

$$\mathcal{F}^{(1)}(x,y;t) = A(t) e^{xt} e^{y\left(e^t-1\right)} = \sum_{j=0}^{\infty} {}_{\mathcal{B}}\mathcal{A}_j(x,y)\frac{t^j}{j!}. \tag{4.24}$$

Observe that

$$\left[\mathcal{F}^{(1)}\left(\frac{x}{r},\frac{y}{r};t\right)\right]^r = \mathcal{F}^{(r)}(x,y;t). \tag{4.25}$$

Now, utilizing (4.25) and using the same idea in [30], we can prove the following closed form summation formula.

Theorem 4.8 *Bell based Appell polynomials* ${}_{\mathcal{B}}\mathcal{A}_j^{[r]}(x,y)$ *satisfy the following summation formula*

$$\sum_{l=0}^{j}\binom{j}{l}\left[\,{}_{\mathcal{B}}\mathcal{A}_{j-l+1}^{[r]}(x,y)\ {}_{\mathcal{B}}\mathcal{A}_l\left(\frac{x}{r},\frac{y}{r}\right) - r\ {}_{\mathcal{B}}\mathcal{A}_{j-l}^{[r]}(x,y)\ {}_{\mathcal{B}}\mathcal{A}_{l+1}\left(\frac{x}{r},\frac{y}{r}\right)\right] = 0. \tag{4.26}$$

Proof By taking the logarithm on both sides of (4.25) and then differentiating with respect to t, we obtain

$$r\mathcal{F}^{(r)}(x,y;t)\frac{\partial\mathcal{F}^{(1)}\left(\frac{x}{r},\frac{y}{r};t\right)}{\partial t} = \mathcal{F}^{(1)}\left(\frac{x}{r},\frac{y}{r};t\right)\frac{\partial\mathcal{F}^{(r)}(x,y;t)}{\partial t}.$$

Substituting the corresponding generating functions, we get

$$\sum_{j=1}^{\infty} j\ {}_{\mathcal{B}}\mathcal{A}_j^{[r]}(x,y)\frac{t^{j-1}}{j!}\sum_{l=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_l\left(\frac{x}{r},\frac{y}{r}\right)\frac{t^l}{l!} = r\sum_{j=0}^{\infty}{}_{\mathcal{B}}\mathcal{A}_j^{[r]}(x,y)\frac{t^j}{j!}\sum_{l=0}^{\infty} l\ {}_{\mathcal{B}}\mathcal{A}_l\left(\frac{x}{r},\frac{y}{r}\right)\frac{t^{l-1}}{l!}.$$

Applying the Cauchy's product rule and then comparing the coefficients of $\frac{t^j}{j!}$ on both sides the proof is completed. □

Corollary 4.9 *In the case* $A(t)=B(t)=\frac{t}{e^t-1}$, *we have the following closed form summation formula for the Bell based Bernoulli polynomials* ${}_{\mathcal{B}}\mathcal{B}_j^{[2]}(x,y)$

$$\sum_{l=0}^{j}\binom{j}{l}\left[\,{}_{\mathcal{B}}\mathcal{B}_{j-l+1}^{[2]}(x,y)\ {}_{\mathcal{B}}\mathcal{B}_l\left(\frac{x}{2},\frac{y}{2}\right) - 2\ {}_{\mathcal{B}}\mathcal{B}_{j-l}^{[2]}(x,y)\ {}_{\mathcal{B}}\mathcal{B}_{l+1}\left(\frac{x}{2},\frac{y}{2}\right)\right] = 0.$$

Corollary 4.10 *In the case* $A(t) = B(t) = \frac{2}{e^t+1}$, *we have the following closed form summation formula for Bell based Euler polynomials* ${}_{\mathcal{B}}\mathcal{E}_j^{[2]}(x,y)$

$$\sum_{l=0}^{j}\binom{j}{l}\left[\ {}_{\mathcal{B}}\mathcal{E}_{j-l+1}^{[2]}(x,y)\ {}_{\mathcal{B}}\mathcal{E}_l\left(\frac{x}{2},\frac{y}{2}\right)-2\ {}_{\mathcal{B}}\mathcal{E}_{j-l}^{[2]}(x,y)\ {}_{\mathcal{B}}\mathcal{E}_{l+1}\left(\frac{x}{2},\frac{y}{2}\right)\right]=0.$$

Corollary 4.11 *In the case* $A(t) = B(t) = \frac{2t}{e^t+1}$, *we have the following closed form summation formula for the Bell based Genocchi polynomials* ${}_{\mathcal{B}}\mathcal{G}_n^{[2]}(x,y)$

$$\sum_{l=0}^{j}\binom{j}{l}\left[\ {}_{\mathcal{B}}\mathcal{G}_{j-l+1}^{[2]}(x,y)\ {}_{\mathcal{B}}\mathcal{G}_l\left(\frac{x}{2},\frac{y}{2}\right)-2\ {}_{\mathcal{B}}\mathcal{G}_{j-l}^{[2]}(x,y)\ {}_{\mathcal{B}}\mathcal{G}_{l+1}\left(\frac{x}{2},\frac{y}{2}\right)\right]=0.$$

Now, without lost of generality in the rest of this section, we consider the case $r=2$.

Theorem 4.12 *We have the following relation between 2-iterated Bell based Appell polynomials and 2-variable Bell polynomials*

$${}_{\mathcal{B}}\mathcal{A}_{j+k}^{[2]}(X,Y)=\sum_{m=0}^{j}\sum_{l=0}^{k}\binom{j}{m}\binom{k}{l}Bel_{m+1}(X-x,Y-y)\ {}_{\mathcal{B}}\mathcal{A}_{j+k-l-m}^{[2]}(x,y). \tag{4.27}$$

Proof If we write $t+\tau$ instead of t in Equation (4.8), we have

$$A(t+\tau)B(t+\tau)e^{x(t+\tau)}e^{y\left(e^{t+\tau}-1\right)}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\ {}_{\mathcal{B}}\mathcal{A}_{j+k}^{[2]}(x,y)\frac{t^j}{j!}\frac{\tau^k}{k!}. \tag{4.28}$$

Hence, we write

$$A(t+\tau)B(t+\tau)=e^{-x(t+\tau)}e^{-y\left(e^{t+\tau}-1\right)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\ {}_{\mathcal{B}}\mathcal{A}_{j+k}^{[2]}(x,y)\frac{t^j}{j!}\frac{\tau^k}{k!}.$$

Multiplying both sides by $e^{X(t+\tau)}e^{Y\left(e^{t+\tau}-1\right)}$, we get

$$A(t+\tau)B(t+\tau)e^{X(t+\tau)}e^{Y\left(e^{t+\tau}-1\right)}=e^{(X-x)(t+\tau)}e^{(Y-y)\left(e^{t+\tau}-1\right)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\ {}_{\mathcal{B}}\mathcal{A}_{j+k}^{[2]}(x,y)\frac{t^j}{j!}\frac{\tau^k}{k!}.$$

Using Equations (4.28) and (1.7), we have

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\ {}_{\mathcal{B}}\mathcal{A}_{j+k}^{[2]}(X,Y)\frac{t^j}{j!}\frac{\tau^k}{k!}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\ {}_{\mathcal{B}}\mathcal{A}_{j+k}^{[2]}(x,y)\frac{t^j}{j!}\frac{\tau^k}{k!}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}Bel_{m+l}(X-x,Y-y)\frac{t^m}{m!}\frac{\tau^l}{l!}.$$

Applying the Cauchy's product rule, we get

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\ {}_{\mathcal{B}}\mathcal{A}_{j+k}^{[2]}(X,Y)\frac{t^j}{j!}\frac{\tau^k}{k!}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{k}\binom{j}{m}\binom{k}{l}Bel_{m+1}(X-x,Y-y)\ {}_{\mathcal{B}}\mathcal{A}_{j+k-l-m}^{[2]}(x,y)\frac{t^j}{j!}\frac{\tau^k}{k!}.$$

Comparing the coefficients of $\frac{t^j}{j!}$ and $\frac{\tau^k}{k!}$ on both sides of the result equation, the proof is completed. □

Corollary 4.13 *In the case* $A(t) = B(t) = \frac{t}{e^t-1}$, *we have the following summation formula for the 2-iterated Bell based Bernoulli polynomials*

$$_{\mathcal{B}}\mathcal{B}_{j+k}^{[2]}(X,Y) = \sum_{m=0}^{j}\sum_{l=0}^{k}\binom{j}{m}\binom{k}{l} Bel_{m+1}(X-x, Y-y)\ _{\mathcal{B}}\mathcal{B}_{j+k-l-m}^{[2]}(x,y).$$

Corollary 4.14 *In the case* $A(t) = B(t) = \frac{2}{e^t+1}$, *we have the following summation formula for the 2-iterated Bell based Euler polynomials*

$$_{\mathcal{B}}\mathcal{E}_{j+k}^{[2]}(X,Y) = \sum_{m=0}^{j}\sum_{l=0}^{k}\binom{j}{m}\binom{k}{l} Bel_{m+1}(X-x, Y-y)\ _{\mathcal{B}}\mathcal{E}_{j+k-l-m}^{[2]}(x,y).$$

Corollary 4.15 *In the case* $A(t) = B(t) = \frac{2t}{e^t+1}$, *we have the following summation formula for the 2-iterated Bell based Genocchi polynomials*

$$_{\mathcal{B}}\mathcal{G}_{j+k}^{[2]}(X,Y) = \sum_{m=0}^{j}\sum_{l=0}^{k}\binom{j}{m}\binom{k}{l} Bel_{m+1}(X-x, Y-y)\ _{\mathcal{B}}\mathcal{G}_{j+k-l-m}^{[2]}(x,y).$$

5. Special cases of the 2-iterated $_{\mathcal{B}}\mathcal{A}_j^{[2]}(x,y)$ polynomials

In this section, we define new hybrid polynomial families with the help of the 2-iterated Bell based Appell polynomials. By introducing the Bell based Bernoulli-Euler polynomials, the Bell based Bernoulli-Genocchi polynomials, the Bell based Euler-Genocchi polynomials and the Bell based Stirling-Appell polynomials of second kind, we obtain the recurrence relations, shift operators, determinantal representations and differential equations of these new polynomial families.

5.1. 2-iterated Bell based Bernoulli polynomials

When $A(t) = B(t) = \frac{t}{e^t-1}$ in (4.8), we have the 2-iterated Bell based Bernoulli polynomials with the generating function

$$\left(\frac{t}{e^t-1}\right)^2 e^{xt}e^{y\left(e^t-1\right)} = \sum_{j=0}^{\infty} {_{\mathcal{B}}\mathcal{B}_j^{[2]}}(x,y)\frac{t^j}{j!}.$$

By taking $\alpha = 2$ in Corollaries 3.1 and 3.2, the recurrence relation, shift operators and differential equation for the 2-iterated Bell based Bernoulli polynomials can be obtained.

The 2-iterated Bell based Bernoulli polynomials ${}_{\mathcal{B}}\mathcal{B}_j^{[2]}(x,y)$ have the following determinantal representation

$$
{}_{\mathcal{B}}\mathcal{B}_j^{[2]}(x,y)=(-1)^j\begin{vmatrix}
{}_{\mathcal{B}}\mathcal{B}_0(x,y) & {}_{\mathcal{B}}\mathcal{B}_1(x,y) & \cdots & {}_{\mathcal{B}}\mathcal{B}_{j-1}(x,y) & {}_{\mathcal{B}}\mathcal{B}_j(x,y)\\
1 & \frac{1}{2} & \cdots & \frac{1}{j} & \frac{1}{j+1}\\
0 & 1 & \cdots & \binom{j-1}{1}\frac{1}{j-1} & \binom{j}{1}\frac{1}{j}\\
0 & 0 & \cdots & \binom{j-1}{2}\frac{1}{j-2} & \binom{j}{2}\frac{1}{j-1}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & \binom{j}{j-1}\frac{1}{2}
\end{vmatrix}.
$$

5.2. 2-iterated Bell based Euler polynomials

When $A(t)=B(t)=\frac{2}{e^t+1}$ in (4.8), we have the 2-iterated Bell based Euler polynomials with the generating function

$$\left(\frac{2}{e^t+1}\right)^2 e^{xt}e^{y\left(e^t-1\right)}=\sum_{j=0}^{\infty}{}_{\mathcal{B}}\mathcal{E}_j^{[2]}(x,y)\frac{t^j}{j!}.$$

By taking $\alpha=2$ in Corollaries 3.4 and 3.5, the recurrence relation, shift operators and differential equation for the 2-iterated Bell based Euler polynomials can be obtained.

The 2-iterated Bell based Euler polynomials ${}_{\mathcal{B}}\mathcal{E}_j^{[2]}(x,y)$ have the following determinantal representation

$$
{}_{\mathcal{B}}\mathcal{E}_j^{[2]}(x,y)=(-1)^j\begin{vmatrix}
{}_{\mathcal{B}}\mathcal{E}_0(x,y) & {}_{\mathcal{B}}\mathcal{E}_1(x,y) & \cdots & {}_{\mathcal{B}}\mathcal{E}_{j-1}(x,y) & {}_{\mathcal{B}}\mathcal{E}_j(x,y)\\
1 & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2}\\
0 & 1 & \cdots & \binom{j-1}{1}\frac{1}{2} & \binom{j}{1}\frac{1}{2}\\
0 & 0 & \cdots & \binom{j-1}{2}\frac{1}{2} & \binom{j}{2}\frac{1}{2}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & \binom{j}{j-1}\frac{1}{2}
\end{vmatrix}.
$$

5.3. 2-iterated Bell based Genocchi polynomials

When $A(t)=B(t)=\frac{2t}{e^t+1}$ in (4.8), we have the 2-iterated Bell based Genocchi polynomials with the generating function

$$\left(\frac{2t}{e^t+1}\right)^2 e^{xt}e^{y\left(e^t-1\right)}=\sum_{j=0}^{\infty}{}_{\mathcal{B}}\mathcal{G}_j^{[2]}(x,y)\frac{t^j}{j!}.$$

By taking $\alpha=2$ in Corollaries 3.7 and 3.8, the recurrence relation, shift operators and differential equation for the 2-iterated Bell based Genocchi polynomials are obtained.

The 2-iterated Bell based Genocchi polynomials ${}_{\mathcal{B}}\mathcal{G}_{j}^{[2]}(x,y)$ have the following determinantal

$$
{}_{\mathcal{B}}\mathcal{G}_{j}^{[2]}(x,y)=(-1)^{j}2^{j+1}\begin{vmatrix}
{}_{\mathcal{B}}\mathcal{G}_{0}(x,y) & {}_{\mathcal{B}}\mathcal{G}_{1}(x,y) & \cdots & {}_{\mathcal{B}}\mathcal{G}_{j-1}(x,y) & {}_{\mathcal{B}}\mathcal{G}_{j}(x,y)\\
\frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2j} & \frac{1}{2(j+1)}\\
0 & \frac{1}{2} & \cdots & \binom{j-1}{1}\frac{1}{2(j-1)} & \binom{j}{1}\frac{1}{2j}\\
0 & 0 & \cdots & \binom{j-1}{2}\frac{1}{2(j-2)} & \binom{j}{2}\frac{1}{2(j-1)}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & \binom{j}{j-1}\frac{1}{4}
\end{vmatrix}.
$$

5.4. 2-iterated Bell based Stirling-Appell polynomials of the second kind

When $A(t)=\frac{(e^{t}-1)^{z}}{z!}$ and $B(t)$ in (4.8), we have the 2-iterated Bell based Stirling-Appell polynomials of the second kind with the generating function

$$
B(t)\frac{(e^{t}-1)^{z}}{z!}e^{xt}e^{y(e^{t}-1)}=\sum_{j=0}^{\infty}{}_{\mathcal{S}}\mathcal{A}_{j}^{[2]}(x,y)\frac{t^{j}}{j!}.
$$

where Stirling numbers of the second kind $S_{2}(l,z)$ in [37] are given by $\frac{(e^{t}-1)^{z}}{z!}=\sum\limits_{l=z}^{\infty}S_{2}(l,z)\frac{t^{l}}{l!}$.

Remark 5.1 *If $B(t)=1$ in the above generating function, we obtain Bell based Stirling polynomials in [11].*

Corollary 5.2 *The recurrence relation satisfied by Bell based Stirling-Appell polynomials of the second kind ${}_{\mathcal{S}}\mathcal{A}_{j}^{[2]}(x,y)$ is given by*

$$
\begin{aligned}
{}_{\mathcal{S}}\mathcal{A}_{j+1}^{[2]}(x,y) &= x\,{}_{\mathcal{S}}\mathcal{A}_{j}^{[2]}(x,y)+y\sum_{l=0}^{j}\binom{j}{l}{}_{\mathcal{S}}\mathcal{A}_{j-l}^{[2]}(x,y)+\sum_{l=0}^{j}\binom{j}{l}\theta_{l}\,{}_{\mathcal{S}}\mathcal{A}_{j-l}^{[2]}(x,y)\\
&\quad +z\sum_{l=-1}^{j}\frac{j!}{(j-l)!\,(l+1)!}b_{l+1}\,\mathcal{A}_{j-l}^{[2]}(x,y)\ ,
\end{aligned}
$$

where $\frac{B'(t)}{B(t)}=\sum\limits_{l=0}^{\infty}\theta_{l}\frac{t^{l}}{l!}$ and $e^{t}\frac{t}{e^{t}-1}=\sum\limits_{l=0}^{\infty}b_{l}\frac{t^{l}}{l!}$.

Corollary 5.3 *The Bell based Stirling-Appell polynomials of the second kind ${}_{\mathcal{S}}\mathcal{A}_{j}^{[2]}(x,y)$ have the following lowering and raising operators*

$$
\begin{aligned}
{}_{x}\sigma_{j}^{-} &:= \frac{1}{j}D_{x},\\
{}_{x}\sigma_{j}^{+} &:= x+y\sum_{l=0}^{j}\frac{D_{x}{}^{l}}{l!}+\sum_{l=0}^{j}\theta_{l}\frac{D_{x}{}^{l}}{l!}+z\sum_{l=-1}^{j}\frac{b_{l+1}}{(l+1)!}D_{x}^{l},
\end{aligned}
$$

and the differential equation satisfied by the Bell based Stirling-Appell polynomials of the second kind ${}_{\mathcal{S}}\mathcal{A}_j^{[2]}(x,y)$ *is given by*

$$\left[xD_x + y\sum_{l=0}^{j}\frac{D_x^{\ l+1}}{l!} + \sum_{l=0}^{j}\theta_l\frac{D_x^{\ l+1}}{l!} + z\sum_{l=-1}^{j}\frac{b_{l+1}}{(l+1)!}D_x^{l+1} - j\right]{}_{\mathcal{S}}\mathcal{A}_j^{[2]}(x,y) = 0.$$

Corollary 5.4 *The Bell based Stirling-Appell polynomials of the second kind* ${}_{\mathcal{S}}\mathcal{A}_j^{[2]}(x,y)$ *have the following determinantal representation*

$${}_{\mathcal{S}}\mathcal{A}_j^{[2]}(x,y) = \frac{(-1)^j}{(\lambda_0)^{j+1}}\begin{vmatrix} {}_{\mathcal{S}}\mathcal{A}_0(x,y) & {}_{\mathcal{S}}\mathcal{A}_1(x,y) & \cdots & {}_{\mathcal{S}}\mathcal{A}_{j-1}(x,y) & {}_{\mathcal{S}}\mathcal{A}_j(x,y) \\ \lambda_0 & \lambda_1 & \cdots & \lambda_{j-1} & \lambda_j \\ 0 & \lambda_0 & \cdots & \binom{j-1}{1}\lambda_{j-2} & \binom{j}{1}\lambda_{j-1} \\ 0 & 0 & \cdots & \binom{j-1}{2}\lambda_{j-3} & \binom{j}{2}\lambda_{j-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_0 & \binom{j}{j-1}\lambda_1 \end{vmatrix},$$

where

$$\frac{1}{B(t)} = \sum_{l=0}^{\infty}\lambda_l\frac{t^l}{l!}.$$

5.5. 2-iterated Bell based Bernoulli-Euler polynomials

When $A(t) = \frac{t}{e^t-1}$ and $B(t) = \frac{2}{e^t+1}$ in (4.8), we have the 2-iterated Bell based Bernoulli-Euler polynomials with the generating function

$$\left(\frac{t}{e^t-1}\right)\left(\frac{2}{e^t+1}\right)e^{xt}e^{y\left(e^t-1\right)} = \sum_{j=0}^{\infty}{}_{\mathcal{E}}\mathcal{B}_j^{[2]}(x,y)\frac{t^j}{j!}.$$

Corollary 5.5 *The recurrence relation satisfied by the Bell based Bernoulli-Euler polynomials* ${}_{\mathcal{E}}\mathcal{B}_j^{[2]}(x,y)$ *is given by*

$$\begin{aligned} {}_{\mathcal{E}}\mathcal{B}_{j+1}^{[2]}(x,y) &= x\,{}_{\mathcal{E}}\mathcal{B}_j^{[2]}(x,y) + y\sum_{l=0}^{j}\binom{j}{l}{}_{\mathcal{E}}\mathcal{B}_{j-l}^{[2]}(x,y) - \sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{j}{l}\binom{l+1}{m}\frac{\mathcal{B}_{l-m+1}}{l+1}{}_{\mathcal{E}}\mathcal{B}_{j-l}^{[2]}(x,y) \\ &\quad -\frac{1}{2}\sum_{l=0}^{j}\sum_{m=0}^{l}\binom{j}{l}\binom{l}{m}E_{l-m}\,{}_{\mathcal{E}}\mathcal{B}_{j-l}^{[2]}(x,y)\ , \end{aligned}$$

where

$$\frac{t}{e^t-1} = \sum_{l=0}^{\infty}\mathcal{B}_l\frac{t^l}{l!} \quad \textit{and} \quad \frac{2}{e^t+1} = \sum_{l=0}^{\infty}E_l\frac{t^l}{l!}.$$

Corollary 5.6 *The Bell based Bernoulli-Euler polynomials ${}_{\mathcal{E}}\mathcal{B}_j^{[2]}(x,y)$ have the lowering and raising operators*

$$\begin{aligned}
{}_x\sigma_j^- &:= \frac{1}{j}D_x, \\
{}_x\sigma_j^+ &:= x + y\sum_{l=0}^{j}\frac{D_x{}^l}{l!} - \sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{l+1}{m}\frac{\mathcal{B}_{l-m+1}}{l+1}\frac{D_x{}^l}{l!} - \frac{1}{2}\sum_{l=0}^{j}\sum_{m=0}^{l}\binom{l}{m}E_{l-m}\frac{D_x{}^l}{l!},
\end{aligned}$$

and the differential equation satisfied by the ${}_{\mathcal{E}}\mathcal{B}_j^{[2]}(x,y)$ Bell based Bernoulli-Euler polynomials is given by

$$\left[xD_x + y\sum_{l=0}^{j}\frac{D_x{}^{l+1}}{l!} - \sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{l+1}{m}\frac{\mathcal{B}_{l-m+1}}{l+1}\frac{D_x{}^{l+1}}{l!} - \frac{1}{2}\sum_{l=0}^{j}\sum_{m=0}^{l}\binom{l}{m}E_{l-m}\frac{D_x{}^{l+1}}{l!} - j\right]{}_{\mathcal{E}}\mathcal{B}_j^{[2]}(x,y) = 0.$$

Corollary 5.7 *The Bell based Bernoulli-Euler polynomials ${}_{\mathcal{E}}\mathcal{B}_j^{[2]}(x,y)$ have the following determinantal representation*

$${}_{\mathcal{E}}\mathcal{B}_j^{[2]}(x,y) = (-1)^j \begin{vmatrix}
{}_{\mathcal{B}}\mathcal{B}_0(x,y) & {}_{\mathcal{B}}\mathcal{B}_1(x,y) & \cdots & {}_{\mathcal{B}}\mathcal{B}_{j-1}(x,y) & {}_{\mathcal{B}}\mathcal{B}_j(x,y) \\
1 & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
0 & 1 & \cdots & \binom{j-1}{1}\frac{1}{2} & \binom{j}{1}\frac{1}{2} \\
0 & 0 & \cdots & \binom{j-1}{2}\frac{1}{2} & \binom{j}{2}\frac{1}{2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & \binom{j}{j-1}\frac{1}{2}
\end{vmatrix}.$$

5.6. 2-iterated Bell based Bernoulli-Genocchi polynomials

When $A(t) = \frac{t}{e^t-1}$ and $B(t) = \frac{2t}{e^t+1}$ in (4.8), we have the 2-iterated Bell based Bernoulli-Genocchi polynomials with the generating function

$$\left(\frac{t}{e^t-1}\right)\left(\frac{2t}{e^t+1}\right)e^{xt}e^{y\left(e^t-1\right)} = \sum_{j=0}^{\infty}{}_{\mathcal{G}}\mathcal{B}_j^{[2]}(x,y)\frac{t^j}{j!}.$$

Corollary 5.8 *The recurrence relation satisfied by Bell based Bernoulli-Genocchi polynomials ${}_{\mathcal{G}}\mathcal{B}_j^{[2]}(x,y)$ is given by*

$$\begin{aligned}
{}_{\mathcal{G}}\mathcal{B}_{j+1}^{[2]}(x,y) &= \left(1+\frac{1}{j}\right)x\,{}_{\mathcal{G}}\mathcal{B}_j^{[2]}(x,y) + y\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\binom{j}{l}{}_{\mathcal{G}}\mathcal{B}_{j-l}^{[2]}(x,y) \\
&\quad - \left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{j}{l}\binom{l+1}{m}\frac{\mathcal{B}_{l-m+1}}{l+1}{}_{\mathcal{G}}\mathcal{B}_{j-l}^{[2]}(x,y) \\
&\quad - \frac{1}{2}\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{j}{l}\binom{l+1}{m}\frac{G_{l-m+1}}{l+1}\,{}_{\mathcal{G}}\mathcal{B}_{j-l}^{[2]}(x,y)\ ,
\end{aligned}$$

where

$$\frac{t}{e^t-1}=\sum_{l=0}^{\infty}\mathcal{B}_l\frac{t^l}{l!} \quad \text{and} \quad \frac{2t}{e^t+1}=\sum_{l=0}^{\infty}G_l\frac{t^l}{l!}.$$

Corollary 5.9 *The Bell based Bernoulli-Genocchi polynomials* ${}_{\mathcal{G}}\mathcal{B}_j^{[2]}(x,y)$ *have the lowering and raising operators*

$$\begin{aligned}
{}_x\sigma_j^- &:= \frac{1}{j}D_x,\\
{}_x\sigma_j^+ &:= \left(1+\frac{1}{j}\right)x+y\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\frac{D_x{}^l}{l!}-\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{l+1}{m}\frac{\mathcal{B}_{l-m+1}}{l+1}\frac{D_x{}^l}{l!}\\
&\quad -\frac{1}{2}\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{l+1}{m}\frac{G_{l-m+1}}{(l+1)}\frac{D_x{}^l}{l!},
\end{aligned}$$

and the differential equation satisfied by the Bell based Bernoulli-Genocchi polynomials ${}_{\mathcal{G}}\mathcal{B}_j^{(\alpha)}(x,y)$ *is given by*

$$\begin{aligned}
&\left[x\left(1+\frac{1}{j}\right)D_x+y\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\frac{D_x{}^{l+1}}{l!}-\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{l+1}{m}\frac{\mathcal{B}_{l-m+1}}{l+1}\frac{D_x{}^{l+1}}{l!}\right.\\
&\left.-\frac{1}{2}\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{l+1}{m}\frac{G_{l-m+1}}{(l+1)}\frac{D_x{}^{l+1}}{l!}+\frac{1}{j}-j\right]{}_{\mathcal{G}}\mathcal{B}_j^{[2]}(x,y)=0.
\end{aligned}$$

Corollary 5.10 *The Bell based Bernoulli-Genocchi polynomials* ${}_{\mathcal{G}}\mathcal{B}_j^{[2]}(x,y)$ *have the following determinantal representation*

$${}_{\mathcal{G}}\mathcal{B}_j^{[2]}(x,y)=(-1)^j2^{j+1}\begin{vmatrix}
{}_{\mathcal{B}}\mathcal{B}_0(x,y) & {}_{\mathcal{B}}\mathcal{B}_1(x,y) & \cdots & {}_{\mathcal{B}}\mathcal{B}_{j-1}(x,y) & {}_{\mathcal{B}}\mathcal{B}_j(x,y)\\
\frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2j} & \frac{1}{2(j+1)}\\
0 & \frac{1}{2} & \cdots & \binom{j-1}{1}\frac{1}{2(j-1)} & \binom{j}{1}\frac{1}{2j}\\
0 & 0 & \cdots & \binom{j-1}{2}\frac{1}{2(j-2)} & \binom{j}{2}\frac{1}{2(j-1)}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & \binom{j}{j-1}\frac{1}{4}
\end{vmatrix}.$$

5.7. 2-iterated Bell based Euler-Genocchi polynomials

When $A(t)=\frac{2}{e^t+1}$ and $B(t)=\frac{2t}{e^t+1}$ in (4.8), we have the 2-iterated Bell based Euler-Genocchi polynomials with the generating function

$$\left(\frac{2}{e^t+1}\right)\left(\frac{2t}{e^t+1}\right)e^{xt}e^{y\left(e^t-1\right)}=\sum_{j=0}^{\infty}{}_{\mathcal{G}}\mathcal{E}_j^{[2]}(x,y)\frac{t^j}{j!}.$$

Corollary 5.11 *The recurrence relation satisfied by Bell based Euler-Genocchi polynomials* ${}_{\mathcal{G}}\mathcal{E}_j^{[2]}(x,y)$ *is given by*

$$\begin{aligned}
{}_{\mathcal{G}}\mathcal{E}_j^{[2]}(x,y) \quad = \quad & \left(1+\frac{1}{j}\right) x {}_{\mathcal{G}}\mathcal{E}_j^{[2]}(x,y) + y\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\binom{j}{l}{}_{\mathcal{G}}\mathcal{E}_{j-l}^{[2]}(x,y) \\
& -\frac{1}{2}\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\sum_{m=0}^{l}\binom{j}{l}\binom{l}{m}E_{l-m}\,{}_{\mathcal{G}}\mathcal{E}_{j-l}^{[2]}(x,y) \\
& -\frac{1}{2}\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{j}{l}\binom{l+1}{m}\frac{G_{l-m+1}}{(l+1)}{}_{\mathcal{G}}\mathcal{E}_{j-l}^{[2]}(x,y) \ ,
\end{aligned}$$

where

$$\frac{2}{e^t+1}=\sum_{l=0}^{\infty}E_l\frac{t^l}{l!} \quad \textit{and} \quad \frac{2t}{e^t+1}=\sum_{l=0}^{\infty}G_l\frac{t^l}{l!}.$$

Corollary 5.12 *The Bell based Euler-Genocchi polynomials* ${}_{\mathcal{G}}\mathcal{E}_j^{[2]}(x,y)$ *have the lowering and raising operators as*

$$\begin{aligned}
{}_x\sigma_j^- := \quad & \frac{1}{j}D_x, \\
{}_x\sigma_j^+ := \quad & x\left(1+\frac{1}{j}\right)+y\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\frac{{D_x}^l}{l!}-\frac{1}{2}\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\sum_{m=0}^{l}\binom{l}{m}E_{l-m}\frac{{D_x}^l}{l!} \\
& -\frac{1}{2}\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{l+1}{m}\frac{G_{l-m+1}}{(l+1)}\frac{{D_x}^l}{l!},
\end{aligned}$$

and the differential equation satisfied by the Bell based Euler-Genocchi polynomials ${}_{\mathcal{G}}\mathcal{E}_j^{[2]}(x,y)$ *is given by*

$$\begin{aligned}
&\left[x\left(1+\frac{1}{j}\right)D_x+y\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\frac{{D_x}^{l+1}}{l!}-\frac{1}{2}\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\sum_{m=0}^{l}\binom{l}{m}E_{l-m}\frac{{D_x}^{l+1}}{l!}\right. \\
&\left.-\frac{1}{2}\left(1+\frac{1}{j}\right)\sum_{l=0}^{j}\sum_{m=0}^{l+1}\binom{l+1}{m}\frac{G_{l-m+1}}{(l+1)}\frac{{D_x}^{l+1}}{l!}+\frac{1}{j}-j\right]\ {}_{\mathcal{G}}\mathcal{E}_j^{[2]}(x,y)=0.
\end{aligned}$$

Corollary 5.13 *The Bell based Euler-Genocchi polynomials* ${}_{\mathcal{G}}\mathcal{E}_j^{[2]}(x,y)$ *have the following determinantal rep-*

resentation

$$
{}_{\mathcal{G}}\mathcal{E}_j^{[2]}(x,y) = (-1)^j 2^{j+1} \begin{vmatrix} {}_{\mathcal{B}}\mathcal{E}_0(x,y) & {}_{\mathcal{B}}\mathcal{E}_1(x,y) & \cdots & {}_{\mathcal{B}}\mathcal{E}_{j-1}(x,y) & {}_{\mathcal{B}}\mathcal{E}_j(x,y) \\ \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2j} & \frac{1}{2(j+1)} \\ 0 & \frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{2(j-1)} & \binom{j}{1}\frac{1}{2j} \\ 0 & 0 & \cdots & \binom{j-1}{2}\frac{1}{2(j-2)} & \binom{j}{2}\frac{1}{2(j-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \binom{j}{j-1}\frac{1}{4} \end{vmatrix}.
$$

6. Conclusion

In our current research, we have defined Bell based Appell polynomials and given the recurrence relation, determinantal representation, shift operators, differential equations and some useful properties. Then, we have exhibited the corresponding properties for Bell based Bernoulli polynomials, Bell based Euler polynomials and Bell based Genocchi polynomials. Later, we have introduced 2-iterated Bell based Appell polynomials and obtained recurrence relation, determinantal representation, shift operators, differential equations and closed form formulae. Finally, by examining special cases, we have presented new hybrid polynomials and stated the corresponding results for them.

According to some ideas, the more general Bell based Appell polynomials can be defined with the help of trigonometric functions. By substituting $x + iz$ for x in (4.8), two new hybrid polynomial families can be constructed whose generating functions contain sine and cosine functions, respectively.

Acknowledgment

We would like to thank the Scientific and Technological Research Council of Türkiye (TÜBİTAK) for the TÜBİTAK BİDEB 2211-A General Domestic Doctorate Scholarship Program that supported the first author.

References

[1] Alam N, Khan WA, Ryoo CS. A note on Bell-based Apostol-type Frobenius-Euler polynomials of complex variable with its certain applications. Mathematics 2022; 10 (12): 1-26. https://doi.org/10.3390/math10122109

[2] Appell P. Sur une classe de pôlynomes. Annales Scientifiques de l'Ecole Normale Supérieure 1980; 9: 119-144 (in French).

[3] Bretti G, Natalini P, Ricci PE. Generalizations of the Bernoulli and Appell polynomials. Abstract and Applied Analysis 2004; 2004 (7): 613-623.

[4] Bretti G, Ricci PE. Multidimensional extension of the Bernoulli and Appell polynomials. Taiwanese Journal of Mathematics 2004; 8 (3): 415-428. https://doi.org/10.11650/twjm/1500407662

[5] Carlitz L. Bernoulli numbers. Fibonacci Quarterly 1968; 6 (3): 71-85.

[6] Carlitz L. Some remarks on the Bell numbers. Fibonacci Quarterly 1980; 18 (1): 66-73.

[7] Comtet L. Advanced Combinatorics: The Art of Finite and Infinite Expansions. Dordrecht, Holland: D. Reidel Publishing Company, 1974.

[8] Costabile FA, Longo E. Δ_h-Appell sequences and related interpolation problem. Numerical Algorithms 2013; 63: 165-186. https://doi.org/ 10.1007/s11075-012-9619-1

[9] Djordjević GB, Milovanović GV. Special Classes of Polynomials. Leskovac: University of Nis, Faculty of Technology, 2014.

[10] Duran U, Acikgoz M. Bell-based Genocchi polynomials. New Trends in Mathematical Sciences 2021; 9 (1): 50-55. https://doi.org/10.20852/ntmsci.2021.428

[11] Duran U, Araci S, Acikgoz M. Bell-based Bernoulli polynomials with applications. Axioms 2021; 10 (1): 29. https://doi.org/10.3390/axioms10010029

[12] He MX, Ricci PE. Differential equation of Appell polynomials via the factorization method. Journal of Computational and Applied Mathematics 2002; 139 (2): 231-237. https://doi.org/10.1016/S0377-0427(01)00423-X

[13] Howard FT. Bell polynomials and degenerate Stirling numbers. Rendiconti del Seminario Matematico della Università di Padova 1979; 61: 203-219.

[14] Kamarujima M, Husain, S. Bell based Apostol-Bernoulli polynomials and its properties. International Journal of Applied and Computational Mathematics 2022; 8 (18): 1-12. https://doi.org/10.1007/s40819-021-01213-0

[15] Khan N, Husain S. Analysis of Bell based Euler polynomials and their application. International Journal of Applied and Computational Mathematics 2021; 7 (5): 1-16. https://doi.org/10.1007/s40819-021-01127-x

[16] Khan N, Husain S. Bell based Apostol type polynomials and its properties. arXiv:2109.10550 [math.NT]. https://doi.org/10.48550/arXiv.2109.10550

[17] Khan S, Ali M. A linear algebra approach to the hybrid Sheffer-Appell polynomials. Mathematical Sciences 2019; 13 (2): 153-164. https://doi.org/10.1007/s40096-019-0286-4

[18] Khan S, Al-Saad MWM, Khan R. Laguerre-based Appell polynomials: properties and applications. Mathematical and Computer Modelling 2010; 52 (1-2): 247-259. https://doi.org/10.1016/j.mcm.2010.02.022

[19] Khan S, Nahid T. Finding non-linear differential equations and certain identities for the Bernoulli–Euler and Bernoulli–Genocchi numbers.SN Applied Sciences 2019; 1(217): 1-9. https://doi.org/10.1007/s42452-019-0222-0

[20] Khan S, Raza N. General-Appell polynomials within the context of monomiality principle. International Journal of Analysis 2013; 1-11. https://doi.org/ 10.1155/2013/328032

[21] Khan S, Raza N. 2-iterated Appell polynomials and related numbers. Applied Mathematics and Computation 2013; 219 (17): 9469-9483. https://doi.org/10.1016/j.amc.2013.03.082

[22] Khan S, Riyaset M. Determinantal approach to certain mixed special polynomials related to Gould–Hopper polynomials. Applied Mathematics and Computation 2015; 251: 599-614. https://doi.org/10.1016/j.amc.2014.11.081

[23] Khan S, Riyaset M. Differential and integral equations for the 2-iterated Bernoulli, 2-iterated Euler and Bernoulli-Euler polynomials. Georgian Mathematical Journal 2020; 27: 375-389. https://doi.org/10.1515/gmj-2018-0062

[24] Khan S, Yasmin G, Ahmad N. A note on truncated exponential based Appell polynomials. Bulletin of the Malaysian Mathematical Sciences Society 2017; 40: 373-388. https://doi.org/10.1007/s40840-016-0343-1

[25] Khan S, Yasmin G, Khan R, Hassan NAM. Hermite-based Appell polynomials: properties and applications. Journal of Mathematical Analysis and Applications 2009; 351 (2): 756-764. https://doi.org/10.1016/j.jmaa.2008.11.002

[26] Kim T, Kim DS, Jang GW. On central complete and incomplete Bell polynomials I. Symmetry 2019; 11 (2): 1-12. https://doi.org/10.3390/sym11020288

[27] Kim T, Kim DS, Jang LC, Lee H, Kim H. Representations of degenerate Hermite polynomials. Advances in Applied Mathematics 2022; 139: 1-18. https://doi.org/ 10.1016/j.aam.2022.102359

[28] Nahid T, Khan S. Construction of some hybrid relatives of Laguerre-Appell polynomials associated with Gould-Hopper matrix polynomials. The Journal of Analysis 2021; 29 (3): 927-946. https://doi.org/10.1007/s41478-020-00288-0

[29] Nahid T, Khan S. Differential equations for certain hybrid special matrix polynomials. Boletim da Sociedade Paranaense de Matemática 2022; 41 (2023): 1-10. https://doi.org/10.5269/bspm.52758

[30] Özarslan MA. Hermite-based unified Apostol-Bernoulli, Euler and Genocchi polynomials. Advances in Difference Equations 2013; 116: 1-13. https://doi.org/10.1186/1687-1847-2013-116

[31] Özarslan MA. Unified Apostol-Bernoulli, Euler and Genocchi polynomials. Computers and Mathematics with Applications 2011; 62 (6): 2452-2462. https://doi.org/10.1016/j.camwa.2011.07.031

[32] Özarslan MA, Yaşar BY. Δ_h-Gould-Hopper Appell polynomials, Acta Mathematica Scientia 2021; 41B (4): 1196-1222. https://doi.org/10.1007/s10473-021-0411-y

[33] Özarslan MA, Yılmaz B. A set of finite order differential equations for the Appell polynomials. Journal of Computational Applied Mathematics 2014; 259: 108-116. https://doi.org/10.1016/j.cam.2013.08.006

[34] Riyasat M, Nahid T, Khan S. An algebraic approach to degenerate Appell polynomials and their hybrid forms via determinants. Acta Mathematica Scientia 2022; 43B: 1-17. https://doi.org/10.1007/s10473-023-0215-3

[35] Roman S. The Umbral Calculus. Pure and Applied Mathematics. New York, NY, USA: Academic Press, Inc., 1984.

[36] Sheffer IM. Note on Appell polynomials. Bulletin of the American Mathematical Society 1945; 51 (10): 739-744.

[37] Srivastava HM, Choi J. Zeta and q-Zeta Functions and Associated Series and Integrals. New York, NY, USA: Elsevier Science Publishers, 2012.

[38] Srivastava HM, Özarslan MA, Yaşar BY. Difference equations for a class of twice iterated Δ_h-Appell sequences of polynomials. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 2019; 113: 1851-1871. https://doi.org/10.1007/s13398-018-0582-0

[39] Srivastava HM, Özarslan MA, Yılmaz B. Some families of differential equations associated with the Hermite-based Appell polynomials and other classes of Hermite-based polynomials. Filomat 2014; 28 (4): 695-708. https://doi.org/10.2298/FIL1404695S

[40] Srivastava HM, Yasmin G, Muhyi A, Araci, S. Certain results for the twice-iterated $2D$ q-Appell polynomials. Symmetry 2019; 11 (10): 1307. https://doi.org/ 10.3390/sym11101307

[41] Varma S, Yaşar BY, Özarslan MA. Hahn-Appell polynomials and their d-orthogonality. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 2019; 113 (3): 2127-2143. https://doi.org/10.1007/s13398-018-0607-8

[42] Wani SA, Nahid T. Finding determinant and integral forms of the 2-iterated $2D$ Appell polynomials. The Journal of Analysis 2022; 30: 731-747. https://doi.org/10.1007/s41478-021-00369-8