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NIOVI KEHAYOPULU



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# On Γ-hypersemigroups

**Niovi KEHAYOPULU***
TBAE, The Scientific and Technological Research Council of Türkiye, Turkey

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**Abstract:** The results on Γ-hypersemigroups are obtained either as corollaries of corresponding results on $\vee e$ or $poe$-semigroups or on the line of the corresponding results on $le$-semigroups. It has come to our attention that Theorem 3.22 in [4] cannot be obtained as corollary to Theorem 2.2 of the same paper as for a Γ-hypersemigroup, $(\mathcal{P}^*(M), \Gamma, \subseteq)$ is a $\vee e$-semigroup and not an $le$-semigroup. Also on p. 1850, l. 12 in [4], the "$le$-semigroup" should be changed to "$\vee e$-semigroup". In the present paper we prove Theorems 3.26 and 3.28 stated without proof in [4]. On this occasion, some further results are given to emphasize what we say. The results on Γ-hypersemigroups are obtained from the more abstract structure of the $poe$-semigroups. Further investigation on $poe$-semigroups and $le$-semigroups is interesting.

**Key words:** Γ-hypersemigroup, $le$-semigroup, intra-regular, right regular, bi-ideal element, bi-ideal

## 1. Introduction

We use the term Γ-hypersemigroup instead of Γ-semihypergroup; hypersemigroup instead of semihypergroup.

The results on Γ-hypersemigroups are obtained from $poe$-semigroups, $\vee e$-semigroups or from the $le$-semigroups (lattice ordered semigroups). On the other hand, according to the introduction of [1], a Γ-semihypergroup is a generalization of semigroup, Γ-semigroup and semihypergroup and the investigation on this structure is very important. This is from the introduction of [1]: "Some motivations for the study of hypersemigroups comes mainly from inside mathematics, computer sciences, biological inheritance, cryptography, theoretical physics, physical phenomenon as the nuclear fission, chemical reactions and redox reactions and a lot of other fields. This wide range of applications in various fields had led to the expansion and generalization of hyperstructures in recent decades, such as $H_v$-structures and Γ-hyperstructures. A lot of work has been done in general on the theory of Γ-hyperstructures, in particular, Γ-semihypergroups by many algebraists, preparing the mathematical background for further applications. Hyperideal theory is important not only for the intrinsic interest and purity of its logical structure but because it is necessary tool in many branches of mathematics and its applications. Several related results have been obtained in Γ-semihypergroups. However, the very fundamental results of Γ-hyperideals in different important classes of Γ-semihypergroups remained yet untouched."

Detailed information about the definition of a Γ-semigroup has been given in [3].

We will give some results on $poe$-semigroups and $le$-semigroups and their analogous for Γ-hypersemigroups that can be obtained either as corollaries or on the line of the proof of the $le$-semigroups. Besides, if we want

*Correspondence: niovi.kehayopulu-tbae@tubitak.gov.tr
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to check if a result on a $\Gamma$-hypersemigroup is true or not, then we always check it for a *poe* or *le*-semigroup. This paper is a continuation of [4]. For any definition not given in the present paper we refer to [4]. We also refer to the introduction of [4].

## 2. Main results

The operation $\Gamma$ in [4, Definition 3.3] is well defined. Indeed: If $(A,B)\in\mathcal{P}^*(M)\times\mathcal{P}^*(M)$ then, by [4, Remark 3.4], $A\Gamma B=\bigcup\limits_{a\in A,b\in B,\gamma\in\Gamma} a\gamma b$. For every $a\in A$, $b\in B$, $\gamma\in\Gamma$, by [4, Definition 3.1(1)], we have $a\gamma b\in\mathcal{P}^*(M)$, thus we get $A\Gamma B\in\mathcal{P}^*(M)$. If $(A,B),(C,D)\in\mathcal{P}^*(M)\times\mathcal{P}^*(M)$ such that $(A,B)=(C,D)$ then, by [4, Remark 3.4], we have $A\Gamma B=\bigcup\limits_{a\in A,b\in B,\gamma\in\Gamma} a\gamma b=\bigcup\limits_{a\in C,b\in D,\gamma\in\Gamma} a\gamma b=C\Gamma D$.

The operation $\overline{\gamma}$ in [4, Definition 3.2] is well defined. Indeed: If $(A,B)\in\mathcal{P}^*(M)\times\mathcal{P}^*(M)$ then, by [4, Definition 3.2], $A\overline{\gamma}B=\bigcup\limits_{a\in A,b\in B} a\gamma b$. For every $a\in A$, $b\in B$, by [4, Definition 3.1], we have $a\gamma b\in\mathcal{P}^*(M)$, thus $A\overline{\gamma}B\in\mathcal{P}^*(M)$. If $(A,B),(C,D)\in\mathcal{P}^*(M)\times\mathcal{P}^*(M)$ such that $(A,B)=(C,D)$, then $A\overline{\gamma}B=\bigcup\limits_{a\in A,b\in B} a\gamma b=\bigcup\limits_{a\in C,b\in D} a\gamma b=C\overline{\gamma}D$.

**Definition 2.1** *[4, Definition 3.25] A $\Gamma$-hypersemigroup $M$ is called intra-regular if for every $a\in M$ there exist $x,y\in M$ and $\gamma,\mu,\rho\in\Gamma$ such that*

$$a\in(x\gamma a)\overline{\mu}(a\rho y).$$

**Theorem 2.2** *[4, Theorem 3.26] A $\Gamma$-hypersemigroup $M$ is intra-regular if and only if, for every right ideal $A$ and every left ideal $B$ of M, we have $A\cap B\subseteq B\Gamma A$.*

To prove this theorem, we need the following proposition.

We write, for short, $a\Gamma M$ instead of $\{a\}\Gamma M$, $M\Gamma a$ instead of $M\Gamma\{a\}$, etc.

**Proposition 2.3** *Let $M$ be a $\Gamma$-hypersemigroup. The following are equivalent:*

(1) *$M$ is intra-regular.*

(2) *For any nonempty subset $A$ of M, we have $A\subseteq M\Gamma A\Gamma A\Gamma M$.*

(3) *For any $a\in M$, we have $a\in M\Gamma a\Gamma a\Gamma M$.*

**Proof** (1) $\Longrightarrow$ (2). Let $\emptyset\neq A\subseteq M$ and $a\in A$. Since $M$ is intra-regular, there exist $x,y\in M$ and $\gamma,\mu,\rho\in\Gamma$ such that $a\in(x\gamma a)\overline{\mu}(a\rho y)$. Since $x\in M$, $\gamma\in\Gamma$, $a\in A$, by [4, Lemma 3.7(2)], we have $x\gamma a\subseteq M\Gamma A$; since $a\in A$, $\rho\in\Gamma$, $y\in M$, we have $a\rho y\subseteq A\Gamma M$. By [4, Lemma 3.6], we have $(x\gamma a)\overline{\mu}(a\rho y)\subseteq(M\Gamma A)\overline{\mu}(A\Gamma M)$. By [4, Definition 3.3], we have $(M\Gamma A)\overline{\mu}(A\Gamma M)\subseteq(M\Gamma A)\Gamma(A\Gamma M)$. By [4, Proposition 3.17], we have $(M\Gamma A)\Gamma(A\Gamma M)=M\Gamma A\Gamma A\Gamma M$ and so $a\in M\Gamma A\Gamma A\Gamma M$.

The implication (2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1). Let $a\in M$. By hypothesis, we have $a\in(M\Gamma a)\Gamma(a\Gamma M)$. By [4, Lemma 3.7(1)], we have $a\in u\mu v$ for some $u\in M\Gamma a$, $\mu\in\Gamma$, $v\in a\Gamma M$, $u\in x\gamma a$ for some $x\in M$, $\gamma\in\Gamma$, $v\in a\rho y$ for some $\rho\in\Gamma$, $y\in M$. By [4, Lemmas 3.5 and 3.6], $a\in u\mu v=\{u\}\overline{\mu}\{v\}\subseteq(x\gamma a)\overline{\mu}(a\rho y)$. Then we have $a\in(x\gamma a)\overline{\mu}(a\rho y)$, where $x,y\in M$ and $\gamma,\mu,\rho\in\Gamma$ and so $M$ is intra-regular. □

**Proof of Theorem 2.2:**
$\Longrightarrow$. Let $A$ be a right ideal and $B$ a left ideal of $M$. By [4, Proposition 3.12], the set $A\cap B$ is nonempty. Since $M$ is intra-regular and $A\cap B\neq\emptyset$, by Proposition 2.3, we have $A\cap B\subseteq M\Gamma(A\cap B)\Gamma(A\cap B)\Gamma M$. Since $A\cap B\subseteq A,B$, by [4, Proposition 3.8], we have $M\Gamma(A\cap B)\Gamma(A\cap B)\Gamma M\subseteq(M\Gamma B)\Gamma(A\Gamma M)\subseteq B\Gamma A$ and so $A\cap B\subseteq B\Gamma A$.
$\Longleftarrow$. Let $a\in M$. By hypothesis, we have

$$\begin{aligned} a\in R(a)\cap L(a) &= L(a)\Gamma R(a)=(a\cup M\Gamma a)\Gamma(a\cup a\Gamma M) \text{ (by [4, Proposition 3.21])} \\ &= a\Gamma a\cup M\Gamma a\Gamma a\cup a\Gamma a\Gamma M\cup M\Gamma a\Gamma a\Gamma M \text{ (by [4, Proposition 3.13])}. \end{aligned}$$

If $a\in a\Gamma a$, then $a\in a\Gamma a\subseteq(a\Gamma a)\Gamma(a\Gamma a)\subseteq M\Gamma a\Gamma a\Gamma M$ (by [4, Lemma 3.8 and Proposition 3.17]).
If $a\in M\Gamma a\Gamma a$, then $a\in M\Gamma a\Gamma a\subseteq M\Gamma(M\Gamma a\Gamma a)\Gamma a=(M\Gamma M)\Gamma(a\Gamma a\Gamma a)\subseteq M\Gamma(a\Gamma a\Gamma M)=M\Gamma a\Gamma a\Gamma M$.
If $a\in a\Gamma a\Gamma M$, in a similar way we get $a\in M\Gamma a\Gamma a\Gamma M$.
In each case, we have $a\in M\Gamma a\Gamma a\Gamma M$. By Proposition 2.3(3) $\Rightarrow$ (1), $M$ is intra-regular.

**Definition 2.4** *[4, Definition 3.27] A $\Gamma$-hypersemigroup M is called right (resp. left) regular if for every $a\in M$ there exist $x\in M$ and $\gamma,\mu\in\Gamma$ such that*

$$a\in(a\gamma a)\overline{\mu}\{x\}\ \textit{(resp. } a\in\{x\}\overline{\gamma}(a\mu a)).$$

**Theorem 2.5** *[4, Theorem 3.28] A $\Gamma$-hypersemigroup M is right (resp. left) regular and right (resp. left) duo if and only if for every right (resp. left) ideals $A,B$ of M we have $A\cap B=A\Gamma B$ (resp. $A\cap B=B\Gamma A$).*

To prove this theorem we need the following proposition.

**Proposition 2.6** *Let M be $\Gamma$-hypersemigroup. The following are equivalent:*

(1) *M is right (resp. left) regular.*

(2) *For every nonempty subset A of M, we have $A\subseteq A\Gamma A\Gamma M$ (resp. $A\subseteq M\Gamma A\Gamma A$).*

(3) *For every $a\in M$, we have $a\in a\Gamma a\Gamma M$ (resp. $a\in M\Gamma a\Gamma a$).*

**Proof** (1) $\Longrightarrow$ (2). Let $\emptyset\neq A\subseteq M$ and $a\in A$. Since $M$ is right regular, there exist $x\in M$ and $\gamma,\mu\in\Gamma$ such that $a\in(a\gamma a)\overline{\mu}\{x\}$. By [4, Definition 3.3], $(a\gamma a)\overline{\mu}\{x\}\subseteq(a\gamma a)\Gamma\{x\}$. By [4, Lemma 3.7(2)], $a\gamma a\subseteq a\Gamma a$. By [4, Lemma 3.8 and Proposition 3.17], $(a\gamma a)\Gamma\{x\}\subseteq(A\Gamma A)\Gamma M=A\Gamma A\Gamma M$ and so $a\in A\Gamma A\Gamma M$.
The implication (2) $\Longrightarrow$ (3) is obvious.
(3) $\Longrightarrow$ (1). Let $a\in M$. By hypothesis and [4, Proposition 3.17], we have $a\in a\Gamma a\Gamma M=(a\Gamma a)\Gamma M$. By [4, Lemma 3.7(1)], $a\in u\mu x$ for some $u\in a\Gamma a$, $\mu\in\Gamma$, $x\in M$ and $u\in a\gamma a$ for some $\gamma\in\Gamma$. By [4, Lemmas 3.5 and 3.6], we have $a\in u\mu x=\{u\}\overline{\mu}\{x\}\subseteq(a\gamma a)\overline{\mu}\{x\}$. We have $a\in(a\gamma a)\overline{\mu}\{x\}$, where $x\in M$ and $\gamma,\mu\in\Gamma$ and so $M$ is right regular. □

**Proof of Theorem 2.5:**
$\Longrightarrow$. Let $M$ be a right regular and right duo and $A,B$ be right ideals of $M$. Then $A\Gamma B\subseteq A\Gamma M\subseteq A$; since $M$ is right duo, $B$ is a right ideal of $M$ as well and so $A\Gamma B\subseteq M\Gamma B\subseteq B$. Thus we have $A\Gamma B\subseteq A\cap B$. Since $A$ is a right ideal and $B$ a left ideal of $M$, by [4, Proposition 3.12], $A\cap B\neq\emptyset$. Since $M$ is right regular and $A\cap B\neq\emptyset$, by Proposition 2.6, $A\cap B\subseteq(A\cap B)\Gamma(A\cap B)\Gamma M$. Since $A\cap B\subseteq A,B$, by [4, Lemma 3.8], we

have $(A\cap B)\Gamma M\subseteq A\Gamma M\cap B\Gamma M\subseteq A\cap B$. Then we have $A\cap B\subseteq (A\cap B)\Gamma(A\cap B)\subseteq A\Gamma B$.
$\Longleftarrow$. Let $A$ be a right ideal of $M$. Since $M$ is a right ideal of $M$, by hypothesis, we have $A=A\cap M=M\Gamma A$ and so $A$ is a left ideal of $M$ and $M$ is right duo. Let now $a\in M$. By hypothesis, we have

$$\begin{aligned} a\in R(a)\cap R(a) &= R(a)\Gamma R(a)=(a\cup a\Gamma M)\Gamma(a\cup a\Gamma M) \text{ (by [4, Proposition 3.21])}\\ &= a\Gamma a\cup a\Gamma M\Gamma a\cup a\Gamma a\Gamma M\cup a\Gamma M\Gamma a\Gamma M. \end{aligned}$$

If $a\in a\Gamma a$, then $a\in a\Gamma a\subseteq a\Gamma a\Gamma a\subseteq a\Gamma a\Gamma M$.
If $a\in a\Gamma M\Gamma a$, then $a\in a\Gamma M\Gamma(a\Gamma M\Gamma a)=a\Gamma M\Gamma(a\Gamma M)\Gamma a$ (by [4, Proposition 3.17]). The set $a\Gamma M$ is a right ideal of $M$ (as $(a\Gamma M)\Gamma M=a\Gamma(M\Gamma M)\subseteq a\Gamma M$). Since $M$ is right duo, $a\Gamma M$ is a left ideal of $M$ as well i.e. $M\Gamma(a\Gamma M)\subseteq a\Gamma M$. Thus we have $a\in a\Gamma(a\Gamma M)\Gamma a=a\Gamma a\Gamma(M\Gamma a)\subseteq a\Gamma a\Gamma(M\Gamma M)\subseteq a\Gamma a\Gamma M$.
Let $a\in a\Gamma M\Gamma a\Gamma M$. Then $a\in(a\Gamma M)\Gamma(a\Gamma M)$. Since $a\Gamma M$ is a right ideal of $M$ and $M$ is right duo, $a\Gamma M$ is a left ideal of $M$ as well and so $M\Gamma(a\Gamma M)\subseteq a\Gamma M$. Then we have $a\in a\Gamma M\Gamma(a\Gamma M)\subseteq a\Gamma a\Gamma M$.
Thus we have $a\in a\Gamma a\Gamma M$ for every $a\in M$ and, by Proposition 2.6(3) $\Rightarrow$ (1), $M$ is right regular.

**Proposition 2.7** *If $(M,\Gamma)$ is a $\Gamma$-hypersemigroup, then $(\mathcal{P}^*(M),\Gamma,\subseteq)$ is a $\vee e$-semigroup and a poe-semigroup.*

**Proof** By [4, Corollary 3.18], $(\mathcal{P}^*(M),\Gamma)$ is a semigroup. The inclusion relation "$\subseteq$" is clearly an order on $\mathcal{P}^*(M)$ and, for any $A,B\in\mathcal{P}^*(M)$, the set $A\cup B$ is the supremum of $A$ and $B$. Moreover, by [4, Proposition 3.13], for any $A,B,C\in\mathcal{P}^*(M)$, we have $(A\cup B)\Gamma C=A\Gamma C\cup B\Gamma C$ and $A\Gamma(B\cup C)=A\Gamma B\cup A\Gamma C$. Thus $(\mathcal{P}^*(M),\Gamma,\subseteq)$ is a $\vee e$-semigroup. Moreover, every $\vee e$-semigroup if a *poe*-semigroup. Indeed: Let $(S,\cdot,\leq)$ be a $\vee e$-semigroup and $a,b,c\in S$ such that $a\leq b$. Then $ac\leq ac\vee bc=(a\vee b)c=bc$ and $ca\leq ca\vee cb=c(a\vee b)=cb$. □

**Definition 2.8** *A poe-semigroup $S$ is called completely regular if it is regular, right regular and left regular. A $\Gamma$-hypersemigroup $M$ is called completely regular if it is regular, right regular and left regular.*

**Proposition 2.9** *A poe-semigroup $(S,\cdot,\leq)$ is completely regular if and only if $a\leq a^2ea^2$ for every $a\in S$.*

**Proof** $\Longrightarrow$. Let $a\in S$. Since $S$ is completely regular, we have $a\leq aea$, $a\leq a^2e$ and $a\leq ea^2$, thus we have $a\leq aea\leq(a^2e)e(ea^2)\leq a^2ea^2$.
$\Longleftarrow$. Let $a\in S$. By hypothesis, we have $a\leq a^2ea^2\leq aea,\ a^2e,\ ea^2$ and so $S$ is completely regular. □

**Proposition 2.10** *A $\Gamma$-hypersemigroup $M$ is completely regular if and only if, for any nonempty subset $A$ of $M$, we have $A\subseteq A\Gamma A\Gamma M\Gamma A\Gamma A$.*

**Proof** (on the line of Proposition 2.9).
$\Longrightarrow$. Let $A$ be a nonempty subset of $S$. Since $M$ is regular and $A\neq\emptyset$, by [4, Proposition 3.20], we have $A\subseteq A\Gamma M\Gamma A$. Since $M$ is right regular and left regular and $A\neq\emptyset$, by Proposition 2.6, we have $A\subseteq A\Gamma A\Gamma M$ and $A\subseteq M\Gamma A\Gamma A$. Thus we have

$$\begin{aligned} A &\subseteq A\Gamma M\Gamma A\subseteq(A\Gamma A\Gamma M)\Gamma M\Gamma(M\Gamma A\Gamma A) \text{ (by [4, Lemma 3.8])}\\ &= A\Gamma A\Gamma(M\Gamma M\Gamma M)\Gamma A\Gamma A \text{ (by [4, Proposition 3.17])}\\ &\subseteq A\Gamma A\Gamma M\Gamma A\Gamma A \text{ (since } M\Gamma M\Gamma M=M\Gamma(M\Gamma M)\subseteq M\Gamma M\subseteq M). \end{aligned}$$

Thus we have $A \subseteq A\Gamma A\Gamma M\Gamma A\Gamma A$.
$\Longleftarrow$. Let $\emptyset \neq A \subseteq M$. By hypothesis, we have

$A \subseteq A\Gamma A\Gamma M\Gamma A\Gamma A \subseteq A\Gamma(M\Gamma M\Gamma M)\Gamma A \subseteq A\Gamma M\Gamma A$,

$A \subseteq A\Gamma A\Gamma M\Gamma A\Gamma A \subseteq A\Gamma A\Gamma(M\Gamma A\Gamma M) \subseteq A\Gamma A\Gamma(M\Gamma M\Gamma M) \subseteq A\Gamma A\Gamma M$,

$A \subseteq A\Gamma A\Gamma M\Gamma A\Gamma A \subseteq (M\Gamma M\Gamma M)\Gamma A\Gamma A \subseteq M\Gamma A\Gamma A$,

and so $M$ is completely regular.
**Second Proof** (as corollary to Proposition 2.9).
$M$ is a completely regular $\Gamma$-hypersemigroup if and only if $(\mathcal{P}^*(M), \Gamma, \subseteq)$ is a completely regular *poe*-semigroup if and only if $A \subseteq A\Gamma A\Gamma M\Gamma A\Gamma A$ for every $A \in \mathcal{P}^*(M)$ (by Proposition 2.9) if and only if $A \subseteq A\Gamma A\Gamma M\Gamma A\Gamma A$ for every $\emptyset \neq A \subseteq M$.

**Proposition 2.11** *A poe-semigroup $S$ is regular and right (resp. left) regular if and only if $a \leq a^2ea$ (resp. $a \leq aea^2$) for every $a \in S$.*

**Proof** $\Longrightarrow$. If $S$ is regular and right regular, then $a \leq aea$ and $a \leq a^2e$ for every $a \in S$, thus $a \leq aea \leq (a^2e)ea = a^2e^2a \leq a^2ea$ for every $a \in S$.
$\Longleftarrow$. If $a \leq a^2ea$ for every $a \in S$, then $a \leq aea$ and $a \leq a^2e$ for every $a \in S$ and so $S$ is regular and right regular. □

**Corollary 2.12** *A $\Gamma$-hypersemigroup is both regular and right (resp. left) regular if and only if, for any nonempty subset A of M, we have $A \subseteq A\Gamma A\Gamma M\Gamma A$ (resp. $A \subseteq A\Gamma M\Gamma A\Gamma A$).*

An element $b$ of a *poe*-semigroup $S$ is called a bi-ideal element of $S$ if $beb \leq b$. A nonempty subset $B$ of a $\Gamma$-hypersemigroup $M$ is called bi-ideal of $M$ if $B\Gamma M\Gamma B \subseteq B$; that is, if $x \in u\gamma b$ for some $u \in B\Gamma M$, $\gamma \in \Gamma$, $b \in B$ and $u \in c\mu s$ for some $c \in B$, $\mu \in \Gamma$, $s \in M$, then $x \in B$. An element $a$ of a *po*-groupoid $S$ is called semiprime if for any $t \in S$ such that $t^2 \leq a$, we have $t \leq a$. A nonempty subset $A$ of a $\Gamma$-hypersemigroup $M$ is called semiprime if for any nonempty subset $T$ of $M$ such that $T\Gamma T \subseteq A$, we have $T \subseteq A$.

**Theorem 2.13** *A poe-semigroup S is completely regular if and only if the bi-ideal elements of S are semiprime.*

**Proof** $\Longrightarrow$. Let $b$ be a bi-ideal element of $S$ and $a \in S$ such that $a^2 \leq b$. Since $S$ is completely regular, by Proposition 2.9, we have $a \leq a^2ea^2 \leq beb \leq b$ and so $a \leq b$.
$\Longleftarrow$. Let $a \in S$. The element $a^2ea^2$ is a bi-ideal element of $S$ as $(a^2ea^2)e(a^2ea^2) \leq a^2ea^2$. Since $(a^4)^2 = a^8 = a^2a^4a^2 \leq a^2ea^2$ and $a^2ea^2$ is semiprime, we have $a^4 \leq a^2ea^2$, $a^2 \leq a^2ea^2$ and $a \leq a^2ea^2$. By Proposition 2.9, $S$ is completely regular. □

**Corollary 2.14** *A $\Gamma$-hypersemigroup M is completely regular if and only if the bi-ideals of M are semiprime.*

**Proposition 2.15** *Let $S$ be an intra-regular poe-semigroup, $i \in S$ and $j$ an ideal element of $S$ such that $i^2 \leq j$. Then $i \leq j$.*

**Proof** Since $S$ is intra-regular and $i \in S$, we have $i \leq ei^2e \leq eje = (ej)e \leq je \leq j$ and so $i \leq j$. □

**Corollary 2.16** *(see [1, Theorem 3.5]) Let M be an intra-regular* $\Gamma$*-hypersemigroup, I a nonempty subset of M and J an ideal of M such that* $I\Gamma I \subseteq J$*. Then* $I \subseteq J$*.*

According to [1, Theorem 3.5] in the above corollary the set $I$ should be an ideal. Proposition 2.15 shows that $I$ is not only an ideal but, more generally, a nonempty subset of $S$.

An element $a$ of a *po*-groupoid $S$ is called idempotent if $a^2 = a$. A nonempty subset $A$ of a $\Gamma$-hypersemigroup $M$ is called idempotent if $A\Gamma A = A$; that is, $x \in A$ if and only if there exist $a, b \in A$ and $\gamma \in \Gamma$ such that $x \in a\gamma b$.

**Proposition 2.17** *If S is a regular poe-semigroup then, for any* $a \in S$*, the elements ae and ea are idempotent. In particular, if S is a regular* $\vee e$*-semigroup then, for every* $a \in S$*, we have*

$$r(a) = ae = r(ae) \text{ and } l(a) = ea = l(ea).$$

**Proof** Let $a \in S$. Since $S$ is regular, we have $a \le aea$. Then

$ae \le (aea)e = (ae)(ae) = a(eae) \le ae$ and $ea \le e(aea) = (ea)(ea) = (eae)a \le ea$

and so $ae$ and $ea$ are idempotent. Let now $S$ be a regular $\vee e$-semigroup and $a \in S$. Since $S$ is regular, we have

$r(a) = a \vee ae \le aea \vee ae = ae \le r(a)$, $l(a) = a \vee ea \le aea \vee ea = ea \le l(a)$

and so $r(a) = ae$ and $l(a) = ea$. In addition,

$r(ae) = ae \vee (ae)e = ae = r(a)$ and $l(ea) = ea \vee e(ea) = ea = l(a)$. □

**Corollary 2.18** *If M is a regular* $\Gamma$*-hypersemigroup then, for any nonempty subset A of M, the sets* $A\Gamma M$ *and* $M\Gamma A$ *are idempotent and we have*

$$R(A) = A\Gamma M = R(A\Gamma M) \text{ and } L(A) = M\Gamma A = L(M\Gamma A).$$

**Theorem 2.19** *Let S be poe-semigroup. If a is a right ideal element of S and* $b \in S$ *(or* $a \in S$ *and b a left ideal element of S), then the element ab is a bi-ideal element of S. In particular, if S is a regular* $\vee e$*-semigroup and x is a bi-ideal element of S, then there exists a right ideal element a and a left ideal element b of S such that* $x = ab$*.*

**Proof** $\Longrightarrow$. If $a$ is a right ideal element of $S$ and $b \in S$, then $(ab)e(ab) = a(bea)b \le (ae)b \le ab$. If $a \in S$ and $b$ is a left ideal element of $S$, then $(ab)e(ab) = a(bea)b \le a(eb) \le ab$.

$\Longleftarrow$. Let $S$ be a regular $\vee e$-semigroup and $x$ a bi-ideal element of $S$. We have

$r(x)l(x) = (x \vee xe)(x \vee ex) = x^2 \vee xex \vee (xe)(ex) = x^2 \vee xex$.

Since $S$ is regular, and $x$ is a bi-ideal element of $S$, we have $x = xex$, then $x^2 = (xex)x = x(ex)x \le xex$. Thus we have $r(x)l(x) = xex = x$, where $r(x)$ is a right ideal element and $l(x)$ is a left ideal element of $S$. □

**Corollary 2.20** *Let M be a regular* $\Gamma$*-hypersemigroup. Then X is a bi-ideal of M if and only if there exists a right ideal A and a left ideal B of M such that* $X = A\Gamma B$*.*

A *po*-groupoid $S$ is called right (resp. left) duo if the right (resp. left) ideal elements of $S$ are left (resp. right) ideal elements of $S$ as well (that is, ideal elements of $S$). A $\Gamma$-hypersemigroup $M$ is said to be right (resp. left) duo if every right (resp. left) ideal of $M$ is an ideal of $M$ [4].

**Proposition 2.21** *If $S$ is a $\vee e$-semigroup regular and right (resp. left) duo, then every bi-ideal element of $S$ is a left (resp. right) ideal element of $S$.*

**Proof** Let $S$ be regular and right duo and $x$ be a bi-ideal element of $S$. Since $S$ is regular, by Theorem 2.19, there exists a right ideal element $a$ and a left ideal element $b$ of $S$ such that $x = ab$. Since $S$ is a duo, $a$ is a left ideal element of $S$ as well. Then $ex = e(ab) = (ea)b \leq ab = x$ and so $x$ is a left ideal element of $S$. □

**Corollary 2.22** *If $M$ is a $\Gamma$-hypesemigroup regular and right (resp. left) duo, then every bi-ideal of $M$ is a left (resp. right) ideal of $M$.*

A right (resp. left) ideal element $a$ of a *po*-groupoid $S$ is called minimal if there is no right (resp. left) ideal element $b$ of $S$ such that $b < a$. That is, if $b$ is a right (resp. left) ideal element of $S$ such that $b \leq a$, then $b = a$. A bi-ideal element $a$ of a *poe*-semigroup $S$ is called minimal if there is no bi-ideal element $b$ of $S$ such that $b < a$. That is, if $b$ is a bi-ideal element of $S$ such that $b \leq a$, then $b = a$. A right ideal $A$ of a $\Gamma$-hypersemigroup $M$ is called minimal if there is no right ideal $B$ of $M$ such that $B \subset A$; that is if $B$ is a right ideal of $M$ such that $B \subseteq A$, then $B = A$. The same if we replace the word "right" by "left" or "bi-ideal".

**Theorem 2.23** *Let $S$ be a poe-semigroup. If $x$ is a minimal bi-ideal element of $S$, then there exists a minimal right ideal element $a$ and a minimal left ideal element $b$ of $S$ such that $x = ab$.*

**Proof** Let $x$ be a minimal bi-ideal element of $S$. The element $xe$ (resp. $ex$) is a right (resp. left) ideal element of $S$. Let $z$ be a right ideal element of $S$ such that $z \leq xe$. Then $z = xe$. Indeed: $zx$ is a bi-ideal element of $S$ (see also Theorem 2.19). Since $zx \leq (xe)x \leq x$ and $x$ is a minimal bi-ideal element of $S$, we have $zx = x$. Then we have $xe = (zx)e \leq ze \leq z$, then $z = xe$ and so $xe$ is a minimal right ideal element of $S$. Let now $t$ be a left ideal element of $S$ such that $t \leq ex$. Then $xt$ is a bi-ideal element of $S$ and $xt \leq x(ex) \leq x$. Since $x$ is a minimal bi-ideal element of $S$, we have $xt = x$. Then $ex = e(xt) \leq et \leq t$, then $t = ex$ and so $ex$ is a minimal left ideal element of $S$. We have $x = (xe)(ex)$. Indeed: Since $xe$ is a right ideal element of $S$, by Theorem 2.19, $(xe)(ex)$ is a bi-ideal element of $S$. Since $(xe)(ex) \leq xex \leq x$ and $x$ is a minimal bi-ideal element of $S$, we have $x = (xe)(ex)$. □

**Corollary 2.24** *If $M$ is $\Gamma$-hypersemigroup and $X$ a minimal bi-ideal of $M$, then the set $B\Gamma M$ is a minimal right ideal of $M$, the set $M\Gamma B$ is a minimal left ideal of $M$, and we have $X = (B\Gamma M)\Gamma(M\Gamma B)$.*

An element $a$ of a *po*-groupoid $S$ is called subidempotent if $a^2 \leq a$. A nonempty subset $A$ of a $\Gamma$-hypersemigroup $M$ is called subidempotent if $A\Gamma A \subseteq A$; that is, if $x \in a\gamma b$ for some $a, b \in A$, $\gamma \in \Gamma$, then $x \in A$. A *poe*-groupoid $S$ is called right (resp. left) simple if the element $e$ is the only right (resp. left) ideal element of $S$; that is if $a$ is a right (resp. left) ideal element of $S$, then $a = e$. A $\Gamma$-hypersemigroup $M$ is called right (resp. left) simple if $M$ is the only right (resp. left) ideal of $M$.

**Proposition 2.25** *Let $S$ be a $\vee e$-semigroup. If $S$ is right (resp. left) simple, then every subidempotent bi-ideal element of $S$ is a left (resp. right) ideal element of $S$.*

**Proof** Let $S$ be right simple and $b$ a subidempotent bi-ideal element of $S$. Since $S$ is right simple and $r(b)$ is a right ideal element of $S$, we have $r(b) = e$. Then $eb = r(b)b = (b \vee be)b = b^2 \vee beb \leq b$ and so $b$ is a left ideal element of $S$. □

**Corollary 2.26** *If $M$ is a right (resp. left) simple $\Gamma$-hypersemigroup, then every subidempotent bi-ideal of $M$ is a left (resp. right) ideal of $M$.*

**Proposition 2.27** *If $S$ is an intra-regular poe-semigroup, then every ideal element of $S$ is idempotent.*

**Proof** Let $a$ be an ideal element of $S$. Then $a^2$ is an ideal element of $S$ as well. Indeed, $a^2e = a(ae) \le a^2$ and $ea^2 = (ea)a \le a^2$. Since $S$ is intra-regular, we have $a \le ea^2e = (ea^2)e \le a^2e \le a^2 \le ea \le a$ and so $a^2 = a$. □

**Corollary 2.28** *If $M$ is an intra-regular $\Gamma$-hypersemigroup, then every ideal of $M$ is idempotent.*

**Theorem 2.29** *Let $S$ be a poe-semigroup. If $S$ is regular and intra-regular, then the bi-ideal elements of $S$ are idempotent. Conversely, if $S$ is an le-semigroup and the bi-ideal elements of $S$ are idempotent, then $S$ is regular and intra-regular.*

**Proof** $\Longrightarrow$. Let $b$ be a bi-ideal element of $S$. Since $S$ is regular, we have $b \le beb$ and so $b = beb$. Then $b^2 = (beb)b = b(eb)b \le beb = b$ and so $b^2 \le b$. Since $S$ is intra-regular, we have $b \le eb^2e$. Thus we have

$$b = beb = be(beb) = bebeb \le be(eb^2e)eb = (be^2b)(be^2b) \le (beb)(beb) = b^2,$$

then $b \le b^2$ and so $b = b^2$.
$\Longleftarrow$. Let now $S$ be an $le$-semigroup, $a$ be a right ideal element and $b$ a left ideal element of $S$. Then $a \wedge b$ is a bi-ideal of $S$. Indeed, $(a \wedge b)e(a \wedge b) \le a(eb) \le ab \le ae \wedge eb \le a \wedge b$. By hypothesis, we have $a \wedge b = (a \wedge b)^2 = (a \wedge b)(a \wedge b) \le ab, ba$. Since $a \wedge b \le ab$, by [4, Theorem 2.2], $S$ is regular. Since $a \wedge b \le ba$, by [4, Theorem 2.4], $S$ is intra-regular. □

According to Theorem 2.29, the following holds.

**Theorem 2.30** *Let $M$ be a $\Gamma$-hypersemigroup. Then $M$ is regular and intra-regular if and only if the bi-ideals of $M$ are idempotent.*

**Proof** The $\Rightarrow$-part can be obtained as a corollary to the first part of Theorem 2.29.
$\Longleftarrow$. On the line of the proof of Theorem 2.29 : Let $A$ be a right ideal and $B$ a left ideal of $M$. Then $A \cap B$ is a bi-ideal of $M$. Indeed, by [4, Proposition 3.12], we have $A \cap B \neq \emptyset$; moreover

$$(A \cap B)\Gamma M\Gamma(A \cap B) \subseteq A\Gamma(M\Gamma B) \subseteq A\Gamma B \subseteq A\Gamma M \cap M\Gamma B \subseteq A \cap B.$$

By hypothesis, we have $A \cap B = (A \cap B)\Gamma(A \cap B) \subseteq A\Gamma B, B\Gamma A$. Since $A \cap B \subseteq A\Gamma B$, by [4, Theorem 3.22], $M$ is regular. Since $A \cap B \subseteq B\Gamma A$, by Theorem 2.2, $M$ is intra-regular. □

An element $q$ of a $poe$-semigroup $S$ is called quasi-ideal element of $S$ if $qe \wedge eq$ exists in $S$ and we have $qe \wedge eq \le q$. A nonempty subset $Q$ of a $\Gamma$-hypersemigroup $M$ is said to be a quasi-ideal of $M$ if $Q\Gamma M \cap M\Gamma Q \subseteq Q$; that is, if $x \in q_1\gamma m_1$ for some $q_1 \in Q$, $\gamma \in \Gamma$, $m_1 \in M$ and $x \in m_2\mu q_2$ for some $m_2 \in M$, $\mu \in \Gamma$, $q_2 \in Q$, then $x \in Q$.

**Proposition 2.31** *In a poe-semigroup, every quasi-ideal element (if it exists) is a bi-ideal element. In a regular le-semigroup, the concepts of bi-ideal elements and quasi-ideal elements are the same.*

**Proof** If $S$ is a *poe*-semigroup and $q$ a quasi-ideal element of $S$, then $qeq \le qe \wedge eq \le q$ and so $q$ is a bi-ideal element of $S$. Let now $S$ be a regular *le*-semigroup and $x$ a bi-ideal element of $S$. The element $xe$ (resp. $ex$) is a right (resp. left) ideal element of $S$. Since $S$ is regular, by [4, Theorem 2.2], we have $xe \wedge ex = (xe)(ex) = xe^2x \le xex \le x$ and so $x$ is a quasi-ideal element of $S$. $\square$

According to Proposition 2.31, we have the following proposition. Let us give its proof to be able to compare it with the proof of Proposition 2.31 to see that both are the same.

**Proposition 2.32** *In a regular $\Gamma$-hypersemigroup $M$, the concepts of bi-ideals and quasi-ideals coincide.*

**Proof** If $Q$ is a quasi-ideal of $M$, then

$$\begin{aligned} Q\Gamma M\Gamma Q &= Q\Gamma(M\Gamma Q)\cap(Q\Gamma M)\Gamma Q \text{ (by [4, Proposition 3.17])} \\ &\subseteq Q\Gamma(M\Gamma M)\cap(M\Gamma M)\Gamma Q \text{ (by [4, Lemma 3.8])} \\ &\subseteq Q\Gamma M\cap M\Gamma Q \text{ (as } M\Gamma M\subseteq M) \\ &\subseteq Q, \end{aligned}$$

and so $Q$ is a bi-ideal of $M$. Let now $M$ be a regular $\Gamma$-hypersemigroup and $X$ a bi-ideal of $M$. The set $X\Gamma M$ is a right ideal of $M$ (as $(X\Gamma M)\Gamma M = X\Gamma(M\Gamma M)\subseteq X\Gamma M$) and $M\Gamma X$ is a left ideal of $M$. Since $M$ is regular, by [4, Theorem 3.22], we have

$$\begin{aligned} X\Gamma M\cap M\Gamma X &= (X\Gamma M)\Gamma(M\Gamma X) = X\Gamma(M\Gamma M)\Gamma X \text{ (by [4, Proposition 3.17])} \\ &\subseteq X\Gamma M\Gamma X\subseteq X, \end{aligned}$$

and so $X$ is a quasi-ideal of $M$. $\square$

**This is Theorem 3.8 in [1]:** In a $\Gamma$-hypersemigroup $S$ the following statements are equivalent:

(1) $S$ is intra-regular.

(2) For any left ideal $I$ and any bi-ideal $B$ of $S$, we have $I\cap B\subseteq I\Gamma B\Gamma S$.

(3) For any left ideal $I$ and any quasi-ideal $Q$ of $S$, we have $I\cap Q\subseteq I\Gamma Q\Gamma S$.

(4) For any right ideal $J$ and any bi-ideal $B$ of $S$, we have $J\cap B\subseteq \Gamma B\Gamma J$.

(5) For any right ideal $J$ and any quasi-ideal $Q$ of $S$, we have $J\cap Q\subseteq S\Gamma Q\Gamma J$.

To check its validity, we prove it for a *poe*-semigroup. The following proposition holds.

**Proposition 2.33** *Let $S$ be an le-semigroup. Then $S$ is intra-regular if and only if, for any $a,b\in S$, we have $a\wedge b\le eabe$.*

**Proof** $\Longrightarrow$. Let $a,b\in S$. Since $S$ is intra-regular, we have $a\wedge b\le e(a\wedge b)(a\wedge b)e\le eabe$.
$\Longleftarrow$. Let $a\in S$. By hypothesis, we have $a=a\wedge a\le ea^2e$ and so $S$ is intra-regular. $\square$

According to Proposition 2.33, the implication (1) $\Rightarrow$ (2) in [1, Theorem 3.8] holds for any nonempty subsets $I$, $B$ of $S$.

**Proposition 2.34** *Let $S$ be a poe-semigroup at the same time semilattice under $\wedge$. If $S$ is both regular and intra-regular then, for every right ideal element $x$, every left ideal element $y$ and every bi-ideal element $b$ of $S$, we have $x\wedge b\wedge y\le ybx$.*

**Proof** We have

$$\begin{aligned} x \wedge b \wedge y &\le (x \wedge b \wedge y)e(x \wedge b \wedge y) \text{ (since } S \text{ is regular)} \\ &\le \Big(e(x \wedge b \wedge y)(x \wedge b \wedge y)e\Big)e\Big(e(x \wedge b \wedge y)(x \wedge b \wedge y)e\Big) \text{ (since } S \text{ is intra-regular)} \\ &= e(x \wedge b \wedge y)(x \wedge b \wedge y)e^3(x \wedge b \wedge y)(x \wedge b \wedge y)e \\ &\le e(x \wedge b \wedge y)\Big((x \wedge b \wedge y)e(x \wedge b \wedge y)\Big)(x \wedge b \wedge y)e \\ &\le ey(beb)(xe) \le ybx. \end{aligned}$$

□

From Proposition 2.34, we have the following proposition.

**Proposition 2.35** *[1, Proposition 4.7]: If $M$ is a $\Gamma$-hypersemigroup both regular and intra-regular, $X$ a right ideal, $Y$ a left ideal and $B$ a bi-ideal of $M$, then $X \cap B \cap Y \subseteq Y\Gamma B\Gamma X$.*

**This is the main part of theorem 4.12 in [1]** (as the equivalence of the other properties are obvious): In a $\Gamma$-hypersemigroup $S$ the following statements are equivalent:

(1) $S$ is both regular and intra-regular.

(2) For any bi-ideals $A, B$ of $S$, $A \cap B \subseteq (A\Gamma B) \cap (B\Gamma A)$.

(3) For any quasi-ideals $A, B$ of $S$, $A \cap B \subseteq (A\Gamma B) \cap (B\Gamma A)$.

To check if it is true or not, we prove it for a *poe*-semigroup. We have the following:

**Theorem 2.36** *Let S be a poe-semigroup at the same time semilattice under $\wedge$. The following are equivalent:*

*(1) S is both regular and intra-regular.*

*(2) $a \wedge b \le ab \wedge ba$ for any bi-ideal elements $a, b$ of S.*

*(3) $a \wedge b \le ab \wedge ba$ for any quasi-ideal elements $a, b$ of S.*

*Then $(1) \Rightarrow 2 \Rightarrow (3)$. In particular, if $S$ is an le-semigroup then the three properties are equivalent:*

**Proof** $(1) \Longrightarrow (2)$. Let $a, b$ be bi-ideal elements of $S$. Then we have

$$\begin{aligned} a \wedge b &\le (a \wedge b)e(a \wedge b) \le \Big((a \wedge b)e(a \wedge b)\Big)e(a \wedge b) \text{ (since } S \text{ is regular)} \\ &= (a \wedge b)e(a \wedge b)e(a \wedge b) \\ &\le (a \wedge b)e\Big(e(a \wedge b)(a \wedge b)e\Big)e(a \wedge b) \text{ (since } S \text{ is intra-regular)} \\ &\le (a \wedge b)e(a \wedge b)(a \wedge b)e(a \wedge b) \le (aea)(beb) \le ab. \end{aligned}$$

By symmetry, we have $b \wedge a \le ba$ and so $a \wedge b \le ab \wedge ba$.
$(2) \Longrightarrow (3)$. Let $a, b$ be quasi-ideal elements of $S$. Then $a, b$ are bi-ideal elements of $S$ as well. By (2), we have $a \wedge b \le ab \wedge ba$.
$(3) \Longrightarrow (1)$. Suppose now that $S$ is an $le$-semigroup. Let $a$ be a right ideal element and $b$ a left ideal element of $S$. Then $a$ and $b$ are quasi-ideal elements of $S$. By (3), we have $a \wedge b \le ab \wedge ba \le ab, ba$. Since $a \wedge b \le ab$, by [4, Theorem 2.2], $S$ is regular. Since $a \wedge b \le ba$, by [4, Theorem 2.4], $S$ is intra-regular. □

**Remark 2.37** If $S$ is an $le$-semigroup both regular and intra-regular then, by [4, Theorem 2.2], we have $a \wedge b \leq ab$ and, by [4, Theorem 2.4], we have $a \wedge b \leq ba$ and so $a \wedge b \leq ab \wedge ba$ and the implication (1) ⇒ (2) of Theorem 2.36 is satisfied.

**Theorem 2.38** *(see also [2]) Let $S$ be a poe-semigroup at the same time semilattice under $\wedge$. If $S$ is regular then, for every right ideal element $x$, every bi-ideal element $b$ and every left ideal element $y$ of $S$, we have*

$$x \wedge b \wedge y \leq xby. \tag{2.1}$$

*Conversely, every le-semigroup having the property (2.1) is regular.*

**Proof** $\Longrightarrow$. Let $x$ be a right ideal element, $b$ a bi-ideal element and $y$ a left ideal element of $S$. Since $S$ is regular, we have

$$\begin{aligned} x \wedge b \wedge y &\leq (x \wedge b \wedge y)e(x \wedge b \wedge y) \leq \Big((x \wedge b \wedge y)e(x \wedge b \wedge y)\Big)e\Big((x \wedge b \wedge y)e(x \wedge b \wedge y)\Big) \\ &= (x \wedge b \wedge y)e\Big((x \wedge b \wedge y)e(x \wedge b \wedge y)\Big)e(x \wedge b \wedge y) \\ &\leq (xe)(beb)ey \leq xby. \end{aligned}$$

$\Longleftarrow$. Let $x \in S$. Since $r(x)$ (resp. $l(x)$) is a right (resp. left) ideal element and $e$ is a bi-ideal element of $S$, by hypothesis, we have

$$\begin{aligned} x \leq r(x) \wedge l(x) &= r(x) \wedge e \wedge l(x) \leq r(x)el(x) = (x \vee xe)e(x \vee ex) \\ &= xex \vee xe^2x \vee xe^3x = xex. \end{aligned}$$

Thus we have $x \leq xex$ for every $x \in S$, and $S$ is regular.

As a modification of the proof of Theorem 2.38 the following theorem holds. Let us prove it to be able to compare its proof with the proof of Theorem 2.38 to see that they are the same.

**Theorem 2.39** *A $\Gamma$-hypersemigroup $M$ is regular if and only if for every right ideal $X$, every left ideal $Y$ and every bi-ideal $B$ of $M$, we have $X \cap B \cap Y \subseteq X\Gamma B\Gamma Y$.*

**Proof** $\Longrightarrow$. If $X \cap B \cap Y = \emptyset$, then this clearly holds. If $X \cap B \cap Y \neq \emptyset$ then, since $M$ is regular, by [4, Proposition 3.20], we have

$$\begin{aligned} X \cap B \cap Y &\subseteq (X \cap B \cap Y)\Gamma M\Gamma(X \cap B \cap Y) \\ &\subseteq \Big((X \cap B \cap Y)\Gamma M\Gamma(X \cap B \cap Y)\Big)\Gamma M\Gamma\Big((X \cap B \cap Y)\Gamma M\Gamma(X \cap B \cap Y)\Big) \\ &= (X \cap B \cap Y)\Gamma M\Gamma\Big((X \cap B \cap Y)\Gamma M\Gamma(X \cap B \cap Y)\Big)\Gamma M\Gamma(X \cap B \cap Y) \text{ (by [4, Proposition 3.17])} \\ &\subseteq (X\Gamma M)\Gamma(B\Gamma M\Gamma B)\Gamma(M\Gamma Y) \text{ (by [4, Lemma 3.8])} \\ &\subseteq X\Gamma B\Gamma Y. \end{aligned}$$

$\Longleftarrow$. Let $\emptyset \neq X \subseteq M$. Since $R(X)$ is a right ideal, $L(X)$ a left ideal and $M$ a bi-ideal of $M$, by hypothesis, we have

$$\begin{aligned} X &\subseteq R(X)\cap L(X) = R(X)\cap M\cap L(X) = R(X)\Gamma M\Gamma L(X) \\ &= (X\cup X\Gamma M)\Gamma M\Gamma(X\cup M\Gamma X) \text{ (by [4, Proposition 3.21])} \\ &= X\Gamma M\Gamma X\cup X\Gamma M\Gamma M\Gamma X\cup X\Gamma M\Gamma M\Gamma M\Gamma X \text{ (by [4, Proposition 3.13])} \\ &= X\Gamma M\Gamma X\cup X\Gamma(M\Gamma M)\Gamma X\cup X\Gamma(M\Gamma M\Gamma M)\Gamma X \text{ (by [4, Proposition 3.17])} \\ &= X\Gamma M\Gamma X \text{ (as } M\Gamma M\subseteq M \text{ and } M\Gamma M\Gamma M\subseteq M). \end{aligned}$$

Thus we have $X\subseteq X\Gamma M\Gamma X$ for every $\emptyset\neq X\subseteq M$ and by [4, Proposition 3.20], $M$ is regular. □

**Note:** We have casually seen that Example 3.5 in [1] is not correct as $\{a\}\overline{\alpha}(b\beta c)=\{a\}\overline{\alpha}\{a\}=a\alpha a=\{a,b\}$ while $(a\alpha b)\overline{\beta}\{c\}=\{b\}\overline{\beta}\{c\}=b\beta c=\{a\}$ that is a further indication that the definition of the $\Gamma$-hypersemigroup should be corrected.

**Note:** Similar results obtained from $poe$, $\vee e$ or $le$-semigroups for an hypersemigroup also hold. It is enough to replace the "$\Gamma$" by "$*$".

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