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Polynomials taking integer values on primes in a function field

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\textbf{Abstract:} Let $F_q[x]$ be the ring of polynomials over a finite field $F_q$ and $F_q(x)$ its quotient field. Let $\mathbb{P}$ be the set of primes in $F_q[x]$, and let $\mathcal{I}$ be the set of all polynomials $f$ over $F_q(x)$ for which $f(\mathbb{P}) \subseteq F_q[x]$. The existence of a basis for $\mathcal{I}$ is established using the notion of characteristic ideal; this shows that $\mathcal{I}$ is a free $F_q[x]$-module. Through localization, explicit shapes of certain bases for the localization of $\mathcal{I}$ are derived, and a well-known procedure is described as to how to obtain explicit forms of some bases of $\mathcal{I}$.

\textbf{Key words:} Integer-valued polynomial, polynomial over finite field, prime, localization

1. Introduction

Let $D$ be an integral domain and $K$ its quotient field. Let $E$ be a subset of $D$. Denote the set of all integer-valued polynomials on $E$ by

\[ \text{Int}(E,D) := \{ f(t) \in K[t] \mid f(E) \subseteq D \}. \]

If $E = D$, write $\text{Int}(D)$ instead of $\text{Int}(D,D)$. In the classical case where $D$ is the ring of rational integers $\mathbb{Z}$, it is well-known that $\text{Int}(\mathbb{Z})$ is a free $\mathbb{Z}$-module. Indeed, for a number field $K$ with the ring of integers $\mathcal{O}_K$, one can show that $\text{Int}(\mathcal{O}_K)$ is a free $\mathcal{O}_K$-module [3, Chapter II, Section 3]. In 1997, Chabert et al. [5] proved that the set $\text{Int}(P,\mathbb{Z})$ is a free $\mathbb{Z}$-module, where $P \subseteq \mathbb{Z}$ is the set of all primes, and described an algorithm which constructs such a free basis. There is a recent work in [6] which gives an interesting application of $\text{Int}(P,\mathbb{Z})$.

Our objective here is to prove results analogous to those in [5] in the function field case. Throughout, we take the domain $D$ to be $F_q[x]$, the ring of all polynomials over $F_q$, a finite field of $q$ elements, so that its quotient field $K$ is $F_q(x)$. In [4, Section 3], it is shown that any polynomial in $F_q[x]$ is uniquely expressible with respect to a certain basis, and, consequently in [4, Theorem 9-10], any element in $\text{Int}(F_q[x])$ is uniquely expressible with respect to such basis with suitably chosen coefficients, and so $\text{Int}(F_q[x])$ is a free $F_q[x]$-module.

Let $\mathbb{P}$ be the set of all monic irreducible polynomials, i.e., primes, in $F_q[x]$. Using the idea of characteristic ideal, we prove in the next section that the set $\mathcal{I} := \text{Int}(\mathbb{P},F_q[x])$ is a free $F_q[x]$-module. In the third section,
we establish certain properties of the module \( I \). In the fourth section, the localization \( I_{(p)} \) \( (p \in \mathbb{P}) \) of \( I \) is investigated, and employing the notion of \( p \)-ordering in \([2]\), a basis for the module \( I_{(p)} \) is derived. Globalizing the localized bases, a procedure to compute explicit bases of the module \( I \) is described in the last section.

2. Existence of a basis

We first introduce the notion of characteristic ideal, \([3, \text{Chapter II, Section 1}]\). For brevity, let \( N_0 := \mathbb{N} \cup \{0\} \).

**Definition 2.1** Let \( B \) be a domain such that \( \mathbb{F}_q[x][t] \subseteq B \subseteq \mathbb{F}_q(x)[t] \). For \( n \in N_0 \), we define the set \( I_B(n) \) to be the union of the element \( 0 \in \mathbb{F}_q \) and the set of leading coefficients of all polynomials in \( B \) of degree \( n \):

\[
I_B(n) = \{0\} \cup \{A \in \mathbb{F}_q(x) \setminus \{0\} \mid \exists f \in B \text{ such that } f = At^n + A_{n-1}t^{n-1} + \cdots + A_0\}.
\]

One can show that if \( B \subseteq \text{Int(}\mathbb{F}_q[x])\), then \( I_B(n) \) is a fractional ideal of \( \mathbb{F}_q[x] \), \([3, \text{Proposition II.1.1}]\), so that there exists \( d(x) \in \mathbb{F}_q[x] \setminus \{0\} \) such that \( d(x)I_B(n) \subseteq \mathbb{F}_q[x] \). We call \( I_B(n) \) the \( n \)-th characteristic ideal of \( B \).

To prove the existence of a basis for \( I \), we make use of the following result, \([3, \text{Proposition II.1.4}]\).

For a domain \( B \) such that \( \mathbb{F}_q[x][t] \subseteq B \subseteq \mathbb{F}_q(x)[t] \), we note that, \([3, \text{Definition II.1.3}]\), a basis \( \{f_n\}_{n \geq 0} \) of the \( \mathbb{F}_q[x] \)-module \( B \) is said to be a regular basis if, for each \( n \), the polynomial \( f_n \) has degree \( n \).

**Proposition 2.2** Let \( B \) be a domain such that \( \mathbb{F}_q[x][t] \subseteq B \subseteq \mathbb{F}_q(x)[t] \). A sequence \( \{f_n\}_{n \geq 0} \subseteq B \) is a regular basis of \( B \) if and only if, for each \( n \in N_0 \), the polynomial \( f_n \) is of degree \( n \) whose leading coefficient generates \( I_B(n) \) as an \( \mathbb{F}_q[x] \)-module.

Taking \( B \) to be \( I = \text{Int}(\mathbb{P}, \mathbb{F}_q[x]) \) which is easily checked to be a domain, the set \( I_I(n) \) has a simple structure.

**Proposition 2.3** The set \( I_I(n) \) is a principal fractional ideal of \( \mathbb{F}_q[x] \).

**Proof** By \([3, \text{Proposition II.1.1}]\), the set \( I_I(n) \) is a fractional ideal of \( \mathbb{F}_q[x] \). To check that it is principal, let \( f(t) = A_0 + A_1t + \cdots + A_nt^n \in I, \quad A_n \neq 0. \)

For \( p_i \in \mathbb{P} \) with \( \deg p_i = i \) \( (1 \leq i \leq n + 1) \), we have

\[
\begin{bmatrix}
A_0 \\
A_1 \\
\vdots \\
A_n
\end{bmatrix} = \begin{bmatrix}
f(p_1) \\
f(p_2) \\
\vdots \\
f(p_{n+1})
\end{bmatrix}, \quad \text{where } A_n := \begin{bmatrix}
1 & p_1 & \cdots & p_1^n \\
1 & p_2 & \cdots & p_2^n \\
\vdots & \vdots & \ddots & \vdots \\
1 & p_{n+1} & \cdots & p_{n+1}^n
\end{bmatrix}.
\]

Since \( A_n \) is a Vandermonde matrix, we have \( \det A_n = \prod_{1 \leq i < j \leq n+1} (p_j - p_i) \in \mathbb{F}_q[x] \setminus \{0\} \) yielding

\[
A_n \det A_n = \begin{bmatrix}
1 & p_1 & \cdots & p_1^{n-1} & f(p_1) \\
1 & p_2 & \cdots & p_2^{n-1} & f(p_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & p_{n+1} & \cdots & p_{n+1}^{n-1} & f(p_{n+1})
\end{bmatrix} \in \mathbb{F}_q[x],
\]

and so \( I_I(n) \) \( \det A_n \subseteq \mathbb{F}_q[x] \). Next, it is easily checked that \( I_I(n) \) \( \det A_n \) is an ideal of \( \mathbb{F}_q[x] \). Since \( \mathbb{F}_q[x] \) is a principal ideal domain, \( I_I(n) \) \( \det A_n \) is a principal ideal implying that \( I_I(n) \) is a principal fractional ideal. \( \square \)
Remarks. In passing, we make some observations about the shape of each $I_2(n)$. Clearly, the sequence \( \{I_2(n)\}_{n \geq 0} \) is increasing with respect to set inclusion. For degree 0, the set of all polynomials of degree 0 in $I$ is $F_q[x]$. For degree 1, consider $f(t) = A_0 + A_1 t \in I$; its leading coefficient is $A_1 = f(x + 1) - f(x) \in F_q[x]$. So, $I_2(0) = I_2(1) = F_q[x]$. In general, for degree $n \geq 2$, the set $I_2(n)$ depends on $q$, the cardinality of $F_q$. For example, if $q = 2$, we have $1/x \in I_2(2)$ because $f(t) = t(t - 1)/x \in \text{Int}(F, F_2[x])$, but if $q > 2$, it is checked that $1/x \notin I_2(2)$; for otherwise, $1/x \in I_2(2)$. Then there exists

$$f(t) = \frac{1}{x} t^2 + \frac{a(x)}{b(x)} t + \frac{c(x)}{d(x)} \in I \ (a(x), b(x), c(x), d(x) \in F_q[x] \text{ and } b(x), d(x) \neq 0).$$

Substituting $t = x$, $x + 1$, $x + \alpha \in F$ with $\alpha \in F_q \setminus \{0, 1\}$, we have $f(x)$, $f(x + 1)$, $f(x + \alpha) \in F_q[x]$. It follows that $\frac{\alpha(a - 1)}{x} = f(x + \alpha) - \alpha f(x + 1) + (\alpha - 1)f(x) \in F_q[x]$, which is a contradiction.

From Proposition 2.3, we know that $I_2(n)$ is a principal fractional ideal so that for each $n \in \mathbb{N}_0$, there is $a_n(x)/b_n(x) \in F_q(x) \setminus \{0\}$ such that $I_2(n) = \frac{a_n(x)}{b_n(x)} F_q[x]$. Thus, $\frac{b_n(x)}{a_n(x)} I_2(n) = F_q[x] = I_2(0) \subseteq I_2(n)$ showing that $b_n(x)/a_n(x) \in F_q[x]$, and so there exists $d_n \in F_q[x]$ such that

$$I_2(n) = d_n^{-1} F_q[x]. \quad (2.1)$$

We are now ready for our first main theorem.

**Theorem 2.4** The set $I$ is a free $F_q[x]$-module with a regular basis.

**Proof** From (2.1), let $I_2(n) = d_n^{-1} F_q[x] \ (n \in \mathbb{N}_0)$. Let $B = \{f_n(t)\}_{n \in \mathbb{N}_0} \subseteq I$ be a sequence of polynomials such that $\deg f_n(t) = n$ and the leading coefficient of $f_n(t)$ is $1/d_n$. By Proposition 2.2, the set $B$ forms a regular basis of $I$. \hfill \Box

3. Auxiliary results

We shall make use of the following function field analogue of Dirichlet’s theorem taken from [9, Theorem 4.8].

**Lemma 3.1** For $a(x), b(x) \in F_q[x]$, if $\gcd(a(x), b(x)) = 1$, then there exists $c(x) \in F_q[x]$ such that $a(x) + b(x)c(x) \in F$ has degree $n$ for a sufficiently large $n \in \mathbb{N}$.

We next describe a classical algorithm to construct a basis for $\text{Int}(F_q[x])$.

Set $F_q := \{a_0 = 0, a_1, \ldots, a_q-1\}$. For $n \in \mathbb{N}$, $n \geq q$, with its unique base-$q$ representation being

$$n = n_0 + n_1 q + \cdots + n_s q^s \quad (0 \leq n_i < q, \ n_s \neq 0),$$

define

$$a_n = a_{n_0} + a_{n_1} x + \cdots + a_{n_s} x^s.$$  

The sequence $\{a_n\}_{n \geq 0}$ so defined is referred to as a polynomial ordering with respect to the base $x$. From the sequence $\{a_n\}_{n \geq 0}$, define the Lagrange type interpolation polynomials $\{C_n(t)\}_{n \geq 0}$ by

$$C_0(t) = 1, \quad C_n(t) = \prod_{i=0}^{n-1} \frac{t - a_i}{a_n - a_i} \quad (n \geq 1). \quad (3.1)$$
The sequence \( \{C_n(t)\}_{n \geq 0} \) forms a basis for \( \text{Int}(F_q[x]) \) as a \( F_q[x] \)-module, [3, Chapter II, Section 2].

For each \( \ell \in \mathbb{P} \), define the \( \ell \)-adic ordinal \( \nu_\ell : F_q[x] \rightarrow \mathbb{R} \cup \{ \infty \} \) by
\[
\nu_\ell(0) = \infty, \quad \nu_\ell(a) = \alpha \quad \text{for} \quad a = \ell^\alpha \cdot f \quad (\alpha \in \mathbb{N}, \ f \in F_q[x], \ \ell \not| f),
\]
and extend it to \( F_q(x) \) in the usual manner. Connecting the sequence \( \{C_n(t)\}_{n \geq 0} \) with Lemma 3.1, we get:

**Proposition 3.2** Let \( \{C_n(t)\}_{n \geq 0} \) be a sequence of polynomials as defined in (3.1), and for a given \( n \in \mathbb{N}_0 \), let \( A_n \in F_q(x) \) be such that \( A_nC_n(t) \in I \). If \( n \not\equiv -1 \pmod{q} \) then, \( A_n \in F_q[x] \).

**Proof** Note that if
\[
C_n(t) = \frac{u_n(x)}{v_n(x)} t^n + \frac{u_{n-1}(x)}{v_{n-1}(x)} t^{n-1} + \ldots + \frac{u_0(x)}{v_0(x)},
\]
where \( u_i(x), v_i(x) \in F_q[x] \), then \( \text{LCM}(v_0, \ldots, v_0)C_n(t) \in F_q[x][t] \), which assures the existence of \( A_n \). Let \( \ell \in \mathbb{P} \). Set \( r = 1 + \sup_{0 \leq i < n} \{ \nu_\ell(a_n - a_i) \} \). If \( \ell \not| a_n \), then, by Lemma 3.1, there exists \( p = a_k\ell^r + a_n \in \mathbb{P} \) for some \( a_k \in F_q[x] \). Then,
\[
C_n(p) = \prod_{i=0}^{n-1} \frac{p - a_i}{a_n - a_i} = \prod_{i=0}^{n-1} \frac{a_k\ell^r + a_n - a_i}{a_n - a_i}.
\]

Since \( \nu_\ell(a_k\ell^r) > \nu_\ell(a_n - a_i) \), we have \( \nu_\ell(a_k\ell^r + a_n - a_i) = \nu_\ell(a_n - a_i) \) for \( 0 \leq i < n \), and so \( \nu_\ell(C_n(p)) = 0 \).

On the other hand, if \( \ell | a_n \), let \( a'_n = a_n - a_{n_0} + a_{q-1} \) where \( n_0 \) is first digit in the \( q \)-adic expansion of \( n = n_0 + n_1q + \cdots + n_sq^s \). Let
\[
r' = 1 + \sup_{0 \leq i < n} \{ \nu_\ell(a'_n - a_i) \}.
\]

Since \( -a_{n_0} + a_{q-1} \in F_q \), we have \( \ell \not| a'_n \). By Lemma 3.1, there exists \( p = a_k\ell^{r'} + a'_n \in \mathbb{P} \) for some \( a_k \in F_q[x] \).

If \( n/q > 1 \), then for \( 0 \leq j < \lfloor n/q \rfloor \) we have
\[
\prod_{i=jq}^{(j+1)q-1} (a_n - a_i) = \prod_{i=jq}^{(j+1)q-1} (a'_n - a_i);
\]  

(3.2)

because \( a_n \) and \( a'_n \) differ only by a constant term and both products in (3.2) run over a complete residue class modulo \( x \). Since \( n \not\equiv -1 \pmod{q} \), \( a'_n - a_{n_0} + a_{q-1} \neq a_i \) for all \( 0 \leq i < n \). Note that \( a'_n - a_i \neq 0 \). As \( \nu_\ell(a_k\ell^{r'}) > \nu_\ell(a'_n - a_i) \) for all \( 0 \leq i < n \),
\[
\nu_\ell \left( \prod_{i=jq}^{(j+1)q-1} (a_k\ell^{r'} + a'_n - a_i) \right) = \nu_\ell \left( \prod_{i=jq}^{(j+1)q-1} (a'_n - a_i) \right) = \nu_\ell \left( \prod_{i=jq}^{(j+1)q-1} (a_n - a_i) \right).
\]

Moreover, \( a_n - a_i \) and \( a'_n - a_i \) are elements in \( F_q \setminus \{0\} \) for all \( n - n_0 \leq i < n \). This implies that
\[
\nu_\ell \left( \prod_{i=n-n_0}^{n-1} (a_k\ell^{r'} + a'_n - a_i) \right) = \nu_\ell \left( \prod_{i=n-n_0}^{n-1} (a'_n - a_i) \right) = 0 = \nu_\ell \left( \prod_{i=n-n_0}^{n-1} (a_n - a_i) \right).
\]
Thus,
\[ \nu_\ell(C_n(p)) = \nu_\ell \left( \prod_{i=0}^{n-1} \frac{a_i \ell^r + a'_i}{a_n - a_i} \right) = 0. \]
Since \( \nu_\ell(A_n C_n(p)) \geq 0 \) for all \( \ell \in \mathbb{P} \).
Since Proposition 3.2 indicates that the sequence \( \{C_n(t)\}_{n \geq 0} \) is a basis of \( \mathcal{I} \) over \( \mathbb{F}_q[x] \), and since the coefficients of each \( C_n(t) \) are of certain special kind, it is then natural to ask for properties that the coefficients of any basis element must have, and this is answered in the next proposition.

**Proposition 3.3** Let \( n \in \mathbb{N} \) and \( \{m_1, m_2, \ldots, m_n\} \subseteq \mathbb{F}_q[x] \). If \( A \in \mathbb{F}_q(x) \) is such that \( A \prod_{i=1}^{n}(t - m_i(x)) \in \mathcal{I} \), then for all \( \ell \in \mathbb{P} \) with \( \deg \ell > \log_q n \), we have \( \nu_\ell(A) \geq 0 \).

**Proof** Let \( \ell \in \mathbb{P} \) be such that \( \deg \ell > \log_q n \). We consider two possible cases.
Case 1: \( \nu_\ell(m_i) = 0 \) for all \( 1 \leq i \leq n \). Then \( \nu_\ell(A) = \nu_\ell(A(\ell - m_1) \cdots (\ell - m_n)) \geq 0 \).
Case 2: there exists \( m_j \) for some \( 1 \leq j \leq n \) such that \( \nu_\ell(m_j) \geq 1 \). Since the number of elements in \( \mathbb{F}_q[x]/(\ell) \) is \( q^{\deg \ell} > n \), there exists \( s \in \mathbb{F}_q[x] \) such that \( \nu_\ell(s - m_j) = 0 \) \((1 \leq i \leq n)\). As \( \nu_\ell(m_j) \geq 1 \) and \( \nu_\ell(s - m_j) = 0 \), we obtain that \( \nu_\ell(s) = 0 \). By Lemma 3.1, there exists \( p = k\ell + s \in \mathbb{P} \) for some \( k \in \mathbb{F}_q[x] \). So,
\[ 0 \leq \nu_\ell(A(p - m_1) \cdots (p - m_n)) = \nu_\ell(A(k_0 \ell + s - m_1) \cdots (k_0 \ell + s - m_n)) = \nu_\ell(A). \]

4. Localization

For each \( p \in \mathbb{P} \), set
- \( X_p = (\mathbb{F}_q[x] \setminus p\mathbb{F}_q[x]) \cup \{p\} \)
- \( \mathbb{F}_q[x]_{(p)} = \{a/b \in \mathbb{F}_q(x) | a \in \mathbb{F}_q[x] \text{ and } b \in \mathbb{F}_q[x] \setminus p\mathbb{F}_q[x]\} \)
- \( \text{Int}(X_p, \mathbb{F}_q[x]_{(p)}) = \{f(t) \in \mathbb{F}_q(x)[t] | f(X_p) \subseteq \mathbb{F}_q[x]_{(p)}\} \).

The set \( \mathbb{F}_q[x]_{(p)} \) is called the localization of \( \mathbb{F}_q[x] \) at the prime \( p \). The next proposition gives a basic result connecting \( \mathcal{I} \) with \( \mathbb{F}_q[x]_{(p)} \).

**Proposition 4.1** Let \( f(t) \in \mathcal{I}, p \in \mathbb{P} \) and \( h \in \mathbb{F}_q[x] \). If \( p \nmid h \), then \( f(h) \in \mathbb{F}_q[x]_{(p)} \).

**Proof** Let \( d \in \mathbb{F}_q[x] \setminus \{0\} \) be such that \( df(t) \in \mathbb{F}_q[x][t] \). Then, \( d = p^{\nu_p(d)} e \) where \( p \nmid e \). Assume that \( p \nmid h \). By Lemma 3.1, there exists \( r = kp^{\nu_p(d)} + h \in \mathbb{P} \) for some \( k \in \mathbb{F}_q[x] \). Since \( df(t) \in \mathbb{F}_q[x][t] \), we can write
\[ df(t) = u_n t^n + u_{n-1} t^{n-1} + \cdots + u_0 \quad (u_i \in \mathbb{F}_q[x]). \]
Thus,
\[ d(f(r) - f(h)) = u_n (r^n - h^n) + u_{n-1} (r^{n-1} - h^{n-1}) + \cdots + u_1 (r - h) \in (r - h)\mathbb{F}_q[x], \]
and so \( cp^{\nu_p(d)}(f(r) - f(h)) \in kp^{\nu_p(d)} \mathbb{F}_q[x] \). This implies that \( f(r) - f(h) \in e^{-1} \mathbb{F}_q[x] \). Since \( e \not\in p\mathbb{F}_q[x] \), we have \( f(r) - f(h) \in \mathbb{F}_q[x]_{(p)} \). As \( f(r) \in \mathbb{F}_q[x] \subseteq \mathbb{F}_q[x]_{(p)} \), the desired result follows. \( \square \)
Proposition 4.2 Let \( p := p(x) \in \mathbb{P} \) and define

\[
\mathcal{I}(p) := \text{Int}(\mathbb{P}, F_q[x](p)) = \{ f(t)/b(x) \in F_q(x)[t] \mid f(t) \in \mathcal{I}, \ b(x) \in X_p \setminus \{ p \} \}.
\]

Then \( \mathcal{I}(p) = \text{Int}(X_p, F_q[x](p)) \).

Proof We first show that \( \mathcal{I} \subseteq \text{Int}(X_p, F_q[x](p)) \). Let \( f(t) \in \mathcal{I} \). Then \( f(p) \in F_q[x] \subseteq F_q[x](p) \), and by Proposition 4.1, \( f(h) \in F_q[x](p) \) for all \( h \in F_q[x] \setminus pF_q[x] \). Thus, \( \mathcal{I} \subseteq \text{Int}(X_p, F_q[x](p)) \). This inclusion and [3, Proposition I.2.7] show that \( \mathcal{I}(p) \subseteq \text{Int}(X_p, F_q[x](p)) \).

Since \( \mathbb{P} \subseteq X_p \), we have \( \text{Int}(X_p, F_q[x](p)) \subseteq \text{Int}(\mathbb{P}, F_q[x](p)) \). By [3, Proposition I.2.7] again, we obtain \( \text{Int}(\mathbb{P}, F_q[x](p)) = \mathcal{I}(p) \). Thus, \( \text{Int}(X_p, F_q[x](p)) \subseteq \mathcal{I}(p) \). \( \square \)

4.1. Localized bases

Throughout this subsection, let \( p := p(x) \) be a prime of degree \( d \). Let

\[
\{ c_0^{(p)} = 0, c_1^{(p)}, \ldots, c_{q^d-1}^{(p)} \}
\]

be a complete set of residue classes of \( \mathbb{F}_q[x] \) modulo \( p \). Define the sequence \( \{ c_n^{(p)} \}_{n \geq q^d} \) by

\[
c_n^{(p)} = c_n^{(p)} + c_1^{(p)} p + \cdots + c_n^{(p)} p^n,
\]

where \( n = n_0 + n_1 q^d + \cdots + n_s q^{ds} \geq q^d \) is the base-\( q^d \) representation of \( n \). Define a corresponding sequence \( \{ b_n^{(p)} \}_{n \geq 0} \subseteq F_q[x] \) by

\[
b_0^{(p)} = p, \quad b_n^{(p)} = c_n^{(p)} + \frac{n}{n+\left \lfloor \frac{n-1}{q^d-1} \right \rfloor} \quad (n \geq 1).
\]

For brevity, throughout this section, since \( p \) is fixed, we replace \( c_n^{(p)} \) by \( c_n \) and \( b_n^{(p)} \) by \( b_n \). For ease of comprehension, we list all the elements in the sequence \( \{ c_n \}_{n \geq 0} \) in the following ordering manner.

\[
\begin{array}{cccccc}
0 & c_1 & \cdots & c_{q^d-1} \\
\hline
 \frac{c_{q^d} = c_1 p}{c_{q^d+1} = c_1 p + c_1} & \cdots & c_{q^d-1} = c_1 p + c_{q^d-1} \\
\frac{c_{q^d+1} = c_2 p}{c_{q^d+2} = c_2 p + c_1} & \cdots & c_{q^d+q^d-1} = c_2 p + c_{q^d-1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{c_{(q^d-1)q^d} = c_{q^d-1} p}{c_{(q^d-1)q^d+1} = c_{q^d-1} p + c_1} & \cdots & c_{(q^d-1)q^d-1} = c_{q^d-1} p + c_{q^d-1} \\
\frac{c_{q^d+1} = c_1 p^2}{c_{q^d+q^d+1} = c_1 p^2 + c_1} & \cdots & c_{q^d+q^d+q^d-1} = c_1 p^2 + c_{q^d-1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{c_{(q^d-1)q^d+(q^d-1)q^d} = c_{(q^d-1)q^d+(q^d-1)q^d+1}}{c_{(q^d-1)q^d+q^d-1} = c_{q^d-1} p^2 + c_{q^d-1} p + c_1} & \cdots & c_{q^d-1} p^2 + c_{q^d-1} p + c_{q^d-1} \\
\vdots & \ddots & \ddots & \ddots \\
\end{array}
\]

From the array, the first line contains all complete residue classes modulo \( p \), while the second and third parts separated by the following two lines contain all complete residue classes modulo \( p^2 \) and \( p^3 \), respectively.
Observe that when we delete the first column of the array, the remaining entries give an ordering of the sequence \( \{b_n\}_{n \geq 1} \).

The sequence \( \{b_n\}_{n \geq 1} \) contains all polynomials which are relatively prime to \( p \). To see this, assume that there exists \( n \in \mathbb{N} \) such that \( b_n = c_{n+(n-1)/(q^d-1)} \) is a multiple of \( p \). This implies that the subscript belongs to the first column of the array, i.e. \( n + [(n-1)/(q^d-1)] = q^d \cdot m \) for some \( m \in \mathbb{N} \). If \( q^d - 1 \mid n \), then \( n = (q^d - 1)\ell \) for some \( \ell \in \mathbb{N} \). We then have \( \ell - 1 = [(n-1)/(q^d-1)] = q^d m - (q^d - 1)\ell \), and so \( q^d \ell = q^d m + 1 \), a contradiction. If \( q^d - 1 \nmid n \), then \( n = (q^d - 1)\ell' + r \) for some \( \ell' \in \mathbb{N} \) and \( r \in \{1, 2, \ldots, q^d - 2\} \). Thus, \( \ell' = [(n-1)/(q^d - 1)] \) is a multiple of \( q^d m - r \), a contradiction.

A useful characterization of the sequence \( \{b_n\}_{n \geq 0} \) is given in the next proposition. To do so, define

\[
w_p(0) = 0, \quad w_p(n) = \sum_{i \geq 0} \left\lfloor \frac{n - 1}{q^d(q^d - 1)} \right\rfloor (n \geq 1). \tag{4.3}
\]

**Proposition 4.3** For each \( n > 0 \), we have \( \nu_p \left( \prod_{k=0}^{n-1} (b_n - b_k) \right) = w_p(n) \).

**Proof** Since \( b_n \not\equiv b_0 = p \mod p \), we have \( \nu_p(b_n - b_0) = 0 \). This implies that

\[
\nu_p \left( \prod_{i=0}^{n-1} (b_n - b_i) \right) = \nu_p \left( \prod_{i=1}^{n-1} (b_n - b_i) \right). \tag{4.4}
\]

Note that the sequence \( \{b_i\}_{1 \leq i \leq q^d-1} \) contains all residue classes modulo \( p \) except the class 0. Moreover, for each \( i \in \{1, \ldots, q^d - 1\} \), we have

\[
b_i = c_i \equiv c_{i+rq^d} = b_{i+r(q^d-1)} \mod p \quad (r \in \mathbb{N}_0).
\]

This implies that the \( \{b_i\}_{1+r(q^d-1) \leq i \leq (r+1)(q^d-1)} \) also contains all residue classes modulo \( p \) except the class 0. Thus, the number of factors \( (b_n - b_i) \) in \( (4.4) \) satisfying \( \nu_p(b_n - b_i) \geq 1 \) \( (1 \leq i \leq n-1) \) is \( [(n-1)/(q^d - 1)] \). In general, consider the case of the modulo \( p^m \) \( (m \in \mathbb{N}) \). We first observe that \( \{c_i\}_{0 \leq i \leq q^d-1} \) is a complete set of residue classes modulo \( p^m \). Since \( \{c_0, c_{q^d}, \ldots, c_{(q^d-1)q^d-1}\} \) is the set of all elements in \( \{c_i\}_{0 \leq i \leq q^d-1} \) each of which is a multiple of \( p \), we deduce that there are \( q^{md} - q^{(m-1)d} = q^{(m-1)d}(q^d - 1) \) elements in \( \{c_i\}_{0 \leq i \leq q^d-1} \) that are relatively prime to \( p \) which in turn shows that \( \{b_i\}_{1 \leq i \leq q^d-1} \) contains all residue classes which are relatively prime to \( p \) modulo \( p^m \). Note also that, for each \( i \in \{1, 2, \ldots, q^{d(m-1)}(q^d - 1)\} \) and \( r \in \mathbb{N} \), modulo \( p^m \) we have

\[
b_i = c_{i+\left\lfloor \frac{i-1}{q^d(q^d - 1)} \right\rfloor} = c_{i+\left\lfloor \frac{i-1}{q^d(q^d - 1)} \right\rfloor + rq^{md}} = c_{i+rq^{md}} = c_{i+rq^{md} + \left\lfloor \frac{i-1}{q^{(m-1)d}(q^d - 1)} \right\rfloor} = b_{i+rq^{md}}.
\]

It follows that \( \{b_i\}_{1+r(q^{md}-1) \leq i \leq (r+1)q^{md}(q^d - 1)} \) contains all residue classes relatively prime to \( p \) modulo \( p^m \). Thus, the number of factors \( (b_n - b_i) \) in \( (4.4) \) such that \( \nu_p(b_n - b_i) \geq m \) \( (1 \leq i \leq n-1) \) for all \( m \in \mathbb{N} \) is
equal to \( \left\lfloor (n-1)/q^{(m-1)d(q^d-1)} \right\rfloor \), and so

\[
\nu_p \left( \prod_{k=0}^{n-1} (b_n - b_k) \right) = \sum_{k \geq 0} \frac{n-1}{q^{dk(q^d-1)}} = w_p(n).
\]

\(\square\)

Now, we recall the notion of \(p\)-ordering due to Bhargava and one of its important consequences, \([1], [2], [7], [8]\).

**Definition 4.4** A sequence \(\{s_n\}_{n \geq 0}\) of elements of \(X_p\) is a \(p\)-ordering of \(X_p\), if for all \(a \in X_p\), one has

\[
\nu_p \left( \prod_{i=0}^{n-1} (s_n - s_i) \right) \leq \nu_p \left( \prod_{i=0}^{n-1} (a - s_i) \right) \quad (n \geq 1),
\]

and \(s_0\) can be chosen arbitrarily in \(X_p\).

**Proposition 4.5** \([7, \text{Section 4.1, Obs 1}]\) Let \(p \in \mathbb{P}\). Any two \(p\)-orderings \(\{s_n\}_{n \geq 0}\) and \(\{s'_n\}_{n \geq 0}\) of \(X_p\) result in the same minimal condition:

\[
\nu_p \left( \prod_{i=0}^{n-1} (s_n - s_i) \right) = \nu_p \left( \prod_{i=0}^{n-1} (s'_n - s'_i) \right) \quad (n \geq 1).
\]

Our sequence \(\{b_n\}_{n \geq 0}\) so constructed in (4.2) above is indeed a \(p\)-ordering as we now verify.

**Proposition 4.6** The sequence \(\{b_n\}_{n \geq 0}\) is a \(p\)-ordering of \(X_p\).

**Proof** Let \(a \in X_p\). Since \(\{b_n\}_{n \geq 0}\) contains the prime \(p\) and all polynomials which are relatively prime to \(p\) and \(X_p = (\mathbb{F}_q[x] \cup \mathbb{P}) \cup \{p\}\), we have \(a = b_m\) for some \(m \geq 0\). Let \(n \in \mathbb{N}\). There are three possible cases.

Case 1: \(m < n\). Then \(\prod_{i=0}^{n-1} (a - b_i) = 0\) and so \(\nu_p \left( \prod_{i=0}^{n-1} (b_n - b_i) \right) < \nu_p \left( \prod_{i=0}^{n-1} (a - b_i) \right)\).

Case 2: \(m = n\). Then \(\prod_{i=0}^{n-1} (a - b_i) = \prod_{i=0}^{n-1} (b_n - b_i)\).

Case 3: \(m > n\). Since the elements \(b_0, b_1, \ldots, b_{n-1}\) precede the element \(a = b_m\) in the above array, we deduce that the number of \(i\) such that \(\nu_p(b_n - b_i) \geq r\) \((1 \leq i \leq n-1)\) is not more than the number of \(j\) such that \(\nu_p(a - b_j) \geq r\) \((1 \leq j \leq n-1)\), and so \(\nu_p \left( \prod_{i=0}^{n-1} (b_n - b_i) \right) \leq \nu_p \left( \prod_{i=0}^{n-1} (a - b_i) \right)\) and the result follows from Definition 4.4. \(\square\)

Combining Propositions 4.3, 4.5, and 4.6, we immediately obtain:

**Proposition 4.7** A sequence \(\{s_n\}_{n \geq 0}\) of elements of \(X_p\) is a \(p\)-ordering if and only if

\[
w_p(n) = \nu_p \left( \prod_{i=0}^{n-1} (s_n - s_i) \right) \quad (n \geq 1).
\]
We are now ready to construct explicit bases of the localized module $\mathcal{I}(p)$ defined in (4.1). For each $p \in \mathbb{P}$, define the polynomials $C_p^b(t)$ and $G_p^b(t)$ by

$$C_p^b(t) = 1, \quad C_p^n(t) = \frac{1}{p^{\nu_p(n)}} \prod_{0 \leq i < n} (t - b_i) \quad (n \geq 1), \quad (4.6)$$

and

$$G_p^b(t) = 1, \quad G_p^n(t) = \prod_{0 \leq i < n} \frac{t - b_i}{b_n - b_i} \quad (n \geq 1). \quad (4.7)$$

**Theorem 4.8** The sequences $\{C_p^n(t)\}_{n \geq 0}$ and $\{G_p^n(t)\}_{n \geq 0}$ are two regular bases of the $\mathbb{F}_q[x]_{(p)}$-module $\mathcal{I}(p)$.

**Proof** By (4.6) and (4.7), $C_p^n(p) = 0 = G_p^n(p)$. By Proposition 4.3 and the inequality (4.5), $C_p^n(a)$ and $G_p^n(a)$ are in $\mathbb{F}_q[x]_{(p)}$ for all $a \in X_p$. It follows that $C_p^n(t)$ and $G_p^n(t)$ belong to $\text{Int}(X_p, \mathbb{F}_q[x]_{(p)})$.

Let $f(t) \in \text{Int}(X_p, \mathbb{F}_q[x]_{(p)})$ with degree $n$. Then, we can write

$$f(t) = \sum_{i=0}^{n} \alpha_i C_i^p(t) = \sum_{i=0}^{n} \gamma_i G_i^p(t) \quad (\alpha_i, \gamma_i \in \mathbb{F}_q(x)). \quad (4.8)$$

It remains to show that $\alpha_i, \gamma_i \in \mathbb{F}_q[x]_{(p)}$. To do so, consider $C_i^p(b_k)$ and $G_i^p(b_k)$. For fixed $i \in \{1, \ldots, n\}$, we have

$$C_i^p(b_k) = 0 = G_i^p(b_k) \quad (0 \leq k < i).$$

Since $G_i^p(b_i) = 1$ and $\nu_p(C_i^p(b_i)) = 0$, we have $(G_i^p(b_i))^{-1}$ and $(C_i^p(b_i))^{-1} \in \mathbb{F}_q[x]_{(p)}$. Substituting $t = b_0, b_1, b_2, \ldots$, we get

$$\alpha_0 = f(b_0) \in \mathbb{F}_q[x]_{(p)}, \quad \alpha_1 = (C_i^p(b_i))^{-1}(f(b_1) - \alpha_0) \in \mathbb{F}_q[x]_{(p)}, \quad \alpha_2 = (C_i^p(b_2))^{-1}(f(b_2) - \alpha_0 - \alpha_1 C_i^p(b_2)) \in \mathbb{F}_q[x]_{(p)}.$$  

Continuing this process, we obtain that $\alpha_i \in \mathbb{F}_q[x]_{(p)}$ for all $i$.

The proof for $\{G_i^p(t)\}_{i \geq 0}$ is similar. \hfill $\square$

The next theorem provides a method of obtaining a new basis from an old one.

**Theorem 4.9** Let $p \in \mathbb{P}$, $n \in \mathbb{N}$, and let $\{b_i\}_{i \geq 0}$ be the sequence defined in (4.2). For fixed $j \in \mathbb{N}_0$ with $0 \leq j < n$, define

$$m = m(p, j, n) := \begin{cases} \max \{\nu_p(b_k - b_j) \mid 0 \leq k \leq n, \ k \neq j\}, & 0 < j < n, \\ 0, & j = 0 \end{cases}.$$  

Let $C_n^p(t)$ be the $n$-th polynomial as defined in (4.6) in the regular basis $\mathcal{B} = \{C_n^p(t)\}_{n \geq 0}$ of the $\mathbb{F}_q[x]_{(p)}$-module $\mathcal{I}(p)$. If $\beta_j \in X_p$ satisfies $\nu_p(\beta_j - b_j) > m$, then the set

$$\mathcal{C} := \left( \mathcal{B} \setminus \{C_n^p(t)\} \right) \cup \left( \frac{t - \beta_j}{t - b_j} \cdot C_n^p(t) \right)$$

is also a regular basis of $\mathbb{F}_q[x]_{(p)}$-module $\mathcal{I}(p)$.
Proof To show that \( C \) is a regular basis whose \( n \)-th polynomial is

\[
C'_n := \frac{t - \beta_j}{t - b_j}, \quad C'_n(p) = \frac{t - \beta_j}{p^{w_p(n)}} \prod_{0 \leq \ell < n \atop \ell \neq j} (t - b_\ell),
\]

by Proposition 2.2, we need to check two assertions (i) \( C'_n(t) \in \mathcal{I}_{(p)} \) and (ii) the leading coefficient of the polynomial \( C'_n(t) \) generates the the \( n \)-th characteristic ideal \( I_\mathcal{I}(n) \).

As \( \nu_p(b_i - \beta_j) > m \), we can write \( \beta_j = b_j + e p^{m+1} \) for some \( e \in \mathbb{F}_q[x] \). For each \( 0 \leq k \leq n \) and \( j \neq k \), we get

\[
\nu_p(b_k - \beta_j) = \nu_p(b_k - b_j - p^{m+1}e) = \min \{ \nu_p(b_k - b_j), \nu_p(p^{m+1}e) \} = \nu_p(b_k - b_j).
\] (4.9)

Consider the sequence \( \{b_0, b_1, \ldots, b_{j-1}, \beta_j, b_{j+1}, \ldots, b_n\} \). Since \( \{b_0, b_1, \ldots, b_{j-1}\} \) contains the first \( j \) elements of some \( p \)-ordering of \( X_p \), it remains to show that \( \{\beta_j, b_{j+1}, \ldots, b_n\} \) is the set of succeeding elements in a \( p \)-ordering. By (4.9), we have

\[
\nu_p \left( \prod_{0 \leq \ell < j - 1} (\beta_j - b_\ell) \right) = \nu_p \left( \prod_{0 \leq \ell < j - 1} (b_j - b_\ell) \right) = w_p(j),
\]

and

\[
\nu_p \left( \prod_{0 \leq \ell < j - 1 \atop \ell \neq j} (b_\ell - b_i) \right) + \nu_p(b_k - \beta_j) = \nu_p \left( \prod_{0 \leq \ell < j - 1 \atop \ell \neq j} (b_\ell - b_i) \right) = w_p(\ell) \quad (i < \ell \leq n).
\]

By Proposition 4.7, the set \( \{b_0, b_1, \ldots, b_{j-1}, \beta_j, b_{j+1}, \ldots, b_n\} \) contains the first \( n+1 \) elements of some \( p \)-ordering on \( X_p \) possibly different from the previous one. This implies that

\[
w_p(n) = \nu_p \left( \prod_{0 \leq \ell < n - 1 \atop \ell \neq j} (b_\ell - b_i) \right) + \nu_p(b_n - \beta_j) \leq \nu_p \left( \prod_{0 \leq \ell < n - 1 \atop \ell \neq j} (a - b_i) \right) + \nu_p(a - \beta_j) \quad (a \in X_p).
\]

Thus, \( C'_n(t) \in \mathcal{I}_{(p)} \), which proves assertion (i).

The leading coefficients of both polynomials \( C'_n(p) \) and \( C'_n(t) \) are equal to \( 1/p^{w_p(n)} \) which is a generator of \( I_\mathcal{I}(n) \), i.e. assertion (ii) holds.

The proof of Theorem 4.8 immediately yields the following result which provides a convenient checking whether a polynomial is an integer-valued polynomial.

Corollary 4.10 Let \( f(t) \in \mathbb{F}_q(x)[t] \) of degree \( n \). Then \( f \) belongs to \( \text{Int}\left(\mathbb{P}, (\mathbb{F}_q[x])_{(p)}\right) \) if and only if \( f(b_i) \) belongs to \( (\mathbb{F}_q[x])_{(p)} \) for all \( i = 0, 1, \ldots, n \).

To illustrate the notation and concepts used in the last section, we work out two examples.

Example 4.11 The following table lists all elements of the bases \( C'_n(t) \) of \( \text{Int}(X_p, \mathbb{F}_3[x]_{(p)}) \) for \( 1 \leq n \leq 4 \) and \( p = x, x + 1, x + 2 \).
### Example 4.12

Keep the above notation and the polynomials in Example 4.11. Recall that $m(p,j,n)$ is defined as in Theorem 4.9.

1. For each $n \in \mathbb{N}, p \in \mathbb{P}$, since $m(p,0,n) = 0$, we can replace $b_0$ in the polynomial $C_n^p(t)$ by $\beta_0$ satisfying the relation
   \[ \beta_0 \equiv b_0 \equiv p \pmod{p^1}. \]

2. Since $m(x,3,4) = 1$, we can replace $b_3$ in the polynomial $C_4^x(t)$ by $\beta_3$ satisfying the relation
   \[ \beta_3 \equiv b_3 \equiv x + 1 \pmod{x^2}. \]

3. Since $m(x + 1,3,4) = 1$, we can replace $b_3$ in the polynomial $C_4^x(t)$ by $\beta_3$ satisfying the relation
   \[ \beta_3 \equiv b_3 \equiv x + 2 \pmod{(x + 1)^2}. \]

4. Since $m(x + 2,3,4) = 1$, we can replace $b_3$ in the polynomial $C_4^x(t)$ by $\beta_3$ satisfying the relation
   \[ \beta_3 \equiv b_3 \equiv x \pmod{(x + 2)^2}. \]

### 5. Explicit bases

To characterize the characteristic ideal of the set $\mathcal{I}$, we need the following lemma.

**Lemma 5.1** For $p \in \mathbb{P}$, one has $I_{\mathcal{I}(p)}(n) = (I_{\mathcal{I}}(n))_{(p)}$ where

\[ (I_{\mathcal{I}}(n))_{(p)} := \left\{ \frac{A}{b} \in \mathbb{F}_q(x) \mid A \in I_{\mathcal{I}}(n), b \in \mathbb{F}_q[x] \setminus p\mathbb{F}_q[x] \right\}, \tag{5.1} \]

is the localization of the characteristic ideal $I_{\mathcal{I}}(n)$ at $p$.

**Proof** Since the characteristic ideal $I_{\mathcal{I}}(n)$ is the set of leading coefficients of all polynomials of degree $n$ in $\mathcal{I}$ including 0, by (5.1), we get

\[ (I_{\mathcal{I}}(n))_{(p)} = \{0\} \cup \left\{ \frac{A}{b} \in \mathbb{F}_q(x) \mid \exists f \in \mathcal{I} \text{ such that } f = At^n + \cdots + A_0, b \in \mathbb{F}_q[x] \setminus p\mathbb{F}_q[x] \right\} \]

\[ = \{0\} \cup \left\{ \frac{A}{b} \in \mathbb{F}_q(x) \mid \exists \frac{f}{b} \in I_{\mathcal{I}(p)} \text{ such that } f = At^n + \cdots + A_0, b \in \mathbb{F}_q[x] \setminus p\mathbb{F}_q[x] \right\} = I_{\mathcal{I}(p)}(n). \]

Returning to the module $\mathcal{I}$, we now prove our final main result.
Theorem 5.2 For \( n \geq 0 \), the set of leading coefficients of the polynomials in \( \mathcal{I} \) with degree \( \leq n \) is the fractional ideal

\[
I_{\mathcal{I}}(n) = \prod_p p^{-w_p(n)} F_q[x],
\]

where the product extends over all \( p \in \mathbb{P} \) such that \( q^{\deg_p} \leq n \), and \( w_p(n) \) is as defined in (4.3).

Proof By Theorem 4.8 and Proposition 2.2, the characteristic ideal of the localized module \( \mathcal{I}(p) \) is

\[
I_{\mathcal{I}(p)}(n) = p^{-w_p(n)} F_q[x] \quad (n \geq 0),
\]

and so by Lemma (5.1), \( (I_{\mathcal{I}}(n))(p) = p^{-w_p(n)} F_q[x] \). Since \( I_{\mathcal{I}}(n) \) is generated by the product of the generators of \( \mathcal{I}(p) \) for all \( p \in \mathbb{P} \), noting that \( w_p(n) = 0 \) when \( q^{\deg_p} > n \), the result follows. \( \square \)

With the globalization result in Theorem 5.2, we show now how to construct an explicit basis of \( \mathcal{I} \) by exhibiting an algorithm to inductively construct elements belonging to a basis of \( \mathcal{I} \). Let \( n \) be a fixed integer \( \geq 0 \), and let

\[
S_n := \{ p \in \mathbb{P} \mid q^{\deg_p} \leq n \}.
\]

For each \( p \in S_n \), let

\[
\mathcal{A}_n^{(p)} = \{ b_0^{(p)}, b_1^{(p)}, \ldots, b_{n-1}^{(p)} \}
\]

be the sequence of polynomials as defined in (4.2) (with \( b_i^{(p)} \) in place of \( b_i \) as in Subsection 4.1) corresponding to the prime \( p \), and let \( C_n^p(t) = p^{-w_p(n)} \prod_{0 \leq i < n} (t - b_i^{(p)}) \) be the \( n \)-th element, as defined in (4.6), in a basis of the localized module \( \mathcal{I}(p) \). The following algorithm determines the \( n \)-th element \( C_n(t) \) belonging to a regular basis of the globalized module \( \mathcal{I} \).

STEP 1: Let \( \mathcal{A}_n := \bigcap_{p \in S_n} \mathcal{A}_n^{(p)} \). Note that for all \( \gamma \in \mathcal{A}_n \), \( t - \gamma \) is a factor of all the numerators of \( C_n^p(t) \) (\( p \in S_n \)). Therefore, the product \( \prod_{\gamma \in \mathcal{A}_n} (t - \gamma) \) is the greatest common factor of the numerators of \( C_n^p(t) \) for all \( p \in S_n \).

STEP 2: If \( \mathcal{A}_n^{(p)} \setminus \mathcal{A}_n \neq \emptyset \), let

\[
\mathcal{A}_n^{(p)} \setminus \mathcal{A}_n := \{ \delta_1^{(p)}, \delta_2^{(p)}, \ldots, \delta_{k_p}^{(p)} \},
\]

and let the integer \( m \) be as defined in Theorem 4.9. For each \( i \in \{1, \ldots, k_p\} \), by the Chinese remainder theorem, [9, Proposition 1.4], the system of congruences

\[
\beta_i \equiv \delta_i^{(p)} \pmod{p^m} \quad (i = 1, \ldots, k_p)
\]

with the primes \( p \) running over the set \( S_n \), is solvable for \( \beta_i \in F_q[x] \).

STEP 3: Set

\[
C_n(t) = \prod_{p \in S_n} p^{-w_p(n)} \prod_{\gamma \in \mathcal{A}_n} (t - \gamma) \prod_{i=1}^{k_p} (t - \beta_i),
\]

where the last product is taken to be 1 if \( \mathcal{A}_n^{(p)} \setminus \mathcal{A}_n = \emptyset \).

There remains to check that the so constructed elements \( C_n(t) \) form a regular basis of \( \mathcal{I} \). By Theorem 4.9, we have \( C_n(t) \in \mathcal{I}(p) \) for all \( p \in S_n \). Since the denominator of \( C_n^p(t) \) is not a multiple of any prime \( \ell \) for
all $\ell \in \mathbb{P} \setminus S_n$, it follows that $C_n^\ell(t)$ also belongs to $I(\ell)$ for all $\ell \in \mathbb{P} \setminus S_n$. Thus, $C_n(t) \in I(p)$ for all $p \in \mathbb{P}$, and so $C_n(t) \in I$. Since the leading coefficient of $C_n(t)$ is $\prod_{p \in S_n} p^{-\nu_p(n)}$, a generator of $I_2(n)$, by Theorem 5.2, the polynomials $C_n(t)$ ($n \geq 0$) form a regular basis of $I$.

We end this paper with an example illustrating our algorithm.

**Example 5.3** Keeping the notation of Example 4.11, for $n = 4$, we have

$$S_4 = \{x, x + 1, x + 2\}.$$  

Thus,

$$A_4^{(x)} = \{x, 1, 2, x + 1\}, \quad A_4^{(x+1)} = \{x + 1, 1, 2, x + 2\}, \quad A_4^{(x+2)} = \{x + 2, 1, 2, x\}.$$  

**STEP 1:** We have $A_4 = \{1, 2\}$. The product $(t - 1)(t - 2)$ is the greatest common factor of the numerators of the polynomials $C_4^x(t)$, $C_4^{x+1}(t)$, $C_4^{x+2}(t)$ displayed in Example 4.11.

**STEP 2:** Since

$$A_n^{(x)} \setminus A_n = \{\delta_1^{(x)} = x, \ \delta_2^{(x)} = x + 1\}$$

$$A_n^{(x+1)} \setminus A_n = \{\delta_1^{(x+1)} = x + 1, \ \delta_2^{(x+1)} = x + 2\}$$

$$A_n^{(x+2)} \setminus A_n = \{\delta_1^{(x+2)} = x + 2, \ \delta_2^{(x+2)} = x\},$$

by Theorem 5.2, we have $I_2(4) = (x(x + 1)(x + 2))^{-1} \mathbb{F}_3(x)$. Using the results in Example 4.12, the polynomial $C_4(t)$ takes the form $C_4(t) = \frac{(t-1)(t-2)(t-\beta_1)(t-\beta_2)}{x(x+1)(x+2)}$, where

$$\beta_1 \equiv x \quad \text{(mod } x\text{)}, \quad \beta_1 \equiv x + 1 \quad \text{(mod } x + 1\text{),} \quad \beta_1 \equiv x + 2 \quad \text{(mod } x + 2\text{)}$$

$$\beta_2 \equiv x + 1 \quad \text{(mod } x^2\text{),} \quad \beta_2 \equiv x + 2 \quad \text{(mod } (x + 1)^2\text{),} \quad \beta_2 \equiv x \quad \text{(mod } (x + 2)^2\text{)}.$$  

Solving for $\beta_1$ and $\beta_2$, we get $\beta_1 = 0$ and $\beta_2 = 2x^3 + x + 1$, and so

$$C_4(t) = \frac{t(t-1)(t-2)(t-2x^3-x-1)}{x(x+1)(x+2)}.$$  

Other globalized polynomials $C_n(t)$ in $\text{Int}(\mathbb{P}, \mathbb{F}_3[x])$ can be constructed in the same manner. For example, when $1 \leq n \leq 3$, we obtain

$$C_1(t) = t, \quad C_2(t) = t(t-1), \quad C_3(t) = \frac{t(t-1)(t-2)}{x(x+1)(x+2)}.$$  

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**References**


