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Nilary group rings and algebras

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Abstract: A ring A is (principally) nilary, denoted (pr-)nilary, if whenever $XY = 0$, then there exists a positive integer n such that either $X^n = 0$ or $Y^n = 0$ for all (principal) ideals X, Y of A . We determine necessary and/or sufficient conditions for the group ring $A[G]$ to be (principally) nilary in terms of conditions on the ring A or the group G . For example, we show that: (1) If $A[G]$ is (pr-)nilary, then A is (pr-)nilary and either G is prime or the order of each finite nontrivial normal subgroup of G is nilpotent in A . (2) Assume that G is finite. Then G is nilpotent and $A[G]$ is (pr-)nilary if and only if G is a p -group, $\text{char}(A) = p^\alpha$ (p is a prime), and A is (pr-)nilary. (3) Let G be a finite supersolvable group such that q is the smallest prime dividing $|G|$, and F is a field with $\text{char}(F) = q$. Then $F[G]$ is nilary if and only if G is a q -group. Examples are provided to illustrate and delimit our results.

Key words: Nilary ring, indecomposable ring, group ring, group algebra, quasi-Frobenius ring

1. Introduction

Throughout, all rings are associative with a nonzero unity; R denotes such a ring; and G denotes a group. Also, we consider all groups to be nontrivial (i.e. of order greater than one) unless indicated otherwise. Recall that a ring R is called Quasi-Frobenius (denoted, QF) if R is right Artinian and right self-injective.

This paper is the second in a series of papers on the classification of indecomposable QF-rings. The following facts are well-known: (1) Every QF-ring is a finite direct sum of indecomposable QF-rings. (2) Let R be a commutative ring. Then R is an indecomposable QF-ring if and only if R is a local Artinian ring with simple socle. (3) R is a prime QF-ring if and only if R is isomorphic to a n -by- n ($n \in \mathbb{N}$) matrix ring over a division ring. Note that a nonprime local QF-ring has a nonzero nilpotent Jacobson radical which is a prime ideal. So from the above facts and other properties of QF-rings, it seems reasonable to investigate the classification of indecomposable QF-rings.

As a first step in our investigation, we recognized that the classes of nilary rings and pr-nilary rings properly contain the class of all rings with a nilpotent prime ideal (e.g., all prime rings and all local rings with nilpotent Jacobson radical).

Recall from [2], a ring R is (principally) nilary, if whenever $XY = 0$ then there exists $n \in \mathbb{N}$ such that either $X^n = 0$ or $Y^n = 0$ for all (principal) ideals X, Y of R . We use pr-nilary to denote principally nilary.

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These classes of rings have the following useful properties (see [2]): (1) Every nilary ring is pr-nilary, but the converse is not true even for commutative rings [2, Example 1.10(ii)]. However, if R has ACC on ideals, then R is pr-nilary if and only if R is nilary [2, Proposition 1.4]. (2) Every pr-nilary ring is indecomposable [2, Proposition 1.4]. (3) The nilary and pr-nilary properties are Morita invariants [2, Theorem 3.5]. (4) If I is a nilpotent ideal of R and $\frac{R}{I}$ is a nilary ring, then R is a nilary ring [2, Proposition 1.6]. Thus, we can classify the indecomposable QF-rings into the class of nilary QF-rings and the class of nonnilary indecomposable QF-rings.

In [1, Theorem 2.2], we characterized the nilary QF-rings. Among the characterizations, the following is useful in this paper: Let R be a QF-ring. Then R is nilary if and only if there exists a set of orthogonal primitive idempotents $\{e_1, \dots, e_m\}$, such that e_1R, \dots, e_mR represent a complete set of isomorphism classes of principal indecomposable modules with each Re_iR right essential in R .

The class of group algebras of the form, $R = F[G]$, where F is a field and G is a finite group are Frobenius algebras (hence QF-rings) which have important applications in group representation theory, algebraic coding theory, and physics through the symmetry group of a physical system. In this paper, after some basic results on (pr-)nilary (i.e. pr-nilary or nilary) rings and an example of an indecomposable QF-ring which is not nilary, we investigate the pr-nilary and nilary conditions on group rings of the form, $R = A[G]$, where A is a ring and G is a group, in Section 2. In particular, we find necessary and/or sufficient conditions on the ring A and the group G for R to be nilary or pr-nilary. For example, we show (let A be a ring, G be a group, and p be a prime): (1) If $A[G]$ is (pr-)nilary, then either G is prime or the order of each finite nontrivial normal subgroup of G is nilpotent in A (Theorem 2.10(i)). (2) Assume that G is finite. Then G is nilpotent and $A[G]$ is (pr-)nilary if and only if G is a p -group, $\text{char}(A) = p^\alpha$ (p is a prime and α is a positive integer), and A is (pr-)nilary (Corollary 2.30). (3) Let G be a finite supersolvable group such that q is the smallest prime dividing $|G|$, and $\text{char}(A) = q^\alpha$. Then $A[G]$ is (pr-)nilary if and only if G is a q -group (Theorem 2.31).

In Section 3, we apply the results of Sections 1 and 2 and [1] to investigate (pr-) nilary group algebras of the form $R = F[G]$ where F is a field and G is a group. In particular, we begin the search for a characterization of the nilary QF-algebra of the form $R = F[G]$ where F is a field and G is a finite group, in terms of the properties of F and G . Also, we characterize the nilary group algebras which have a nilpotent prime ideal (Theorem 3.5).

We use \mathbb{N} , \mathbb{Z} , and \mathbb{Z}_n ($n > 1$) to denote the set of positive integers, the ring of integers, and the ring of integers modulo n , respectively; $I \trianglelefteq A$ means that I is an ideal (a two-sided) of the ring A , $K \leq R_R$ and $K \leq^{ess} R_R$ denote that K is a right ideal of R and K is right essential in R (i.e. $K \cap Y \neq 0$ for each nonzero right ideal Y of R), respectively; and we use $\langle X \rangle$ for the ideal generated by the nonempty subset X of A . $P(A)$, $J(A)$, $\text{gcd}(a, b)$, $U(A)$, $\text{char}(A)$, and $\text{cent}(A)$ denote the prime radical, the Jacobson radical of a ring A , the greatest common divisor of a and b , the units of A , the characteristic of A , and the center of A , respectively. The left (right) annihilator of the subset X of the ring A is denoted by $\ell_A(X) = \{a \in A \mid aX = 0\}$ ($r_A(X) = \{a \in A \mid Xa = 0\}$). For other terminology see [9, 10, 11].

2. nilary rings

Definition 2.1 [2, Definition 1.1 (i)-(iii) and Definition 2.1]

- (i) An ideal I of a ring R is said to be a (principally) right primary ideal if whenever X and Y are (principal) ideals of R with $XY \subseteq I$, then either $X \subseteq I$ or $Y^n \subseteq I$ for some positive integer n depending

on X and Y .

- (ii) An ideal I is called a (principally) nilary ideal if whenever X and Y are (principal) ideals of R with $XY \subseteq I$, then either $X^n \subseteq I$ or $Y^n \subseteq I$ for some positive integer n depending on X and Y .
- (iii) A ring R is said to be a (principally) right primary ring or (principally) nilary ring if the zero ideal is a (principal) right primary or a (principal) nilary ideal of R , respectively.
- (iv) An ideal I of R is called a strongly pr-nilary ideal if \sqrt{I} is a prime ideal of R . R is called a strongly pr-nilary ring if the zero ideal of R is a strongly pr-nilary. Every strongly pr-nilary ideal (ring) is a pr-nilary ideal (ring) [2, Proposition 2.4(i)].

In this paper, p-groups are used in conjunction with the principally nilary concept. In order to avoid confusion, we use "pr-" as an abbreviation for "principally". Thus pr-nilary ring denotes principally nilary ring. Note that in [2] a principally nilary ring (ideal) is denoted as a p-nilary ring (ideal). Also, we use "(pr-) nilary" to denote "principally nilary or nilary, respectively".

In parts (i) and (iii) above, the left-sided version is defined analogously. Define $\sqrt{I} = \sum \{V \trianglelefteq R \mid V^n \subseteq I \text{ for some } n \in \mathbb{N}\}$; \sqrt{I} is called the pseudo radical of I . Let $\sqrt{0_R}$, and $\sqrt{0_{A[G]}}$ denote the pseudo radical (i.e. Wedderburn radical) of R , and $A[G]$, respectively.

Observe, from Proposition 2.8, that if R has a nilpotent prime ideal, then R is nilary and strongly pr-nilary. For example, all prime rings and all local rings with nilpotent Jacobson radical are nilary and strongly pr-nilary. See [2] for more examples.

Lemma 2.2 *Let I be an ideal of a ring R . The following conditions are equivalent:*

- (i) I is a nilary ideal of R .
- (ii) $AB \subseteq I$ implies that $A^m \subseteq I$ or $B^m \subseteq I$ for some $m \in \mathbb{N}$ and for all left ideals A, B of R .
- (iii) Let B be any left ideal of R and $(I : B) = \{r \in R \mid rB \subseteq I\}$. Then $B^m \subseteq I$ or $(I : B)^m \subseteq I$ for some $m \in \mathbb{N}$.
- (iv) $\frac{R}{I}$ is a nilary ring.

Proof

(i) \Rightarrow (ii) Assume I is nilary and $AB \subseteq I$. Then $(A(RB))R \subseteq I$. So $(AR)(BR) \subseteq I$. Hence, $(AR)^m \subseteq I$ or $(BR)^m \subseteq I$, for some $m \in \mathbb{N}$. Therefore, $A^m \subseteq I$ or $B^m \subseteq I$.

(ii) \Rightarrow (iii) Observe that $(B : I)$ is a two sided ideal of R and $(I : B)B \subseteq I$. So $(I : B)^m \subseteq I$ or $B^m \subseteq I$ for some positive integer m .

(iii) \Rightarrow (i) Let A, B be ideals of R such that $AB \subseteq I$. Then $A \subseteq (I : B)$. Therefore, $A^m \subseteq I$ or $B^m \subseteq I$ for some positive integer m .

(vi) \Leftrightarrow (i) The proof of this implication is routine.

□

Corollary 2.3 *The following conditions are equivalent:*

- (i) R is a nilary ring.
- (ii) $IJ = 0$ implies that $I^m = 0$ or $J^m = 0$ for some $m \in \mathbb{N}$ and for all left ideals I, J of R .
- (iii) For any left ideal J of R either $J^m = 0$ or $\ell(J)^m = 0$ for some $m \in \mathbb{N}$.
- (iv) For any right ideal I of R either $I^m = 0$ or $r(I)^m = 0$ for some $m \in \mathbb{N}$.

Results similar to Lemma 2.2 and Corollary 2.3 hold for pr-nilary ideals and rings, respectively. These results are used implicitly throughout the paper.

Lemma 2.4

- (i) If R/I is a nilary ring and I is a nilpotent ideal, then R is a nilary ring.
- (ii) If R/I is a pr-nilary ring and \sqrt{I} is a sum of nilpotent ideals, then R is a pr-nilary ring.
- (iii) $\sqrt{0_R}$ is nilpotent if and only if $P(R)$ is nilpotent, in either case, $\sqrt{0_R} = P(R)$.

Proof

- (i) This is [2, Proposition 1.6(iii)].
- (ii) This is [2, Proposition 1.6(iv)].
- (iii) Clearly $\sqrt{0_R} \subseteq P(R)$; hence, $P(R)$ nilpotent implies $\sqrt{0_R} = P(R)$. So $\sqrt{0_R}$ is nilpotent. Now assume $\sqrt{0_R}$ is nilpotent. Then [7, P.184] yields $\sqrt{0_R} = P(R)$. So $P(R)$ is nilpotent.

□

Proposition 2.5 (i) $P(R)$ is a (pr-)nilary ideal if and only if $P(R)$ is a prime ideal.

(ii) $P(R) = \sqrt{0_R}$ and $P(R)$ is a pr-nilary ideal if and only if R is a strongly pr-nilary ring.

(iii) Assume $\sqrt{0_R}$ is a nilpotent pr-nilary ideal. Then $P(R) = \sqrt{0_R}$, $P(R)$ is a prime ideal, R is a nilary and strongly pr-nilary ring.

(iv) R has a nilpotent prime ideal if and only if $\sqrt{0_R}$ is a nilpotent pr-nilary ideal if and only if $P(R)$ is a nilpotent prime ideal.

Proof (i) Since $P(R)$ is a semiprime ideal, this part follows from [2, Proposition 1.3(i)].

(ii) Part(i) and [2, Proposition 2.3] yields this part.

(iii) Assume $\sqrt{0_R}$ is a nilpotent pr-nilary ideal. From Lemma 2.4(iii), $P(R) = \sqrt{0_R}$. By part (i), $P(R)$ is a prime ideal and R is strongly pr-nilary. From Lemma 2.4(i), R is a nilary ring.

(iv) Assume X is a nilpotent prime ideal. Then $X \subseteq \sqrt{0_R} \subseteq P(R) \subseteq X$. The remainder of the proof follows from Lemma 2.4(iii) and part(i).

□

Proposition 2.6 *Let R be a commutative ring. The following are equivalent.*

- (i) R is pr-nilary.
- (ii) $P(R)$ is a prime ideal.
- (iii) R is a strongly pr-nilary.

Proof (i) \Rightarrow (ii) Suppose R is pr-nilary. Let $x, y \in R$ such that $xy \in P(R)$. Since $P(R)$ is the set of nilpotent elements of R , there exists a positive integer m such that $(xy)^m = x^m y^m = 0$. Then $(x^m R)(y^m R) = 0$. Since R is pr-nilary, there exists a positive integer k such that $(x^m R)^k = 0$ or $(y^m R)^k = 0$. Then $x^{mk} = 0$ or $y^{mk} = 0$. Therefore, either $x \in P(R)$ or $y \in P(R)$, so $P(R)$ is a prime ideal of R .

(ii) \Rightarrow (iii) Since R is commutative, $\sqrt{0_R} = P(R)$. So R is strongly pr-nilary.

(iii) \Rightarrow (i) This implication follows from [2, Proposition 2.4(i)]

□

Note that Proposition 2.6 is not true for noncommutative rings. In Example 4.13 or [1, Example 2.13], it is shown that $R = \mathbb{Z}_3[S_3]$ is a nilary ring such that $P(R) = J(R)$ is nilpotent; but $\frac{R}{P(R)} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Therefore, $P(R)$ is not a prime ideal of R .

Corollary 2.7 *Assume R is (pr-)nilary and S is a subring of R .*

- (i) *If $S \subseteq \text{cent}(R)$, then S is a (pr-)nilary ring and $\frac{S}{P(S)}$ is a domain.*
- (ii) *If R is right duo (i.e. every right ideal is an ideal), then S is (pr-)nilary.*

Proof

(i) Assume that R is nilary and $X, Y \trianglelefteq S$ such that $XY = 0$. Then $(XR)(YR) = 0$. So there exist $m \in \mathbb{N}$ such that $(XR)^m = 0$ or $(YR)^m = 0$. Hence, $X^m = 0$ or $Y^m = 0$. Therefore, S is nilary. The pr-nilary part is similar. By Proposition 2.6, $\frac{S}{P(S)}$ is a domain.

(ii) We show the nilary case. The pr-nilary case is similar. Let $X, Y \trianglelefteq S$ such that $XY = 0$. Then $XR, YR \leq R_R$. Hence, $(XR)(YR) = X(RYR) = XYR = 0$. So $(XR)^m = 0$ or $(YR)^m = 0$ for some $m \in \mathbb{N}$. Therefore, $X^m = 0$ or $Y^m = 0$.

□

Proposition 2.8 (i) *If R has a nilpotent prime ideal, then R is nilary and strongly pr-nilary. In particular, prime rings and local rings with nilpotent Jacobson radical are nilary and strongly pr-nilary.*

(ii) *Let M be an (R, R) -bimodule and $\mathcal{T}(R, M)$ the trivial (also called the split-null) extension of M by R . If R is (pr-)nilary, then $\mathcal{T}(R, M)$ is (pr-)nilary.*

Proof (i) This part follows from Proposition 2.5 (iii) and (iv).

(ii) Let $(0, M) = \{(0, m) \mid m \in M\}$. Then $(0, M)$ is a nilpotent ideal of $\mathcal{T}(R, M)$ and $\frac{\mathcal{T}(R, M)}{(0, M)} \cong R$. The remainder of the proof follows from Lemmas 2.2 and 2.4.

□

Theorem 2.9 *Let R be a (pr-)nilary ring. Then either $\text{char}(R) = 0$ or $\text{char}(R) = p^\beta$ for some positive integer β , where p is a prime number. If R is semiprime and $\text{char}(R) \neq 0$, then $\text{char}(R) = p$.*

Proof Assume that $\text{char}(R) > 0$ and $\text{char}(R) = p^\alpha m$ for some $\alpha, m \in \mathbb{N}$, where p is a prime number such that $\text{gcd}(p, m) = 1$. Since $\langle p^\alpha \rangle \langle m \rangle = 0$ and R is a (pr-)nilary ring, either p or m is nilpotent in R . In case p is nilpotent in R , then $p \in J(R)$; this implies that $m \in U(R)$ because $\text{gcd}(p, m) = 1$. Now, we have

$$0 = (\text{char}(R) \cdot 1)m^{-1} = p^\alpha mm^{-1} = p^\alpha;$$

thus, $p^\alpha = 0$ in R ; but $\text{char}(R) = p^\alpha m$, so $m = 1$. In case m is nilpotent in R , then $m \in J(R)$; this implies that $p \in U(R)$ and $p^\alpha \in U(R)$ because $\text{gcd}(p^\alpha, m) = 1$. Now, we have

$$0 = p^{-\alpha}(1 \cdot \text{char}(R)) = p^{-\alpha}p^\alpha m = m;$$

hence, $m = 0$ in R ; but $\text{char}(R) = p^\alpha m$, a contradiction. So either $\text{char}(R) = 0$ or $\text{char}(R) = p^\alpha$ for some $\alpha \in \mathbb{N}$. In the last part of the result, since R is semiprime and $p \in \text{cent}(R)$, $\text{char}(R) = p$. \square

Example 2.10 This example is an indecomposable Frobenius basic ring R with $u.\dim(R_R) = 3$ which is not a nilary ring (see [11, Example 16.19(4)]). In fact, this example can be extended to an indecomposable Frobenius basic ring R with $u.\dim(R_R) = n$ for $n \geq 3$ which is not a nilary ring. Let K be a division ring, and

$$R = \left\{ \begin{bmatrix} a_1 & x_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & x_2 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 & x_3 \\ 0 & 0 & 0 & 0 & 0 & a_1 \end{bmatrix} \mid a_i, x_i \in K \text{ for } i = 1, 2, 3 \right\}.$$

Let $E = \{e_1, e_2, e_3\}$ where $e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, and $e_3 =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then E is a complete set of orthogonal primitive idempotents for R . To see that

R is indecomposable, assume c is a nontrivial central idempotent of R . Then either c or $1 - c$ is in E . Since no element of E is central, R is indecomposable. To see that R is not nilary, let $I = Re_1R$. Then

$$I = \left\{ \begin{bmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mid a, x_i \in K \text{ for } i = 1, 2 \right\} \text{ and}$$

$$r(I) = Re_2R \left\{ \left[\begin{array}{cccccc} b & y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 & 0 & b \end{array} \right] \mid b, y_i \in K \text{ for } i = 1, 2 \right\}.$$

Since neither I nor $r(I)$ are nilpotent, R cannot be nilary.

3. Group rings that are (pr-)nilary

If A is a ring and G is a group, $A[G]$ will denote the group ring of G over A . Consider the function $\varepsilon : A[G] \rightarrow A$ defined by $\varepsilon(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g$. This function is called the augmentation map, and ε is a ring homomorphism that maps $A[G]$ onto A . $\text{Ker}(\varepsilon) = \{\alpha = \sum_{g \in G} a_g g \in A[G] \mid \varepsilon(\alpha) = \sum_{g \in G} a_g = 0\}$. $\text{Ker}(\varepsilon)$ is a nontrivial ideal called the augmentation (fundamental) ideal of $A[G]$ and is denoted by $\Delta(G)$. The ideal $\Delta(G)$ consists of the elements of the form $a_1(1 - g_1) + \dots + a_k(1 - g_k)$ with each $a_i \in A$, each $g_i \in G$, and k a positive integer. From the above, it is clear that $A[G]/\Delta(G) \cong A$. Let H be a normal subgroup of G . Then the natural homomorphism $G \rightarrow G/H$ mapping g to gH induces a ring homomorphism $\varepsilon_H : A[G] \rightarrow A[G/H]$ defined by $\varepsilon_H(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g gH \in A[G/H]$. Also, $\text{Ker}(\varepsilon_H) = \Delta(G, H)$ for the kernel of this homomorphism. The ideal $\Delta(G, H)$ consists of the elements of the form $a_1 g_1(1 - h_1) + \dots + a_k g_k(1 - h_k)$ with each $a_i \in A$, each $g_i \in G$, $h_i \in H$, and k a positive integer. $A[G]\Delta(H)$ is the kernel of ε_H (i.e. $\Delta(G, H) = A[G]\Delta(H)$). In particular, if $H = G$, then $\varepsilon = \varepsilon_G$, and we write $\Delta(G) = \Delta(G, G)$. For a nonempty subset I of A , we have I is a right ideal of A if and only if $I[G]$ ($IA[G] = I[G]$) is a right ideal of $A[G]$; if I is an ideal, then $I[G]$ is an ideal and $A[G]/I[G] \cong (A/I)[G]$. We use $\nu(G)$ to denote the set of orders of all finite normal subgroups; $Z(G)$ to denote the center of a group G ; $\rho(G)$ is the set of $g \in G$ which have only a finite number of conjugates; $\sigma(G)$ denotes the set of $g \in \rho(G)$ of finite order; groups with $\sigma(G) = G$ are called locally normal, an equivalent definition being every finite subset is contained in a finite normal subgroup; S_n is the symmetric group and A_n its alternating subgroup; C_n is used for the cyclic group of order n ($n \geq 1$); the subgroup $\langle g \rangle$ is called the cyclic subgroup of G generated by g ; a group G is called a p -group if the order of each element of G is a power of p ; $|G|$ denotes the order of G ; the order of an element g is denoted by $o(g)$; G is called a torsion group if, for every $g \in G$ there exists a nonzero $n \in \mathbb{N}$ with $o(g) = n$; G is prime if it satisfies either one of the following two equivalent conditions: (i) $\sigma(G) = 1$, (ii) $\nu(G) = \{1\}$, i.e. G contains no finite normal subgroup except 1; G is called a Dedekind group if every subgroup of G is normal. Also, for a prime p and a finite group G we denote by $O_p(G)$ the maximal normal subgroup of G such that its order is divisible by p ; and $O_{p'}(G)$ the maximal normal subgroup of G such that its order is not divisible by p . These definitions and concepts may be found in [5, 11, 15].

This section is devoted to obtaining and investigating results related to nilary group rings. The results of Connell, in [5], which relate to prime rings are (partially) generalized to the class of nilary rings. We start with finding necessary conditions on A and G so that $A[G]$ is nilary.

Lemma 3.1 *Let A be a ring and G be a group. The following statements are equivalent:*

- (i) $\sqrt{0_{A[G]}} = \Delta(G)$;
- (ii) $P(A[G]) = \sqrt{0_{A[G]}} = \Delta(G)$;

(iii) G is a locally normal p -group, A is semiprime, and $p = 0$ in A .

Proof See [5, p. 682, Theorem 10]. □

Recall that if G is a group and H is a finite subgroup, then $\hat{H} = \sum_{h \in H} h$.

Lemma 3.2 [11, Lemma 3.4.3] and [5, p. 651, Proposition 1]

Let H be a subgroup of a group G and let A be a ring. Then $\ell_{A[G]}(\Delta(G, H)) \neq 0$ if and only if H is finite. In this case, we have

$$\ell_{A[G]}(\Delta(G, H)) = A[G]\hat{H}.$$

Furthermore, if H is normal in G , then the element \hat{H} is central in $A[G]$ and we have

$$\ell_{A[G]}(\Delta(G, H)) = r_{A[G]}(\Delta(G, H)) = \hat{H}A[G].$$

Lemma 3.3 [5, p. 656, Proposition 4(ii)] The left and right annihilator ideals of $\Delta(G)$ coincide and are given by

$$(\Delta(G))^* = \begin{cases} 0 & \text{if } G \text{ is infinite,} \\ A \sum_{g \in G} g & \text{if } G = \{g_1, g_2, \dots, g_n\}. \end{cases}$$

In the latter case

$$\Delta(G) \cap (\Delta(G))^* = \{a \sum_{g \in G} g \mid a \in A, na = 0\}.$$

From Lemma 3.3, observe that if $R = A[G]$ is nilary, then either $\ell(\Delta(G)) = r(\Delta(G)) = 0$, $\Delta(G)$ is nilpotent, or \hat{G} is a central nilpotent element of R .

Proposition 3.4 Let A be a ring, G be a group, and $H \trianglelefteq G$.

(i) $\Delta(G, H)$ is a (pr-)nilary ideal if and only if $A[G/H]$ is a (pr-)nilary ring.

(ii) $\Delta(G)$ is a (pr-)nilary ideal if and only if A is a (pr-)nilary ring.

Proof (i) This part follows from Lemma 2.2 and the fact that $A[G/H] \cong A[G]/\Delta(G, H)$ for any normal subgroup H of G [11, Corollary 3.3.5].

(ii) Put $H = G$ in part(i). □

Lemma 3.5 [5, p. 681, Theorem 9] Let A be a ring and G be a group. Then $\Delta(G)$ is nilpotent if and only if

(i) G is a finite p -group, and

(ii) p is nilpotent in A .

Proposition 3.6 *Let A be a ring, G a group and $H \trianglelefteq G$.*

(i) *H is a finite p -subgroup and p is nilpotent in A if and only if $\Delta(G, H)$ is nilpotent.*

(ii) *Assume H is a finite p -subgroup and p is nilpotent in A . If $A[\frac{G}{H}]$ is (pr-)nilary, then $A[G]$ is (pr-) nilary.*

(iii) *Let $G = H \rtimes K$ where H is a finite p -subgroup and p is nilpotent in A . If $A[K]$ is (pr-)nilary, then $A[G]$ is (pr-)nilary.*

Proof (i) (\Rightarrow) Since

$$\Delta(G, H) = A[G]\Delta(H) = \Delta(H)A[G],$$

we have $\Delta(G, H)^n = (A[G]\Delta(H))^n = A[G](\Delta(H))^n$. The result follows from Lemma 3.5.

(\Leftarrow) Since $\Delta(H) \subseteq \Delta(G, H)$. If $\Delta(G, H)$ is nilpotent, then $\Delta(H)$ is nilpotent. By Lemma 3.5, then H is a finite p -subgroup and p is nilpotent in A .

(ii) By part(i), $\Delta(G, H)$ is nilpotent. From [11, Corollary 3.3.5], $\frac{A[G]}{\Delta(G, H)} = A[\frac{G}{H}]$. Now, Lemma 2.4 yields the result.

(iii) This part follows from part(ii). □

From [1, Example 2.13 or Example 4.3], $\mathbb{Z}_3[S_3]$ is nilary. By Proposition 3.6(iii), $\mathbb{Z}_3[C_{3^n} \times S_3]$ is nilary for each positive integer n .

Lemma 3.7 *Let A be a ring and G be a group. Then, $A[G]$ is prime if and only if A is prime and G is prime.*

Proof See [5, p. 675, Theorem 8]. □

Lemma 3.8 *Let A be a ring and G be a group. If G is prime, then $A[G]$ is semiprime if and only if A is semiprime.*

Proof See [5, p. 676]. □

Theorem 3.9 *Let A be a ring and H a subgroup of G such that $H \subseteq Z(G)$.*

(i) *If J is a nilary ideal of $A[G]$, then $J \cap A[H]$ is a nilary ideal of $A[H]$.*

(ii) *If J is a pr-nilary ideal of $A[G]$, then $J \cap A[H]$ is a pr-nilary ideal of $A[H]$.*

(iii) *If $A[G]$ is a (pr-)nilary ring, then $A[H]$ is a (pr-)nilary ring.*

(iv) *If $A[G]$ is a (pr-)nilary ring, then A is a (pr-)nilary ring.*

(v) *If $A[G]$ is a (pr-)nilary, then $A[Z(G)]$ is a (pr-)nilary ring.*

Proof

(i) Let $R = A[G]$ and let $R' = A[H]$. Assume that $I, K \trianglelefteq R'$ with $IK \subseteq J \cap R'$. Now, we have that $(IK)R \subseteq (J \cap R')R \subseteq JR = J$. Since $H \subseteq Z(G)$, we get, $RK = KR$ and so $(IK)R = (IR)(KR)$. Then $(IR)(KR) \subseteq J$. Hence, $(IR)^n \subseteq J$ or $(KR)^n \subseteq J$ for some $n \in \mathbb{N}$ because J is a nilary ideal. Assume that $(IR)^n \subseteq J$. Since R is a unitary ring, $I^n \subseteq (IR)^n \subseteq J$. Therefore, $I^n \subseteq J \cap R'$. If $(KR)^n \subseteq J$, then $K^n \subseteq J \cap R'$. Hence, $J \cap R'$ is a nilary ideal of R' .

- (ii) Notice that if I is a (principal) finitely generated ideal of $A[H]$, then $IA[G]$ is a (principal) finitely generated ideal of $A[G]$. By using part(i) and [2, Proposition 1.3(iii)], we have the result.
- (iii) Put $J = \{0\}$ in Part(i) and (ii).
- (iv) Put $H = \{1\}$ in Part(iii).
- (v) Put $H = Z(G)$ in Part(iii).

□

Theorem 3.10 *Let A be a ring, p be a prime, and G be a group.*

- (i) *If $A[G]$ is (pr-)nilary, then either G is prime or the order of each finite nontrivial normal subgroup of G is nilpotent in A .*
- (ii) *If $\text{char}(A) = 0$ and $A[G]$ is (pr-)nilary, then G is prime (i.e. $\nu(G) = 1$).*
- (iii) *If $\text{char}(A) = p^\alpha$, and $A[G]$ is (pr-)nilary, then p divides $|H|$ for each nontrivial finite normal subgroup H of G .*
- (iv) *If G is finite and $\text{char}(A) = p^\alpha$, and $A[G]$ is (pr-)nilary, then $O_{p'}(G) = 1$.*

Proof

- (i) Assume that G is not prime. Then there exists a finite nontrivial normal subgroup H of G . Now, we have that $\Delta(G, H)$ is a finitely generated ideal of $A[G]$, and by Lemma 3.2 $0 \neq r_{A[G]}(\Delta(G, H))$ is a principal ideal of $A[G]$ generated by $\hat{H} = \sum_{i=1}^n h_i$, with $|H| = n$. Now, we have

$$(\Delta(G, H)) r_{A[G]}(\Delta(G, H)) = 0.$$

Since $A[G]$ is (pr-)nilary, either $\Delta(G, H)$ or $r_{A[G]}(\Delta(G, H))$ is nilpotent (see [2, Proposition 1.3(iii)]), so either $(\Delta(G, H))^m = 0$ or $(r_{A[G]}(\Delta(G, H)))^m = 0$ for some $m \in \mathbb{N}$, by Lemma 2.3(ii).

Suppose $(\Delta(G, H))^m = 0$. By Proposition 3.6, H is a p -group and p is a nilpotent in A . Hence, n is nilpotent in A . Since n was arbitrary, this implies it is nilpotent in A for each $1 \neq n \in \nu(G)$.

Now suppose $(r_{A[G]}(\Delta(G, H)))^m = 0$. Since $x = \hat{H} \in r_{A[G]}(\Delta(G, H))$, $x^m = 0$. Also, $x^2 = \hat{H}x = \sum_{i=1}^n h_i x = nx$; hence, $x^3 = (x^2)x = (nx)x = n(x^2) = n(nx) = (n \cdot n)x = (n^2)x = n^2x$. Therefore, $x^m = n^{m-1}x$, since $x^m = 0$. Then $n^{m-1}x = 0$. So $n^{m-1}\hat{H} = n^{m-1}(h_1 + h_2 + h_3 + \dots + h_n) = 0$. Since H is linearly independent over A , $n^{m-1} = 0$ in A . Since n is arbitrary, this implies that the order of each finite nontrivial normal subgroup of G is nilpotent in A .

- (ii) Assume that G is not prime. There is a nontrivial finite normal subgroup H of G . Since $A[G]$ is (pr-)nilary, either $\Delta(G, H)$ or $r_{A[G]}(\Delta(G, H))$ is nilpotent. By using part(i), we find $|H|$ is nilpotent in A . However, $|H| \neq 0$, and $\text{char}(A) = 0$, a contradiction. Hence, $\sigma(G) = 1$. Therefore, G is prime.

- (iii) Assume that G has a nontrivial finite normal subgroup H . By part(i) we have $|H|$ is nilpotent in A . However, $|H| \neq 0$, and $\text{char}(A) = p^\alpha$. Hence, $o(1) = p^\alpha$ in the additive group A . So $p^\alpha \parallel |H|$. Therefore, $p \parallel |H|$, since p is a prime.
- (iv) Assume that $O_{p'}(G)$ is nontrivial. Hence, $O_{p'}(G)$ is a nontrivial normal subgroup of G . Then p divides $|O_{p'}(G)|$. By Cauchy's Theorem, we find that $O_{p'}(G)$ contains an element of order p , a contradiction.

□

The following examples illustrate and delimit Theorem 3.10.

Example 3.11 Let A be a domain (e.g., $A = \mathbb{Q}$, the rational numbers) and $G = S_\infty$ (the infinite symmetric group). Notice that A is prime and G is prime, and hence $A[G]$ is a prime ring by Lemma 3.7. Therefore, $A[G]$ is (pr-)nilary. However, G has infinitely many nontrivial finite subgroups which are not normal and A contains no nonzero nilpotent elements.

Example 3.12 From [1, Proposition 1.4(iii)], a (pr-) nilary ring is indecomposable. Then it is free of nontrivial central idempotents. Hence, for any (pr-)nilary group algebra $F[G]$, the principal block is the unique block of G over the field F . So if the group algebra $F[G]$ has more than one block, then it is not (pr-)nilary. By using [6, Theorem 4], we find that $\mathbb{Z}_2[A_5]$ is not nilary because it has two blocks. Therefore, the converse of Theorem 3.10 part(i) is false, since A_5 has only one nontrivial normal subgroup and its order is nilpotent in \mathbb{Z}_2 .

Corollary 3.13 Let A be a ring and G be a finite group.

If $A[G]$ is (pr-)nilary, then A is (pr-)nilary and $|G|$ is nilpotent in A .

Proof The proof follows from Theorems 3.9 part(iv) and 3.10(i). □

Corollary 3.14 Let A be a semiprime ring and G be a group. If either G is prime or $\text{char}(A) = 0$, then $A[G]$ is a (pr-)nilary ring if and only if $A[G]$ is a prime ring.

Proof Suppose $A[G]$ is (pr-)nilary. First, assume that $\text{char}(A) = 0$. From Theorem 3.10(ii), G is prime. Since $A[G]$ is a (pr-)nilary ring, A is a (pr-)nilary ring, by Theorem 3.9(iv). From [2, Proposition 1.3(i)] A is prime. Hence, $A[G]$ is prime because A is prime and G is prime, (see, Lemma 3.7). The converse is clear. Next, assume that G is prime. This proof is similar to that used for $\text{char}(A) = 0$. The converse is routine. □

Remark 3.15 Let $A = \mathbb{Z}_{p^n}$ with $n \in \mathbb{N}$, and p be a prime number.

(i) If $n = 1$ then A is prime and hence nilary.

(ii) If $n > 1$ then A is nilary, but it is not prime.

The next result provides examples of (pr-)nilary group rings $A[G]$ where G is a prime group, but $A[G]$ is not a prime ring.

Proposition 3.16 Assume that G is prime. Let A be a commutative ring with a nonzero nilpotent prime ideal I . Then $A[G]$ is (pr-)nilary which is not prime.

Proof Put $B = A/I$, since I is prime then B is a prime ring. Hence, $B[G] = (A/I)[G] \cong A[G]/I[G]$, by [5, p. 654, (9)]. Since B is prime and G is prime, $B[G]$ is prime, by Lemma 3.7. Therefore, $B[G] \cong A[G]/I[G]$ is prime. Hence, $A[G]/I[G]$ is (pr-)nilary. Since $I[G]$ is nilpotent, $A[G]$ is (pr-)nilary, by Lemma 2.4. Since A is not prime, $A[G]$ is not prime, by Lemma 3.7. □

For a particular example, take $A = \mathbb{Z}_{p^m}$ and $G = S_\infty$, for some $m \in \mathbb{N}$ and $m > 1$.

Proposition 3.17 *Let A be a ring, G be a group, $H \triangleleft G$, and $R = A[G]$.*

(i) *If $|G| = \infty$, then $\Delta(G) \leq^{ess} R_R$ and $\Delta(G) \leq^{ess} R_R$.*

(ii) *If $|G| = m < \infty$, R is pr-nilary, and $A[\frac{G}{H}]$ is semiprime, then A is a prime ring, $char(A) = p$, $\Delta(G, H) \leq^{ess} R_R$ and $\Delta(G, H) \leq^{ess} R_R$.*

Proof Let $X \leq R_R$ such that $X \cap \Delta(G) = 0$. Then $X \subseteq \ell(\Delta(G, H)) = \hat{H}R$ by Lemma 3.2.

(i) Since $|G| = \infty$, $X = 0$ by Lemma 3.3. Therefore, $\Delta(G, H) \leq^{ess} R_R$. Similarly, $\Delta(G, H) \leq^{ess} R_R$.

(ii) Since $\frac{R}{\Delta(G, H)} \cong A[\frac{G}{H}]$, A is semiprime and $\Delta(G, H)$ is a semiprime ideal of R . Hence, $P(R) \subseteq \Delta(G, H)$.

By Theorem 3.9(iv), A is pr-nilary ring. From [2, definition 1.1(iii) and Proposition 1.3(i)], A is a prime ring. Theorem 2.9 yields that $char(A) = 0$ or $char(A) = p$ for some prime p . If $char(A) = 0$, then G is prime by Theorem 3.10(ii). This is a contradiction to $|G| = m < \infty$. So $char(A) = p$. By Theorem 3.10(iii), $p \mid |H|$. Then $X^2 \subseteq (\hat{H}R)^2 = |H|\hat{H}R = 0$. So $X \subseteq X \cap P(R) \subseteq X \cap \Delta(G, H) = 0$. Then $\Delta(G, H) \leq^{ess} R_R$. A similar argument yields $\Delta(G, H) \leq^{ess} R_R$. □

Proposition 3.18 *Let A be a ring and G be a Dedekind group. Assume $A[G]$ is (pr-)nilary, then $T(G)$ is trivial or $T(G)$ is a p -group where p is a prime number.*

Proof From Theorem 2.9, if A is nilary then either $char(A) = 0$ or $char(A) = p^\alpha$ for some prime p and positive integer α . If $char(A) = 0$, then by Theorem 3.10 G is prime. Hence, $T(G)$ is trivial. Assume $char(A) = p^\alpha$ and $T(G)$ is nontrivial. Let $g \in G$ of finite order and $g \neq 1$. Let $H = \langle g \rangle$. Since G is Dedekind, then H is a finite normal subgroup of G . By Theorem 3.10, p divides $|H|$. Assume that there is a prime number $q \neq p$ such that q divides $|H|$. Notice that H is cyclic, so abelian; thus, there is a subgroup K of H such that the order of K is q . Also, K is a subgroup G ; therefore, K is normal because G is Dedekind. Again, by Theorem 3.10, p divides $|K|$, a contradiction. Therefore, $T(G)$ is a p -group. □

A group G is hypercentral if there exists a smallest ordinal α such that $Z_\alpha = G$, where $Z_0 = 1, Z_1 = Z(G)$ if λ is a limit ordinal $Z_\lambda = \bigcup_{\beta < \lambda} Z_\beta$ and $\frac{Z_{\beta+1}}{Z_\beta} = Z(\frac{G}{Z_\beta})$. Furthermore, α is called the class of G . If α is finite, G is also called nilpotent.

If π is a set of primes, a group is said to be π -free if it contains no nontrivial elements whose order is a π -number (i.e. a product of primes in π).

Lemma 3.19 [13, Theorem 2.2.12] *Let G be a group with π -free center. Then each upper central factor, and therefore the hypercenter of G , $h(G)$, is π -free.*

Theorem 3.20 *Let A be a ring and G be any group.*

- (i) If $A[G]$ is (pr-)nilary, then $T(h(G))$ is either trivial or a p -group for some prime number p .
- (ii) If $\text{char}(A) = 0$ and $A[G]$ is (pr-)nilary, then $h(G)$ is either trivial or torsion-free.

Proof

- (i) Assume that $A[G]$ is (pr-)nilary. Then, by Theorem 3.9(v), $A[Z(G)]$ is (pr-)nilary. By using Proposition 3.18, if $T(Z(G))$ is nontrivial, then it is a p -group for some prime number p . Assume that $T(h(G))$ is nontrivial. Now, if $T(Z(G))$ is trivial, then it is q -free for any prime q , by Lemma 3.19. $T(h(G))$ is also q -free for any prime q . Therefore, $T(h(G))$ is trivial which is contrary to our assumption. Thus, $T(Z(G))$ is nontrivial. So we have that $T(Z(G))$ is a p -group for some prime number p . From Lemma 3.19, we have that $T(h(G))$ is a p -group.
- (ii) Assume that $Z(G) \neq \{1\}$. If $Z(G)$ is not torsion-free, then there is $g \in Z(G)$ with $o(g) < \infty$. Put $H = \langle g \rangle$. Then $H \trianglelefteq Z(G)$. By using Theorem 3.10, we find that $|H|$ is nilpotent in A , contrary to $\text{char}(A) = 0$. So $Z(G)$ is torsion-free. Hence, it is q -free for each prime number q . From Lemma 3.19, we have that $T(h(G))$ is torsion-free.

□

Theorem 3.21 Let A be a ring and G be a nontrivial nilpotent group.

- (i) If G is torsion and $A[G]$ is (pr-)nilary, then $\text{char}(A) = p^\alpha$ and G is a p -group for some prime number p .
- (ii) If $\text{char}(A) = p^\alpha$ and $A[G]$ is a (pr-)nilary ring, then $T(G)$ is either trivial or a p -group.
- (iii) If $\text{char}(A) = 0$ and $A[G]$ is a (pr-)nilary ring, then $T(G)$ is trivial and G is torsion-free.

Proof

- (i) Let G be a torsion nilpotent group; hence, G is not prime. By Theorem 3.10(iii) and Theorem 3.9, we find that $\text{char}(A) = p^\alpha$ for some prime number p . Since G is torsion and nilpotent, $h(G) = T(h(G)) = G$. From Theorem 3.20(ii), we find that G is a p -group.
- (ii) Since G is a nilpotent group, $T(G) = T(h(G))$. By using Theorem 3.20 (ii), we find that $T(G)$ is either trivial or a p -group.
- (iii) This part follows directly from Theorem 3.20 part(iii).

□

Remark 3.22 Theorem 2.21 (i) and (ii) are false, if we replace the condition "G is nontrivial nilpotent" with "G is nontrivial solvable." To see this, observe that for $A = \mathbb{Z}_3$ and $G = S_3$, then $A[G]$ is nilary (see Example 3.13), G is solvable, $T(G) = G$, $\text{char}(A) = 3$; but G is not a p -group.

Now, we give some sufficient conditions on A and G so that $A[G]$ is (pr-)nilary ring.

Lemma 3.23 *Let G be a nontrivial locally normal p -group, and A a pr-nilary ring such that p is nilpotent in A . Then*

(i) $\Delta(G) \subseteq \sqrt{0_{A[G]}}$.

(ii) $\sqrt{0_{A[G]}} \subseteq \Delta(G) \Leftrightarrow A$ is semiprime $\Leftrightarrow A$ is prime.

Proof (i) Let $g \in G$ and H be the normal closure $\langle g \rangle$ (i.e. the smallest normal subgroup of G containing g). Then, $\Delta_A(H)$ is nilpotent, say $(\Delta_A(H))^m = 0$, by Lemma 3.5. Now an element of $\Delta(G, H)$ is a sum of terms of the form $(1 - g_1)r_1(1 - g_2)r_2 \cdots (1 - g_m)r_m$, $g_i \in H, r_i \in A[G]$. From the normality of H , $x \in \Delta(G, H)$ is a sum of terms of the form

$$y = (1 - g'_1)(1 - g'_2) \cdots (1 - g'_n)r, \quad g'_i \in H, r \in A[G];$$

hence, $y = 0$, since $(1 - g'_1) \cdots (1 - g'_n) \in (\Delta_A(H))^n$. Thus, $\Delta_A(G, H)$ is nilpotent and $1 - g \in \Delta(G, H)$. Therefore, $\Delta(G) \subseteq \sqrt{0_{A[G]}}$.

(ii) Assume $\sqrt{0_{A[G]}} \subseteq \Delta(G)$. From part(i), $\Delta(G) = \sqrt{0_{A[G]}}$. By Lemma 3.1, A is semiprime. Now assume A is semiprime, by [2, Proposition 1.3(i)], A is a prime ring.

Finally, assume A is a prime ring. Since p is a central nilpotent element of A , $p = 0$ in A . From Lemma 3.1, $\Delta(G) = \sqrt{0_{A[G]}}$.

□

Theorem 3.24 *Let G be a nontrivial locally normal p -group, A be a ring such that p is nilpotent in A , and either $\sqrt{0_{A[G]}} \subseteq \Delta(G)$ or A is semiprime. Consider the following conditions:*

(i) $A[G]$ is pr-nilary.

(ii) A is pr-nilary.

(iii) A is prime.

(iv) $P(A[G])$ is a prime ideal.

Then, (i) \Leftrightarrow (ii) \Leftrightarrow (iii) and (iii) \Rightarrow (iv). If A is semiprime, then (iv) \Rightarrow (iii).

Proof (i) \Rightarrow (ii) This implication follows from, Theorem 3.9(iv).

(ii) \Leftrightarrow (iii) This equivalence is a consequence of Lemma 3.23.

(ii) \Rightarrow (i) and (iv) Since A is pr-nilary, A is a prime ring by Lemma 3.23. Hence, $\Delta(G)$ is a prime ideal of $A[G]$. By Lemma 3.1 and Lemma 3.23, $\Delta(G) = \sqrt{0_{A[G]}} = P(A[G])$ is a prime ideal. Let X, Y be ideals of $A[G]$ such that $XY = 0$. Then $XY \subseteq \sqrt{0_{A[G]}}$. Hence, $X \subseteq \sqrt{0_{A[G]}}$ or $Y \subseteq \sqrt{0_{A[G]}}$. From [2, Proposition 1.3(ii)], $A[G]$ is pr-nilary.

(iii) \Rightarrow (iv) By Lemma 3.1, $P(A[G]) = \Delta(G)$. Hence, $\frac{A[G]}{P(A[G])} = \frac{A[G]}{\Delta(G)} \cong A$. Therefore, $P(A[G])$ is a prime ideal.

(iv) \Rightarrow (iii) Assume A is semiprime and $P(A[G])$ is a prime ideal. From Lemma 3.1, $P(A[G]) = \Delta(G)$. Then $A \cong \frac{A[G]}{\Delta(G)} = \frac{A[G]}{P(A[G])}$. Therefore, A is a prime ring. □

Example 3.25 *Let F be a field such that $\text{char}(F) = p$, A be the ring of $n \times n$ matrices over F , and $G \cong C_p \oplus C_{p^2} + C_{p^3} \oplus \cdots \oplus C_{p^n} \oplus \cdots$. Then $A[G]$ is a pr-nilary ring by Theorem 3.24.*

Theorem 3.26 *Let G be a finite p -group and p be a nilpotent in A . Assume that $I \trianglelefteq A$. Then I is a (pr-)nilary ideal of A , if and only if $I[G]$ is a (pr-)nilary ideal of $A[G]$.*

Proof (\Rightarrow) Assume that I is a (pr-)nilary ideal of A , then $\bar{A} = \frac{A}{I}$ is a (pr-)nilary ring. Since p is nilpotent in A . Then $p^k = 0$ in A for some $k \in \mathbb{N}$. This implies that $p^k A = 0$. Hence, $p^k \bar{A} = 0$. Thus, $\Delta_{\bar{A}}(G)$ is nilpotent because G is a finite p -group and $p^k \bar{A} = 0$, by Lemma 3.5. Since $\bar{A} \cong \frac{A[G]}{\Delta_{\bar{A}}(G)}$, $\bar{A}[G]$ is a (pr-)nilary ring, by Lemma 2.4. Since $\bar{A}[G] = (\frac{A}{I})[G] \cong \frac{A[G]}{I[G]}$. $\frac{A[G]}{I[G]}$ is a (pr-)nilary ring. Then $I[G]$ is a (pr-)nilary ideal of $A[G]$.

(\Leftarrow) Assume that $I[G]$ is a (pr-)nilary ideal of $A[G]$. Then $\frac{A[G]}{I[G]} \cong (\frac{A}{I})[G]$ is (pr-)nilary. From Theorem 3.9(iv), $\frac{A}{I}$ is (pr-)nilary. Therefore, I is (pr-)nilary. \square

Corollary 3.27 *Let A be a ring and G a finite p -group such that p is nilpotent in A . Then, $A[G]$ is a (pr-)nilary ring if and only if A is a (pr-)nilary ring.*

Proof Use Theorem 3.26 with $I = 0$. \square

Theorem 3.28 *Let G be a locally normal p -group, A a ring such that p is nilpotent in A , $I \trianglelefteq A$, and either $\sqrt{0_{\bar{A}[G]}} \subseteq \Delta_{\bar{A}}(G)$ or \bar{A} is semiprime, where $\bar{A} = \frac{A}{I}$. Then I is a pr-nilary ideal of A if and only if $I[G]$ is a pr-nilary ideal of $A[G]$.*

Proof (\Rightarrow) Assume I is a pr-nilary ideal of A . Then $\bar{A} = \frac{A}{I}$ is pr-nilary. Since p is nilpotent in A , $p^k = 0$ for some $k \in \mathbb{N}$. As in the proof of Theorem 3.26, $p^k \bar{A} = 0$. By Theorem 3.24, $\bar{A}[G] = \frac{A}{I}[G] = \frac{A[G]}{I[G]}$ is pr-nilary. Therefore, $I[G]$ is a pr-nilary ideal of $A[G]$.

(\Leftarrow) Assume $I[G]$ is a pr-nilary ideal of $A[G]$. Then $\frac{A[G]}{I[G]} = \frac{A}{I}[G] = \bar{A}[G]$ is pr-nilary. By Theorem 3.9(iv), $\bar{A} = \frac{A}{I}$ is pr-nilary. Therefore, 3.9(iv), I is a pr-nilary ideal of A . \square

Theorem 3.29 *Let A be a ring with $\text{char}(A) = p^n$ and G be a nontrivial nilpotent group.*

- (i) *If G is finite, then $A[G]$ is a (pr-)nilary ring if and only if A is (pr-)nilary and G is a p -group.*
- (ii) *If $T(G)$ is finite, and A is prime, then $A[G]$ is a (pr-)nilary ring if and only if $T(G)$ is a p -group.*
- (iii) *Let G be a locally normal torsion group such that $\sqrt{0_{A[G]}} \subseteq \Delta(G)$ or A is semiprime. Then $A[G]$ is a pr-nilary ring if and only if A is (pr-)nilary and G is a p -group.*

Proof

(i) Assume that G is a finite p -group and A is (pr-)nilary. Then by Corollary 3.27, it follows that $A[G]$ is a (pr-)nilary ring. Conversely, assume that $A[G]$ is a (pr-)nilary ring. Hence, A is (pr-)nilary, by Theorem 3.9(iv). Since G is nilpotent, by Theorem 3.21 it follows that G is a p -group.

(ii) Let A be a prime ring and G be a nilpotent group such that $T(G)$ is a finite p -group. If $\frac{G}{T(G)}$ is trivial, then G is finite p -group. From part(i), it follows that $A[G]$ is a (pr-)nilary ring. If $\frac{G}{T(G)}$ is nontrivial, then

$\frac{G}{T(G)}$ is torsion-free, hence a prime group. So $A[\frac{G}{T(G)}]$ is a prime ring, by Lemma 3.7. Hence, $A[\frac{G}{T(G)}]$ is (pr-)nilary. Now we have $A[\frac{G}{T(G)}] \cong \frac{A[G]}{\Delta(G, T(G))}$. Since $T(G)$ is a finite p -group, by using Proposition 3.6(i), it follows that the ideal $\Delta(G, T(G))$ is nilpotent. By Lemma 2.4, we conclude that $A[G]$ is a nilary ring. Conversely, if $A[G]$ is a nilary ring, then $T(G)$ is a finite p -group by Theorem 3.21.

(iii) Assume that $A[G]$ is pr-nilary. By Theorem 3.9(iv), A is pr-nilary. Then Theorem 3.21(i) yields that G is a p -group. Conversely, assume that A is pr-nilary and G is a p -group. By Lemma 3.23, A is prime. From Theorem 3.24, $A[G]$ is a pr-nilary ring.

□

Corollary 3.30 *Assume that G is a finite group and A is a ring. Then G is nilpotent and $A[G]$ is (pr-)nilary if and only if G is a p -group, $\text{char}(A) = p^\alpha$ (p is a prime), and A is (pr-)nilary.*

Proof This result follows from Theorems 3.9(iv), 3.21(i), 3.29(i). □

Theorem 3.31 *Let G be a finite supersolvable group such that q is the smallest prime dividing $|G|$; and A be a ring such that $\text{char}(A) = q^\alpha$ for some positive integer α . Then, $A[G]$ is (pr-)nilary if and only if A is (pr-)nilary and G is a p -group for some prime p .*

Proof (\Rightarrow) Assume $A[G]$ is (pr-)nilary. Then A is (pr-)nilary by Theorem 3.9(iv). Suppose G is not a p -group. Then G is not a q -group. From (p.16, Theorem 4.24, Subgroup series 2, <https://kconrad.math.uconn.edu/blurbs/grouptheory/subgppseries2.pdf>), the set N of all elements of order prime to q forms a normal subgroup of G . Since G is not a q -group, $|N| > 1$. By Theorem 3.10(i), $|N|$ is nilpotent in A . Hence, q divides $|N|$, a contradiction. Therefore, G is a p -group.

(\Leftarrow) Assume A is (pr-)nilary and G is a p -group. Then G is a q -group. The result follows from Corollary 3.27. □

Observe that the condition, $\text{char}(A) = q^\alpha$, where q is the smallest prime divisor of $|G|$ is not superflous in Theorem 3.31. For example, S_3 is a finite supersolvable group. Let $R_1 = \mathbb{Z}_3[S_3]$. From Example 4.13 or [2, Example 2.13], R_1 is nilary; but S_3 is not a p -group. Also, from Theorem 3.31, we can conclude that $R_2 = \mathbb{Z}_{2^m}[S_3]$ is not nilary for any positive integer m .

4. Nilary group algebra

In this section, we determine necessary and/or sufficient conditions for a group algebra, $R = F[G]$, to be pr-nilary or nilary in terms of properties of the field F and the group G . For example, we show: (1) If $\text{char}(F) = 0$, then R is prime $\Leftrightarrow R$ is (pr-)nilary $\Leftrightarrow G$ is prime (Proposition 4.3). (2) If G is finite, then R is local $\Leftrightarrow \text{char}(F) = p$ and G is a p -group $\Leftrightarrow J(R) = \Delta(G) \Leftrightarrow R$ is strongly pr-nilary $\Leftrightarrow G$ is nilpotent and $F[G]$ is nilary (Theorem 4.5). In the remainder of the section, we consider R where $\text{char}(F) = p$ and G is a finite solvable group. We apply our results to show that $\mathbb{Z}_3[S_3]$, $\mathbb{Z}_2[A_4]$, and $\mathbb{Z}_2[S_4]$ are nilary group rings which are not rings with a nilpotent prime ideal, hence neither prime nor local.

Proposition 4.1 *Let F be a field with $\text{char}(F) = 2$ and G be a finite simple nonabelian group. If $F[G]$ is nilary, then G is either M_{22} or M_{24} .*

Proof

Since any nilary ring is indecomposable as a ring, then it is free of central idempotents. Hence, for any group algebra $F[G]$, the principal block is the unique block of G over F . So if the group algebra $F[G]$ has more than one block, then it is not nilary. By using [6, Theorem 4], we find that G is either M_{22} or M_{24} . \square

Proposition 4.2 *Let F be a field, $\text{char}(F) = 0$; and let G be a group. Then the following conditions are equivalent:*

- (i) G is a prime group;
- (ii) $F[G]$ is a (pr-)nilary ring;
- (iii) $F[G]$ is a prime ring.

Proof (iii) \Rightarrow (ii) Clear. (ii) \Rightarrow (i) Since $\text{char}(F) = 0$, $F[G]$ is semiprime. By [2, Proposition 1.3(i)], $F[G]$ is prime. From Lemma 3.7, G is prime. (i) \Rightarrow (iii) Use Lemma 3.7. \square

Proposition 4.3 *Let G be a finite group, F a field and $R = F[G]$.*

- (i) R is a prime ring if and only if $|G| = 1$.
- (ii) Assume $|G| > 1$. If R is nilary then $\text{char}(F) = p$, a prime integer, and p divides $|G|$. Also, $O_{p'}(G) = 1$.
- (iii) If R is indecomposable and $|G| > 1$, then $\text{Soc}(R_R) \leq^{ess} J(R) \leq^{ess} \Delta(G) \leq^{ess} R_R$ and $\text{Soc}({}_R R) \leq^{ess} J(R) \leq^{ess} \Delta(G) \leq^{ess} {}_R R$.

Proof (i) This part follows from [5, p.675].

(ii) This part is a consequence of Theorem 3.10.

(iii) Since $R = \frac{R}{\Delta(G)} \cong F$, $\Delta(G)$ is a maximal left and right ideal. Hence, $J(R) \subseteq \Delta(G)$. From [10, Example 16.56], R is a Frobenius algebra (hence a QF-ring). Now the result follows from [1, Lemma 1.6]. \square

Theorem 4.4 *Let F be a field.*

- (i) Assume that $F[G]$ is (pr-)nilary and G is a nilpotent torsion group. Then $\text{char}(F) = p$ (a prime) and G is a p -group.
- (ii) Assume that $\text{char}(F) = p$ (a prime), and G is a locally normal p -group. Then $F[G]$ is a pr-nilary local ring.

Proof (i) This part follows from Theorem 3.21.

(ii) By Theorem 3.24, $F[G]$ is a pr-nilary ring. From Lemma 3.1, $\sqrt{0_{F[G]}} = P(F[G]) = \Delta(G)$. Since $\frac{F[G]}{\Delta(G)} \cong F$, $\Delta(G)$ is a maximal left and right ideal of $F[G]$. Then $P(F[G]) \subseteq J(F[G]) \subseteq \Delta(G) \subseteq P(F[G])$, so $J(F[G]) = \Delta(G)$. By [9, Theorem 19.1], $F[G]$ is a local ring. \square

From [4], a ring is called an idempotent fine ring (denoted IF-ring) if each of its nonzero idempotents is a sum of a nilpotent element and a unit (e.g., any ring with trivial idempotents such as local rings and domains).

Theorem 4.5 *Let $R = F[G]$ where F is a field and G is a finite group such that $|G| > 1$. The following statements are equivalent.*

- (i) R is a local ring.
- (ii) R_R is indecomposable.
- (iii) $\frac{R}{J(R)}$ is a simple ring.
- (iv) $\text{char}(F) = p$ and G is a p -group, where p is a prime.
- (v) $J(R) = \Delta(G)$.
- (vi) R is an IF-ring.
- (vii) R is strongly pr-nilary.
- (viii) R has a nilpotent prime ideal.
- (ix) G is nilpotent and R is nilary.
- (x) G is nilpotent and R is right primary.

Proof (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) These equivalences are in [9, p. 294, Excercise 19.4].

(i) \Rightarrow (v) From Proposition 4.3(iii), $J(R) \subseteq \Delta(G)$. Since R is local, $J(R) \subseteq \Delta(G)$.

(v) \Rightarrow (i) Assume $J(R) = \Delta(G)$. Then $\frac{R}{J(R)} = \frac{R}{\Delta(G)} \cong F$. By [9, Theorem 19.1], R is local.

(i) \Rightarrow (vi) Clear.

(vi) \Rightarrow (vii) From [4, Proposition 7], every IF-ring is idempotent simple (i.e. $R = ReR$ for each $0 \neq e = e^2 \in R$). The concept of an idempotent-simple ring has been previously defined in [1, Definition 2.4] as a full ring. Since R is a QF-ring [10, Example 16.56], [1, Theorem 2.10] yields that R is strongly pr-nilary.

(vii) \Rightarrow (viii) Assume that R is strongly pr-nilary. Since R is right Artinian, $\sqrt{0_R}$ is nilpotent. From Proposition 2.5(ii), R has a nilpotent prime ideal.

(viii) \Rightarrow (vii) This implication follows from Proposition 2.8(i).

(vii) \Rightarrow (iii) Since R is strongly pr-nilary, $\sqrt{0_R}$ is a prime ideal. Because R is right Artinian, $J(R)$ is nilpotent. So $\sqrt{0_R} \subseteq P(R) \subseteq J(R) \subseteq \sqrt{0_R}$; hence, $J(R)$ is a prime ideal. Therefore, $\frac{R}{J(R)}$ is simple.

(iv) \Rightarrow (ix) This implication follows from Theorem 4.4(ii), since G is a finite p -group.

(ix) \Rightarrow (iv) This implication follows from Theorem 3.21(ii).

(vii) \Rightarrow (x) This implication follows from [2, Lemma 3.13].

(x) \Rightarrow (ix) This implication is clear from the definitions of right primary and nilary rings. □

Note that in Theorem 4.5 if R is local, then $\frac{R}{J(R)} \cong F$, since $J(R) = \Delta(G)$.

Corollary 4.6 *Let F be a field, G be a group and H a nontrivial locally finite subgroup of $Z(G)$. If $F[G]$ or $F[Z(G)]$ is (pr-)nilary, then H is a p -group, $\text{char}(F) = p$ (a prime) and $F[H]$ is a (pr-)nilary local ring.*

Proof

From Corollary 3.7, $F[H]$ is (pr-)nilary. Let $1 \neq x \in H$, and X be the subgroup generated by x . From Theorem 4.5, $\text{char}(F) = p$ and X is a p -group. Since x was arbitrary, H is a p -group. By Theorem 4.4, $F[H]$ is local. □

Proposition 4.7 *Let G be a group, H be a finite normal p -subgroup of G , K be a group such that $\frac{G}{H} \cong K$ and F be a field such that $\text{char}(F) = p$, where p is a prime. If $F[K]$ is (pr-)nilary, then $F[G]$ is (pr-)nilary.*

Proof This proof is a consequence of Proposition 3.6(ii). □

Corollary 4.8 *Let $G = H \rtimes K$ where H is a finite normal p -subgroup of G and F is a field such that $\text{char}(F) = p$, where p is a prime. If $F[K]$, is (pr-)nilary, then $F[G]$ is (pr-)nilary.*

Example 4.9 *Let $G = C_{3^m} \times S_3$ where $m \in \mathbb{N}$, and $F = \mathbb{Z}_3$. From [1, Example 2.13], $F[S_3]$ is nilary. Therefore, $F[G]$ is nilary by Corollary 4.8.*

Proposition 4.10 *Let G be a finite supersolvable group such that q is the smallest prime dividing $|G|$, and F is a field with $\text{char}(F) = q$. Then $F[G]$ is nilary if and only if G is a q -group.*

Proof This result is a corollary of Theorem 3.31. □

Theorem 4.11 *Let $R = F[G]$, where F is a field; and G is a group. Assume H is a finite subnormal subgroup of G such that $F[H]$ is nilary.*

(i) *If $H \neq \{1\}$, then $J(F[H])R \subseteq J(R)$ and $J(F[H])R$ is left and right essential in R .*

(ii) *If I is a nonnilpotent ideal of $F[H]$, then IR is right essential in R . In particular, ReR is left and right essential in R for all $0 \neq e = e^2 \in F[H]$.*

(iii) *If $\{e_i | 1 \leq i \leq m\} \subseteq F[H]$ is an orthogonal set of idempotents such that e_1R, \dots, e_mR represents a complete set of isomorphism classes of the principal indecomposable modules of R , then R is nilary.*

Proof Note that $F[H]$ is a Frobenius algebra (hence a QF-ring) [10, pp. 442-443, Example 16.56].

(i) Using the subnormality of H , Proposition 4.3(iii) and an induction argument with [9, p.137, Excercise 8.5] and [12, p.467, Excercise 27], we obtain the result.

(ii) Since $F[H]$ is a QF-ring, [1, Theorem 2.2] yields that each nonnilpotent ideal I of $F[H]$ is left and right essential in $F[H]$. Now the result follows from the subnormality of H and [12, p 467, Excercise 27]

(iii) This part is a consequence of part (ii) and [1, Theorem 2.2]. □

For the ring $R = F[G]$ where F is a field and G is a finite solvable group, our next result provides a method for determining nilary subrings of R of the form $F[H]$ where H is a subnormal subgroup of G . The three examples following this result illustrate this method. Moreover, the examples are nilary group algebras which are not rings with a nilpotent prime ideal, hence neither prime nor local.

Theorem 4.12 *Let G be a finite solvable group which is not a p -group, p a prime, H_0 be a nontrivial subnormal p -subgroup of G , F be a field such that $\text{char}(F) = p$ and $R = F[G]$. Then there exists a composition series,*

$$\{1\} \trianglelefteq \dots \trianglelefteq H_0 \trianglelefteq \dots \trianglelefteq H_k \trianglelefteq \dots \trianglelefteq H_m \trianglelefteq \dots \trianglelefteq H_n = G$$

such that $R_i = F[H_i]$ and:

(i) H_0, \dots, H_{k-1} are p -groups;

(ii) R_i is a nilary local ring for all $i = 0, 1, \dots, k - 1$;

(iii) $|\frac{H_k}{H_{k-1}}| = q$, where q is a prime and $q \neq p$;

(iv) $\frac{F[H_k]}{\Delta(H_k, H_{k-1})} \cong F[\frac{H_k}{H_{k-1}}] \cong F[C_q] \cong \bigoplus_{j=1}^t F(j)$ where each $F(j)$ is a cyclotomic extension of F ;

(v) there exists a complete set, $\{e_1, \dots, e_t\}$, of orthogonal primitive idempotents of $F[H_k]$, so, $u.\text{dim}(R_k) = t$;

(vi) $J(R_k) = \Delta(H_k, H_{k-1})$, and R_k is a basic ring.

(vii) If $R_k e_j R_k$ is right essential in R_k for each e_j and each e_j is primitive in R_m for $j = 1, \dots, t$ where $m \geq k$, then R_m is nilary.

Proof It is well known that the indicated composition series exists, see [14, p.80], and that it satisfies condition(i).

(ii) This part follows from Theorem 4.5.

(iii) This part is a property of the composition series [14, p.75].

(iv) The first isomorphism is due to [11, Corollary 3.3.5]. The second isomorphism follows from part (iii). For the third, see [11, p.144-145].

(v) From Proposition 3.6, $\Delta(H_k, H_{k-1})$ is nilpotent. Hence, the complete set of orthogonal primitive idempotents from the decomposition of $F[C_q]$ lift to $F[H_k]$, see [9, pp.319-321]. From [10, pp.442-443, Example 16.56], R_k is a symmetric algebra, hence a QF-ring. So each $e_f R_k$ is the injective hull of a minimal right ideal and $Soc(R_k R_k)$ is right essential in R_k . Therefore, $u.dim(R_k) = t$.

(vi) Since $\Delta(H_k, H_{k-1})$ is nilpotent and $J(\frac{R_k}{\Delta(H_k, H_{k-1})}) = 0$ by part(iv), then $\Delta(H_k, H_{k-1}) = J(R_k)$, see [9, p.51, Proposition 4.6]. From [9, Proposition 25.10], R_k is a basic ring.

(vii) This part follows from Theorem 4.11(iii). □

The following three examples of nilary nonlocal group algebras are applications of Theorem 4.12. Throughout these examples, we use the notation of Theorem 4.12, and σ denotes the permutation (123). Recall from [1, Definition 2.4], R is antifull, if R has a nontrivial idempotent and for each $1 \neq e = e^2 \in R$, $ReR \neq R$.

Example 4.13 Let $R = \mathbb{Z}_3[S_3]$. Then R is a nilary nonlocal basic antifull group algebra. To see this, observe that $H_0 = \langle \sigma \rangle = \{1, \sigma, \sigma^2\}$, and $R_0 = \mathbb{Z}_3[C_3]$ is a nilary local ring by Theorem 4.5. Next, we have

$$\frac{\mathbb{Z}_3[S_3]}{\Delta(S_3, H_0)} \cong \mathbb{Z}_3[C_2] \cong \frac{\mathbb{Z}_3[x]}{(x+1)} \oplus \frac{\mathbb{Z}_3[x]}{(x-1)} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3.$$

Then $\{e, 1 - e\}$ is a complete set of orthogonal primitive idempotents of R , where $e = \sigma + \sigma^2$; and $u.dim(R) = 2$. $J(R) = \Delta(S_3, H_0)$, and R is basic by [9, Proposition 25.10]. Since $ReR = eR \oplus (1 - e)R \cap ReR$ and $(1 - e)R \cap ReR \neq 0$, $ReR \leq^{ess} R_R$. Similarly, $R(1 - e)R \leq^{ess} R_R$. So R is nilary. Alternatively, one can show that e is not central, so $1 - e$ is not central; hence, R is indecomposable. By [1, Corollary 2.3], R is nilary. R is antifull from [1, Proposition 2.5].

Example 4.14 Let $R = \mathbb{Z}_2[A_4]$. Then R is a nilary nonlocal basic antifull group algebra. To see this, observe that $H_0 = \{1, (12)(34), (13)(24), (14)(23)\} \cong K_4$, and $R_0 = \mathbb{Z}_2[K_4]$ is a nilary local ring by Theorem 4.5. Next,

$$\frac{\mathbb{Z}_2[A_4]}{\Delta(A_4, H_0)} \cong \mathbb{Z}_2[C_3] \cong \frac{\mathbb{Z}_2[x]}{(x-1)} \oplus \frac{\mathbb{Z}_2[x]}{(x^2+x+1)}$$

Then $\{e, 1 - e\}$ is a complete set of orthogonal primitive idempotents of R where $e = \sigma + \sigma^2$; and $u.dim(R) = 2$. $J(R) = \Delta(A_4, H_0)$, and R is basic by [9, Proposition 25.10]. Again neither e nor $1 - e$ is central in R . By [1, Corollary 2.3], R is nilary, and R is antifull from [1, Proposition 2.5].

Example 4.15 Let $R = \mathbb{Z}_2[S_4]$. Then, R is a nilary non-local group algebra. To see this, observe

$$\{1\} \trianglelefteq \{1, (12)(34)\} \trianglelefteq H_0 \trianglelefteq H_1 \trianglelefteq H_2 = S_4$$

is a composition series where H_0 is as in Example 4.14, $H_1 = A_4$ and $H_2 = S_4$. Then,

$$\frac{\mathbb{Z}_2[S_4]}{\Delta(S_4, H_0)} \cong \mathbb{Z}_2\left[\frac{S_4}{H_0}\right] \cong \mathbb{Z}_2[S_3]$$

It can be shown by computer or hand calculations that $\{e, 1 - e\}$ is a complete set of orthogonal primitive idempotents of $\mathbb{Z}_2[S_3]$, where $e = \sigma + \sigma^2$. Since $\Delta(S_4, H_0)$ is nilpotent by Proposition 3.6, $\{e, 1 - e\}$ is a complete set of orthogonal primitive idempotents of $\mathbb{Z}_2[S_4]$. From Example 3.14, $R_1 = \mathbb{Z}_2[A_4]$ is nilary, so $R_1 e R_1 \leq^{ess} R_{1R_1}$ and $R_1(1 - e)R_1 \leq^{ess} R_{1R_1}$, from [1, Theorem 2.2]. By Theorem 4.12(vii), $R_2 = R$ is nilary and nonlocal.

Corollary 4.16 Let F be a field with $\text{char}(F) = p$, and G be a finite group such that $G = C_2 \times G_p$ where G_p is a p -group and C_2 is not normal in G . Then $R = F[G]$ is a nilary ring.

Proof If $p = 2$ then G is a p -group. By Theorem 4.5, R is a nilary ring. Assume $p > 2$. Then G is solvable. By using the notation of Theorem 4.12, $O_p(G) = H_0 = H_{k-1} = G_p$, $H_1 = H_k = G$, $q = 2$. Then,

$$\frac{F[H_k]}{\Delta(H_k, H_{k-1})} = \frac{F[G]}{\Delta(G, G_p)} \cong F[C_2].$$

Hence, $u.\dim(F[C_2]) \leq \dim(F[C_2]) = 2$. So $F[G]$ has a complete set $\{e, 1 - e\}$, of orthogonal primitive idempotents, by Theorem 4.12. Now, we claim that $F[G]$ is indecomposable. Indeed, since G has order $2p^n$ and G has a normal p -Sylow subgroup, then G is p -constrained and it is clear that $O_{p'}(G) = 1$. Therefore, by using [8, p.112, Proposition 1.12], we have that $F[G]$ is indecomposable. Since $F[G]$ is QF-ring, by our claim and by using [1, Corollary 2.3], we get that $F[G]$ is nilary. \square

Note that for $p \neq 2$, R is neither local nor prime.

Corollary 4.17 Let F be a field with $\text{char}(F) = p$, and G be the dihedral group $G = D_{p^n}$ for a positive integer n and a prime p . Then $F[G]$ is a nilary ring. In particular, the ring $F[S_3]$ is nilary, for any field F with $\text{char}(F) = 3$.

Open Problem:

Characterize the nilary group algebras, $F[G]$, where $\text{char}(F) = p$ (a prime) in terms of properties of G and F . Note that the case for $\text{char}(F) = 0$ is included in Proposition 4.2.

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