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SAMED ÖZKAN

MUAMMER KULA

SÜMEYYE KULA

TESNİM MERYEM BARAN

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Closure operators, irreducibility, Urysohn's lemma, and Tietze extension theorem for proximity spaces

Samed ÖZKAN[1], **Muammer KULA**[2,*], **Sümeyye KULA**[2], **Tesnim Meryem BARAN**[3]
[1]Department of Mathematics, Nevşehir Hacı Bektaş Veli University, Nevşehir, Turkey
[2]Department of Mathematics, Erciyes University, Kayseri, Turkey
[3]Republic of Türkiye Ministry of National Education (MEB), Kayseri, Turkey

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Abstract: In this paper, we introduce two notions of closure in the category of proximity spaces which satisfy (weak) hereditariness, productivity, and idempotency, and we characterize each of $T_i, i = 0, 1, 2$, proximity spaces by using these closure operators and show how these subcategories are related. Furthermore, we characterize the irreducible proximity spaces and investigate the relationship among each of irreducible, connected and $T_i, i = 1, 2$, proximity spaces. Finally, we present Tietze extension theorem and Urysohn's lemma for proximity spaces.

Key words: Topological category, proximity space, closure operators, irreducible objects

1. Introduction

The concept of proximity was described by Efremovich during the first part of 1930s and axiomatized in 1951 [16] under the name of infinitesimal space. Smirnov [25] was responsible for a lot of the early work on proximity spaces. He was also the first to discover the relationship between uniformity and proximity which are two important notions close to topology and have rich topological properties. Proximity spaces provide a level of structure in between topologies and uniformities. On the theory of proximity spaces, the most comprehensive study was done by Naimpally and Warrack [22].

For arbitrary set-based topological categories, the concepts of closedness and strong closedness are presented by Baran [2] and he used these notions to generalize some fundamental topological concepts such as compactness [5], separation axioms T_i, $i = 0, 1, 2, 3, 4$ [2], and connectedness [6].

In general topology, one of the most important uses of separation axioms is theorems such as Tietze extension theorem and Urysohn's lemma. In view of this, these results are presented in **pqsMet**, the category of extended pseudo-quasi-semi metric spaces [13] and **ConFCO**, the category of constant filter convergence spaces [12].

The aim of this paper is stated below:

(i) to introduce two notions of closure in **Prox**, the category of proximity spaces and to characterize each of $T_i, i = 0, 1, 2$ proximity spaces by using these closure operators as well as to show how these subcategories are related,

*Correspondence: kulam@erciyes.edu.tr
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(ii) to characterize the irreducible proximity spaces and to investigate the relationship among each of $T_i, i = 0,1,2$, irreducible, connected proximity spaces and the subcategories $\mathbf{Prox}_{ic}$ and $\mathbf{Prox}_{isc}$, $i = 1,2$, where $\mathbf{c}$ and $\mathbf{sc}$ are closure operators defined in Definition 3.7,

(iii) to present Tietze extension theorem and Urysohn's lemma for proximity spaces.

2. Preliminaries

Definition 2.1 *[22] Let X be a set and δ be a binary relation on $P(X)$, the power set. A proximity (Efremovich-proximity) space is a pair (X,δ) that provides the following conditions: For $A,B \subset X$,*

(1_P) $(A,B)\in\delta$ iff $(B,A)\in\delta$,

(2_P) $(A,B\cup C)\in\delta$ iff $(A,B)\in\delta$ or $(A,C)\in\delta$,

(3_P) $(A,B)\in\delta$ implies $A,B\neq\emptyset$,

(4_P) $A\cap B\neq\emptyset$ implies $(A,B)\in\delta$,

(5_P) $(A,B)\notin\delta$ implies there is an $E\subseteq X$ such that $(A,E)\notin\delta$ and $(X-E,B)\notin\delta$,

A function $f:(X,\delta)\to(Y,\delta')$ is called a p-map (or proximity mapping) iff $(f(A),f(B))\in\delta'$ whenever $(A,B)\in\delta$ for $A,B\subset X$. Equivalently, f is a proximity mapping if and only if $(f^{-1}(M),f^{-1}(N))\notin\delta$ whenever $(M,N)\notin\delta'$ for $M,N\subset Y$.

Remark 2.2 *The category of proximity spaces and p-maps denoted by **Prox**, and it is a topological category [1, 19] over **Set**, the category of sets and functions.*

In a proximity space (X,δ), we write $A\ll B$ if and only if $(A,X-B)\notin\delta$. The relation $\ll$ is called p-neighborhood relation or the strong inclusion. When $A\ll B$, we say that B is a p-neighborhood of A or A is strongly contained in B [18] or [22].

Definition 2.3 *([22] p. 9) Let X be a nonempty set and $A,B\subset X$.*

The discrete proximity relation δ_D on X is stated by

$$(A,B)\in\delta_D \iff A\cap B\neq\emptyset.$$

The indiscrete proximity relation δ_I on X is stated by

$$(A,B)\in\delta_I \iff A\neq\emptyset \text{ and } B\neq\emptyset.$$

Lemma 2.4 *Let $X\neq\emptyset$ be a set, $\{(X_i,\delta_i): i\in I\}$ be a family of proximity spaces and $f_i : X\to X_i$ be a source in* **Set***. A relation β on $P(X)$ is defined as: For $A,B\in P(X)$, $(A,B)\in\beta$ if and only if $(f_i(A),f_i(B))\in\delta_i$ for all $i\in I$, where β is a proximity base on X (theorem 3.8, [24]). The initial proximity relation δ on X generated by β is stated by:*

For $A,B\in P(X)$, $(A,B)\in\delta$ if and only if for any finite covers $\{A_j : 1\leq j\leq m\}$ and $\{B_k : 1\leq k\leq n\}$ of A and B respectively, then there exists a pair (j,k) such that $(A_j,B_k)\in\beta$ [24].

Definition 2.5 *Let (X,δ) be a proximity space and f be a function from (X,δ) onto a nonempty set Y. The strong inclusion $\ll^*$ induced by the finest proximity δ^* (the quotient proximity) on Y making f proximally continuous is stated by:*

For every $A,B\subset Y$, $A\ll^* B$ if and only if there is some $C_s\subset Y$ for each binary rational s in $[0,1]$ such that $C_0=A$, $C_1=B$ and $s<t$ implies $f^{-1}(C_s)\ll_\delta f^{-1}(C_t)$ ([18] or [26] p. 276), where $\ll_\delta$ represents the strong inclusion induced by the relation δ on X. In addition, if $f:(X,\delta)\to(X,\delta^*)$ be a one-to-one p-quotient map, then $(A,B)\in\delta^*$ iff $(f^{-1}(A),f^{-1}(B))\in\delta$ ([18] p. 591).

Lemma 2.6 *Let X be a set, $p\in X$ and $X\vee_p X$ be the wedge at p [2]. An epi sink $\{i_1,i_2:(X,\delta)\to(X\vee_p X,\delta')\}$, i_1,i_2 are p-maps and the canonical injections, in* **Prox** *is a final lift iff the following statement is provided: For $A,B\in X\vee_p X$,*

(i) If A and B are in the different component of $X\vee_p X$, $(A,B)\in\delta'$ if and only if there exist sets C,D in X such that $(C,\{p\})\in\delta$ and $(\{p\},D)\in\delta$ with $i_k^{-1}(A)=C$ and $i_j^{-1}(B)=D$ for $k,j=1,2$ and $k\neq j$.

(ii) If A and B are in the same component of the wedge, then $(A,B)\in\delta'$ if and only if there exist sets C,D in X such that $(C,D)\in\delta$ and $i_k^{-1}(A)=C$ and $i_k^{-1}(B)=D$ for some $k=1,2$.

(iii) Specially, if $i_k(C)=A$ and $i_k(D)=B$, then $(i_k(C),i_k(D))\in\delta'$ iff $(i_k^{-1}(i_k(C)),i_k^{-1}(i_k(D)))=(C,D)\in\delta$.

3. Closure operators

Let E be a set, $p\in E$ and $E^\infty=E\times E\times ...$ be the countable cartesian product of E. The infinite wedge product $\vee_p^\infty E$ is defined by identifying countably many disjoint copies of E at the point p.

Define the infinite principal p-axis map [2, 3],

$$A_p^\infty:\vee_p^\infty E\to E^\infty,\qquad A_p^\infty(x_i)=(p,p,...,p,x,p,...),$$

where x_i is in the i-th component of $\vee_p^\infty E$ and x is in the i-th place in $A_p^\infty(x_i)$, and the infinite fold map [2, 3],

$$\nabla_p^\infty:\vee_p^\infty E\to E,\qquad \bigtriangledown_p^\infty(x_i)=x,\ \ \forall i\in I.$$

Note that the map A_p^∞ is the unique map arising from the multiple pushout of $p:1\to E$ for which $A_p^\infty i_j=(p,p,...,p,id,p,...):E\to E^\infty$, where id in the j-th place is the identity map and 1 is the terminal object in **Set** [4].

Let $\mathcal{U}:\mathcal{E}\to\textbf{Set}$ be a topological functor, $X\in Ob(\mathcal{E})$ and $\mathcal{U}(X)=E$. Let $\emptyset\neq F\subset E$. The final lift of the epi $\mathcal{U}$-sink

$$q:\mathcal{U}(X)=E\to E/F=(E\backslash F)\cup\{*\}$$

is denoted by X/F, where q is the epi map that identifying F with a point $*$ and is the identity on $E\backslash F$ [2].

Definition 3.1 *(cf. [2, 3])*

1. *A point $p\in E$ is closed provided that the initial lift of the $\mathcal{U}$-source*

$$\{A_p^\infty:\vee_p^\infty E\to\mathcal{U}(X^\infty)=E^\infty \text{ and } \nabla_p^\infty:\vee_p^\infty E\to\mathcal{U}\mathcal{D}(E)=E\}$$

is discrete.

2. $K \subset X$ *is closed provided that the image of* K, $\{*\}$, *is closed in* X/K.

3. $K \subset X$ *is strongly closed provided that* X/K *is* T_1 *at* $\{*\}$.

4. *If* $E = K = \emptyset$, *then* K *is to be both closed and strongly closed.*

Remark 3.2 *In* **Top**, *the notion of closedness coincides with the usual one [2, 3] and* K *is strongly closed iff* K *is closed and there exists a neighbourhood of* K *missing* x *for each* $x \notin K$. *For a* T_1 *topological space, the notions of closedness and strong closedness coincide [2, 3].*

Theorem 3.3 *[20] Let* (X, δ) *be an object in* **Prox**, $p \in X$ *and* $\emptyset \neq K \subset X$.

1. $\{p\}$ *is closed in* X *provided that for any* $E \subset X$, *if* $(\{p\}, E) \in \delta$, *then* $p \in E$.

2. *The following expressions are equivalent.*

   (a) K *is closed.*

   (b) K *is strongly closed.*

   (c) $x \in K$ *whenever* $(\{x\}, K) \in \delta$ *for all* $x \in X$.

3. K *is (strongly) open provided that* $x \in K^c$ *whenever* $(\{x\}, K^c) \in \delta$ *for all* $x \in X$.

Example 3.4 *Let* $X = \{a, b, c\}$. *The following relations* δ_i $(i = 1, 2, 3)$ *on* $P(X)$ *are proximity relations.*

$$\delta_1 = \{(A, B) \in P^2(X) \mid A \cap B \neq \emptyset\}$$

$$\delta_2 = \delta_1 \;\cup\; \{(\{a\}, \{b\}), (\{b\}, \{a\}), (\{a\}, \{b, c\}), (\{b, c\}, \{a\}), (\{b\}, \{a, c\}), (\{a, c\}, \{b\})\}$$

$$\begin{aligned}\delta_3 = \delta_1 \;\cup\; \{&(\{a\}, \{b\}), (\{b\}, \{a\}), (\{a\}, \{c\}), (\{c\}, \{a\}), (\{b\}, \{c\}), (\{c\}, \{b\}), (\{a\}, \{b, c\}), (\{b, c\}, \{a\}),\\ &(\{b\}, \{a, c\}), (\{a, c\}, \{b\}), (\{c\}, \{a, b\}), (\{a, b\}, \{c\})\}\end{aligned}$$

For (X, δ_1), *all subsets of* X *are (strongly) closed since* δ_1 *is discrete.*
For (X, δ_2), *the family of all (strongly) closed subsets is given below:*

$$\{\emptyset, X, \{c\}, \{a, b\}\}.$$

For (X, δ_3), *the only (strongly) closed subsets are* $\emptyset$ *and* X *since* δ_3 *is indiscrete.*

Theorem 3.5

1. *Let* (X, δ_1) *and* (Y, δ_2) *be proximity spaces and* $f : (X, \delta_1) \to (Y, \delta_2)$ *be a* p*-map. If* $K \subset Y$ *is (strongly) closed, then* $f^{-1}(K) \subset X$ *is (strongly) closed.*

2. *If* $K \subset L$ *and* $L \subset X$ *are (strongly) closed in a proximity space* (X, δ), *so also is* $K \subset X$.

Proof

1. Suppose $K \subset Y$ is (strongly) closed and for any $x \in X$, $(\{x\}, f^{-1}(K)) \in \delta_1$. We will show that $x \in f^{-1}(K)$. Since f is a proximity map and $(\{x\}, f^{-1}(K)) \in \delta_1$, then $(f(\{x\}), f(f^{-1}(K))) \in \delta_2$ and by the condition (2_P) of Definition 2.1, $(f(\{x\}), K) \in \delta_2$. We have $f(x) \in K$ since K is (strongly) closed. It follows that $x \in f^{-1}(K)$, and we obtain $f^{-1}(K) \subset X$ is (strongly) closed.

2. Suppose $K \subset L$ and $L \subset X$ are (strongly) closed. We will show that $K \subset X$ is (strongly) closed. Let δ_L be the subspace structure on L induced by the inclusion map $i : L \to (X, \delta)$. Suppose δ_K is the subproximity structure on K induced by the inclusion map $i : K \to (L, \delta_L)$. Let $a \in K$. Since $a \in L$ and $L \subset X$ is (strongly) closed, by Theorem 3.3, $(\{a\}, L) \in \delta_L$. It follows $(i^{-1}(\{a\}), i^{-1}(L)) = (\{a\}, K) \in \delta_K$. Since $a \in K$, by Theorem 3.3, $K \subset L$ is (strongly) closed. Hence, $K \subset X$ is (strongly) closed by Theorem 3.3.

□

Let $\mathcal{E}$ be a set-based topological category and cl be a closure operator of $\mathcal{E}$.

1. $\mathcal{E}_{0cl} = \{X \in \mathcal{E} \mid x \in cl(\{y\}) \text{ and } y \in cl(\{x\}) \implies x = y \text{ with } x, y \in X\}$ [15].

2. $\mathcal{E}_{1cl} = \{X \in \mathcal{E} \mid cl(\{x\}) = \{x\}, \text{ for each } x \in X\}$ [15].

3. $\mathcal{E}_{2cl} = \{X \in \mathcal{E} \mid cl(\Delta) = \Delta, \text{ the diagonal}\}$ [15].

Remark 3.6 *For $\mathcal{E} = \mathbf{Top}$ and cl, the ordinary closure, $\mathbf{Top}_{icl}$ reduce to the class of T_i spaces for $i = 0, 1, 2$.*

Definition 3.7 *Let (X, δ) be a proximity space and $K \subset X$.*

(i) $\mathbf{c}(K) = \bigcap \{U \subset X \mid K \subset U \text{ and } U \text{ is closed}\}$ *is called the closure of K.*

(ii) $\mathbf{sc}(K) = \bigcap \{U \subset X \mid K \subset U \text{ and } U \text{ is strongly closed}\}$ *is called the strong closure of K.*

It is shown that the notion of closedness forms closure operator [14] in some topological categories [5, 8, 11, 17, 23].

Theorem 3.8 $\mathbf{c}$ *and* $\mathbf{sc}$ *are (weakly) hereditary, productive and idempotent closure operators in* $\mathbf{Prox}$.

Proof Combine Theorem 3.5, Definition 3.7, and theorems 2.3, 2.4, proposition 2.5, exercise 2.D of [15]. □

Theorem 3.9 *$(X, \delta) \in \mathbf{Prox_{0c}}$ iff for any $x, y \in X$ with $x \neq y$, there exists a closed subset $K \subset X$ such that $x \notin K$ and $y \in K$ or a closed subset $L \subset X$ such that $x \in L$ and $y \notin L$.*

Proof Suppose $(X, \delta) \in \mathbf{Prox_{0c}}$ and $x, y \in X$ with $x \neq y$. It follows that $x \notin \mathbf{c}(\{y\})$ or $y \notin \mathbf{c}(\{x\})$. Suppose $x \notin \mathbf{c}(\{y\})$. Then, there exists a closed subset $K \subset X$ such that $x \notin K$ and $y \in K$. Similarly, if $y \notin \mathbf{c}(\{x\})$, then there exists a closed subset $L \subset X$ such that $x \in L$ and $y \notin L$.

Conversely, suppose for any $x, y \in X$ with $x \neq y$, there exists a closed subset $K \subset X$ such that $x \notin K$ and $y \in K$ or a closed subset $L \subset X$ such that $x \in L$ and $y \notin L$. If the first case holds, then $x \notin \mathbf{c}(\{y\})$. If the second case holds, then $y \notin \mathbf{c}(\{x\})$. Hence, we have $(X, \delta) \in \mathbf{Prox_{0c}}$. □

Theorem 3.10 *$(X, \delta) \in \mathbf{Prox_{1c}}$ iff $(\{x\}, \{y\}) \notin \delta$ for any $x, y \in X$ with $x \neq y$.*

Proof Suppose $(X,\delta) \in \mathbf{Prox_{1c}}$ and $x, y \in X$ with $x \neq y$. We have $\mathbf{c}(\{x\}) = \{x\}$ for all $x \in X$, i.e. $\{x\}$ is closed. By Theorem 3.3 (1), if $(\{x\},\{y\}) \in \delta$ for some $x, y \in X$ with $x \neq y$, then $x \in \{y\}$. This is a contradiction since $\{x\}$ is closed. Hence, we have $(\{x\},\{y\}) \notin \delta$ for any distinct pair $x, y \in X$.

Conversely, suppose for any $x, y \in X$ with $x \neq y$, $(\{x\},\{y\}) \notin \delta$. It follows that for any $E \subset X$, if $(\{x\}, E) \in \delta$, then $x \in E$. By Theorem 3.3 (1), $\{x\}$ is closed, i.e. $\mathbf{c}(\{x\}) = \{x\}$, and consequently $(X,\delta) \in \mathbf{Prox_{1c}}$. □

Theorem 3.11 *$(X,\delta) \in \mathbf{Prox_{2c}}$ iff $(\{x\},\{y\}) \notin \delta$ for any $x, y \in X$ with $x \neq y$.*

Proof Suppose $(X,\delta) \in \mathbf{Prox_{2c}}$ and $x, y \in X$ with $x \neq y$. Note that $(x,y) \notin \Delta$. Since Δ is (strongly) closed, by Theorem 3.3 (2), $(\{(x,y)\},\Delta) \notin \delta^2$ where δ^2 is the product proximity relation on X^2. By Lemma 2.4, for any $x, y \in X$ with $x \neq y$, $(\{x\},\{y\}) \notin \delta$.

Conversely, assume that the condition is satisfied and $(x,y) \in X^2$ with $x \neq y$. Then $(x,y) \notin \Delta$ and by assumption, we have $(\{(x,y)\},\Delta) \notin \delta^2$. We will show that if $(\{(*,*)\}, B^2) \in \delta'^2$, then $(*,*) \in B^2$ for any $B^2 \subset X^2/\Delta$, i.e. Δ is (strongly) closed. Suppose $(*,*) \notin B^2$ for some $B^2 \subset X^2/\Delta$. Since $(\{(*,*)\}, B^2) \in \delta'^2$, there is some $C_s^2 \subset X^2/\Delta$ for each binary rational s in $[0,1]$ such that $C_0^2 = \{*\} \times \{*\}$, $C_1^2 = B \times B$ and $s < t$ implies $((q^{-1} \times q^{-1})(C_s^2), (q^{-1} \times q^{-1})(C_t^2)) \in \delta^2$. It follows that $((q^{-1} \times q^{-1})(\{*\} \times \{*\}), (q^{-1} \times q^{-1})(B \times B)) = (\Delta, B^2) \in \delta^2$ by definition of q-map and Definition 2.5. Since $(\Delta, B^2) \in \delta^2$, there exists $(x,y) \in B^2$ $(x, y \in B$ and $x \neq y)$ such that $(\Delta, \{(x,y)\}) \in \delta^2$ by the condition (2_P) of Definition 2.1. However, for all $(x,y) \in B^2$, $(x,y) \notin \Delta$ since $(*,*) \notin B^2$. Since $(\{x\},\{y\}) \notin \delta$ for any $x, y \in X$ with $x \neq y$, this is a contradiction. Hence, Δ is (strongly) closed, and consequently $(X,\delta) \in \mathbf{Prox_{2c}}$. □

Remark 3.12 *A proximity space $(X,\delta) \in \mathbf{Prox}_{i\mathbf{c}}, i = 1,2$ iff (X,δ) is discrete.*

Theorem 3.13 *If a proximity space $(X,\delta) \in \mathbf{Prox}_{i\mathbf{c}}, i = 1,2$, then $(X,\delta) \in \mathbf{Prox_{0c}}$.*

Proof Suppose $(X,\delta) \in \mathbf{Prox}_{i\mathbf{c}}, i = 1,2$, i.e. $(\{x\},\{y\}) \notin \delta$ for any $x, y \in X$ with $x \neq y$. By Theorem 3.3 (1), $\{x\}$ and $\{y\}$ are closed. Let $K = \{y\}$ or $L = \{x\}$. It follows that $x \notin K$ and $y \in K$ or $x \in L$ and $y \notin L$, and consequently $(X,\delta) \in \mathbf{Prox_{0c}}$. □

Theorem 3.14 *A proximity space $(X,\delta) \in \mathbf{Prox}_{i\mathbf{c}}$ iff $(X,\delta) \in \mathbf{Prox}_{i\mathbf{sc}}$ for $i = 0,1,2$.*

Proof It follows from Theorem 3.3 and Definition 3.7. □

Example 3.15 *The proximity space (X,δ_1) defined in Example 3.4 is in $\mathbf{Prox}_{ik}$, $i = 0,1,2$ and $k = \mathbf{c}$ or $\mathbf{sc}$.*

Remark 3.16 *$\mathbb{T}\,\mathbf{Prox}$ is the full subcategory of $\mathbf{Prox}$ consisting of all $\mathbb{T}$ objects, where $\mathbb{T}$ is $\overline{T}_0$ (resp. T_1, $PreT_2'$, $\overline{T}_2$, T_2') which were defined in [2].*

Theorem 3.17 *[21] Let (X,δ) be a proximity space, then the following are equivalent.*

1. *(X,δ) is $\overline{T}_0$.*

2. (X,δ) *is* T_1.

3. (X,δ) *is* $PreT_2'$.

4. (X,δ) *is* $\overline{T}_2$.

5. (X,δ) *is* T_2'.

6. *For any pair* $x,y \in X$ *with* $x \neq y$, $(\{x\},\{y\}) \notin \delta$.

Theorem 3.18 *The following categories are isomorphic.*

1. $\mathbf{Prox}_{ik}$ *for* $i = 1,2$ *and* $k = \mathbf{c}$ *or* $\mathbf{sc}$.

2. $\mathbb{T}\,\mathbf{Prox}$ *for* $\mathbb{T} = \overline{T}_0, T_1, PreT_2', \overline{T}_2, T_2'$.

Proof It follows from Theorems 3.13, 3.14, 3.17 and Remarks 3.12, 3.16. □

Remark 3.19 1. *By Remark 3.12 and Theorems 3.13, 3.14, we have*

$$\mathbf{Prox}_{2\mathbf{c}} = \mathbf{Prox}_{2\mathbf{sc}} = \mathbf{Prox}_{1\mathbf{c}} = \mathbf{Prox}_{1\mathbf{sc}} \subset \mathbf{Prox}_{0\mathbf{c}} = \mathbf{Prox}_{0\mathbf{sc}}.$$

2. *For the category* **Top**, *by remark 3.5 of [5],*

$$\mathbf{Top}_{2cl} = \mathbf{Top}_{2scl} \subset \mathbf{Top}_{1cl} = \mathbf{Top}_{1scl} \subset \mathbf{Top}_{0cl} = \mathbf{Top}_{0scl}.$$

3. *For the category of preordered spaces,* **Prord**, *by theorem 4.5 of [7],*

$$\mathbf{Prord}_{2cl} = \mathbf{Prord}_{2scl} \subset \mathbf{Prord}_{0cl} = \mathbf{Prord}_{0scl},$$

$$\mathbf{Prord}_{1cl} = \mathbf{Prord}_{1scl} \subset \mathbf{Prord}_{0cl} = \mathbf{Prord}_{0scl}.$$

4. *For the category of bornological spaces,* **Born**, *by Lemma 2.11 of [4],*

$$\mathbf{Born}_{0cl} = \mathbf{Born}_{1cl} = \mathbf{Born}_{2cl} \subset \mathbf{Born}_{0scl} = \mathbf{Born}_{1scl} = \mathbf{Born}_{2scl}.$$

5. *For the category of filter convergence spaces,* **FCO**, *by theorem 2.9 of [4],*

$$\mathbf{FCO}_{2scl} \subset \mathbf{FCO}_{2cl} = \mathbf{FCO}_{1scl} = \mathbf{FCO}_{1cl} \subset \mathbf{FCO}_{0scl} = \mathbf{FCO}_{0cl}.$$

6. *For* **ConFCO**, *the category of constant filter convergence spaces, by remark 4.8 of [12],*

$$\mathbf{ConFCO}_{2cl} = \mathbf{ConFCO}_{2scl} \subset \mathbf{ConFCO}_{1cl} = \mathbf{ConFCO}_{1scl} \subset \mathbf{ConFCO}_{0cl} = \mathbf{ConFCO}_{0scl}.$$

7. *For the category of extended pseudo-quasi-semi metric spaces,* **pqsMet**, *by remark 3.12 of [13],*

$$\mathbf{pqsMet}_{1scl} = \mathbf{pqsMet}_{2scl} \subset \mathbf{pqsMet}_{1cl} = \mathbf{pqsMet}_{2cl},$$

$$\mathbf{pqsMet}_{0scl} \subset \mathbf{pqsMet}_{0cl}.$$

8. *For the category of convergence approach spaces,* **CApp**, *by remark 4.15 of [23],*

$$\mathbf{CApp}_{2scl} \subset \mathbf{CApp}_{1scl} \subset \mathbf{CApp}_{0scl}$$

$$\mathbf{CApp}_{2cl} \subset \mathbf{CApp}_{1cl} \subset \mathbf{CApp}_{0cl}.$$

4. Irreducible proximity spaces

Definition 4.1 *[11] Let $\mathcal{U} : \mathcal{E} \to$ **Set** be a topological functor, X be an object in $\mathcal{E}$.*

1. *X is said to be irreducible if A, B are closed subobjects of X and $X = A \cup B$, then $A = X$ or $B = X$.*
2. *X is said to be strongly irreducible if A, B are strongly closed subobjects of X and $X = A \cup B$, then $A = X$ or $B = X$.*

Irreducibility plays an important role in algebraic geometry. For example, according to a fundamental theorem of classical algebraic geometry, every algebraic set can be expressed in a unique way as a finite union of irreducible components. Also, the Zariski topologies are irreducible.

In **Top**, the notion of irreducibility coincides with the usual irreducibility [11]. Note that if a topological space (X, τ) is irreducible, then (X, τ) is connected, and if (X, τ) is T_1, then the notions of irreducible spaces and strongly irreducible spaces coincide [11]. Additionally, if (X, τ) is nonempty irreducible and T_2, then (X, τ) must be a one-point space [11].

Theorem 4.2 *A proximity space (X, δ) is irreducible if and only if for any nonempty proper subset $K \subset X$, either the condition (1) or (2) holds:*

1. *For some $x \in X$, $x \notin K$ whenever $(\{x\}, K) \in \delta$.*
2. *For some $x \in X$, $x \notin K^c$ whenever $(\{x\}, K^c) \in \delta$.*

Proof Assume that (X, δ) is irreducible but for some $\emptyset \neq K \subset X$, neither of the conditions is satisfied. We have $x \in K$ whenever $(\{x\}, K) \in \delta$ for all $x \in X$, since the first condition is not satisfied. By Theorem 3.3 (2), this means that K is closed. Similarly, since the second condition does not hold, we have for all $x \in X$, $x \in K^c$ whenever $(\{x\}, K^c) \in \delta$, which means that K^c is closed. Hence, $X = K \cup K^c$, but $X \neq K$ and $X \neq K^c$. This is a contradiction.

Conversely, assume that the first condition is satisfied. We have $x \notin K$ whenever $(\{x\}, K) \in \delta$ for some $x \in X$ and by Theorem 3.3 (2), K is not closed. Similarly, assume that the second condition is satisfied. Then we get $x \notin K^c$ whenever $(\{x\}, K^c) \in \delta$ for some $x \in X$, i.e., K^c is not closed. Hence, both open and closed subsets of X are only $\emptyset$ and X. It follows that if A, B are closed subset of X and $X = A \cup B$, then $A = X$ or $B = X$. Thus, (X, δ) is irreducible. □

Theorem 4.3 *A proximity space (X, δ) is irreducible iff (X, δ) is strongly irreducible.*

Proof It follows directly from Theorem 3.3 (2) and Definition 4.1. □

Example 4.4 *Let (X, δ_1), (X, δ_2), and (X, δ_3) be the proximity spaces defined in Example 3.4. Then (X, δ_3) is (strongly) irreducible, but (X, δ_1) and (X, δ_2) are not (strongly) irreducible because neither of the conditions in Theorem 4.2 is held for $F = \{c\}$.*

Theorem 4.5 *A proximity space (X, δ) is (strongly) irreducible iff (X, δ) is (strongly) connected.*

Proof It follows from Theorems 4.2 and 4.3, definition 3.1 of [6] and theorem 4.13 of [20]. □

Remark 4.6 *Let (X,δ) be a proximity space. Then, the following are equivalent:*

1. *For any pair $x,y \in X$ with $x \neq y$, $(\{x\},\{y\}) \notin \delta$.*
2. *(X,δ) is discrete.*
3. *Each subset of X is (strongly) closed.*

Theorem 4.7 *Let (X,δ) be a nonempty (strongly) irreducible proximity space.*

1. *If $(X,\delta) \in \mathbf{Prox}_{ik}$, $i = 1,2$ and $k = \mathbf{c}$ or $\mathbf{sc}$, then (X,δ) must be a one-point space.*
2. *If (X,δ) is $\overline{T}_0$ (resp. T_1 or $PreT_2'$ or $\overline{T}_2$ or T_2'), then (X,δ) must be a one-point space.*
3. *If (X,δ) is T_0' (resp. $Pre\overline{T}_2$), then (X,δ) may not be a one-point space.*

Proof

1. Suppose that (X,δ) is nonempty (strongly) irreducible, $(X,\delta) \in \mathbf{Prox}_{ik}$, $i = 1,2$ and $k = \mathbf{c}$ or $\mathbf{sc}$, and X has least two points, x and y. By Theorem 3.14 and Remarks 3.12, 4.6, all subsets of X are (strongly) closed. It follows that $\{x\}$ and $\{y\}$ are proper (strongly) closed subsets and $X = \{x\} \cup \{y\}$. This is a contradiction.
2. The proof is similar to the proof of (1) by using Theorem 3.17 and Remark 4.6.
3. By theorems 3.4 and 3.7 of [21], a proximity space is T_0' and $Pre\overline{T}_2$. Let (X,δ_3) be the proximity space defined in Example 3.4. Then (X,δ_3) is (strongly) irreducible and T_0' (resp. $Pre\overline{T}_2$), but (X,δ_3) is not a one-point space.

□

Let $\mathbb{IR}\mathcal{E}$ be the full subcategory of $\mathcal{E}$ consisting of all irreducible objects, and $\mathbf{T}\mathcal{E}$ be the full subcategory of $\mathcal{E}$ consisting of all $\mathbf{T}$ objects, where $\mathbf{T} = T_0, \overline{T}_0, T_0', T_1, \overline{T}_2, T_2'$.

Remark 4.8

1. *By Theorem 4.7, for $i = 1,2$ and $k = \mathbf{c}$ or $\mathbf{sc}$, we have*

$$\mathbb{IR}\mathbf{Prox}_{ik} = \overline{T}_0\mathbb{IR}\mathbf{Prox} = T_1\mathbb{IR}\mathbf{Prox} = \overline{T}_2\mathbb{IR}\mathbf{Prox} = T_2'\mathbb{IR}\mathbf{Prox} \subset T_0'\mathbb{IR}\mathbf{Prox}.$$

2. *For the category $\mathbf{Top}$, by remark 3.5 of [5] and theorem 3.12 of [9],*

$$T_2\mathbb{IR}\mathbf{Top} = \mathbb{IR}\mathbf{Top}_{2cl} = \mathbb{IR}\mathbf{Top}_{2scl} \subset T_2\mathbf{Top} \cap \mathbb{IR}\mathbf{Top} \subset T_0\mathbf{Top}.$$

3. *For the category of prebornological spaces, $\mathbf{PBorn}$, by theorems 3.6 and 3.9 of [9] and theorem 3.7 of [10], for $i = 1,2$,*

$$\mathbf{PBorn}_{icl} = T_0\mathbf{PBorn} = T_2'\mathbf{PBorn} \subset \overline{T}_2\mathbf{PBorn} \subset \mathbb{IR}\mathbf{PBorn} =$$

$$\overline{T}_0\mathbf{PBorn} = T_0'\mathbf{PBorn} = T_1\mathbf{PBorn} = \mathbf{PBorn}_{iscl}.$$

4. *For the category of pair spaces,* **CP***, by [10] and theorems 3.7 and 3.10 of [9], for* $i = 0, 1, 2$*,*

$$T_0\mathbf{CP} = \mathbb{IR}\mathbf{CP} \subset \overline{T}_0\mathbf{CP} = T_0'\mathbf{CP} = T_1\mathbf{CP} = \overline{T}_2\mathbf{CP} = T_2'\mathbf{CP} = \mathbf{CP}_{icl}.$$

5. *For the category of reflexive spaces,* **RRel***, by theorems 3.8 and 3.11 of [9] and theorem 3.7 of [10], for* $i = 1, 2$*,*

$$\mathbb{IR}\mathbf{RRel}_{iscl} = T_1\mathbb{IR}\mathbf{RRel} = \overline{T}_2\mathbb{IR}\mathbf{RRel} = T_2'\mathbb{IR}\mathbf{RRel} \subset \overline{T}_0\mathbf{RRel} \subset T_0\mathbf{RRel} = \mathbf{RRel}_{icl} \subset T_0'\mathbf{RRel}.$$

6. *For the category of extended pseudo quasi-semi-metric spaces,* **pqsMet***, by [10] and theorem 3.10 of [11], for* $i = 1, 2$*,*

$$\mathbb{IR}\mathbf{pqsMet}_{icl} = T_1\mathbb{IR}\mathbf{pqsMet} = \overline{T}_2\mathbb{IR}\mathbf{pqsMet} = T_2'\mathbb{IR}\mathbf{pqsMet} \subset T_0'\mathbb{IR}\mathbf{pqsMet}.$$

5. Urysohn's lemma and Tietze extension theorem

In this section, we present Urysohn's lemma and Tietze extension theorem for the proximity spaces.

Theorem 5.1 *(Urysohn's lemma) Let* (X, δ) *be a* $\overline{T}_0$ *(resp.* T_1*,* $PreT_2'$*,* $\overline{T}_2$ *or* T_2'*) proximity space and* $K, L \subset X$ *be nonempty disjoint (strongly) closed subsets of* X*. Then, there exists a proximity mapping* $f : (X, \delta) \to ([0,1], \delta_{[0,1]})$*, where* $\delta_{[0,1]}$ *is any proximity structure on* $[0,1]$*, such that* $f(K) = \{0\}$ *and* $f(L) = \{1\}$*.*

Proof Let (X, δ) be a $\overline{T}_0$ (resp. T_1, $PreT_2'$, $\overline{T}_2$ or T_2') proximity space and $K, L \subset X$ be nonempty disjoint closed subsets of X. Note that each subset of X is (strongly) closed by Theorem 3.17 and Remark 4.6. Define $f : (X, \delta) \to ([0,1], \delta_{[0,1]})$, where $\delta_{[0,1]}$ is any proximity structure on $[0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in K \\ 1 & \text{if } x \notin K \end{cases}$$

for $x \in X$.

Note that $f(K) = \{0\}$ and $f(L) = \{1\}$. We show that f is a proximity mapping.

Let $x, y \in X$ and suppose $(\{x\}, \{y\}) \in \delta$.

(i) If $x, y \in K$, then $(\{f(x)\}, \{f(y)\}) = (\{0\}, \{0\}) \in \delta_{[0,1]}$.

(ii) If $x, y \in K^c$, then $(\{f(x)\}, \{f(y)\}) = (\{1\}, \{1\}) \in \delta_{[0,1]}$.

(iii) If $x \in K$ and $y \in K^c$ (resp. $y \in K$ and $x \in K^c$), then $x \neq y$ and $(\{x\}, \{y\}) \notin \delta$ since (X, δ) is $\overline{T}_0$ (resp. T_1, $PreT_2'$, $\overline{T}_2$ or T_2'). This is a contradiction since $(\{x\}, \{y\}) \in \delta$. Hence, if $(\{x\}, \{y\}) \in \delta$, then $x, y \in K$ or $x, y \in K^c$.

Consequently, f is a proximity mapping. □

Theorem 5.2 *Let* (X, δ) *be a proximity space and* $K, L \subset X$ *be nonempty disjoint (strongly) closed subset of* X*. Then, there exists a proximity mapping* $f : (X, \delta) \to ([0,1], \delta_{[0,1]})$*, where* $\delta_{[0,1]}$ *is any proximity structure on* $[0,1]$*, such that* $f(K) = \{0\}$ *and* $f(L) = \{1\}$*.*

Proof The proof is similar to the proof of Theorem 5.1, where the case (iii) should be given as follows:

(iii) If $x \in K$ and $y \in K^c$ (resp. $y \in K$ and $x \in K^c$), then $(\{y\}, K) \in \delta$ (resp. $(\{x\}, K) \in \delta$) by the condition (2_P) of Definition 2.1. Since K is (strongly) closed we have $y \in K$ (resp. $x \in K$). This is a contradiction since $y \in K^c$ (resp. $x \in K^c$). Hence, if $(\{x\}, \{y\}) \in \delta$, then $x, y \in K$ or $x, y \in K^c$.

□

Theorem 5.3 *(Tietze Extension Theorem) Let (X, δ) be a $\overline{T}_0$ (resp. T_1, $PreT_2'$, $\overline{T}_2$ or T_2') proximity space and $K \subset X$ be nonempty (strongly) closed subspace of X. Then, every proximity mapping $f : (K, \delta_K) \to ([0,1], \delta_{[0,1]})$, where $\delta_{[0,1]}$ is any proximity structure on $[0,1]$, has a proximity mapping $g : (X, \delta) \to ([0,1], \delta_{[0,1]})$.*

Proof Suppose (X, δ) is a $\overline{T}_0$ (resp. T_1, $PreT_2'$, $\overline{T}_2$ or T_2') proximity space, K is nonempty (strongly) closed subspace of X. Note that each subset of X is (strongly) closed by Theorem 3.17 and Remark 4.6. Let $f : (K, \delta_K) \to ([0,1], \delta_{[0,1]})$ be a proximity mapping, where δ_K is the initial proximity structure on K induced by the inclusion map $i : K \to (X, \delta)$ and where $\delta_{[0,1]}$ is any proximity structure on $[0,1]$.

$$\begin{tikzcd}
(K, \delta_K) \arrow[r, "f\ (p-map)"] \arrow[d, "i"'] & ([0,1], \delta_{[0,1]}) \\
(X, \delta) \arrow[ur, "g"']
\end{tikzcd}$$

Define $g : (X, \delta) \to ([0,1], \delta_{[0,1]})$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in K \\ 0 & \text{if } x \notin K \end{cases}$$

for $x \in X$.

Note that $g(x) = f(x)$ for all $x \in K$. We will show that g is a proximity mapping.

Let $U, V \subset X$ and suppose $(U, V) \in \delta$.

(i) If $U, V \subset K$, then $(U, V) \in \delta_K$ and by definition of the map g, $(g(U), g(V)) = (f(U), f(V)) \in \delta_{[0,1]}$ since f is a proximity mapping.

(ii) If $U, V \subset K^c$, then $(g(U), g(V)) = (\{0\}, \{0\}) \in \delta_{[0,1]}$.

(iii) If $U \subset K$ and $V \subset K^c$ (resp. $V \subset K$ and $U \subset K^c$), then there exist $x \in U$ and $y \in V$ such that $x \neq y$ and $(\{x\}, \{y\}) \notin \delta$ since (X, δ) is $\overline{T}_0$ (resp. T_1, $PreT_2'$, $\overline{T}_2$ or T_2'). It follows that $(U, V) \notin \delta$ by the condition (2_P) of Definition 2.1. This is a contradiction since $(U, V) \in \delta$. Hence, if $(U, V) \in \delta$, then $U, V \subset K$ or $U, V \subset K^c$.

Hence, g is a proximity mapping. □

Theorem 5.4 *Let (X, δ) be a proximity space and $K \subset X$ be nonempty (strongly) closed subspace of X. Then, every proximity mapping $f : (K, \delta_K) \to ([0,1], \delta_{[0,1]})$, where $\delta_{[0,1]}$ is any proximity structure on $[0,1]$, has a proximity mapping $g : (X, \delta) \to ([0,1], \delta_{[0,1]})$.*

Proof The proof is similar to the proof of Theorem 5.3, where the case (iii) should be given as follows:

(iii) If $U \subset K$ and $V \subset K^c$ (resp. $V \subset K$ and $U \subset K^c$), then there exists $y \in V$ (resp. $y \in U$) such that $(\{y\}, K) \in \delta$ by the condition (2_P) of Definition 2.1. Since K is (strongly) closed, we have $y \in K$. This is a contradiction since $y \in V \subset K^c$ (resp. $y \in U \subset K^c$). Hence, if $(U,V) \in \delta$, then $U,V \subset K$ or $U,V \subset K^c$. □

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