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Generating functions for reciprocal Catalan-type sums: approach to linear differentiation equation and (p-adic) integral equations

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Abstract: This article is inspired by the reciprocal Catalan sums associated with problem 11765, proposed by David Beckwith and Sag Harbor. For this reason, partial derivative equations, the first-order linear differentiation equation and integral representations for series and generating functions for reciprocal Catalan-type sums containing the Catalan-type numbers are constructed. Some special values of these series and generating functions, which are given solutions of problem 11765, are found. Partial derivative equations of the generating function for the Catalan-type numbers are given. By using these equations, recurrence relations and derivative formulas involving these numbers are found. Finally, applying the p-adic Volkenborn integral to the Catalan-type polynomials, some combinatorial sums and identities involving the Bernoulli numbers, the Stirling numbers and the Catalan-type numbers are derived.

Key words: Generating functions, reciprocal Catalan-type sums, derivative operator, Bernoulli numbers and polynomials, Stirling numbers, Catalan numbers

1. Introduction

It has been seen in recent years that not only generating functions, but also the numbers and polynomials produced by them are used in many different disciplines, especially mathematics [1–31]. In this paper, we investigate some families of generating functions including the Bernoulli numbers and polynomials of higher order, the Stirling numbers, and the Catalan numbers. Using generating functions with their functional equations, gamma and beta functions, we give some identities and relations including the Bernoulli numbers of higher order, the Stirling numbers, and the Catalan numbers. We give integral representations for these numbers. We also give some inequalities including binomial coefficients, the Bernoulli numbers of negative higher order, the Stirling numbers, and the Catalan numbers.

In this paper, the following definitions, relations, and notations can be used.

The Stirling numbers of the first kind are given by

$$u_{(a)} = \sum_{s=0}^{a} S_1(a,s)u^s,$$

where

$$u_{(a)} = \begin{cases} u(u-1)\dots(u-a+1), & a \in \mathbb{N} \\ 1, & a=0 \end{cases}$$

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[29, p. 76, Eq. (1)].

The Catalan numbers C_n are defined by means of the following ordinary generating function:

$$C(t) = \frac{1 - \sqrt{1 - 4t}}{2t} = \sum_{n=0}^{\infty} C_n t^n, \tag{1.1}$$

where $0 < |t| \leq \frac{1}{4}$ [1, 22, 26, 29, 30].

By using (1.1), for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we have

$$C_n = \frac{1}{n+1} \binom{2n}{n} \tag{1.2}$$

and

$$(n+2) C_{n+1} = 2(2n+1) C_n, \tag{1.3}$$

where $C_0 = 1$ (see, for details, [22, 26, 29, 30]).

Using the above formula for the Catalan numbers, the first eight values of these numbers are given by

$$C_0 = 1, \quad C_1 = 1, \quad C_2 = 2, \quad C_3 = 5, \quad C_4 = 14, \quad C_5 = 42, \quad C_6 = 132, \quad C_7 = 429,$$

and so on.

Remark 1.1 *The Catalan numbers appear in various mathematical areas and and combinatorial problems. The numbers C_n are also related to the following problems: the binary bracketings of n letters, the solution to the Ballot problem, the number of trivalent planted planar trees, the number of states possible in an n-flexagon, the number of different diagonals possible in a frieze pattern with $n+1$ rows, the number of Dyck paths with n strokes, the number of ways of forming an n-fold exponential, the number of rooted plane bushes with n graph edges, and the number of extended binary trees with n internal nodes* [3, 4, 8, 9, 15–17, 21–23, 30]. *Recently, Kim et al. gave relations between the Catalan numbers and the degenerate Whitney numbers associated with the Dowling lattice, which is the geometric lattice of flats associated with a Dowling geometry* [8, 9, 15–17, 23].

In [21, eqs. (2.1) and (2.4)], Kucukoglu et al. constructed the following generating functions for new classes of Catalan-type numbers and polynomials, respectively:

$$F_v(t, \lambda) = \frac{1 - \lambda + \sqrt{(\lambda - 1)^2 + 8\lambda^2 t}}{2\lambda^2 t} = \sum_{n=0}^{\infty} V_n(\lambda) t^n, \tag{1.4}$$

where

$$0 < \left| \frac{\lambda^2 t}{(\lambda - 1)^2} \right| \leq \frac{1}{8}$$

and

$$F_v(t, x; \lambda) = F_v(t, \lambda)(1+t)^{\frac{x}{2}} = \sum_{n=0}^{\infty} V_n(x; \lambda) t^n. \tag{1.5}$$

By using (1.4), we have

$$V_n(\lambda) = (-1)^n C_n \frac{2^{n+1}\lambda^{2n}}{(\lambda-1)^{2n+1}} \tag{1.6}$$

(see [21, eq. (2.2)] and also [20]).

Putting $n = 0$ in (1.6), we have

$$V_0(\lambda) = \frac{2}{\lambda-1}.$$

A relation between the numbers C_n and $V_n(\lambda)$ is given as follows:

$$C_n = \frac{(-1)^n (\lambda-1)^{2n+1} V_n(\lambda)}{2^{n+1}\lambda^{2n}}$$

[21, eq. (2.2)].

In order to give generating function for the reciprocal of the new classes of Catalan-type number, we also need the following novel formula

$$\frac{V_{n+1}(\lambda)}{V_n(\lambda)} = -\frac{8n+4}{n+2}\left(\frac{\lambda}{\lambda-1}\right)^2 \tag{1.7}$$

[21, Eqs. (2.1) and (2.4)].

2. Generating function for reciprocal of the Catalan-type numbers

In this section, using not only the partial derivative operator $\frac{\partial}{\partial z}$, and the first-order linear differentiation equation, but also the Euler gamma function and the beta function, we construct generating functions for the reciprocal of the Catalan-type numbers $V_n(\lambda)$.

We think that there are other methods, which are used to prove other formulas for the generating functions of the reciprocal of the Catalan-type numbers. Using similar methods associated with Abel [5, problem 11765] and Amdeberhan et al. [1], some novel formulas and relations are given.

We give special values of the first-order linear differentiation equation arising from the generating function for reciprocal of the Catalan-type numbers $V_n(\lambda)$.

We define a generating function for the reciprocal of the Catalan-type numbers as follows:

$$G(z,\lambda) = \sum_{n=0}^{\infty} \frac{z^n}{V_n(\lambda)}. \tag{2.1}$$

In order to give a set of converges for the above series, we use the ratio test. Thus, we have

$$\lim_{n\to\infty} \left|\frac{z^{n+1}}{V_{n+1}(\lambda)}\right| \left|\frac{V_n(\lambda)}{z^n}\right|.$$

Combining the above equation with (1.7), after some elementary calculations, for $\lambda \neq 1$, we get

$$|z| < 8\left|\frac{\lambda}{\lambda-1}\right|^2.$$

By the following theorem, we give an explicit formula for the function $G(z,\lambda)$. Firstly, we introduce the following well-known the Gauss's hypergeometric series:

$$ {}_2F_1(x,y;t;z) = \sum_{n=0}^{\infty} \frac{x^{(n)}y^{(n)}}{t^{(n)}} \frac{z^n}{n!}, $$

where $t \notin \{0,-1,-2,-3,\ldots\}$ and $x^{(n)} = x(x+1)\ldots(x+n-1)$. When $|z|=1$, the hypergeometric series in the above equation satisfies the following conditions:

1) Absolutely convergent, if $Re(t-x-y)>0$,

2) Conditionally convergent, if $-1<Re(t-x-y)\leq 0$ and $z\neq 1$,

3) Divergent, if $Re(t-x-y)\leq -1$.

It is clear that for $|z|<1$,

$$ {}_2F_1(x,1;x;z) = {}_2F_1(1,y;y;z) = \frac{1}{1-z} $$

[29, p. 64].

For $t \notin \{0,-1,-2,-3,\ldots\}$, arbitrary constants β and η, the function

$$ u = \beta_2F_1(x,y;t;z) + z^{1-t}\eta_2F_1(x-t+1,y-t+1;2-t;z). $$

is a solution of the following second-order linear ordinary differential equation, which is also called the Gauss hypergeometric equation:

$$ \{\vartheta(\vartheta+t-1) - z(\vartheta+x)(\vartheta+y)\}\, u = 0, $$

where x, y, and t are real or complex parameters and ϑ denotes following the well-known Euler operator:

$$ \vartheta := z\frac{d}{dz} $$

[29, p. 63].

Theorem 2.1 *Let $\lambda \neq 1,0$. Then we have*

$$ G(z,\lambda) = -\frac{32\sqrt{2}\,(\lambda-1)^3\left(\frac{z\lambda^2}{8(\lambda-1)^2}\right)^{\frac{3}{2\lambda}}}{\sqrt{z}\lambda^2\,(\lambda-3)}\ {}_2F_1\left(\frac{3-\lambda}{2\lambda},\frac{3-3\lambda}{2\lambda};\frac{3+\lambda}{2\lambda};-\frac{z}{8}\left(\frac{\lambda}{\lambda-1}\right)^2\right). \tag{2.2} $$

Proof By using (1.7), we get

$$ \frac{1}{V_n(\lambda)} = -\frac{4\lambda^2\,(2n+1)}{(\lambda-1)^2\,(n+2)}\frac{1}{V_{n+1}(\lambda)}. $$

Combining the above equation with (2.1), we obtain

$$ G(z,\lambda) = \sum_{n=0}^{\infty}\frac{z^n}{V_n(\lambda)} = -\frac{4\lambda^2}{(\lambda-1)^2}\sum_{n=0}^{\infty}\left(2-\frac{3}{(n+2)}\right)\frac{z^n}{V_{n+1}(\lambda)}. $$

After some elementary calculations, we have

$$G(z,\lambda) = -\frac{4\lambda^2}{(\lambda-1)^2}\left(\frac{2}{z}G(z,\lambda) - \frac{2}{zV_0(\lambda)} - \frac{3}{z}\sum_{n=1}^{\infty}\frac{z^n}{(n+1)V_n(\lambda)}\right).$$

Substituting

$$V_0(\lambda) = \frac{2}{\lambda-1}$$

into the above equation, we get

$$\left(\frac{z^2}{3}\left(\frac{\lambda-1}{2\lambda}\right)^2 + \frac{2}{3}z\right)G(z,\lambda) - \frac{(\lambda-1)z}{3} = \sum_{n=1}^{\infty}\frac{z^{n+1}}{(n+1)V_n(\lambda)}. \tag{2.3}$$

Applying partial derivative operator $\frac{\partial}{\partial z}$ to the above equation, we obtain

$$\left(\frac{2z}{3}\left(\frac{\lambda-1}{2\lambda}\right)^2 + \frac{2}{3}\right)G(z,\lambda) + \left(\frac{z^2}{3}\left(\frac{\lambda-1}{2\lambda}\right)^2 + \frac{2}{3}z\right)\frac{\partial}{\partial z}\{G(z,\lambda)\} + \frac{(\lambda-1)}{6} = G(z,\lambda).$$

Therefore,

$$2\left(z\left(\frac{\lambda-1}{\lambda}\right)^2 - 2\right)G(z,\lambda) + \left(z^2\left(\frac{\lambda-1}{\lambda}\right)^2 + 8z\right)\frac{\partial}{\partial z}\{G(z,\lambda)\} = 2(1-\lambda). \tag{2.4}$$

After some elementary calculations in Equation (2.4), we obtain the following standard form of the first-order linear differentiation equation:

$$\frac{\partial}{\partial z}\{G(z,\lambda)\} + \frac{2\left(z\left(\frac{\lambda-1}{\lambda}\right)^2 - 2\right)}{z^2\left(\frac{\lambda-1}{\lambda}\right)^2 + 8z}G(z,\lambda) = \frac{2(1-\lambda)}{z^2\left(\frac{\lambda-1}{\lambda}\right)^2 + 8z}. \tag{2.5}$$

In order to solve Equation (2.5), we need the following integrating factor:

$$\begin{aligned} u(z,\lambda) &= e^{\int \frac{2\left(z\left(\frac{\lambda-1}{\lambda}\right)^2 - 2\right)}{z^2\left(\frac{\lambda-1}{\lambda}\right)^2 + 8z}dz} \\ &= \left(z^2\left(\frac{\lambda-1}{\lambda}\right)^2 + 8z\right)\frac{\left(\frac{\lambda-1}{\lambda}z + \frac{8\lambda}{\lambda-1}\right)^{\frac{3}{2}\left(\frac{\lambda-1}{\lambda}\right)}}{\left(\frac{\lambda-1}{\lambda}z\right)^{\frac{3}{2}\left(\frac{\lambda-1}{\lambda}\right)}}. \end{aligned}$$

Hence, multiplying both sides of the standard form of Equation (2.5) by $u(z,\lambda)$ yields the following separable differential equation:

$$\frac{\partial}{\partial z}\left\{\left(z^2\left(\frac{\lambda-1}{\lambda}\right)^2 + 8z\right)\left(\frac{\frac{\lambda-1}{\lambda}z + \frac{8\lambda}{\lambda-1}}{\frac{\lambda-1}{\lambda}z}\right)^{\frac{3}{2}\left(\frac{\lambda-1}{\lambda}\right)}G(z,\lambda)\right\} = \frac{2(1-\lambda)}{\left(\frac{\lambda-1}{\lambda}z\right)^{\frac{3}{2}\left(\frac{\lambda-1}{\lambda}\right)}}\left(\frac{\lambda-1}{\lambda}z + \frac{8\lambda}{\lambda-1}\right)^{\frac{3}{2}\left(\frac{\lambda-1}{\lambda}\right)}.$$

The general solution to the above equation is given by

$$G(z,\lambda)\left(z^2\left(\frac{\lambda-1}{\lambda}\right)^2 + 8z\right)\left(\frac{\frac{\lambda-1}{\lambda}z + \frac{8\lambda}{\lambda-1}}{\frac{\lambda-1}{\lambda}z}\right)^{\frac{3}{2}\left(\frac{\lambda-1}{\lambda}\right)} = 2(1-\lambda)\int\left(1 + \frac{8}{z}\left(\frac{\lambda}{\lambda-1}\right)^2\right)^{\frac{3}{2}\left(\frac{\lambda-1}{\lambda}\right)}dz. \tag{2.6}$$

Integrate the right-hand side of the above equation (by parts) with respect to z, we arrive at the desired result. $\square$

Remark 2.2 *Here we note that integrating both sides of (2.1), we get*

$$\int f(z,\lambda)\,dz = \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)V_n(\lambda)}.$$

By applying the "Second Fundamental Theorem of Calculus (Part 2)" to the above equation and using

$$V_0(\lambda) = \frac{2}{\lambda-1},$$

we also arrive at Equation (2.3).

Remark 2.3 *Note that for different values of λ, the above first-order linear differentiation may be used in many important real-world problems and mathematical models involving fluid flow and other problems. This type of equation can also be used in real-world problems associated with decay and growth involving radioactive decay, compound interest, population growth, etc. Putting $\lambda = 1$ in (2.5), we get the following separable differential equation:*

$$\frac{d}{dz}\{G(z,1)\} - \frac{1}{2z}G(z,1) = 0.$$

Solution of this equation is given by

$$G(z,1) = c\sqrt{z},$$

where c is a constant. Thus, Equation (2.5) will be potentially used not only to get combinatorial identities but also to contribute to new mathematical models.

We simplify Equation (2.2) as follows:

Corollary 2.4 *Let $\lambda \neq 1, 0$. Then we have*

$$G(z,\lambda) = -z^2\lambda^3 \left(\frac{8+\lambda(-16+(8+z)\lambda)}{z\lambda^2}\right)^{\frac{5}{2}-\frac{3}{2\lambda}} \cdot \frac{{}_2F_1\left(1,2,\frac{3+\lambda}{2\lambda},-\frac{z}{8}\left(\frac{\lambda}{\lambda-1}\right)^2\right)}{4(\lambda-3)(\lambda-1)^2}.$$

By using (2.4), we get

$$2\left(z\left(\frac{\lambda-1}{\lambda}\right)^2 - 2\right)\sum_{n=0}^{\infty}\frac{z^n}{V_n(\lambda)} + \left(z\left(\frac{\lambda-1}{\lambda}\right)^2 + 8\right)\sum_{n=1}^{\infty}\frac{nz^n}{V_n(\lambda)} = 2(1-\lambda).$$

After some elementary calculations in the above equation, we have

$$2(1-\lambda) = 2\left(\frac{\lambda-1}{\lambda}\right)^2\sum_{n=0}^{\infty}\frac{z^{n+1}}{V_n(\lambda)} - 4\sum_{n=0}^{\infty}\frac{z^n}{V_n(\lambda)} + \left(\frac{\lambda-1}{\lambda}\right)^2\sum_{n=1}^{\infty}\frac{nz^{n+1}}{V_n(\lambda)} + 8\sum_{n=1}^{\infty}\frac{nz^n}{V_n(\lambda)}.$$

Therefore,

$$0=\sum_{n=1}^{\infty}\left(\left(\frac{\lambda-1}{\lambda}\right)^{2}\frac{n+1}{V_{n-1}(\lambda)}+\frac{4(2n-1)}{V_{n}(\lambda)}\right)z^{n}.$$

The above equation gives another proof of Equation (1.7).

Theorem 2.5 *Let $\lambda\neq 1,0$. Then we have*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{z^{n}}{V_{n}(\lambda)} &= \frac{\lambda-1}{2}-\frac{z(\lambda-1)^{3}\left(z(\lambda-1)^{2}+20\lambda^{2}\right)}{2(z(\lambda-1)^{2}+8\lambda^{2})^{2}} \\ &\quad +\frac{48\lambda^{4}z(\lambda-1)^{2}\arctan\left(\frac{\sqrt{z}(\lambda-1)}{\sqrt{-z(\lambda-1)^{2}-8\lambda^{2}}}\right)}{\sqrt{z}\left(-z(\lambda-1)^{2}-8\lambda^{2}\right)^{5/2}}.\end{aligned}\tag{2.7}$$

Proof Substituting (1.6) into $\sum_{n=0}^{\infty}\frac{z^{n}}{V_{n}(\lambda)}$, using direct calculation, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{z^{n}}{V_{n}(\lambda)} &= \sum_{n=0}^{\infty}\frac{(n+1)(\lambda-1)^{2n+1}(-z)^{n}}{\binom{2n}{n}2^{n+1}\lambda^{2n}} \\ &= \frac{\lambda-1}{2}+\frac{\lambda-1}{2}\sum_{n=1}^{\infty}\frac{(n+1)(n!)^{2}}{(2n)!}\left(-\frac{z}{2}\left(\frac{\lambda-1}{\lambda}\right)^{2}\right)^{n}.\end{aligned}$$

Combining the above equation with the following well-known formula Beta and Euler gamma functions:

$$\begin{aligned}\int_{0}^{1}u^{n-1}(1-u)^{n}du &= B(n,n+1)=\frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)} \\ &= \frac{(n-1)!\,(n)!}{(2n)!},\end{aligned}$$

we obtain

$$\sum_{n=0}^{\infty}\frac{z^{n}}{V_{n}(\lambda)}=\frac{\lambda-1}{2}+\frac{\lambda-1}{2}\sum_{n=1}^{\infty}(n^{2}+n)\left(-\frac{z}{2}\left(\frac{\lambda-1}{\lambda}\right)^{2}\right)^{n}\int_{0}^{1}u^{n-1}(1-u)^{n}du.$$

It is easy to see that a series on the right-hand side is uniformly convergent. So, it is possible to interchange of summation and integration. Thus, we have

$$\sum_{n=0}^{\infty}\frac{z^{n}}{V_{n}(\lambda)}=\frac{\lambda-1}{2}+\frac{\lambda-1}{2}\int_{0}^{1}\frac{1}{u}\sum_{n=1}^{\infty}(n^{2}+n)\left(-\frac{zu(1-u)}{2}\left(\frac{\lambda-1}{\lambda}\right)^{2}\right)^{n}du.$$

Using the following series

$$\sum_{n=1}^{\infty}nw^{n}=\frac{w}{(1-w)^{2}}$$

and

$$\sum_{n=1}^{\infty} n^2 w^n = \frac{w(1+w)}{(1-w)^3},$$

where $|w| < 1$, we get

$$\sum_{n=0}^{\infty} \frac{z^n}{V_n(\lambda)} = \frac{\lambda-1}{2}\left(1 - \int_0^1 \frac{z\left(\frac{\lambda-1}{\lambda}\right)^2 (1-u)}{\left(1+\frac{z}{2}\left(\frac{\lambda-1}{\lambda}\right)^2 u(1-u)\right)^3} du\right).$$

Making some elementary calculations of the above integral, we get

$$\int_0^1 \frac{z\left(\frac{\lambda-1}{\lambda}\right)^2 (1-u)}{\left(1+\frac{z}{2}\left(\frac{\lambda-1}{\lambda}\right)^2 u(1-u)\right)^3} du$$

$$= \frac{z(\lambda-1)^2}{\lambda^2}\left(\frac{\lambda^2\left(z(\lambda-1)^2+20\lambda^2\right)}{(z(\lambda-1)^2+8\lambda^2)^2} - \frac{96\lambda^6 \arctan\left(\frac{\sqrt{z}(\lambda-1)}{\sqrt{-z(\lambda-1)^2-8\lambda^2}}\right)}{\sqrt{z}(\lambda-1)\left(-z(\lambda-1)^2-8\lambda^2\right)^{5/2}}\right).$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{z^n}{V_n(\lambda)} - \frac{\lambda-1}{2} = -\frac{z(\lambda-1)^3\left(z(\lambda-1)^2+20\lambda^2\right)}{2(z(\lambda-1)^2+8\lambda^2)^2} + \frac{48\lambda^4 z(\lambda-1)^2 \arctan\left(\frac{\sqrt{z}(\lambda-1)}{\sqrt{-z(\lambda-1)^2-8\lambda^2}}\right)}{\sqrt{z}\left(-z(\lambda-1)^2-8\lambda^2\right)^{5/2}}.$$

After some elementary calculations, we arrive at the desired result. □

We now give some numerical examples.

Example 2.6 *Setting $z=-1$ and $\lambda=2$ in (2.7), we have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{V_n(2)} = \sum_{n=0}^{\infty} \frac{n+1}{2^{3n+1}\binom{2n}{n}} = \frac{520}{961} + \frac{768 \arctan\left(\frac{1}{\sqrt{31}}\right)}{961\sqrt{31}}.$$

Example 2.7 *Substituting $z=-2\left(\frac{\lambda}{\lambda-1}\right)^2$ into (2.7), we get*

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{V_n(\lambda)}\left(\frac{\lambda}{\lambda-1}\right)^{2n} = (\lambda-1)\left(1+\frac{8}{3\sqrt{12}}\arctan\left(\frac{1}{\sqrt{3}}\right)\right).$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{V_n(\lambda)}\left(\frac{\lambda}{\lambda-1}\right)^{2n} = (\lambda-1)\left(1+\frac{2\pi}{9\sqrt{3}}\right).$$

Combining the above equation with (1.6), after some elementary calculations, we arrive at the reciprocal Catalan sum involving

$$\sum_{n=0}^{\infty}\frac{1}{C_n}=2+\frac{4}{9\sqrt{3}}\pi,$$

and also using the Wolfram Mathematica 12.0, we have the following hypergeometric series representation for $\sum_{n=0}^{\infty}\frac{1}{C_n}$*:*

$$\sum_{n=0}^{\infty}\frac{1}{C_n}=\sum_{n=0}^{\infty}\frac{1^{(n)}2^{(n)}}{\left(\frac{1}{2}\right)^{(n)}}\frac{\left(\frac{1}{4}\right)^{(n)}}{n!}\simeq 2,806.$$

Example 2.8 *Substituting* $z=-\left(\frac{2\lambda}{\lambda-1}\right)^2$ *into (2.7), we obtain*

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{V_n(\lambda)}4^n\left(\frac{\lambda}{\lambda-1}\right)^{2n}=\frac{\lambda-1}{2}\left(5+6\arctan(1)\right).$$

Combining the above equation with (1.6), after some elementary calculations, we arrive at the reciprocal Catalan sum involving

$$\sum_{n=0}^{\infty}\frac{2^n}{C_n}=5+\frac{3\pi}{2}. \tag{2.8}$$

By using the Wolfram Mathematica 12.0, we also have the following series representations for $\sum_{n=0}^{\infty}\frac{2^n}{C_n}$*:*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{2^n}{C_n} &= 5+6\sum_{n=0}^{\infty}\frac{(-1)^n}{1+2n}\\ &= 5+\frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n\left(\frac{1}{1+2n}+\frac{2}{1+4n}+\frac{1}{3+4n}\right)\\ &\simeq 9,7124.\end{aligned}$$

Remark 2.9 *Equation (2.8) gives solution of problem 11765 (a), which was proposed by David Beckwith, Sag Harbor [5]. The first solution of this problem was given by Abel; for details, see [5]. Problem 11765 was also solved by Amdeberhan et al. [1].*

Example 2.10 *Substituting* $z=-6\left(\frac{\lambda}{\lambda-1}\right)^2$ *into (2.7), we get*

$$\sum_{n=0}^{\infty}\frac{(-6)^n}{V_n(\lambda)}\left(\frac{\lambda}{\lambda-1}\right)^{2n}=\frac{\lambda-1}{2}\left(22+24\sqrt{3}\arctan\left(\sqrt{3}\right)\right).$$

Combining the above equation with (1.6), after some elementary calculations, we arrive at the reciprocal Catalan sum involving

$$\sum_{n=0}^{\infty}\frac{3^n}{C_n}=22+8\sqrt{3}\pi. \tag{2.9}$$

By using the Wolfram Mathematica 12.0, we have the following hypergeometric series representation for $\sum_{n=0}^{\infty} \frac{3^n}{C_n}$ *:*

$$\sum_{n=0}^{\infty} \frac{3^n}{C_n} = \sum_{n=0}^{\infty} \frac{1^{(n)} 2^{(n)}}{\left(\frac{1}{2}\right)^{(n)}} \frac{\left(\frac{3}{4}\right)^{(n)}}{n!} \simeq 65,53.$$

Remark 2.11 *Equation (2.9) gives solution of problem 11765 (b), which was proposed by David Beckwith, Sag Harbor; for details, see [5]. The first solution of this problem was given by Abel [5], see also [1].*

3. Identity arising from partial derivative equation of the function $F_v(t, \lambda)$

In this section, we give a recurrence relation for the Catalan-type numbers $V_k(\lambda)$ and derivative formula for $\frac{d}{d\lambda}\{V_n(\lambda)\}$, using partial derivative equation of the function $F_v(t, \lambda)$. Our recurrence relation is different from that of the work of Kucukoglu et al. [19, theorem 3.1].

Theorem 3.1 (Recurrence relations for the numbers $V_n(\lambda)$). *Let* $n \in \mathbb{N}_0$. *Then we have*

$$(n+1) V_{n+1}(\lambda) = \sum_{j=0}^{n} \sum_{k=0}^{j} V_k(\lambda) V_{j-k}(\lambda) \binom{-\frac{1}{2}}{n-j} \frac{8^{n-j} \lambda^{2n-2j+2}}{(1-\lambda)^{2n-2j+1}}. \tag{3.1}$$

Proof Assuming that $1 - \lambda < 0$. We modify Equation (1.4) as follows:

$$F_v(t, \lambda) = \frac{-4}{(1-\lambda)\left(1 + \sqrt{1 + \frac{8\lambda^2}{(\lambda-1)^2} t}\right)} = \sum_{n=0}^{\infty} V_n(\lambda) t^n. \tag{3.2}$$

Taking partial derivative of the function $F_v(t, \lambda)$ with respect to t, we get the following equation:

$$\frac{\partial}{\partial t}\{F_v(t, \lambda)\} = F_v\left(\frac{8\lambda^2}{(\lambda-1)^2} t, -1; \lambda\right) F_v(t, \lambda) \frac{\lambda^2}{(1-\lambda)}.$$

By using the above equation, we obtain

$$\sum_{n=1}^{\infty} n V_n(\lambda) t^{n-1} = \frac{\lambda^2}{1-\lambda} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{j} V_k(\lambda) V_{j-k}(\lambda) \binom{-\frac{1}{2}}{n-j} \frac{8^{n-j} \lambda^{2n-2j}}{(\lambda-1)^{2n-2j}} t^n.$$

Comparing the coefficients of t^n on both sides of the above equation, we get the desired result. □

Substituting the following well-known identity

$$\binom{-\frac{1}{2}}{n} = \frac{(-1)^n (2n)!}{(n!)^2 4^n} \tag{3.3}$$

into (3.1), we arrive at the following corollary:

Corollary 3.2 *Let* $n \in \mathbb{N}_0$. *Then we have*

$$V_{n+1}(\lambda) = \sum_{j=0}^{n} \sum_{k=0}^{j} (-2)^{n-j} C_{n-j} V_k(\lambda) V_{j-k}(\lambda) \frac{(n-j+1)\lambda^{2n-2j+2}}{(n+1)(\lambda-1)^{2n-2j+1}}.$$

Theorem 3.3 *Let $n \in \mathbb{N}_0$. Then we have*

$$\frac{d}{d\lambda}\left\{V_n\left(\lambda\right)\right\}+\sum_{k=0}^{n}(-1)^k\frac{2^k\lambda^{2k}\left(2k\right)!V_{n-k}\left(\lambda\right)}{\left(\lambda-1\right)^{2k+1}\left(k!\right)^2}+\frac{2}{\lambda}V_n\left(\lambda\right)=(-1)^n\frac{2^{n+2}\left(2n\right)!\lambda^{2n-1}}{\left(\lambda-1\right)^{2n+1}\left(n!\right)^2}.$$

Proof Assuming that $\lambda-1>0$. Taking partial derivative of the function $F_v\left(t,\lambda\right)$ with respect to λ, we get the following equation:

$$\frac{\partial}{\partial\lambda}\left\{F_v\left(t,\lambda\right)\right\}+\frac{F_v\left(t,\lambda\right)}{\left(\lambda-1\right)\sqrt{1+8t\left(\frac{\lambda}{\lambda-1}\right)^2}}+\frac{2}{\lambda}F_v\left(t,\lambda\right)=\frac{4}{\lambda\left(\lambda-1\right)\sqrt{1+8t\left(\frac{\lambda}{\lambda-1}\right)^2}}.$$

By applying binomial series representation in the above equation yields

$$\sum_{n=0}^{\infty}\frac{d}{d\lambda}\left\{V_n\left(\lambda\right)\right\}t^n+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{\frac{-1}{2}}{k}\frac{8^k\lambda^{2k}V_{n-k}\left(\lambda\right)}{\left(\lambda-1\right)^{2k+1}}t^n+\frac{2}{\lambda}\sum_{n=0}^{\infty}V_n\left(\lambda\right)t^n=4\sum_{n=0}^{\infty}\binom{\frac{-1}{2}}{n}\frac{8^n\lambda^{2n-1}}{\left(\lambda-1\right)^{2n+1}}t^n.$$

Comparing the coefficients of t^n on both sides of the above equation, we get

$$\frac{d}{d\lambda}\left\{V_n\left(\lambda\right)\right\}+\sum_{k=0}^{n}\binom{\frac{-1}{2}}{k}\frac{8^k\lambda^{2k}V_{n-k}\left(\lambda\right)}{\left(\lambda-1\right)^{2k+1}}+\frac{2}{\lambda}V_n\left(\lambda\right)=\frac{8^n4}{\lambda\left(\lambda-1\right)}\binom{\frac{-1}{2}}{n}\left(\frac{\lambda}{\lambda-1}\right)^{2n}.$$

Substituting (3.3) into the above equation, we get the desired result. □

4. Identities and combinatorial sums derive from p-adic Volkenborn integral

The p-adic Volkenborn integral (or the p-adic bosonic integral) has many applications in mathematics and mathematical physics. It is known that this integral has been used p-adic mathematical analysis and quantum physicists. Therefore, by applying this integral to uniformly differential function on $\mathbb{Z}_p$, generating functions for the Bernoulli-type numbers and polynomials and other special numbers and polynomials can be constructed [13], see also [7, 11, 12, 27, 28, 31]. In 2007, Kim [13] defined the fermionic integral on $\mathbb{Z}_p$. By using these integrals, he gave many new formulas and relations involving the Changhee numbers and polynomials, the Daehee numbers and polynomials, the Catalan-Daehee polynomials, and other special numbers and polynomials. These integrals are also used to give generating functions for certain families of the special numbers and polynomials.

4.1. p-adic Volkenborn integral represantations for special polynomials and numbers

Let $\mathbb{Z}_p$ be the set of p-adic integers. Let $u\in\mathbb{Z}_p$ and $\ell(u)$ be a uniformly differential function on $\mathbb{Z}_p$. The Volkenborn integral (or bosonic integral) of the function $\ell(u)$ is given by

$$\int_{\mathbb{Z}_p}\ell(u)d\mu_1\left(u\right)=\lim_{M\to\infty}\frac{1}{p^M}\sum_{d=0}^{p^M-1}\ell(d), \tag{4.1}$$

where p is a prime number and

$$\mu_1\left(u\right)=\frac{1}{p^M}$$

[11, 27, 28, 31].

By applying (4.1) to the function $\ell(u) = (u+a)^v$ with $v \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $u, a \in \mathbb{Z}_p$, we have the following p-adic bosonic integral representations for the Bernoulli polynomials, $B_v(a)$, and the Bernoulli numbers, B_v, respectively

$$B_v(a) = \int_{\mathbb{Z}_p} (a+u)^v \, d\mu_1(u) \tag{4.2}$$

and

$$B_v := B_v(0) = \int_{\mathbb{Z}_p} u^v d\mu_1(u) \tag{4.3}$$

(see [27], and also [11, 12, 28]).

By applying (4.1) to the function $\ell(u) = u_{(v)}$ with $v \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $u \in \mathbb{Z}_p$, Kim et al. [7] gave the following p-adic bosonic integral representation for the Daehee numbers D_v,

$$D_v = \int_{\mathbb{Z}_p} u_{(v)} d\mu_1(u), \tag{4.4}$$

where

$$D_v = \frac{(-1)^v v!}{v+1}.$$

We also have

$$\int_{\mathbb{Z}_p} u_{(v)} d\mu_1(u) = \sum_{d=0}^{v} S_1(v,d) B_d \tag{4.5}$$

and

$$\int_{\mathbb{Z}_p} u_{(v)} d\mu_1(u) = \frac{(-1)^v v!}{v+1} \tag{4.6}$$

[4, 7, 27, 28].

4.2. Formulas and combinatorial sums derive from p-adic integrals of Catalan-type polynomials

Here, using p-adic integrals representations of Catalan-type polynomials and the polynomials $\ell(u) = u_{(v)}$, we give some novel formulas and combinatorial sums involving the Bernoulli, and Euler numbers and polynomials, the Stirling numbers, and the Catalan-type numbers.

Substituting x with $u+v$ (1.5), we obtain

$$V_n(u+v;\lambda) = \sum_{k=0}^{n} \binom{\frac{u}{2}}{k} V_{n-k}(v;\lambda) \tag{4.7}$$

and

$$V_n(u+v;\lambda) = \sum_{k=0}^{n} \binom{\frac{u+v}{2}}{k} V_{n-k}(\lambda). \tag{4.8}$$

Combining the well-known Chu-Vandermonde identity with the above equation, we get

$$V_n\left(u+v;\lambda\right)=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{\frac{u}{2}}{j}\binom{\frac{v}{2}}{k-j}V_{n-k}\left(\lambda\right). \tag{4.9}$$

By applying (4.1) with respect to u and v to Equations (4.7), (4.8), and (4.9), we obtain

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}V_a\left(u+v;\lambda\right)d\mu_1\left(u\right)d\mu_1\left(v\right)=\sum_{k=0}^{a}\int_{\mathbb{Z}_p}V_{a-k}\left(v;\lambda\right)d\mu_1\left(v\right)\int_{\mathbb{Z}_p}\binom{\frac{u}{2}}{k}d\mu_1\left(u\right),$$

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}V_a\left(u+v;\lambda\right)d\mu_1\left(u\right)d\mu_1\left(v\right)=\sum_{k=0}^{a}V_{a-k}\left(\lambda\right)\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\binom{\frac{u+v}{2}}{k}d\mu_1\left(u\right)d\mu_1\left(v\right) \tag{4.10}$$

and

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}V_a\left(u+v;\lambda\right)d\mu_1\left(u\right)d\mu_1\left(v\right)=\sum_{k=0}^{a}\sum_{j=0}^{k}V_{a-k}\left(\lambda\right)\int_{\mathbb{Z}_p}\binom{\frac{u}{2}}{j}d\mu_1\left(u\right)\int_{\mathbb{Z}_p}\binom{\frac{v}{2}}{k-j}d\mu_1\left(v\right),$$

where $a\in\mathbb{N}_0$. Combining the above p-adic integrals of the polynomials $V_a\left(x;\lambda\right)$ with

$$d_a=\frac{(-1)^a4^a}{a!}\int_{\mathbb{Z}_p}\left(\frac{u}{2}\right)_{(a)}d\mu_1\left(u\right),$$

$$\int_{\mathbb{Z}_p}\left(\frac{u}{2}\right)_{(a)}d\mu_1\left(u\right)=\sum_{k=0}^{a}\frac{S_1(a,k)B_k}{2^k}$$

([4], see also [19, theorem 5.4]) and the following formula, which was proven by Kucukoglu et al. [19, theorem 5.4]:

$$\int_{\mathbb{Z}_p}V_a\left(u;\lambda\right)d\mu_1\left(u\right)=\sum_{j=0}^{a}\sum_{c=0}^{j}\frac{V_{a-j}\left(\lambda\right)S_1(j,c)B_c}{j!2^c},$$

where $a\in\mathbb{N}_0$, we obtain the following interesting formulas:

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}V_a\left(u+v;\lambda\right)d\mu_1\left(u\right)d\mu_1\left(v\right)=\sum_{k=0}^{a}\sum_{j=0}^{a-k}\sum_{c=0}^{j}\sum_{s=0}^{k}\frac{B_cB_sV_{a-k-j}\left(\lambda\right)S_1(j,c)S_1(k,s)}{j!k!2^{c+s}},$$

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}V_a\left(u+v;\lambda\right)d\mu_1\left(u\right)d\mu_1\left(v\right)=\sum_{k=0}^{a}\sum_{j=0}^{a-k}\sum_{c=0}^{j}(-1)^k\frac{d_kB_cV_{a-k-j}\left(\lambda\right)S_1(j,c)}{j!2^{c+2k}},$$

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}V_a\left(u+v;\lambda\right)d\mu_1\left(u\right)d\mu_1\left(v\right)=\sum_{k=0}^{a}\sum_{j=0}^{k}(-1)^k\frac{d_jd_{k-j}V_{a-k}\left(\lambda\right)}{4^k},$$

and

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} V_a\left(u+v;\lambda\right)d\mu_1\left(u\right)d\mu_1\left(v\right)=\sum_{k=0}^{a}\sum_{j=0}^{k}\sum_{s=0}^{j}\sum_{m=0}^{k-j}\binom{k}{j}\frac{V_{a-k}\left(\lambda\right)S_1(j,s)S_1(k-j,m)B_sB_m}{k!2^{s+m}}.$$

Combining (4.10) with the following formula

$$\binom{\frac{u+v}{2}}{k}=\frac{1}{k!}\sum_{s=0}^{k}\frac{S_1(k,s)}{2^s}\sum_{m=0}^{s}\binom{s}{m}u^m v^{s-m}$$

and using (4.3), we obtain

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} V_a\left(u+v;\lambda\right)d\mu_1\left(u\right)d\mu_1\left(v\right)=\sum_{k=0}^{a}\sum_{s=0}^{k}\sum_{m=0}^{s}\binom{s}{m}\frac{B_mB_{s-m}V_{a-k}\left(\lambda\right)S_1(k,s)}{2^sk!}.$$

By combining the right-hand sides of the above equation, we arrive at the following theorems, respectively:

Theorem 4.1 *Let $a\in\mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{k=0}^{a}\sum_{s=0}^{k}\sum_{m=0}^{s}\binom{s}{m}\frac{B_mB_{s-m}V_{a-k}\left(\lambda\right)S_1(k,s)}{2^sk!}\\ =\ &\sum_{k=0}^{a}\sum_{j=0}^{k}\sum_{s=0}^{j}\sum_{m=0}^{k-j}\binom{k}{j}\frac{V_{a-k}\left(\lambda\right)S_1(j,s)S_1(k-j,m)B_sB_m}{k!2^{s+m}}.\end{aligned}$$

Theorem 4.2 *Let $a\in\mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{a}\sum_{s=0}^{k}\sum_{m=0}^{s}\binom{s}{m}\frac{B_mB_{s-m}V_{a-k}\left(\lambda\right)S_1(k,s)}{2^sk!}=\sum_{k=0}^{a}\sum_{j=0}^{k}(-1)^k\frac{d_jd_{k-j}V_{a-k}\left(\lambda\right)}{4^k}.$$

Theorem 4.3 *Let $a\in\mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{a}\sum_{s=0}^{k}\sum_{m=0}^{s}\binom{s}{m}\frac{B_mB_{s-m}V_{a-k}\left(\lambda\right)S_1(k,s)}{2^sk!}=\sum_{k=0}^{a}\sum_{j=0}^{a-k}\sum_{c=0}^{j}(-1)^k\frac{d_kB_cV_{a-k-j}\left(\lambda\right)S_1(j,c)}{j!2^{c+2k}}.$$

Theorem 4.4 *Let $a\in\mathbb{N}_0$. Then we have*

$$\begin{aligned}&\sum_{k=0}^{a}\sum_{s=0}^{k}\sum_{m=0}^{s}\binom{s}{m}\frac{B_mB_{s-m}V_{a-k}\left(\lambda\right)S_1(k,s)}{2^sk!}\\ =\ &\sum_{k=0}^{a}\sum_{j=0}^{a-k}\sum_{c=0}^{j}\sum_{s=0}^{k}\frac{B_cB_sV_{a-k-j}\left(\lambda\right)S_1(j,c)S_1(k,s)}{j!k!2^{c+s}}.\end{aligned}$$

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