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Clairaut Riemannian maps

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Abstract: In this paper, first we define Clairaut Riemannian map between Riemannian manifolds by using a geodesic curve on the base space and find necessary and sufficient conditions for a Riemannian map to be Clairaut with a nontrivial example. We also obtain necessary and sufficient condition for a Clairaut Riemannian map to be harmonic. Thereafter, we study Clairaut Riemannian map from Riemannian manifold to Ricci soliton with a nontrivial example. We obtain scalar curvatures of $\text{range} F_*$ and $(\text{range} F_*)^\perp$ by using Ricci soliton. Further, we obtain necessary conditions for the leaves of $\text{range} F_*$ to be almost Ricci soliton and Einstein. We also obtain necessary condition for the vector field $\dot{\beta}$ to be conformal on $\text{range} F_*$ and necessary and sufficient condition for the vector field $\dot{\beta}$ to be Killing on $(\text{range} F_*)^\perp$, where $\beta$ is a geodesic curve on the base space of Clairaut Riemannian map. Also, we obtain necessary condition for the mean curvature vector field of $\text{range} F_*$ to be constant. Finally, we introduce Clairaut antiinvariant Riemannian map from Riemannian manifold to Kähler manifold, and obtain necessary and sufficient condition for an antiinvariant Riemannian map to be Clairaut with a nontrivial example. Further, we find necessary condition for $\text{range} F_*$ to be minimal and totally geodesic. We also obtain necessary and sufficient condition for Clairaut antiinvariant Riemannian maps to be harmonic.

Key words: Riemannian manifold, Kähler manifold, Riemannian map, Clairaut Riemannian map, antiinvariant Riemannian map, Ricci soliton

1. Introduction

The geometry of Riemannian submersions has been discussed widely in [8]. In 1992, Fischer introduced Riemannian map between Riemannian manifolds as a generalization of an isometric immersion and Riemannian submersion that satisfies the well-known generalized eikonal equation $\|F_*\|^2 = \text{rank} F_*$, which is a bridge between geometric optics and physical optics [9]. Further, the geometry of Riemannian maps was investigated in [2, 3, 20–26].

An important Clairaut’s relation states that $\tilde{r}\sin \theta$ is constant, where $\theta$ is the angle between the velocity vector of a geodesic and a meridian, and $\tilde{r}$ is the distance to the axis of a surface of revolution. In 1972, Bishop defined Clairaut Riemannian submersion with connected fibers and gave a necessary and sufficient condition for a Riemannian submersion to be Clairaut Riemannian submersion [5]. Further, Clairaut submersions were studied in [1, 12, 14]. In [25], Şahin introduced Clairaut Riemannian map by using a geodesic curve on the total space and obtained necessary and sufficient conditions for Riemannian map to be Clairaut Riemannian map.

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Further, Şahin gave an open problem to find characterizations for Clairaut Riemannian maps (see [26], page 165, open problem 2). In Section 3, we introduce a new type of Clairaut Riemannian map by using a geodesic curve on the base space and obtain necessary and sufficient conditions for a Riemannian map to be Clairaut Riemannian map.

A Riemannian manifold $(N, g_2)$ is called a Ricci soliton [11] if there exists a smooth vector field $Z_1$ (called potential vector field) on $N$ such that $\frac{1}{2}(L_{Z_1}g_2)(X_1, Y_1) + \text{Ric}(X_1, Y_1) + \lambda g_2(X_1, Y_1) = 0$, where $L_{Z_1}$ is the Lie derivative of the metric tensor of $g_2$ with respect to $Z_1$, $\text{Ric}$ is the Ricci tensor of $(N, g_2)$, $\lambda$ is a constant function and $X_1, Y_1$ are arbitrary vector fields on $N$. We shall denote a Ricci soliton by $(N, g_2, Z_1, \lambda)$. The Ricci soliton $(N, g_2, Z_1, \lambda)$ is said to be shrinking, steady or expanding accordingly as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. It is obvious that a trivial Ricci soliton is an Einstein manifold [4] with $Z_1$ zero or Killing (Lie derivative of metric tensor $g_2$ with respect to $Z_1$ is vanishes). Ricci soliton can be used to solve the Poincaré conjecture [17]. A Ricci soliton $(N, g_2, Z_1, \lambda)$ becomes an almost Ricci soliton [18] if the function $\lambda$ is a variable. The Ricci soliton $(N, g_2, Z_1, \lambda)$ is said to be a gradient Ricci soliton if the potential vector field $Z_1$ is the gradient of some smooth function $f$ on $N$, which is denoted by $(N, g_2, f, \lambda)$. Moreover, a non-Killing tangent vector field $Z_1$ on a Riemannian manifold $(N, g_2)$ is called conformal [7] if it satisfies $L_{Z_1}g_2 = 2fg_2$, where $f$ is called the potential function of $Z_1$. The submersions and Riemannian maps from a Ricci soliton to a Riemannian manifold were studied in [10, 13, 15, 29, 30]. In [32], present authors introduced Riemannian map from a Riemannian manifold to a Ricci soliton. In Section 4, we introduce Clairaut Riemannian map from a Riemannian manifold to a Ricci soliton.

In [28], Watson studied almost Hermitian submersions. In [23], Şahin introduced holomorphic Riemannian map as generalization of holomorphic submersion and holomorphic submanifold. In [2, 3, 20, 22] invariant, antiinvariant and semiinvariant Riemannian maps were studied from a Riemannian manifold to a Kähler manifold. Recently, present authors introduced Clairaut invariant Riemannian map from a Riemannian manifold to a Kähler manifold in [31]. In Section 5, we introduce Clairaut antiinvariant Riemannian map from a Riemannian manifold to a Kähler manifold.

2. Preliminaries

In this section, we recall the notion of Riemannian map between Riemannian manifolds and give a brief review of basic facts.

Let $F : (M^m, g_1) \to (N^n, g_2)$ be a smooth map between Riemannian manifolds such that $0 < \text{rank}F \leq \min\{m, n\}$, where $\text{dim}(M) = m$ and $\text{dim}(N) = n$. We denote the kernel space of $F_*$ by $\nu_p = \text{ker}F_{*p}$ at $p \in M$ and consider the orthogonal complementary space $\mathcal{H}_p = (\text{ker}F_{*p})^\perp$ to $\text{ker}F_{*p}$ in $T_pM$. Then the tangent space $T_pM$ of $M$ at $p$ has the decomposition $T_pM = (\text{ker}F_{*p}) \oplus (\text{ker}F_{*p})^\perp = \nu_p \oplus \mathcal{H}_p$. We denote the range of $F_*$ by $\text{range}F_*$ at $p \in M$ and consider the orthogonal complementary space $(\text{range}F_{*p})^\perp$ to $\text{range}F_{*p}$ in the tangent space $T_{F(p)}N$ of $N$ at $F(p) \in N$. Since $\text{rank}F \leq \min\{m, n\}$, we have $(\text{range}F_*)^\perp \neq \{0\}$. Thus the tangent space $T_{F(p)}N$ of $N$ at $F(p) \in N$ has the decomposition $T_{F(p)}N = (\text{range}F_{*p}) \oplus (\text{range}F_{*p})^\perp$. Then $F$ is called Riemannian map at $p \in M$ if the horizontal restriction $F^h_p : (\text{ker}F_{*p})^\perp \to (\text{range}F_{*p})$ is a linear isometry between the spaces $((\text{ker}F_{*p})^\perp, g_1(p)|(\text{ker}F_{*p})^\perp)$ and $(\text{range}F_{*p}, g_2(p_1)|(\text{range}F_{*p}))$, where $F(p) = p_1$. In other words, $F_*$ satisfies

$$g_2(F_*X, F_*Y) = g_1(X, Y), \quad (2.1)$$
for all $X,Y$ vector field tangent to $\Gamma(\ker F_p)^\perp$. It follows that isometric immersions and Riemannian submersions are particular Riemannian maps with $\ker F_e = \{0\}$ and $(\text{range } F_e)^\perp = \{0\}$, respectively. The differential map $F_*$ of $F$ can be viewed as a section of bundle $\text{Hom}(TM, F^{-1}TN) \to M$, where $F^{-1}TN$ is the pullback bundle whose fibers at $p \in M$ is $(F^{-1}TN)_p = T_{F(p)}N$, $p \in M$. The bundle $\text{Hom}(TM, F^{-1}TN)$ has a connection $\nabla$ induced from the Levi-Civita connection $\nabla^M$ and the pullback connection $\nabla^F$. Then the second fundamental form of $F$ is given by [16]
\[(\nabla F_*)(X,Y) = \nabla^N_{F_*} F_* Y - F_* (\nabla^M_N Y),\tag{2.2}\]
for all $X,Y \in \Gamma(TM)$, where $\nabla^N_{F_*} F_* Y \circ F = \nabla^N_{F_*X} F_* Y$. It is known that the second fundamental form is symmetric. In [20] Sahin proved that $(\nabla F_*)(X,Y)$ has no component in $\text{range } F_*$ for all $X,Y \in \Gamma(\ker F_*)^\perp$. More precisely, we have
\[(\nabla F_*)(X,Y) \in \Gamma(\text{range } F_* )^\perp.\tag{2.3}\]

The tension field of $F$ is defined to be the trace of the second fundamental form of $F$, i.e. $\tau(F) = \text{trace}(\nabla F_*) = \sum_{i=1}^m (\nabla F_*)(e_i, e_i)$, where $m = \dim(M)$ and $\{e_1, e_2, \ldots, e_m\}$ is the orthonormal frame on $M$. Moreover, a map $F : (M^m, g_1) \to (N^n, g_2)$ between Riemannian manifolds is harmonic if and only if the tension field of $F$ vanishes at each point $p \in M$.

**Lemma 2.1** [21] Let $F : (M^m, g_1) \to (N^n, g_2)$ be a Riemannian map between Riemannian manifolds. Then the tension field of $F$ is given by $\tau(F) = -r F_*(H) + (m - r) H_2$, where $r = \dim(\ker F_*)$, $(m - r) = \text{rank } F$, $H$ and $H_2$ are the mean curvature vector fields of the distribution $\ker F_*$ and $\text{range } F_*$, respectively.

**Lemma 2.2** [22] Let $F : (M, g_1) \to (N, g_2)$ be a Riemannian map between Riemannian manifolds. Then $F$ is umbilical Riemannian map if and only if
\[(\nabla F_*)(X,Y) = g_1(X,Y) H_2,\]
for $X,Y \in \Gamma(\ker F_*)^\perp$ and $H_2$ is the mean curvature vector field of $\text{range } F_*$. For any vector field $X$ on $M$ and any section $V$ of $(\text{range } F_*)^\perp$, we have $\nabla^F X V$, which is the orthogonal projection of $\nabla^N_X V$ on $(\text{range } F_*)^\perp$, where $\nabla^F$ is linear connection on $(\text{range } F_*)^\perp$ such that $\nabla^F g_2 = 0$.

Now, for a Riemannian map $F$ we define $S_V$ as ([24], p. 188)
\[\nabla^N_{F_*X} V = -S_V F_* X + \nabla^F X V,\tag{2.4}\]
where $\nabla^N$ is Levi-Civita connection on $N$, $S_V F_* X$ is the tangential component (a vector field along $F$) of $\nabla^N_{F_*X} V$. Thus at $p \in M$, we have $\nabla^N_{F_*X} V(p) \in T_{F(p)}N$, $S_V F_* X \in F_{*p}(T_p M)$ and $\nabla^F X V(p) \in (F_{*p}(T_p M))^\perp$. It is easy to see that $S_V F_* X$ is bilinear in $V$, and $F_* X$ at $p$ depends only on $V_p$ and $F_{*p} X_p$. Hence from (2.2) and (2.4), we obtain
\[g_2(S_V F_* X, F_* Y) = g_2(V, (\nabla F_*)(X,Y)),\tag{2.5}\]
for $X,Y \in \Gamma(\ker F_*)^\perp$ and $V \in \Gamma(\text{range } F_*)^\perp$, where $S_V$ is self-adjoint operator.
3. Clairaut Riemannian map between Riemannian manifolds

In this section, we define Clairaut Riemannian map between Riemannian manifolds by using a geodesic curve on the base space and investigate geometry.

The notion of Clairaut Riemannian map was defined by Şahin in [25]. According to the definition, a Riemannian map $F : (M, g_1) \rightarrow (N, g_2)$ between Riemannian manifolds is called Clairaut Riemannian map if there is a function $\tilde{s} : N \rightarrow \mathbb{R}^+$ such that for every geodesic $\beta$ on $N$, the function $(\tilde{s} \circ \beta)\sin \omega(t)$ is constant, where, for all $t$, $\theta(t)$ is the angle between $\dot{\alpha}(t)$ and the horizontal space at $\alpha(t)$.

Thus, the notion of Clairaut Riemannian map comes from a geodesic curve on a surface of revolution. Therefore, we are going to give a definition of Clairaut Riemannian map by using geodesic curve on the base space.

**Definition 3.1** A Riemannian map $F : (M, g_1) \rightarrow (N, g_2)$ between Riemannian manifolds is called Clairaut Riemannian map if there is a function $\tilde{s} : N \rightarrow \mathbb{R}^+$ such that for every geodesic $\beta$ on $N$, the function $(\tilde{s} \circ \beta)\sin \omega(t)$ is constant, where, $F \circ X, F X \in \Gamma(\text{range}F_\ast)$ for $X \in \Gamma(\text{ker}F_\ast)$ and $V \in \Gamma(\text{range}F_\ast)^\perp$ are components of $\dot{\beta}(t)$, and $\omega(t)$ is the angle between $\dot{\beta}(t)$ and $V$ for all $t$.

**Note:** For all $U, V \in \Gamma(\text{range}F_\ast)^\perp$ we define

$$\nabla^N_U V = \mathcal{R}(\nabla^N_U V) + \nabla^{F\perp}_U V,$$

where $\mathcal{R}(\nabla^N_U V)$ and $\nabla^{F\perp}_U V$ denote $\text{range}F_\ast$ and $(\text{range}F_\ast)^\perp$ part of $\nabla^N_U V$, respectively. Therefore $(\text{range}F_\ast)^\perp$ is totally geodesic if and only if

$$\nabla^N_U V = \nabla^{F\perp}_U V.$$

Note that from now, throughout the paper, we are assuming $(\text{range}F_\ast)^\perp$ is totally geodesic.

**Lemma 3.2** Let $F : (M, g_1) \rightarrow (N, g_2)$ be a Riemannian map between Riemannian manifolds and $\alpha : I \rightarrow M$ be a geodesic curve on $M$. Then the curve $\beta = F \circ \alpha$ is geodesic curve on $N$ if and only if

$$\nabla_\ast F(X, X) + \nabla^{F\perp}_{X V} V + \nabla^{F\perp}_{V X} X = 0, \quad (3.1)$$

$$-S_{V X} F_\ast X + F_\ast(\nabla^{N\perp}_X X) + \nabla^{N F}_{V X} X = 0, \quad (3.2)$$

where $F_\ast X \in \Gamma(\text{range}F_\ast), V \in \Gamma(\text{range}F_\ast)^\perp$ are components of $\dot{\beta}(t)$ and $\nabla^N$ is Levi-Civita connection on $N$ and $\nabla^{F\perp}$ is a linear connection on $(\text{range}F_\ast)^\perp$.

**Proof** Let $\alpha : I \rightarrow M$ be a geodesic on $M$ with $U(t) = \nu \dot{\alpha}(t)$ and $X(t) = \mathcal{H} \dot{\alpha}(t)$. Let $\beta = F \circ \alpha$ be a geodesic on $N$ with $F_\ast X \in \Gamma(\text{range}F_\ast)$ and $V \in \Gamma(\text{range}F_\ast)^\perp$ are components of $\dot{\beta}(t)$.

Now,

$$\nabla^{N F}_{\beta X + V} (F_\ast X + V),$$

which implies

$$\nabla^{N F}_{\beta X + V} (F_\ast X + V) = \nabla^{N F}_{\beta X} F_\ast X + \nabla^{N F}_{\beta V} F_\ast X + \nabla^{N F}_V F_\ast X + \nabla^{N F}_V V.$$
Using (2.4) in above equation, we get
\[ \nabla^N_{\beta} \dot{\beta} = \nabla^F_{\beta} F_\alpha X \circ F + (-S_{\beta} F_\alpha X + \nabla^F_{\alpha} V) + \nabla^N_{\beta} F_\alpha X + \nabla^N_{\beta} V. \]

Using (2.2) in above equation, we get
\[ \nabla^N_{\beta} \dot{\beta} = (\nabla F_\alpha)(X, X) + F_\alpha (\nabla^M_{\alpha} X) - S_{\alpha} F_\alpha X + \nabla^F_{\alpha} V + \nabla^N_{\alpha} F_\alpha X + \nabla^N_{\alpha} V. \]  

Since \((\text{range}F_\alpha)^{\perp}\) is totally geodesic, (3.3) can be written as
\[ \nabla^N_{\beta} \dot{\beta} = (\nabla F_\alpha)(X, X) + F_\alpha (\nabla^M_{\alpha} X) - S_{\alpha} F_\alpha X + \nabla^F_{\alpha} V + \nabla^N_{\alpha} F_\alpha X + \nabla^N_{\alpha} V. \]

Now \(\beta\) is geodesic on \(N\) if and only if \(\nabla^N_{\beta} \dot{\beta} = 0\). Then (3.4) implies \((\nabla F_\alpha)(X, X) + F_\alpha (\nabla^M_{\alpha} X) - S_{\alpha} F_\alpha X + \nabla^F_{\alpha} V + \nabla^N_{\alpha} F_\alpha X + \nabla^N_{\alpha} V = 0\), which completes the proof. \(\square\)

**Theorem 3.3** Let \(F : (M, g_1) \to (N, g_2)\) be a Riemannian map between Riemannian manifolds such that \(\text{range}F_\alpha\) is connected and \(\alpha, \beta = F \circ \alpha\) are geodesic curves on \(M\) and \(N\), respectively. Then \(F\) is Clairaut Riemannian map with \(\dot{s} = e^g\) if and only if any one of the following conditions holds:

(i) \(S_{\alpha} F_\alpha X = -V(g) F_\alpha X\), where \(F_\alpha X \in \Gamma(\text{range}F_\alpha), V \in \Gamma(\text{range}F_\alpha)^{\perp}\) are components of \(\dot{\beta}(t)\).

(ii) \(F\) is umbilical map, and has \(H_2 = -\nabla^N g\), where \(g\) is a smooth function on \(N\) and \(H_2\) is the mean curvature vector field of \(\text{range}F_\alpha\).

**Proof** First we prove \(F\) is a Clairaut Riemannian map with \(\dot{s} = e^g\) if and only if for any geodesic \(\beta : I \to N\) with tangential components \(F_\alpha X \in \Gamma(\text{range}F_\alpha)\) and \(V \in \Gamma(\text{range}F_\alpha)^{\perp}\), \(t \in I\) the equation
\[ g_{2\beta(t)}(F_\alpha X(t), F_\alpha X(t))g_{2}(\dot{\beta}(t), (\nabla^N g)) + g_{2}(S_{\alpha} F_\alpha X(t), F_\alpha X(t)) = 0, \tag{3.5} \]

is satisfied. To prove this, let \(\beta\) be a geodesic on \(N\) with \(\dot{\beta}(t) = F_\alpha X(t) + V(t)\) and let \(\omega(t) \in [0, \pi]\) denote the angle between \(\dot{\beta}(t)\) and \(V(t)\). If \(\dot{\beta}(t) \in \Gamma(\text{range}F_\alpha)^{\perp}\), then we have \(F_\alpha X(t_0) = 0\) (i.e. (3.5) is satisfied), which implies \(\sin \omega(t) = 0\) at point \(\beta(t_0)\). Thus for any function \(\dot{s} = e^g\) on \(M\), \((\dot{s}((\beta(t))\) \sin \omega(t) identically vanishes. Therefore, the statement holds trivially in this case. Now, we consider the case \(\sin \omega(t) \neq 0\), i.e. \(\dot{\beta}(t)\) does not belongs only in \(\Gamma(\text{range}F_\alpha)^{\perp}\). Since \(\beta\) is geodesic, its speed is constant \(b = ||\dot{\beta}||^2\) (say). Then
\[ g_{2\beta(t)}(V, V) = b \cos^2 \omega(t), \tag{3.6} \]
\[ g_{2\beta(t)}(F_\alpha X, F_\alpha X) = b \sin^2 \omega(t). \tag{3.7} \]

Now differentiating (3.7) along \(\beta\), we get
\[ \frac{d}{dt} g_{2}(F_\alpha X, F_\alpha X) = 2b \sin \omega \cos \omega \frac{d\omega}{dt}. \tag{3.8} \]

On the other hand,
\[ \frac{d}{dt} g_{2}(F_\alpha X, F_\alpha X) = 2g_{2}(\nabla^N_{\beta} F_\alpha X, F_\alpha X). \]

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By putting $\dot{\gamma} = F_*X + V$ in above equation, we get
\[
\frac{d}{dt} g_2(F_*X, F_*X) = 2g_2(\nabla^N_{F_*X} F_*X + \nabla^N_{F_*X} F_*X, F_*X),
\]
which implies
\[
\frac{d}{dt} g_2(F_*X, F_*X) = 2g_2(\nabla^F_{F_*X} F_*X \circ F + \nabla^N_{F_*X} F_*X, F_*X). \tag{3.9}
\]
Using (2.2) and (3.2) in (3.9), we get
\[
\frac{d}{dt} g_2(F_*X, F_*X) = 2g_2((\nabla F)_*(X, X) + F_*(\nabla^M_{F_*X} X + \mathcal{S}_V F_*X - F_*(\nabla^M_{F_*X} X), F_*X). \tag{3.10}
\]
Using (2.3) in above equation, we get
\[
\frac{d}{dt} g_2(F_*X, F_*X) = 2g_2(\mathcal{S}_V F_*X, F_*X). \tag{3.10}
\]
Now from (3.8) and (3.10), we get
\[
g_2(\mathcal{S}_V F_*X, F_*X) = b\sin \omega \cos \omega \frac{d\omega}{dt}. \tag{3.11}
\]
Moreover, $F$ is a Clairaut Riemannian map with $\tilde{s} = e^g$ if and only if $\frac{d}{dt}(e^{g \circ \beta} \sin \omega) = 0$, that is, $e^{g \circ \beta} \sin \omega \frac{d(e^{g \circ \beta})}{dt} + e^{g \circ \beta} \cos \omega \frac{d\omega}{dt} = 0$. By multiplying this with nonzero factor $b\sin \omega$ and using (3.7), we get
\[
g_2(F_*X, F_*X) \frac{d(g \circ \beta)}{dt} = -b\sin \omega \cos \omega \frac{d\omega}{dt}. \tag{3.12}
\]
Now from (3.11) and (3.12), we get
\[
g_2(\mathcal{S}_V F_*X, F_*X) = -g_2(F_*X, F_*X) \frac{d(g \circ \beta)}{dt},
\]
which means
\[
g_2(\mathcal{S}_V F_*X, F_*X) = -g_2(F_*X, F_*X) g_2(\nabla^N g, \dot{\beta}). \tag{3.13}
\]
Indeed assuming (3.5) and considering any geodesic $\beta$ on $N$ with initial tangent vector which belongs in $\Gamma(range F_*)$, then by using $V(t_0) = 0$ in (3.13), we get $g$ is constant on $range F_*$ and since $range F_*$ is connected, $\nabla^N g \in \Gamma(range F_*)^+$. Then by (3.13), we get
\[
g_2(\mathcal{S}_V F_*X, F_*X) = -g_2(F_*X, F_*X) g_2(\nabla^N g, V). \tag{3.14}
\]
Thus $\mathcal{S}_V F_*X = -V(g)F_*X$, where $V(g)$ is a smooth function on $N$, which implies the proof of (i). Now, by using (2.5) in (3.14), we get
\[
g_2(V, (\nabla F_*)(X, X)) = -g_2(F_*X, F_*X) g_2(\nabla^N g, V), \tag{3.15}
\]
for $F_*X \in \Gamma(\text{range}F_*)$ and $V \in \Gamma(\text{range}F_*)^\perp$. Now using (2.2) in (3.15), we get
\[ g_2(\nabla_X^NF_*X) = -g_2(\nabla^Ng)g_2(F_*X,F_*X). \]
Thus by comparing, we get
\[ \nabla_X^NF_*X = -(\nabla^Ng)g_2(F_*X,F_*X). \tag{3.16} \]
Taking trace of (3.16), we get
\[ \sum_{j=r+1}^m \nabla_{X_j}^NF_*X_j = -(\nabla^Ng)(m-r), \tag{3.17} \]
where $\{X_{r+1},X_{r+2},...,X_m\}$ and $\{F_*X_{r+1},F_*X_{r+2},...,F_*X_m\}$ are orthonormal bases of $(\ker F_*)^\perp$ and $\text{range}F_*$, respectively.
Moreover, the mean curvature vector field of $\text{range}F_*$ is defined by (21), page 199
\[ H_2 = \frac{1}{m-r} \sum_{j=r+1}^m \nabla_{X_j}^NF_*X_j, \tag{3.18} \]
where $\{X_j\}_{r+1 \leq j \leq m}$ is an orthonormal basis of $(\ker F_*)^\perp$. Then from (3.17) and (3.18), we get
\[ H_2 = -\nabla^Ng. \tag{3.19} \]
Also, by (3.15), we get
\[ (\nabla F_*)(X,X) = -g_2(F_*X,F_*X)(\nabla^Ng). \tag{3.20} \]
Since $F$ is Riemannian map, using (2.1) in (3.20), we get
\[ (\nabla F_*)(X,X) = -g_1(X,X)(\nabla^Ng). \tag{3.21} \]
From (3.19) and (3.21), we get
\[ (\nabla F_*)(X,X) = g_1(X,X)H_2. \]
Thus by Lemma 2.2 $F$ is umbilical map, which completes the proof.

**Remark 3.4** In [25], Sahin considered geodesic curve on the total manifold of a Riemannian map $F$, then by using Clairaut relation fibers of $F$ are totally umbilical. On the other hand, in Definition 3.1, we considered geodesic curve on the base manifold of $F$, then by using Clairaut’s relation $F$ becomes totally umbilical.

**Theorem 3.5** Let $F : (M^m,g_1) \to (N^n,g_2)$ be a Clairaut Riemannian map with $\tilde{s} = e^g$ between Riemannian manifolds such that $\ker F_*$ is minimal. Then $F$ is harmonic if and only if $g$ is constant function on $N$.

**Proof** Since $H = 0$, then by Lemma 2.1 $F$ is harmonic if and only if $H_2 = 0$ if and only if $\nabla^Ng = 0$, which completes the proof.

**Theorem 3.6** Let $F : (M^m,g_1) \to (N^n,g_2)$ be a Clairaut Riemannian map with $\tilde{s} = e^g$ between Riemannian manifolds. Then $N = N_{(\text{range}F_*)^\perp} \times_f N_{\text{range}F_*}$ is a twisted product manifold.
Then, we get
\[ \nabla^X F_Y = g_1(X, Y)H_2 \]
for \( X, Y \in \Gamma(\ker F_*) \), which implies \( \text{range} F_* \) is totally umbilical. Then proof follows by [19].

**Proof** By (3.20), (3.21) and Theorem 3.3, we have \( \nabla^X F_Y = g_1(X, Y)H_2 \) for \( X, Y \in \Gamma(\ker F_*) \), which implies \( \text{range} F_* \) is totally umbilical. Then proof follows by [19].

**Example 3.7** Let \( M = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0\} \) be a Riemannian manifold with Riemannian metric \( g_1 = e^{2x_2}dx_1^2 + dx_2^2 \) on \( M \). Let \( N = \{(y_1, y_2) \in \mathbb{R}^2\} \) be a Riemannian manifold with Riemannian metric \( g_2 = e^{2x_2}dy_1^2 + dy_2^2 \) on \( N \). Consider a map \( F : (M, g_1) \to (N, g_2) \) defined by

\[ F(x_1, x_2) = (x_1, 0). \]

Then, we get
\[ \ker F_* = \text{span}\{U = e_2\} \] and \( \text{range} F_* = \text{span}\{X = e_1\}, \]
where \( \{e_1 = e^{-x_2} \frac{\partial}{\partial x_2}, e_2 = \frac{\partial}{\partial x_1}\} \) and \( \{e'_1 = e^{-x_2} \frac{\partial}{\partial y_2}, e'_2 = \frac{\partial}{\partial y_1}\} \) are bases on \( T_pM \) and \( T_{F(p)}N \), respectively, for all \( p \in M \). By easy computations, we see that \( F_*(X) = e'_1 \) and \( g_1(X, X) = g_2(F_*X, F_*X) \) for \( X \in \Gamma(\ker F_*) \). Thus \( F \) is Riemannian map with range \( \text{range} F_* \). By easy computations, we see that \( g_2(X, F_*X) = 1 \)
and \( g_2(F_*X, F_*X) = 1 \).

Now to show \( F \) is Clairaut Riemannian map we will verify Theorem 3.3, for this we will verify (3.14). Since \( V \) and \( (\nabla F_*)(X, X) \in \Gamma(\text{range} F_*) \), here we can write \( V = ae'_2 \) and \( (\nabla F_*)(X, X) = be'_2 \) for some \( a, b \in \mathbb{R} \).

Then we get
\[ g_2(V, (\nabla F_*)(X, X)) = g_2(ae'_2, be'_2) = ab, \]
and
\[ g_2(F_*X, F_*X) = g_2(e'_1, e'_1) = 1. \]

Since \( \nabla^N g = \sum_{i,j=1}^{2} \frac{\partial^2 g}{\partial y_i \partial y_j} \frac{\partial}{\partial y_i} \). Therefore for the function \( g = -by_2 \)

\[ g_2(\nabla^N g, V) = -ab. \]

Thus by using (2.5), (3.22), (3.23) and (3.24) we see that (3.14) holds. Thus \( F \) is a Clairaut Riemannian map.

4. Clairaut Riemannian map from Riemannian manifold to Ricci soliton

In this section, we study Clairaut Riemannian map \( F : (M, g_1) \to (N, g_2) \) from a Riemannian manifold to a Ricci soliton and give some characterizations.

**Lemma 4.1** [32] Let \( F : (M^m, g_1) \to (N^n, g_2) \) be a Riemannian map between Riemannian manifolds. Then the Ricci tensor on \( (N, g_2) \) given by

\[
\text{Ric}(F_*X, F_*Y) = \text{Ric}^{\text{range} F_*}(F_*X, F_*Y) - \sum_{k=1}^{m} \left\{ g_2(S_{F_*X} F_*X, F_*X) - g_2(\nabla^N_{e_k} S_{F_*X} F_*X, F_*X) + g_2(\nabla^N_{e_k} F_*X, S_{F_*X} F_*X) + g_2(\nabla^N_{e_k} F_*X, S_{F_*X} F_*X)\right\},
\]

\[
\text{Ric}(V, W) = \text{Ric}^{\text{range} F_*}(V, W) - \sum_{j=r+1}^{m} \left\{ g_2(S_{F_*X} F_*X, F_*X) + g_2(S_{F_*X} F_*X, S_{F_*X} F_*X) - \nabla^N_V (g_2(S_{F_*X} F_*X, F_*X)) + 2g_2(S_{F_*X} F_*X, \nabla^N_F X, F_*X)\right\},
\]

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\[
Ric(F, X, V) = \sum_{j=r+1}^{m} \left\{ g_2((\tilde{\nabla}_X S)\nu F, X_j, X_j) - g_2((\tilde{\nabla}_X S)\nu F, X, X_j) \right\} - \sum_{k=1}^{n_1} g_2(R^F_{\perp}(F, X, e_k)V, e_k), \tag{4.3}
\]

for \(X, Y \in \Gamma(\ker F)^\perp\), \(V, W \in \Gamma(\text{range} F)^\perp\) and \(F, X, F, Y \in \Gamma(\text{range} F_{\ast})\), where \(\{F, X_j\}_{r+1 \leq j \leq m}\) and \(\{e_k\}_{1 \leq k \leq n_1}\) are orthonormal bases of \(\text{range} F_{\ast}\) and \(\text{range} F_{\ast}^\perp\), respectively.

**Theorem 4.2** Let \(F : (M^m, g_1) \to (N^n, g_2)\) be a Clairaut Riemannian map with \(\tilde{s} = e^g\) between Riemannian manifolds. Then the Ricci tensor on \((N, g_2)\) given by

\[
Ric(F, X, F, Y) = Ric^{\text{range} F_{\ast}}(F, X, F, Y) - \sum_{k=1}^{n_1} (e_k(g))^2 g_2(F, X, F, Y) + \sum_{k=1}^{n_1} g_2(\nabla_{e_k} e_k, \nabla N g) g_2(F, X, F, Y) - \sum_{k=1}^{n_1} (\nabla^N e_k(g)) g_2(F, X, F, Y),
\]

\[
Ric(V, W) = Ric^{\text{range} F_{\ast}^\perp}(V, W) + (m - r) g_2(\nabla^N g, \nabla F^\perp_{\nu} W) - (m - r) V(g) W(g) - (m - r) \nabla^N W(g),
\]

\[
Ric(F, X, V) = \sum_{j=r+1}^{m} g_2((\tilde{\nabla}_X S)\nu F, X_j, X_j) - \sum_{j=r+1}^{m} g_2((\tilde{\nabla}_X S)\nu F, X, X_j) - \sum_{k=1}^{n_1} g_2(R^F_{\perp}(F, X, e_k)V, e_k),
\]

for \(X, Y \in \Gamma(\ker F)^\perp\), \(V, W \in \Gamma(\text{range} F)^\perp\) and \(F, X, F, Y \in \Gamma(\text{range} F_{\ast})\), where \(\{F, X_j\}_{r+1 \leq j \leq m}\) and \(\{e_k\}_{1 \leq k \leq n_1}\) are orthonormal bases of \(\text{range} F_{\ast}\) and \(\text{range} F_{\ast}^\perp\), respectively.

**Proof** Using Theorem 3.3 and (3.14) in (4.1), we get

\[
Ric(F, X, F, Y) = Ric^{\text{range} F_{\ast}}(F, X, F, Y) - \sum_{k=1}^{n_1} (e_k(g))^2 g_2(F, X, F, Y) + \sum_{k=1}^{n_1} g_2(\nabla^N e_k, \nabla^N g) g_2(F, X, F, Y) - \sum_{k=1}^{n_1} g_2(\nabla^N F e_k, F, Y) + \sum_{k=1}^{n_1} g_2(\nabla^N F, e_k(g) F, Y),
\]

which implies (4.4). Also using Theorem 3.3 and (3.14) in (4.2), we get

\[
Ric(V, W) = Ric^{\text{range} F_{\ast}^\perp}(V, W) + \sum_{j=r+1}^{m} g_2(\nabla^N_{\nu} W, \nabla^N g) g_2(F, X_j, F, X_j) \sum_{j=r+1}^{m} g_2(V(g) F, X_j, W(g) F, X_j) - \sum_{j=r+1}^{m} \nabla^N g(2 g_2(W(g) F, X_j, \nabla^N F, X_j) + 2 \sum_{j=r+1}^{m} g_2(W(g) F, X_j, \nabla^N F, X_j),
\]

which implies (4.5). Also the proof of (4.3) and (4.6) is same. \(\square\)

**Theorem 4.3** Let \((N, g_2, H_2, \lambda)\) be a Ricci soliton with the potential vector field \(H_2 \in \Gamma(\text{range} F_{\ast})^\perp\) and \(F : (M, g_1) \to (N, g_2)\) be a Clairaut Riemannian map with \(\tilde{s} = e^g\) between Riemannian manifolds. Then

\[
s^{\text{range} F_{\ast}} = -\lambda (m - r) + (m - r) \Delta g - (m - r)(m - r - 2)\|\nabla^N g\|^2,
\]

where \(s^{\text{range} F_{\ast}}\) is the scalar curvature of \(\text{range} F_{\ast}\) and \((m - r) = \dim(\text{range} F_{\ast})\).
Proof Since \((N, g_2, H_2, \lambda)\) admit Ricci soliton with the potential vector field \(H_2 \in \Gamma(\text{range} F_2)^{\perp}\) then, we have
\[
\frac{1}{2}(L_{H_2} g_2)(F_2 X, F_2 Y) + \text{Ric}(F_2 X, F_2 Y) + \lambda g_2(F_2 X, F_2 Y) = 0,
\]
for \(F_2 X, F_2 Y \in \Gamma(\text{range} F_2)\), which implies
\[
\frac{1}{2}\{g_2(\nabla^N_{F_2 X} H_2, F_2 Y) + g_2(\nabla^N_{F_2 Y} H_2, F_2 X)\} + \text{Ric}(F_2 X, F_2 Y) + \lambda g_2(F_2 X, F_2 Y) = 0.
\]
Using (2.4) in above equation, we get
\[
\frac{1}{2}\{g_2(\mathcal{S}_{H_2} F_2 X, F_2 Y) + g_2(\mathcal{S}_{H_2} F_2 Y, F_2 X)\} + \text{Ric}(F_2 X, F_2 Y) + \lambda g_2(F_2 X, F_2 Y) = 0.
\]
Since \(\mathcal{S}_{H_2}\) is self-adjoint, above equation can be written as
\[
-g_2(\mathcal{S}_{H_2} F_2 X, F_2 Y) + \text{Ric}(F_2 X, F_2 Y) + \lambda g_2(F_2 X, F_2 Y) = 0.
\]
Using (3.14), (3.19) and (4.4) in (4.7), we get
\[
-g_2(\nabla^N g, \nabla^N g)g_2(F_2 X, F_2 Y) + \text{Ric}^{\text{range} F_2}(F_2 X, F_2 Y) - \sum_{k=1}^{n_1} (e_k(g))^2 g_2(X, F_2 Y)
\]
\[
+ \sum_{k=1}^{n_1} g_2(\nabla^N_{\perp e_k} e_k, \nabla^N g)g_2(F_2 X, F_2 Y) - \sum_{k=1}^{n_1} \nabla^N_{\perp e_k} e_k(g)g_2(F_2 X, F_2 Y) + \lambda g_2(F_2 X, F_2 Y) = 0,
\]
where \(\{e_k\}_{1 \leq k \leq n_1}\) is an orthonormal basis of \((\text{range} F_2)\). This implies
\[
-2\|\nabla^N g\|^2 g_2(F_2 X, F_2 Y) + \text{Ric}^{\text{range} F_2}(F_2 X, F_2 Y)
\]
\[
- \sum_{k=1}^{n_1} g_2(\nabla^N_{\perp e_k} e_k, \nabla^N g)g_2(F_2 X, F_2 Y) + \lambda g_2(F_2 X, F_2 Y) = 0.
\]
Taking trace of (4.8) for \(\text{range} F_2\), we get
\[
s^{\text{range} F_2} - 2(m - r)\|\nabla^N g\|^2 - (m - r) \sum_{k=1}^{n_1} g_2(\nabla^N_{\perp e_k} e_k, e_k) + \lambda(m - r) = 0.
\]
Using definition of Hessian form of \(g\) (i.e. \(H^g(X_1, Y_1) = g_2(\nabla^N_{X_1} \nabla^N g, Y_1)\) for all \(X_1, Y_1 \in \Gamma(T N)\)) from [8] in above equation, we get
\[
s^{\text{range} F_2} + (m - r)\{-2\|\nabla^N g\|^2 - \sum_{k=1}^{n_1} H^g(e_k, e_k) + \lambda\} = 0.
\]
Since we know that
\[
\Delta g = \sum_{j=r+1}^{m} H^g(F_2 X_j, F_2 X_j) + \sum_{k=1}^{n_1} H^g(e_k, e_k),
\]
where \(\{F_2 X_j\}_{r+1 \leq j \leq m}\) and \(\{e_k\}_{1 \leq k \leq n_1}\) are orthonormal bases of \(\text{range} F_2\) and \((\text{range} F_2)^{\perp}\), respectively. Then by using definition of Hessian form of \(g\) in (4.10), we get
\[
\Delta g = \sum_{j=r+1}^{m} g_2(\nabla^N_{F_2 X_j} \nabla^N g, F_2 X_j) + \sum_{k=1}^{n_1} H^g(e_k, e_k).
\]
Using (2.4) in (4.11), we get
\[
\Delta g = - \sum_{j=r+1}^{m} g_2(\mathcal{S}_v \nabla g X_j, F_* X_j) + \sum_{k=1}^{n_1} H^g(e_k, e_k).
\]
Using Theorem 3.3 in above equation, we get
\[
\Delta g - (m - r)\|\nabla^N g\|^2 = \sum_{k=1}^{n_1} H^g(e_k, e_k).
\] (4.12)

Thus (4.9) and (4.12) implies the proof. \(\square\)

**Theorem 4.4** Let \((N, g_2, H_2, \lambda)\) be a Ricci soliton with the potential vector field \(H_2 \in \Gamma(\text{range } F_* \uparrow)\) and \(F : (M^m, g_1) \to (N^n, g_2)\) be a Clairaut Riemannian map with \(\bar{s} = \frac{e^3}{1}\) between Riemannian manifolds. Then
\[
s(\text{range } F_* \uparrow) = -\lambda n_1 + (m - r + 1)\Delta g - (m - r)^2\|\nabla^N g\|^2,
\]
where \(s(\text{range } F_* \uparrow)\) denotes the scalar curvature of \((\text{range } F_* \uparrow)\) and \((m - r) = \text{dim}(\text{range } F_*), n_1 = \text{dim}(\text{range } F_* \uparrow)\).

**Proof** Since \((N, g_2, H_2, \lambda)\) admit Ricci soliton with the potential vector field \(H_2 \in \Gamma(\text{range } F_* \uparrow)\) then, we have
\[
\frac{1}{2}(L_{H_2} g_2)(V, W) + \text{Ric}(V, W) + \lambda g_2(V, W) = 0,
\]
for \(V, W \in \Gamma(\text{range } F_* \uparrow),\) which implies
\[
\frac{1}{2}\{g_2(\nabla^N g H, W) + g_2(\nabla^N g H, V)\} + \text{Ric}(V, W) + \lambda g_2(V, W) = 0.
\]
Putting \(H_2 = -\nabla^N g\) in above equation, we get
\[
\frac{1}{2}\{g_2(\nabla^N g, W) + g_2(\nabla^N g, V)\} + \text{Ric}(V, W) + \lambda g_2(V, W) = 0. \tag{4.13}
\]
Using definition of Hessian form of \(g\) and (4.5) in (4.13), we get
\[
-H^g(V, W) + \text{Ric}(\text{range } F_* \uparrow)(V, W) + (m - r)g_2(\nabla^N g, \nabla^N g \nabla^N g (V)) - (m - r)\|\nabla^N g\|^2 g_2(V, W) + \lambda g_2(V, W) = 0. \tag{4.14}
\]
Taking trace of (4.14) for \((\text{range } F_* \uparrow)\), we get
\[
-\sum_{k=1}^{n_1} H^g(e_k, e_k) + s(\text{range } F_* \uparrow) + \sum_{k=1}^{n_1} (m - r)g_2(\nabla^N g, \nabla^N g e_k) - (m - r)\sum_{k=1}^{n_1} (e_k(g))^2 - (m - r)\sum_{k=1}^{n_1} \nabla^N g e_k(g) + \lambda n_1 = 0,
\]
where \(\{e_k\}_{1 \leq k \leq n_1}\) is an orthonormal basis of \((\text{range } F_* \uparrow)\), which implies
\[
s(\text{range } F_* \uparrow) + \lambda n_1 - (m - r)\sum_{k=1}^{n_1} (e_k(g))^2 - (m - r + 1)\sum_{k=1}^{n_1} H^g(e_k, e_k) = 0.
\]
Using (4.12) and \((e_k(g))^2 = g_2(\nabla^N g, e_k)^2 = g_2(\nabla^N g, \nabla^N g)\) in above equation, we get the proof. \(\square\)
Remark 4.5 Since $\text{range}F_*$ and $(\text{range}F_*)^\perp$ are subbundles of $TN$, they define distributions on $N$. Then for $F, X, Y \in \Gamma(\text{range}F_*)$, we have

$$[F_*X, F_*Y] = \nabla^N_{F_*X}F_*Y - \nabla^N_{F_*Y}F_*X$$

$$= \nabla^N_XF_*Y \circ F - \nabla^N_YF_*X \circ F.$$ 

Using (2.2) in above equation, we get

$$[F_*X, F_*Y] = F_*(\nabla_XY) - F_*(\nabla_YX) = F_*(\nabla_XY - \nabla_YX) \in \Gamma(\text{range}F_*).$$

Thus $\text{range}F_*$ is an integrable distribution. Then for any point $F(p) \in N$ there exists maximal integral manifold or a leaf of $\text{range}F_*$ containing $F(p)$.

Theorem 4.6 Let $(N, g_2, F_*Z, \lambda)$ be a Ricci soliton with the potential vector field $F_*Z \in \Gamma(\text{range}F_*)$ and $F : (M, g_1) \to (N, g_2)$ be a Clairaut Riemannian map with $\tilde{s} = e^g$ between Riemannian manifolds. Then a leaf of $\text{range}F_*$ is an almost Ricci soliton.

Proof Since $(N, g_2, F_*Z, \lambda)$ admit Ricci soliton with the potential vector field $F_*Z \in \Gamma(\text{range}F_*)$ then, we have

$$\frac{1}{2}(L_{F_*Z}g_2)(F_*X, F_*Y) + \text{Ric}(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0,$$ 

for $F_*X, F_*Y, F_*Z \in \Gamma(\text{range}F_*)$. Using (4.4) in (4.15), we get

$$\frac{1}{2}(L_{F_*Z}g_2)(F_*X, F_*Y) + \text{Ric}^{\text{range}F_*}(F_*X, F_*Y) - \sum_{k=1}^{n_1}(e_k(g))^2g_2(F_*X, F_*Y)$$

$$+ \sum_{k=1}^{n_1}g_2(\nabla_{e_k}e_k, \nabla^N g)g_2(F_*X, F_*Y) - \sum_{k=1}^{n_1}\nabla^N_{e_k}e_k(g)g_2(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0,$$ 

where $\{e_k\}_{1 \leq k \leq n_1}$ is an orthonormal basis of $(\text{range}F_*)^\perp$, which implies

$$\frac{1}{2}(L_{F_*Z}g_2)(F_*X, F_*Y) + \text{Ric}^{\text{range}F_*}(F_*X, F_*Y) + \tilde{\lambda} g_2(F_*X, F_*Y) = 0,$$ 

where $\tilde{\lambda} = -\sum_{k=1}^{n_1}(e_k(g))^2 + \sum_{k=1}^{n_1}g_2(\nabla_{e_k}e_k, \nabla^N g) - \sum_{k=1}^{n_1}e_k(e_k(g)) + \lambda$ is a smooth function on $N$. Thus a leaf of $\text{range}F_*$ is an almost Ricci soliton, which completes the proof.

Theorem 4.7 Let $(N, g_2, V, \lambda)$ be a Ricci soliton with the potential vector field $V \in \Gamma(\text{range}F_*)^\perp$ and $F : (M, g_1) \to (N, g_2)$ be a Clairaut Riemannian map with $\tilde{s} = e^g$ between Riemannian manifolds. Then a leaf of $\text{range}F_*$ is an Einstein.

Proof Since $(N, g_2, F_*Z, \lambda)$ admit Ricci soliton with the potential vector field $F_*Z \in \Gamma(\text{range}F_*)$ then, we have

$$\frac{1}{2}(L_Vg_2)(F_*X, F_*Y) + \text{Ric}(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0,$$ 

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for $F, X, Y \in \Gamma(\text{range} F_*)$, which implies

$$\frac{1}{2} \left\{ g_2(\nabla^N_{F_*} V, F_* Y) + g_2(\nabla^N_{F_*} V, F_* X) \right\} + \text{Ric}(F_* X, F_* Y) + \lambda g_2(F_* X, F_* Y) = 0.$$  

Using (2.4) in above equation, we get

$$\frac{1}{2} \left\{ g_2(-S_\beta F_* X, F_* Y) + g_2(-S_\beta F_* Y, F_* X) \right\} + \text{Ric}(F_* X, F_* Y) + \lambda g_2(F_* X, F_* Y) = 0.$$  

Since $S_\beta$ is self-adjoint, above equation can be written as

$$-g_2(S_\beta F_* X, F_* Y) + \text{Ric}(F_* X, F_* Y) + \lambda g_2(F_* X, F_* Y) = 0.$$  

(4.16)

Since $F$ is Clairaut Riemannian map, using $S_\beta F_* X = -V(g)F_* X$ and (4.4) in (4.16), we get

$$V(g)g_2(F_* X, F_* Y) + \text{Ric}^{\text{range} F_*}(F_* X, F_* Y) - \sum_{k=1}^{n_1} (e_k(g))^2 g_2(F_* X, F_* Y)$$

$$+ \sum_{k=1}^{n_1} g_2(\nabla^N_{e_k} e_k, \nabla^N g) g_2(F_* X, F_* Y) - \sum_{k=1}^{n_1} \nabla^N_{e_k} e_k(g) g_2(F_* X, F_* Y) + \lambda g_2(F_* X, F_* Y) = 0,$$

where $\{e_k\}_{1 \leq k \leq n_1}$ is an orthonormal basis of $(\text{range} F_*)^\perp$, which implies

$$\text{Ric}^{\text{range} F_*}(F_* X, F_* Y) = \lambda g_2(F_* X, F_* Y),$$

where $\lambda = \sum_{k=1}^{n_1} (e_k(g))^2 - \sum_{k=1}^{n_1} g_2(\nabla^N_{e_k} e_k, \nabla^N g) + \sum_{k=1}^{n_1} e_k(e_k(g)) - \lambda - V(g)$ is a smooth function on $N$. Thus a leaf of $\text{range} F_*$ is an Einstein, which completes the proof. \qed

**Theorem 4.8** Let $\beta$ be a geodesic curve on $N$ and $(N, g_2, \beta, \lambda)$ be a Ricci soliton with the potential vector field $\beta \in \Gamma(TN)$. Let $F : (M, g_1) \to (N, g_2)$ be a Clairaut Riemannian map with $\beta = e^g$ from a Riemannian manifold $M$ to an Einstein manifold $N$. Then the following statements are true:

(i) $\beta$ is a conformal vector field on $\text{range} F_*$.  

(ii) $\beta$ is Killing vector field on $(\text{range} F_*)^\perp$ if and only if $V(g)W(g) = -H^g(V, W)$ for all $V, W \in \Gamma(\text{range} F_*)^\perp$.

**Proof** Since $(N, g_2, \beta, \lambda)$ is a Ricci soliton then, we have

$$\frac{1}{2} \left( L_\beta g_2 \right)(F_* X, F_* Y) + \text{Ric}(F_* X, F_* Y) + \lambda g_2(F_* X, F_* Y) = 0,$$  

(4.17)

for $F_* X, F_* Y \in \Gamma(\text{range} F_*)$. Using (4.4) in (4.17), we get

$$\frac{1}{2} \left( L_\beta g_2 \right)(F_* X, F_* Y) + \text{Ric}^{\text{range} F_*}(F_* X, F_* Y) - \sum_{k=1}^{n_1} (e_k(g))^2 g_2(F_* X, F_* Y)$$

$$+ \sum_{k=1}^{n_1} g_2(\nabla^N_{e_k} e_k, \nabla^N g) g_2(F_* X, F_* Y) - \sum_{k=1}^{n_1} \nabla^N_{e_k} e_k(g) g_2(F_* X, F_* Y) + \lambda g_2(F_* X, F_* Y) = 0,$$  

(4.18)
where \( \{e_k\}_{1 \leq k \leq n_1} \) is an orthonormal basis of \((\text{range}_F)^\perp\). Since \( N \) is Einstein, putting \( \text{Ric}^{\text{range}_F}(F_* X, F_* Y) = -\lambda g_2(F_* X, F_* Y) \) in (4.18), we get
\[
\frac{1}{2}(L_{\beta}g_2)(F_* X, F_* Y) + \mu g_2(F_* X, F_* Y) = 0,
\]
where \( \mu = -\sum_{k=1}^{n_1} (e_k(g))^2 + \sum_{k=1}^{n_1} g_2(\nabla_{e_k} F, \nabla_N g) - \sum_{k=1}^{n_1} e_k(e_k(g)) \) is a smooth function on \( N \). Thus \( \beta \) is a conformal vector field on \( \text{range}_F \). On the other hand, since \((N, g_2, \hat{\beta}, \lambda)\) is a Ricci soliton then, we have
\[
\frac{1}{2}(L_{\beta}g_2)(V, W) + \text{Ric}(V, W) + \lambda g_2(V, W) = 0, \tag{4.19}
\]
for any \( V, W \in \Gamma(\text{range}_F)^\perp \). Using (4.5) in (4.19), we get
\[
\frac{1}{2}(L_{\beta}g_2)(V, W) + \text{Ric}^{(\text{range}_F)^\perp}(V, W) + (m - r)g_2(\nabla^N g, \nabla^F_{\beta} W) - (m - r)V(g)W(g) - (m - r)\nabla^N_{V}W(g) + \lambda g_2(V, W) = 0. \tag{4.20}
\]
Since \( N \) is Einstein, putting \( \text{Ric}^{(\text{range}_F)^\perp}(V, W) = -\lambda g_2(V, W) \) in (4.20), we get
\[
\frac{1}{2}(L_{\beta}g_2)(V, W) + \{g_2(\nabla^N_{V}g, \nabla^F_{\beta} W) - V(g)W(g) - \nabla^N_{V}W(g)\}(m - r) = 0.
\]
Then by using \( \nabla^N_{V}W(g) = \nabla^F_{\beta}(g_2(W, \nabla^N g)) = g_2(\nabla^F_{\beta} W, \nabla^N g) + H^g(V, W) = g_2(\nabla^F_{\beta} W, \nabla^N g) + H^g(V, W) \) in above equation, we get \( \frac{1}{2}(L_{\beta}g_2)(V, W) = 0 \) if and only if \( V(g)W(g) = -H^g(V, W) \). This completes the proof.

**Lemma 4.9** Let \((N, g_2, X_1, \lambda)\) be a Ricci soliton with the potential vector field \( X_1 \in \Gamma(TN) \) and \( F : (M^m, g_1) \to (N^n, g_2) \) be a Clairaut Riemannian map with \( s = e^g \) between Riemannian manifolds. Then
\[
s = -\lambda n, \tag{4.21}
\]
where \( s \) denotes the scalar curvature of \( N \).

**Proof** The proof is similar to remark 9 of [30]; therefore, we are omitting it.

**Theorem 4.10** Let \((N, g_2, -H_2, \lambda)\) be a Ricci soliton with the potential vector field \(-H_2 \in \Gamma(\text{range}_F)^\perp \) and \( F : (M, g_1) \to (N, g_2) \) be a Clairaut Riemannian map with \( s = e^g \) between Riemannian manifolds. Then following statements are true:

(i) \( N \) admits a gradient Ricci soliton.

(ii) The mean curvature vector field of \( \text{range}_F \) is constant.

**Proof** By similar proof as theorem 10 of [30], we get
\[
\Delta g = 0.
\]
Hence \( \nabla^N(\nabla^N g) = 0 \), i.e. \( \nabla^N H_2 = 0 \), which means \( H_2 \) is constant. This completes the proof.

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Example 4.11 The map $F : M \to N$ given in Example 3.7 is Clairaut Riemannian map. Now, we will show that $N$ admits a Ricci soliton, i.e.

$$\frac{1}{2}(L_{Z_1}g_2)(X_1, Y_1) + \text{Ric}(X_1, Y_1) + \lambda g_2(X_1, Y_1) = 0,$$  \hfill (4.22)

for any $X_1, Y_1, Z_1 \in \Gamma(TN)$. By similar computations as example 6.1 of [32], we get

$$\frac{1}{2}(L_{Z_1}g_2)(X_1, Y_1) = 0,$$  \hfill (4.23)

and

$$g_2(X_1, Y_1) = (a_1a_3 + a_2a_4),$$  \hfill (4.24)

and

$$\text{Ric}(X_1, Y_1) = a_1a_3\text{Ric}(e_1', e_1') + (a_1a_4 + a_2a_3)\text{Ric}(e_1', e_2') + a_2a_4\text{Ric}(e_2', e_2').$$  \hfill (4.25)

By (4.4), we get

$$\text{Ric}(e_1', e_1') = \text{Ric}^{\text{range } F_*}(e_1', e_1') - (g_2(\nabla^N g, e_2'))^2 + g_2(\nabla^{T\perp}_e e_2', \nabla^N g) - \nabla^{N}_{e_2'}(g_2(e_2', \nabla^N g)).$$

Since dimension of $\text{range } F_*$ is one, $\text{Ric}^{\text{range } F_*}(e_1', e_1') = 0$ and we have $\nabla^N g = -be_2'$ for some $b \in \mathbb{R}$. So

$$\text{Ric}(e_1', e_1') = -b^2,$$  \hfill (4.26)

By (4.5), we get

$$\text{Ric}(e_2', e_2') = \text{Ric}^{\text{range } F_*}(e_2', e_2') - (g_2(\nabla^N g, e_2'))^2 + g_2(\nabla^{T\perp}_e e_2', \nabla^N g) - \nabla^{N}_{e_2'}(g_2(e_2', \nabla^N g)).$$

Since dimension of $\text{range } F_* \perp$ is one, $\text{Ric}^{\text{range } F_*}(e_2', e_2') = 0$ and putting $\nabla^N g = -be_2'$ for some $b \in \mathbb{R}$, we get

$$\text{Ric}(e_2', e_2') = -b^2.$$  \hfill (4.27)

And by similar computation as example 6.1 of [32], we get

$$\text{Ric}(e_1', e_2') = 0.$$  \hfill (4.28)

Using (4.26), (4.27) and (4.28) in (4.25), we get

$$\text{Ric}(X_1, Y_1) = -(a_1a_3 + a_2a_4)b^2.$$  \hfill (4.29)

Now, using (4.23), (4.24) and (4.29) in (4.22), we obtain that metric $g_2$ admits Ricci soliton for

$$\lambda = b^2.$$

Since $b \in \mathbb{R}$, for some choices of $b$ Ricci soliton $(N, g_2)$ will be expanding or steady according to $\lambda > 0$ or $\lambda = 0$.  

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5. Clairaut antiinvariant Riemannian map from Riemannian manifold to Kähler manifold

In this section, we introduce Clairaut antiinvariant Riemannian map from a Riemannian manifold to a Kähler manifold and investigate the geometry with a nontrivial example.

Let \((N, g_2)\) be an almost Hermitian manifold \([33]\), then \(N\) admits a tensor \(J\) of type \((1, 1)\) on \(N\) such that \(J^2 = -I\) and

\[
g_2(JX_1, JY_1) = g_2(X_1, Y_1),
\]

for all \(X_1, Y_1 \in \Gamma(TN)\). An almost Hermitian manifold \(N\) is called Kähler manifold if

\[
(\nabla^N_X J) Y_1 = 0,
\]

for all \(X, Y_1 \in \Gamma(TN)\), where \(\nabla^N\) is the Levi-Civita connection on \(N\).

**Definition 5.1** \([20]\) Let \(F : (M, g_1) \to (N, g_2)\) be a proper Riemannian map from a Riemannian manifold \(M\) to an almost Hermitian manifold \(N\) with almost complex structure \(J\). We say that \(F\) is an antiinvariant Riemannian map at \(p \in M\) if \(J(\text{range} F_p) \subset (\text{range} F_p)^\perp\). If \(F\) is an antiinvariant Riemannian map for every \(p \in M\) then \(F\) is called an antiinvariant Riemannian map.

In this case we denote the orthogonal subbundle to \(J(\text{range} F)\) in \((\text{range} F)^\perp\) by \(\mu\), i.e. \((\text{range} F)^\perp = J(\text{range} F) \oplus \mu\). For any \(V \in \Gamma(\text{range} F)^\perp\), we have

\[
JV = BV + CV,
\]

where \(BV \in \Gamma(\text{range} F)\) and \(CV \in \mu\). Note that if \(\mu = 0\) then \(F\) is called Lagrangian Riemannian map \([27]\).

**Lemma 5.2** Let \(F : (M, g_1) \to (N, g_2, J)\) be an antiinvariant Riemannian map from a Riemannian manifold \(M\) to a Kähler manifold \(N\) and \(\alpha : I \to M\) be a geodesic curve on \(M\). Then the curve \(\beta = F \circ \alpha\) is geodesic on \(N\) if and only if

\[
-S_{JF_*X} F_* X - S_{CV} F_* X + \nabla^N_V BV + F_*(\nabla^M_X F_* BV) = 0,
\]

(5.3)

\[
(\nabla F_*)(X, *F_* BV) + \nabla^N_{F_* X} F_* X + \nabla^F_{F_* X} CV + \nabla^F_{F_* X} CV = 0,
\]

(5.4)

where \(F_* X, V \in \Gamma(\text{range} F)\), \(V \in \Gamma(\text{range} F)^\perp\) are components of \(\dot{\beta}(t)\) and \(*F_*\) is the adjoint map of \(F_*\), and \(\nabla^N\) is the Levi-Civita connection on \(N\), and \(\nabla^F\) is a linear connection on \((\text{range} F)^\perp\).

**Proof** Let \(\alpha : I \to M\) be a geodesic on \(M\) and let \(\beta = F \circ \alpha\) be a geodesic on \(N\) with \(F_* X \in \Gamma(\text{range} F)\) and \(V \in \Gamma(\text{range} F)^\perp\) are components of \(\dot{\beta}(t)\). Since \(N\) is Kähler manifold, \(\nabla^N_{\dot{\beta}} = -J \nabla^N_{\dot{\beta}} J \dot{\beta}\). Thus

\[
\nabla^N_{\dot{\beta}} = -J \nabla^N_{\dot{\beta}} J \dot{\beta} = -J \nabla^N_{F_* X + V} J(F_* X + V),
\]

which implies

\[
\nabla^N_{\dot{\beta}} = -J(\nabla^N_{F_* X} JF_* X + \nabla^N_{F_* X} JV + \nabla^N_{JF_* X} + \nabla^N_{JV}).
\]

(5.5)

Using (2.4) and (5.2) in (5.5), we get

\[
\nabla^N_{\dot{\beta}} = -J(-S_{JF_* X} F_* X - S_{CV} F_* X + \nabla^N_V BV + \nabla^N_{F_* X} BV + \nabla^F_{F_* X} JF_* X + \nabla^F_{F_* X} CV + \nabla^F_{F_* X} CV).
\]

(5.6)
Since $\nabla^N$ is Levi-Civita connection on $N$ and $g_2(\nabla^N_BV,U) = 0$ for any $U \in \Gamma(\text{range}F_*)$, $\nabla^N_BV \in \Gamma(\text{range}F_*)$ and using (2.2), we get $\nabla^N_BV = \nabla^N_BV \circ F = (\nabla F_*)(X, F_*BV) + F_*(\nabla^M_F*F_*BV)$. Then by (5.6), we get
\[
\nabla^N_B\dot{\beta} = -J(-S_{\dot{F}_*F_*X} - \nabla^N_BV + \nabla^N_BV + (\nabla F_*)(X, F_*BV) + F_*(\nabla^M_F*F_*BV) + \nabla^F_LF_*X + \nabla^F_LF_*X + \nabla^F_LCV + \nabla^F_LCV).
\]

Now $\beta$ is geodesic on $N$ $\iff$ $\nabla^N_B\dot{\beta} = 0$ $\iff$ $-S_{\dot{F}_*F_*X} - \nabla^N_BV + \nabla^N_BV + (\nabla F_*)(X, F_*BV) + F_*(\nabla^M_F*F_*BV) + \nabla^F_LF_*X + \nabla^F_LF_*X + \nabla^F_LCV + \nabla^F_LCV = 0$, which completes the proof. $\square$

**Definition 5.3** An antiinvariant Riemannian map from a Riemannian manifold to a Kähler manifold is called Clairaut antiinvariant Riemannian map if it satisfies the condition of Clairaut Riemannian map.

**Theorem 5.4** Let $F : (M, g_1) \to (N, g_2, J)$ be an antiinvariant Riemannian map from a Riemannian manifold $M$ to a Kähler manifold $N$ and $\alpha, \beta = F \circ \alpha$ are geodesic curves on $M$ and $N$, respectively. Then $F$ is Clairaut antiinvariant Riemannian map with $\dot{s} = e^g$ if and only if $g_2(S_{\dot{F}_*F_*X} + \nabla^F_LF_*X, BV) - g_2((\nabla F_*)(X, F_*BV) + \nabla^F_LF_*X + \nabla^F_LF_*X, CV) - g_2(F_2F_*X, F_*X)\frac{d(g_2\beta)}{dt} = 0$, where $g$ is a smooth function on $N$ and $F_*X \in \Gamma(\text{range}F_*)$, $V \in \Gamma(\text{range}F_*)^\perp$ are components of $\dot{\alpha}(t)$.

**Proof** Let $\alpha : I \to M$ be a geodesic on $M$ and let $F = F \circ \alpha$ be a geodesic on $N$ with $F_*X \in \Gamma(\text{range}F_*)$ and $V \in \Gamma(\text{range}F_*)^\perp$ are components of $\dot{\beta}(t)$ and $\omega(t)$ denote the angle in $[0,\pi]$ between $\dot{\beta}$ and $V$. Assuming $b = ||\dot{\beta}(t)||^2$, then we get
\[
g_2\beta(t)(V, V) = bcos^2(\omega(t)), \quad (5.7)
\]
\[
g_2\beta(t)(F_*X, F_*X) = bsin^2(\omega(t)). \quad (5.8)
\]
Now differentiating (5.7) along $\beta$, we get
\[
\frac{d}{dt}g_2(V, V) = -2bsin(\omega(t))cos(\omega(t))\frac{d\omega}{dt}. \quad (5.9)
\]
On the other hand by (5.1), we get
\[
\frac{d}{dt}g_2(V, V) = \frac{d}{dt}g_2(JV, JV).
\]
Using (5.2) in above equation, we get
\[
\frac{d}{dt}g_2(V, V) = \frac{d}{dt}
\left(g_2(BV, BV) + g_2(CV, CV)\right),
\]
which implies
\[
\frac{d}{dt}g_2(V, V) = 2g_2(\nabla_B^{N*}BV, BV) + 2g_2(\nabla_B^{N*}CV, CV). \quad (5.10)
\]
Putting $\dot{\beta} = F_*X + V$ in (5.10), we get

$$\frac{d}{dt} g_2(V, V) = 2g_2(\nabla^N_X BV, BV) + 2g_2(\nabla^N_N X CV, CV) + 2g_2(\nabla^N_N BV, BV) + 2g_2(\nabla^N_N CV, CV).$$

Since $(\text{range} F_*)^\perp$ is totally geodesic, above equation can be written as

$$\frac{d}{dt} g_2(V, V) = 2g_2(\nabla^N_X BV \circ F, BV) + 2g_2(\nabla^N_N X CV, CV) + 2g_2(\nabla^N_N BV, BV) + 2g_2(\nabla^N_N CV, CV).$$

Using (2.2), (2.3) and (2.4) in (5.11), we get

$$\frac{d}{dt} g_2(V, V) = 2g_2(F_*(\nabla^M_X^* F_*)^\perp + \nabla^N_X BV, BV) + 2g_2(\nabla^N_N X CV + \nabla^N_N CV, CV).$$

Using (5.3) and (5.4) in (5.12), we get

$$\frac{d}{dt} g_2(V, V) = 2g_2(S_{F_*, X} F_*X + S_{CV} F_*X, BV) - 2g_2((\nabla F_*)(X, *F_*BV) + \nabla^N_X^* X V_1 + \nabla^N_N X CV).$$

Now from (5.9) and (5.13), we get

$$g_2(S_{F_*, X} F_*X + S_{CV} F_*X, BV) - 2g_2((\nabla F_*)(X, *F_*BV) + \nabla^N_X^* X V_1 + \nabla^N_N X CV) = -bsin\omega cos\omega \frac{d\omega}{dt}.\quad (5.14)$$

Moreover, $F$ is a Clairaut Riemannian map with $\tilde{s} = e^{\beta}$ if and only if $\frac{d}{dt}(e^{\beta}s\sin \omega) = 0$, that is, $e^{\beta}s\sin \omega \frac{d(e^{\beta}s\sin \omega)}{dt} + e^{\beta}\cos \omega \frac{d\omega}{dt} = 0$. By multiplying this with nonzero factor $bsin\omega$ and using (5.8), we get

$$g_2(F_*X, F_*X) \frac{d(g \circ \beta)}{dt} = -bsin\omega cos\omega \frac{d\omega}{dt}.\quad (5.15)$$

Thus (5.14) and (5.15) complete the proof. \hfill \Box

**Theorem 5.5** Let $F : (M^n, g_1) \to (N^n, g_2, J)$ be a Clairaut antiinvariant Riemannian map with $\tilde{s} = e^{\beta}$ from a Riemannian manifold $M$ to a Kähler manifold $N$. Then at least one of the following statement is true:

(i) dim(range $F_*) = 1$,

(ii) $g$ is constant on $J$(range $F_*$), where $g$ is a smooth function on $N$.

**Proof** Since $F$ is Clairaut Riemannian map with $\tilde{s} = e^{\beta}$ then using (2.2) in (3.21), we get

$$\nabla^N_X F_*Y - F_*(\nabla^M_X Y) = -g_1(X, Y) \nabla^N g,$$ \quad (5.16)

for $F_*Y \in \Gamma$(range $F_*$) and $X, Y \in \Gamma$(ker$F_*)^\perp$. Taking inner product of (5.16) with $JF_*Z \in \Gamma$(range $F_*)^\perp$, we get

$$g_2(\nabla^N_X F_*Y - F_*(\nabla^M_X Y), JF_*Z) = -g_1(X, Y) g_2(\nabla^N g, JF_*Z).$$ \quad (5.17)

Since $\nabla^F$ is pullback connection of the Levi-Civita connection $\nabla^N$. Therefore $\nabla^F$ is also Levi-Civita connection. Then using metric compatibility condition in (5.17), we get

$$-g_2(\nabla^F_X JF_*Z, F_*Y) = -g_1(X, Y)g_2(\nabla^F g, JF_*Z),$$

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which implies
\[ g_2(J\nabla^N_X F_* Z, F_* Y) = g_1(X, Y)g_2(\nabla^N g, JF_* Z). \] (5.18)

Using (5.1) in (5.18), we get
\[ -g_2(\nabla^N_X F_* Z, JF_* Y) = g_1(X, Y)g_2(\nabla^N g, JF_* Z). \] (5.19)

Using (5.16) in above equation, we get
\[ g_1(X, Z)g_2(\nabla^N g, JF_* X) = g_1(X, Y)g_2(\nabla^N g, JF_* Z). \] (5.20)

Now putting \( X = Y \) in (5.19), we get
\[ g_1(X, Z)g_2(\nabla^N g, JF_* X) = g_1(X, X)g_2(\nabla^N g, JF_* Z). \] (5.21)

Now interchanging \( X \) and \( Z \) in (5.20), we get
\[ g_1(X, Z)g_2(\nabla^N g, JF_* Z) = g_1(Z, Z)g_2(\nabla^N g, JF_* X). \] (5.22)

From (5.20) and (5.21), we get
\[ g_2(\nabla^N g, JF_* X) \left( 1 - \frac{g_1(X, X)g_1(Z, Z)}{g_1(X, Z)g_1(X, Z)} \right) = 0, \]
which implies either \( \dim((\ker F_*)^\perp) = 1 \) or \( g_2(\nabla^N g, JF_* X) = 0 \), which means \( (JF_* X)(g) = 0 \), which completes the proof. \( \square \)

**Theorem 5.6** Let \( F : (M^m, g_1) \to (N^n, g_2, J) \) be a Clairaut Lagrangian Riemannian map with \( \tilde{s} = e^g \) from a Riemannian manifold \( M \) to a Kähler manifold \( N \) such that \( \dim(\text{range} F_*) > 1 \). Then following statements are true:

(i) \( \text{range} F_* \) is minimal.

(ii) \( \text{range} F_* \) is totally geodesic.

**Proof** Since \( F \) is Clairaut Riemannian map then from (3.21) and Theorem 3.3, we have
\[ (\nabla F_*)(X, X) = g_1(X, X)H_2, \]
for \( X \in \Gamma(\ker F_*)^\perp \) and \( H_2 \) is the mean curvature vector field of \( \text{range} F_* \). Now multiply above equation by \( U \in \Gamma(\text{range} F_*)^\perp \), we get
\[ g_2((\nabla F_*)(X, X), U) = g_1(X, X)g_2(H_2, U). \] (5.22)

Using (2.2) in (5.22), we get
\[ g_2(\nabla^N_X F_* X, U) = g_1(X, X)g_2(H_2, U). \] (5.23)

Since \( N \) is Kähler manifold, using (5.1) in (5.23), we get
\[ g_2(\nabla^N_X JF_* X, JU) = g_1(X, X)g_2(H_2, U). \] (5.24)
Since $\nabla^N$ is Levi-Civita connection on $N$, using metric compatibility condition in (5.24), we get

$$-g_2(JF_\star X, \nabla^N_X JU) = g_1(X, X)g_2(H_2, U). \tag{5.25}$$

Using (5.23) in (5.25), we get

$$-g_2(JF_\star X, g_1(X, *F_\star JU)H_2) = g_1(X, X)g_2(H_2, U), \tag{5.26}$$

where $*F_\star$ is the adjoint map of $F_\star$. Now using $H_2 = -\nabla^N g$ in (5.26), we get

$$g_1(X, *F_\star JU)g_2(JF_\star X, \nabla^N g) = g_1(X, X)g_2(H_2, U),$$

which implies

$$g_1(X, *F_\star JU)JF_\star X(g) = g_1(X, X)g_2(H_2, U). \tag{5.27}$$

Since $\dim(\text{range}F_\star) > 1$ then by Theorem 5.5, $g$ is constant on $J(\text{range}F_\star)$, which means $JF_\star X(g) = 0$. Then (5.27) implies $g_2(H_2, U) = 0$. Thus

$$H_2 = 0, \tag{5.28}$$

which implies (i).

Since $H_2 = \text{trace}(\nabla^N_X F_\star Y)$. Then by (5.28), we get $\nabla^N_X F_\star Y = 0$, which implies (ii).

\[\square\]

**Theorem 5.7** Let $F : (M^n, g_1) \to (N^n, g_2, J)$ be a Clairaut Lagrangian Riemannian map with $\tilde{s} = e^g$ from a Riemannian manifold $M$ to a Kähler manifold $N$ such that $\dim(\text{range}F_\star) > 1$. Then $F$ is harmonic if and only if mean curvature vector field of $\ker F_\star$ is constant.

**Proof** Let $F : (M^n, g_1) \to (N^n, g_2)$ be a smooth map between Riemannian manifolds. Then $F$ is harmonic if and only if the tension field $\tau(F)$ of map $F$ vanishes. Then proof follows by Lemma 2.1 and Theorem 5.6. \[\square\]

**Theorem 5.8** Let $F : (M^n, g_1) \to (N^n, g_2, J)$ be a Clairaut Lagrangian Riemannian map with $\tilde{s} = e^g$ from a Riemannian manifold $M$ to a Kähler manifold $N$ such that $\dim(\text{range}F_\star) > 1$. Then $N = N_{\text{range}F_\star} \times N_{\text{range}F_\star}^\perp$ is a usual product manifold.

**Proof** The proof follows by [19] and Theorem 5.6. \[\square\]

**Example 5.9** Let $M = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0\}$ be a Riemannian manifold with Riemannian metric $g_1 = e^{2x_2}dx_1^2 + e^{2x_2}dx_2^2$ on $M$. Let $N = \{(y_1, y_2) \in \mathbb{R}^2\}$ be a Riemannian manifold with Riemannian metric $g_2 = e^{2y_1}dy_1^2 + dy_2^2$ on $N$ and the complex structure $J$ on $N$ defined as $J(y_1, y_2) = (-y_2, y_1)$. Consider a map $F : (M, g_1) \to (N, g_2, J)$ defined by

$$F(x_1, x_2) = \left(\frac{x_1 - x_2}{\sqrt{2}}, 0\right).$$

Then

$$\ker F_\star = \text{span}\left\{U = \frac{e_1 + e_2}{\sqrt{2}}\right\} \text{ and } (\ker F_\star)^\perp = \text{span}\left\{X = \frac{e_1 - e_2}{\sqrt{2}}\right\},$$

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Hence $\nabla_{\partial x^2}g_{1} \neq 0$ and $\nabla_{\partial y}g_{2} \neq 0$. Thus $F$ is a Riemannian map with range $F = \text{span}\{F_1(X,F_2,X)\}$ for $X \in \Gamma(\ker F)$.

Moreover it is easy to see that $JF_1X = Je_1 = -e_2$. Thus $F$ is an anti-invariant Riemannian map.

Now to show $F$ is Clairaut Riemannian map we will find a smooth function $g$ on $N$ satisfying $(\nabla F_1)(X,F) = -g_1(X,F)\nabla^N g$ for $X \in \Gamma(\ker F)_{N}$. Since $(\nabla F_1)(X,F) \in \Gamma(\text{range} F_1)$ for any $X \in \Gamma(\ker F)_{N}$. So here we can write $(\nabla F_1)(X,F) = ae_2'$, for some $a \in \mathbb{R}$. Since $\nabla^N g = e^{-2x^2} \frac{\partial}{\partial y^1} \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \frac{\partial}{\partial y_2}$.

Hence $\nabla^N g = -a \frac{\partial}{\partial y_2} = -ae_2'$ for the function $g = -ay_2$. Then it is easy to verify that $(\nabla F_1)(X,F) = -g_1(X,F)\nabla^N g$, where $g_1(X,F) = 1$, for vector field $X \in \Gamma(\ker F)_{N}$ and we can easily see that $\nabla e_2' e_2' = 0$.

Thus by Theorem 3.3, $F$ is Clairaut anti-invariant Riemannian map.

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