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Scattering solutions and scattering function of a Klein-Gordon s-wave equation with jump conditions

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Abstract: In this work, we are interested in a boundary value problem (BVP) generated by a Klein -Gordon equation (KG) with Jump conditions and a boundary condition. First, we introduce scattering solutions and Jost solution of the problem. Then, we give the scattering function and we prove some properties of it. Lastly, we conclude the paper by a special example.

Key words: Klein-Gordon equation, eigenvalue, scattering function, scattering solution

1. Introduction

The Klein-Gordon equation (KG) is a relativistic wave equation which is related to the Schrödinger equation. KG equation can be put into the form of a Schrödinger equation. In this form, it is expressed as two coupled differential equations, each of first order in time. The solutions have two components, reflecting the change degree of freedom in relativity [17]. Klein-Gordon equations with any potential play an important role in relativistic quantum mechanics [27] and so there are many studies on the exact solutions of the KG_s with various type of potentials by using different methods. Some of these methods are formal variable separation method [18], algebraic method [13, 16, 23], Nikiforov-Uvarov method [19], and asymptotic iteration method [14]. By using these exact solutions of the KG_s, various authors investigate the spectral analysis of these equations [1, 6–9, 11, 15]. In general, mentioned studies present the properties of eigenvalues and spectral singularities if they exist under the various assumptions with different boundary conditions. In [8], the authors give a condition which guarantees that the KG including complex potential and a simple boundary condition has a finite number of eigenvalues and spectral singularities with finite multiplicities. They also investigate the discrete spectrum and principal functions of the problem. Later, the same problem is examined on the whole axis in [7]. On the other hand, one can find some researches about the scattering analysis of KG_s in the literature [2, 10, 25, 28, 29]. Although the problems about KG_s have attracted the attention of many researchers since 1926, it is still an open problem to investigate scattering analysis of such problems whenever it has jump conditions. Differential equations with jump conditions involve discontinuties at one or more than one point in an interval. In that occasion, it makes an impulsive effect which appear as a natural description of observed evolution phenomena of several real-world problems studied in physics, biotechnology, chemical technology, population dynamics and economics, see [12, 21]. These points with discontinuties are called jump conditions or impulsive conditions,

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sometimes they are also called transmission or interface conditions in the literature. For the large literature concerning differential equations with jump conditions, we refer to [3–5, 12, 21, 26] and the references therein. The goal of this paper is to present Jost and scattering solutions of a BVP generated by Klein-Gordon s-wave equation with jump conditions. By using these solutions, defining scattering function and proving its properties are another aim of the paper. In Section 2, some general information about KG_s without any jump condition and required preliminaries are given. Section 3 presents main results about Jost solution, scattering function and eigenvalues of the problem with jump conditions and with general boundary condition. Furthermore, an example to illustrate the main results is given in this section. At the end, a conclusion part is presented in Section 4.

2. Preliminaries for Klein-Gordon equations

In this section, we give some useful information and results for KG_s with a general boundary condition on the half-axis. Let us consider the following BVP in $L_2(\mathbb{R}^+)$:

$$y^{''} + [\lambda \ - Q(x)]^2 \, y = 0, \ x \in \mathbb{R}^+ = [0, \infty) \, , \tag{2.1}$$

$$y(0) = 0. \tag{2.2}$$

Equation (2.1) is called the Klein-Gordon s-wave equation for a particle of zero mass with static potential Q [8], here λ is a spectral parameter, Q is complex-valued, bounded, and absolutely continuous in each finite subinterval of $\mathbb{R}^+$, which satisfies

$$\int_0^\infty x\left[|Q(x)| + \left|Q^{'}(x)\right|\right] dx < \infty. \tag{2.3}$$

It is known from [8] that the BVP (2.1)-(2.2) has the following solutions under the condition (2.3):

$$e^+(x,\lambda) = e^{i\gamma(x)+i\lambda x} + \int_x^\infty K^+(x,t)e^{i\lambda t}dt \tag{2.4}$$

$$\overline{e^+(x,\lambda)} = e^{-i\gamma(x)-i\lambda x} + \int_x^\infty K^+(x,t)e^{-i\lambda t}dt$$

for $\lambda \in \mathbb{R}$, where $\gamma(x) = \int_x^\infty Q(t)dt$ and $K^+(x,t)$ is solution of Volterra type integral equations which has continuous derivatives with respect to their arguments. On the other hand, it is well-known from [20] that $K^+(x,t)$, $\dfrac{d}{dx}K^+(x,t)$ and $\dfrac{d}{dt}K^+(x,t)$ satisfy the following inequalities:

$$\left|K^+(x,t)\right| \le Cw(\frac{x+t}{2}) \exp\left\{g(x)\right\},$$

$$\left|K_x^+(x,t)\right|, \ \left|K_t^+(x,t)\right| \le C[w^2(\frac{x+t}{2}) + \theta(\frac{x+t}{2})],$$

where

$$w(x) = \int_x^\infty \left[|Q(t)|^2 + \left| Q^{'}(t) \right| \right] dt$$

$$g(x) = \int_x^\infty \left[t\,|Q(t)|^2 + 2\,|Q(t)| \right] dt$$

$$\theta(x) = \frac{1}{4} \left\{ 2\,|Q(x)|^2 + \left| Q^{'}(x) \right| \right\}$$

and C is a constant. Therefore, $e^+(x,\lambda)$ and $\overline{e^+(x,\lambda)}$ are analytic with respect to λ in $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda > 0\}$ and in $\mathbb{C}_- := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda < 0\}$, respectively, and continuous up to the real axis. The solutions $e^+(x,\lambda)$ and $\overline{e^+(x,\lambda)}$ are called the Jost solution of (2.1)-(2.2). The solution $e^+(x,\lambda)$ satisfies the following asymptotic equalities [11]:

$$e^+(x,\lambda) = e^{i\lambda x}[1 + o(1)], \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \geq 0\}, \ x \longrightarrow \infty \tag{2.5}$$

$$e_x^+(x,\lambda) = e^{i\lambda x}[i\lambda + o(1)], \quad \lambda \in \overline{\mathbb{C}}_+, \quad x \longrightarrow \infty \tag{2.6}$$

and

$$e^+(x,\lambda) = e^{i[\gamma(x) + \lambda x]} + o(1), \ \lambda \in \overline{\mathbb{C}}_+, \ |\lambda| \longrightarrow \infty. \tag{2.7}$$

By using these asymptotic equations, it is easy to write the Wronskian

$$W[e^+(x,\lambda), \overline{e^+(x,\lambda)}] = \lim_{x \longrightarrow \infty} W[e^+(x,\lambda), \overline{e^+(x,\lambda)}] = -2i\lambda$$

for $\lambda \in \mathbb{R}$, so $e^+(x,\lambda)$ and $\overline{e^+(x,\lambda)}$ are the fundamental solutions of (2.1) for $\lambda \in \mathbb{R}\backslash\{0\}$. Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ denote the solution of (2.1) satisfying the following initial conditions which are entire functions of λ

$$\varphi(0,\lambda) = 0 \ , \ \varphi_x(0,\lambda) = 1$$

$$\psi(0,\lambda) = 1 \ , \ \psi_x(0,\lambda) = 0.$$

It is clear from that the Wronskian of the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ is

$$W[\varphi(x,\lambda), \psi(x,\lambda)] = -1 \tag{2.8}$$

for $\lambda \in \mathbb{R}$.

3. Main results

In this section, we first determine the main problem generated by KG and jump conditions. We define Jost solution and scattering function of this problem by using scattering solutions. After investigating the properties of scattering function, we examine the eigenvalues of them problem. At the end, we present an example. Let us call the BVP (2.1)-(2.2) with the following jump conditions

$$y(1^+) = \alpha y(1^-), \quad y^{'}(1^+) = \beta y^{'}(1^-), \tag{3.1}$$

where α, β are real numbers with $\alpha\beta \neq 0$. Throughout the paper, we shortly call the BVP (2.1)-(2.2) with jump conditions BVP (3.1) by JBVP. Note that (3.1) is the jump (impulsive) conditions of JBVP and $x = 1$ is the impulsive point of this problem. At first, we need to consider the following solution of JBVP:

$$E^{+}(x,\lambda) = \begin{cases} c_1\varphi(x,\lambda) + c_2\psi(x,\lambda) & ; \quad x \in [0,1) \\ e^{+}(x,\lambda) & ; \quad x \in (1,\infty) \end{cases} \tag{3.2}$$

for $\lambda \in \mathbb{R}$, where $e^{+}(x,\lambda)$, $\varphi(x,\lambda)$, and $\psi(x,\lambda)$ are the solutions given in Section 2. The jump conditions (3.1) imply that

$$e^{+}(1,\lambda) = \alpha c_1\varphi(1,\lambda) + \alpha c_2\psi(1,\lambda)$$

and

$$e^{+'}(1,\lambda) = \beta c_1\varphi^{'}(1,\lambda) + \beta c_2\psi^{'}(1,\lambda).$$

By using these equations and (2.8), we find the coefficients c_1 and c_2 as

$$c_1 = \frac{\alpha e^{+'}(1,\lambda)\psi(1,\lambda) - \beta e^{+}(1,\lambda)\psi^{'}(1,\lambda)}{\alpha\beta} \tag{3.3}$$

and

$$c_2 = \frac{\beta\varphi^{'}(1,\lambda)e^{+}(1,\lambda) - \alpha\varphi(1,\lambda)e^{+'}(1,\lambda)}{\alpha\beta} \tag{3.4}$$

for $\lambda \in \mathbb{R}$. Substituting (3.3) and (3.4) into (3.2), we obtain that $E^{+}(x,\lambda)$ is the Jost solution of JBVP. Hence, $E^{+}(0,\lambda) = c_2(\lambda)$ is the Jost function of JBVP. It follows from (2.3) and (3.2) that $E^{+}(x,\lambda)$ has an analytic continuation from real axis to the upper complex half-plain and from real axis to the lower complex half-plain. Furthermore, the representation of Jost solution can also be shown by $E^{+}(x,\lambda)$ for $\lambda \in \overline{\mathbb{C}}_{+}$. As a result of this representation, the function $c_2(\lambda)$ is analytic in $\overline{\mathbb{C}}_{+}$ and continuous up to the real axis. Next, we will consider another solution of JBVP by

$$G^{+}(x,\lambda) = \begin{cases} \varphi(x,\lambda) & ; \quad x \in [0,1) \\ c_3e^{+}(x,\lambda) + c_4\overline{e^{+}(x,\lambda)} & ; \quad x \in (1,\infty) \end{cases}$$

for $\lambda \in \mathbb{R}\backslash\{0\}$. By using (3.1) jump conditions and the Wronskian of the solutions $e^{+}(x,\lambda)$, $\overline{e^{+}(x,\lambda)}$, we get

$$c_3 = \frac{-\alpha\varphi(1,\lambda)\overline{e^{+'}(1,\lambda)} + \beta\varphi^{'}(1,\lambda)\overline{e^{+}(1,\lambda)}}{2i\lambda} \tag{3.5}$$

and

$$c_4 = \frac{-\beta e^{+}(1,\lambda)\varphi^{'}(1,\lambda) + \alpha e^{+'}(1,\lambda)\varphi(1,\lambda)}{2i\lambda} \tag{3.6}$$

for $\lambda \in \mathbb{R}\backslash\{0\}$.

Lemma 3.1 *The Wronskian of the solutions $E^{+}(x,\lambda)$ and $G^{+}(x,\lambda)$ is given by the following equation for $\lambda \in \mathbb{R}\backslash\{0\}$*

$$W[E^{+}(x,\lambda), G^{+}(x,\lambda)] = \begin{cases} -\dfrac{2i\lambda}{\alpha\beta}c_4 & ; \quad x \in [0,1) \\ -2i\lambda c_4 & ; \quad x \in (1,\infty). \end{cases}$$

Proof By using the definition of Wronskian, $E^{+}(x,\lambda)$, $G^{+}(x,\lambda)$ and the results given in Section 2, we obtain $W[E^{+}(x,\lambda),G^{+}(x,\lambda)]=c_2$ for $x\in[0,1)$ and

$$W[E^{+}(x,\lambda),\ G^{+}(x,\lambda)]=(-2i\lambda)c_4$$

for $x\in(1,\infty)$. On the other hand, if we get the relation between the coefficients c_2 and c_4, we find $c_4=-\dfrac{\alpha\beta}{2i\lambda}c_2$ for $\lambda\in\mathbb{R}\backslash\{0\}$ from (3.4) and (3.6). It completes the proof of Lemma 3.1. □

Theorem 3.2 *Assume* $\lambda\in\mathbb{R}\backslash\{0\}$. *Then* $c_2(\lambda)\neq 0$.

Proof Assume that there exists a $\widetilde{\lambda}\in\mathbb{R}\backslash\{0\}$ such that $c_2(\widetilde{\lambda})=0$. It means that $E^{+}(0,\widetilde{\lambda})=0$. By using (3.4), (3.5), and (3.6), we get $c_3(\widetilde{\lambda})=c_4(\widetilde{\lambda})=0$. It gives that the solution $G^{+}(x,\widetilde{\lambda})$ is equal to zero identically. Since $G^{+}(x,\widetilde{\lambda})$ becomes a trivial solution of JBVP in that condition, it is a contradiction. As a result of this contradiction, we can say that $c_2(\lambda)\neq 0$ for all $\lambda\in\mathbb{R}\backslash\{0\}$. □

The function

$$S(\lambda):=\frac{\overline{E^{+}(0,\lambda)}}{E^{+}(0,\lambda)},\ \lambda\in\mathbb{R}\backslash\{0\}$$

is called the scattering function of the JBVP. It is obvious from $E^{+}(x,\lambda)$ that

$$S(\lambda)=\frac{\overline{c_2}}{c_2}=\frac{\alpha\varphi(0,\lambda)\overline{e^{+\prime}(0,\lambda)}-\beta\varphi^{\prime}(0,\lambda)\overline{e^{+}(0,\lambda)}}{\alpha\varphi(0,\lambda)e^{+\prime}(0,\lambda)-\beta\varphi^{\prime}(0,\lambda)e^{+}(0,\lambda)} \tag{3.7}$$

for all $\lambda\in\mathbb{R}\backslash\{0\}$. It follows from (3.7) that

$$S(0)=\lim_{\lambda\longrightarrow 0}S(\lambda)=1.$$

Theorem 3.3 *The scattering function satisfies the following equation for all* $\lambda\in\mathbb{R}\backslash\{0\}$:

$$\overline{S(\lambda)}=S^{-1}(\lambda).$$

Proof From the definition of $S(\lambda)$, we have

$$\overline{S(\lambda)}=\frac{E^{+}(0,\lambda)}{\overline{E^{+}(0,\lambda)}}=S^{-1}(\lambda)$$

for $\lambda\in\mathbb{R}\backslash\{0\}$. □

In the following, we will suppose another solution of the JBVP for $\lambda\in\overline{\mathbb{C}}_{-}$

$$E^{-}(x,\lambda)=\begin{cases} c_5\varphi(x,\lambda)+c_6\psi(x,\lambda) & ;\quad x\in[0,1) \\ \overline{e^{+}(x,\lambda)} & ;\quad x\in(1,\infty). \end{cases} \tag{3.8}$$

Similar to previous coefficients, we can find $c_5(\lambda)$ and $c_6(\lambda)$ uniquely. Using (3.1), we obtain

$$\overline{e^{+}(1,\lambda)}=\alpha c_5\varphi(1,\lambda)+\alpha c_6\psi(1,\lambda)$$

and

$$\overline{e^{+'}(1,\lambda)} = \beta c_5 \varphi^{'}(1,\lambda) + \beta c_6 \psi^{'}(1,\lambda).$$

Since $W[\varphi(x,\lambda), \psi(x,\lambda)] = -1,$ we clearly obtain

$$c_5(\lambda) = \frac{1}{\alpha\beta}\left[\alpha\overline{e^{+'}(1,\lambda)}\psi(1,\lambda) - \beta\overline{e^{+}(1,\lambda)}\psi^{'}(1,\lambda)\right]$$

$$c_6(\lambda) = \frac{1}{\alpha\beta}\left[\beta\varphi^{'}(1,\lambda)\overline{e^{+}(1,\lambda)} - \alpha\varphi(1,\lambda)\overline{e^{+'}(1,\lambda)}\right]$$

for $\lambda \in \mathbb{R}$.

Corollary 3.4 *The following equations can be written for the coefficients* $c_1(\lambda),\ c_2(\lambda),\ c_3(\lambda),\ c_4(\lambda),\ c_5(\lambda),$ *and* $c_6(\lambda)$ *for all* $\lambda \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned} c_1(\lambda) &= \overline{c_5(\lambda)}, \quad c_2(\lambda) = \frac{-2i\lambda}{\alpha\beta}c_4(\lambda), \quad c_3(\lambda) = \frac{\alpha\beta}{2i\lambda}c_6(\lambda), \\ \overline{c_2(\lambda)} &= c_6(\lambda), \quad c_1(\lambda)c_6(\lambda) - c_2(\lambda)c_5(\lambda) = \frac{2i\lambda}{\alpha\beta}. \end{aligned}$$

As a consequence of Corollary 3.4 and the definition of Wronskian, we can eventually get the following:

$$W[E^{+}(x,\lambda), E^{-}(x,\lambda)] = \begin{cases} -\dfrac{2i\lambda}{\alpha\beta} & ; \quad x \in [0,1) \\ -2i\lambda & ; \quad x \in (1,\infty) \end{cases} = \begin{cases} \dfrac{c_2}{c_4} & ; \quad x \in [0,1) \\ \alpha\beta\dfrac{c_2}{c_4} & ; \quad x \in (1,\infty) \end{cases}$$

and

$$W[G^{+}(x,\lambda), E^{-}(x,\lambda)] = \begin{cases} -c_6 & ; \quad x \in [0,1) \\ -\alpha\beta c_6 & ; \quad x \in (1,\infty) \end{cases}$$

for all $\lambda \in \mathbb{R}\setminus\{0\}$.

Theorem 3.5 *The set of eigenvalues of (2.1)-(2.2) with (3.1) can be given by*

$$\sigma_d = \{\lambda \in \mathbb{C}_+ : \ c_2(\lambda) = 0\} \cup \left\{\lambda \in \mathbb{C}_- : \ \overline{c_2(\lambda)} = 0\right\}.$$

Proof The function $E^{-}(x,\lambda)$ is the unbounded solution of given JBVP. It follows from (3.8), the explicit forms of $c_2(\lambda)$, $c_6(\lambda)$ and the definition of eigenvalues [24] that

$$\sigma_d = \left\{\lambda \in \mathbb{C}_+ : \ \overline{c_6(\lambda)} = 0\right\} \cup \{\lambda \in \mathbb{C}_- : \ c_6(\lambda) = 0\}$$

or

$$\sigma_d = \{\lambda \in \mathbb{C}_+ : \ c_2(\lambda) = 0\} \cup \left\{\lambda \in \mathbb{C}_- \ : \ \overline{c_2(\lambda)} = 0\right\}.$$

□

It is clear from Theorem 3.5 that to investigate the quantitative properties of the eigenvalues of related JBVP, it is necessary to study the quantitative properties of the zeros of the functions $c_2(\lambda)$ and $c_6(\lambda)$ in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively.

Theorem 3.6 *Under the condition (2.3), following asymptotic equation is satisfied for* $c_2(\lambda)$ *of JBVP*

$$c_2(\lambda) = B\left(\frac{\alpha+\beta}{\alpha\beta}\right) + O(\frac{1}{\lambda}),\ \lambda \in \overline{\mathbb{C}}_+,\ \ |\lambda| \to \infty.$$

Proof It can be easily obtained that

$$\varphi(x,\lambda) = D\frac{e^{-i(\lambda-1)x}}{\lambda-1}(\frac{i}{2}+o(1)),\ \lambda \in \overline{\mathbb{C}}_+,\ \ |\lambda| \to \infty$$

and

$$\varphi^{'}(x,\lambda) = De^{-i(\lambda-1)x}(\frac{i}{2}+o(1)),\ \lambda \in \overline{\mathbb{C}}_+,\ \ |\lambda| \to \infty,$$

where D is a constant. It follows from that

$$\varphi(1,\lambda) = D\frac{e^{-i(\lambda-1)}}{\lambda-1}(\frac{i}{2}+o(1)),\ \lambda \in \overline{\mathbb{C}}_+,\ \ |\lambda| \to \infty \tag{3.10}$$

and

$$\varphi^{'}(1,\lambda) = De^{-i(\lambda-1)}(\frac{i}{2}+o(1)),\ \lambda \in \overline{\mathbb{C}}_+,\ \ |\lambda| \to \infty. \tag{3.11}$$

On the other hand, by using (2.7), we get

$$e^{+}(1,\lambda) = Fe^{i\lambda}[1+o(1)],\ \lambda \in \overline{\mathbb{C}}_+,\ \ |\lambda| \to \infty \tag{3.12}$$

and

$$e^{+'}(1,\lambda) = Fe^{i\lambda}[i\lambda+o(1)],\ \lambda \in \overline{\mathbb{C}}_+,\ \ |\lambda| \to \infty, \tag{3.13}$$

where F is a constant. We get the result considering (3.10), (3.11), (3.12), and (3.13) in (3.4). Note that the constant $B \neq 0$ is given by the constants D and F, and we can get similar asymptotic equation for $\overline{c_2(\lambda)}$, whenever $\lambda \in \overline{\mathbb{C}}_-$ and $|\lambda| \to \infty$. □

Example 3.7 *Let us consider the following boundary value problem for* $0 \le x < \infty$

$$\left\{ \begin{array}{c} y^{''} + [\lambda - Q(x)]^2 y = 0 \\ y(0) = 0, \end{array} \right. \tag{3.14}$$

where

$$Q(x) = \left\{ \begin{array}{lcl} 1 & ; & x \in [0,1) \\ 0 & ; & x \in (1,\infty). \end{array} \right. \tag{3.15}$$

It can be easily seen that (3.14)-(3.15) boundary value problem can be formed as a Sturm-Liouville boundary problem generated by

$$\left\{ \begin{array}{lcl} y^{''} + (1-\lambda)^2 y = 0 & ; & x \in [0,1) \\ y^{''} + \lambda^2 y = 0 & ; & x \in (1,\infty) \end{array} \right. \tag{3.16}$$

with boundary condition

$$y(0) = 0 \tag{3.17}$$

and the jump conditions

$$\begin{aligned} y(1^+) &= \alpha y(1^-) \\ y^{'}(1^+) &= \beta y^{'}(1^-), \end{aligned} \tag{3.18}$$

where $\alpha, \beta \in \mathbb{R}$ *and* $\alpha\beta \neq 0$ *like (2.1)-(2.2) with (3.2). It follows from that* $e^+(x,\lambda) = e^{i\lambda x}$, $\varphi(x,\lambda) = \dfrac{\sin(\lambda-1)x}{\lambda-1}$ *and* $\psi(x,\lambda) = \cos(\lambda-1)x$ *for this problem. By using these solutions, we obtain the Jost solution of (3.16)-(3.18) as*

$$E^+(x,\lambda) = \begin{cases} m(\lambda)\cos(\lambda-1)x + n(\lambda)\dfrac{\sin(\lambda-1)x}{\lambda-1} & ; \quad x \in [0,1) \\ e^{i\lambda x} & ; \quad x \in (1,\infty), \end{cases}$$

where

$$m(\lambda) = e^{i\lambda}\left(\frac{\cos(\lambda-1)}{\alpha} - i\lambda\frac{\sin(\lambda-1)}{\beta(\lambda-1)}\right)$$

and

$$n(\lambda) = e^{i\lambda}\left(\frac{i\lambda\cos(\lambda-1)}{\beta} + \frac{(\lambda-1)\sin(\lambda-1)}{\alpha}\right).$$

From the definition of Jost solution of (3.16)-(3.18) and the above equations of $m(\lambda)$, $n(\lambda)$, *we obtain the Jost function and the scattering function of (3.16)-(3.18) as*

$$E^+(0,\lambda) = e^{i\lambda}\left(\frac{\cos(\lambda-1)}{\alpha} - i\lambda\frac{\sin(\lambda-1)}{\beta(\lambda-1)}\right), \lambda \in \overline{\mathbb{C}}_+$$

and

$$S(\lambda) = e^{-2i\lambda}\left(\frac{\dfrac{\cos(\lambda-1)}{\alpha} + i\lambda\dfrac{\sin(\lambda-1)}{\beta(\lambda-1)}}{\dfrac{\cos(\lambda-1)}{\alpha} - i\lambda\dfrac{\sin(\lambda-1)}{\beta(\lambda-1)}}\right), \lambda \in \mathbb{R}\backslash\{0\},$$

respectively. Now, we can write the set of eigenvalues of (3.16) by using Theorem 3.5

$$\sigma_d = \left\{\lambda \in \mathbb{C}_+ : \ E^+(0,\lambda) = 0\right\} \cup \left\{\lambda \in \mathbb{C}_- : \ \overline{E^+(0,\lambda)} = 0\right\}.$$

To get the set of eigenvalues of this problem, it is necessary to find the zeros of $E^+(0,\lambda)$ *and to consider the conjugate values of obtained eigenvalues. If*

$$E^+(0,\lambda) = 0$$

it follows from (3.18) that $e^{i\lambda}\left(\dfrac{\cos(\lambda-1)}{\alpha} - i\lambda\dfrac{\sin(\lambda-1)}{\beta(\lambda-1)}\right) = 0$. *Using the last equation, we find*

$$\lambda_k = -\frac{i}{2}\ln\left|\frac{1+A}{1-A}\right| + \frac{1}{2}Arg\left|\frac{1+A}{1-A}\right| + 1 + k\pi, k \in \mathbb{Z}, \text{ where } A = \frac{\beta(\lambda-1)}{\alpha\lambda}.$$

Case1: *For* $0 < A < 1$, *we see that*

$$\lambda_k = -\frac{i}{2}\ln\left(\frac{1+A}{1-A}\right) + 1 + k\pi, \ k \in \mathbb{Z},$$

since $\lambda_k \in \mathbb{C}_- := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda < 0\}$ *in this case, (3.16)-(3.18) has no eigenvalues and* $\overline{\lambda_k}$ *are not eigenvalues of the problem. Because these values are not the zeros of* $\overline{E^+(0,\lambda)} = 0$ *whenever* $\lambda \in \mathbb{C}_-$.

Case2: *For* $1 < A < \infty$, *we find*

$$\lambda_k = -\frac{i}{2}\ln\left|\frac{1+A}{1-A}\right| + 1 + \left(k + \frac{1}{2}\right)\pi,\ k \in \mathbb{Z},$$

Similarly to the Case1, here again $\lambda_k \in \mathbb{C}_-$ *and the problem (3.16)-(3.18) has no eigenvalues.*

Case3: *For* $A \in (-1,0)$, *we obtain that*

$$\lambda_k = \frac{i}{2}\ln\left(\frac{1-A}{1+A}\right) + 1 + k\pi,\ k \in \mathbb{Z},$$

here $\lambda_k \in \mathbb{C}_+$ *and they are the eigenvalues of the impulsive boundary value problem (3.16)-(3.18). Furthermore,* $\overline{\lambda_k} \in \mathbb{C}_-$ *are the zeros of* $\overline{E^+(0,\lambda)} = 0$, *as a result of this by using the definition of* σ_d *for this problem,* $\overline{\lambda_k} \in \mathbb{C}_-$, $k \in \mathbb{Z}$ *become eigenvalues of (3.16)-(3.18).*

Case4: *For* $A \in (-\infty,-1)$, *we find*

$$\lambda_k = \frac{i}{2}\ln\left|\frac{1-A}{1+A}\right| + 1 + \left(k + \frac{1}{2}\right)\pi,\ k \in \mathbb{Z},$$

and similar to the Case3, the numbers λ_k, $k \in \mathbb{Z}$ *and* $\overline{\lambda_k}$, $k \in \mathbb{Z}$ *are the eigenvalues of (3.16)-(3.18).*

4. Conclusion

In this study, scattering analysis of Klein-Gordon equation, which is an important differential equation of differential theory and applied sciences, is investigated with jump conditions. Scattering solutions and Jost solution of this problem are given and by using these basic solutions, we get the scattering function and its basic properties. This paper is the first that studies the scattering solutions and function of a Klein-Gordon equation with jump (impulsive) conditions. Furthermore, the paper consists some spectral properties of the same problem. We give an asymptotic equation for Jost function which is related to the coefficients of the Jost solution of impulsive problem and we determine the set of eigenvalues of the related problem. We hope that this work will open a new field for mathematicians and be the basis for many different fields in applied mathematics. It is possible to extend this results in terms of the different properties of potential function by using different methods.

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