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Contiguity distance between simplicial maps

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Abstract: For simplicial complexes and simplicial maps, the notion of being in the same contiguity class is defined as the discrete version of homotopy. In this paper, we study the contiguity distance, SD, between two simplicial maps adapted from the homotopic distance. In particular, we show that simplicial versions of LS-category and topological complexity are particular cases of this more general notion. Moreover, we present the behaviour of SD under the barycentric subdivision, and its relation with strong collapsibility of a simplicial complex.

Key words: Contiguity distance, homotopic distance, topological complexity, Lusternik-Schnirelmann category

1. Introduction

The Lusternik-Schnirelmann category, introduced by Lusternik and Schnirelmann [12], is an important numerical invariant concerning the critical points of smooth functions on manifolds.

Definition 1.1 *[3, 12] Lusternik Schnirelmann category of a space X, denoted by $cat(X)$, is the least nonnegative integer k if there are open subsets $U_0, U_1, \ldots, U_k$ which cover X such that each inclusion map $\iota_i : U_i \hookrightarrow X$ is null-homotopic in X for $i = 0, 1, \ldots, k$.*

Topological complexity of a topological space introduced by Farber [4] is another numerical invariant closely related to motion planning problems.

Definition 1.2 *[4] Let $\pi : PX \to X \times X$ be the path fibration. Topological complexity of a space X, denoted by $TC(X)$, is the least nonnegative integer k if there are open subsets $U_0, U_1, \ldots, U_k$ which cover $X \times X$ such that on each U_i there exists a continuous section of π for $i = 0, 1, \ldots, k$.*

Although these invariants, cat and TC, seem to be independent, they are similar in nature both being homotopy invariants. Macias-Virgos and Mosquera-Lois [13] introduced homotopic distance, a notion generalizing both cat and TC. However, unlike cat and TC which are related to spaces, the homotopic distance is a number related to functions. Hence, we have the opportunity to investigate the behaviour of the homotopic distance under compositions which is not possible to do with cat and TC. This feature also leads us

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to prove the known TC- and cat-related theorems in simpler ways. For example, one may compare our proof of Theorem 2.25 with the proof of theorem 4.3 in [7].

Definition 1.3 *[13] Let $f, g : X \to Y$ be continuous maps. The homotopic distance between f and g, denoted by $D(f, g)$, is the least nonnegative integer k if there are open subsets $U_0, U_1, \ldots, U_k$ which cover X such that f and g restricted to U_i are homotopic, $f|_{U_i} \simeq g|_{U_i}$, for all $i = 0, 1, \ldots, k$.*

In this paper, we consider simplicial complexes and study the distance between two simplicial maps adapted from homotopic distance. Note that one can consider a geometric realization of a simplicial complex and study the ordinary homotopic distance between continuous maps induced by the geometric realization of simplicial maps. However, we opt to stay in the simplicial category in order not to lose the combinatorial aspects. To do this, we consider a simplicial analogue of homotopic distance between simplicial maps which relies on the contiguity. Then the simplicial analogues of cat and TC of a simplicial complex can be defined in terms of this distance. However, we want to remark that the contiguity distance between simplicial maps and the homotopic distance between their corresponding geometric realizations might differ, see Example 2.6.

Given a set V, an abstract simplicial complex with a vertex set V is a set K of finite subsets of V such that the elements of V belong to K and for any $\sigma \in K$ any subset of σ belongs to K. The elements of K are called the faces or the simplices of K. The dimension of an abstract simplex is just its cardinality minus 1 and the dimension of K is the largest dimension of its simplices. For further details on abstract simplicial complexes, we refer to [11, 16].

The combinatorial description of any geometric simplicial complex $\tilde{K}$ obviously gives rise to an abstract simplicial complex K. One can always associate a geometric simplicial complex $\tilde{K}$ to an abstract simplicial complex K in such a way that the combinatorial description of $\tilde{K}$ is the same as K so that the underlying space of $\tilde{K}$ is homeomorphic to the geometric realization $|K|$. As a consequence, abstract simplicial complexes can be seen as topological spaces and geometric complexes can be seen as geometric realizations of their underlying combinatorial structure, so one can consider simplicial complexes at the same time as combinatorial objects that are well-suited for effective computations and as topological spaces from which topological properties can be inferred.

It is a classical result that an arbitrary continuous map between geometric realizations of simplicial complexes can be deformed (after sufficiently many subdivisions) to a simplicial map, known as the simplicial approximation theorem. However, in general, simplicial approximations to a given continuous map are not unique. An analogue of homotopy, called contiguity, is defined for simplicial maps so that different simplicial approximations to the same continuous map are contiguous.

Definition 1.4 *Let $\varphi, \psi \colon K \to K'$ be two simplicial maps between simplicial complexes. We say that φ and ψ are* contiguous*, denoted by $\varphi \sim_c \psi$, provided for a simplex $\sigma = \{v_0, \ldots, v_n\}$ in K, the set of vertices $\varphi(\sigma) \cup \psi(\sigma) = \{\varphi(v_0), \ldots, \varphi(v_n), \psi(v_0), \ldots, \psi(v_n)\}$ constitutes a simplex in K'.*

For simplicial complexes and simplicial maps, the notion of being in the same contiguity class can be considered the discrete version of homotopy. Being contiguous is a combinatorial condition which defines a reflexive and symmetric relation among simplicial maps. On the other hand, this relation is not transitive.

There is, however, an equivalence relation in the set of simplicial maps and the corresponding equivalence classes are called contiguity classes.

Definition 1.5 *We say that two simplicial maps $\varphi, \psi \colon K \to K'$ are in the same contiguity class, denoted by $\varphi \sim \psi$, provided there exists a finite sequence of simplicial maps $\varphi_i \colon K \to K'$, $i = 1, \ldots, m$, such that $\varphi = \varphi_1 \sim_c \varphi_2 \sim_c \cdots \sim_c \varphi_m = \psi$.*

Barmak and Minian [1, 2] introduced the notion of strong collapse, a particular type of collapse which is specially adapted to the simplicial structure. Actually, it can be modelled as a simplicial map, in contrast with the standard concept of collapse which is not a simplicial map in general: For a simplicial complex K, suppose that there is a pair of simplices $\sigma < \tau$ in K such that σ is a face of τ, and σ has no other cofaces. Such a simplex σ is called a free face of τ. Then the simplicial complex $K - \{\sigma, \tau\}$ is complex called an elementary collapse of K (see Figure 1). The action of collapsing is denoted by $K \searrow K - \{\sigma, \tau\}$. The inverse of an elementary collapse is called an elementary expansion.

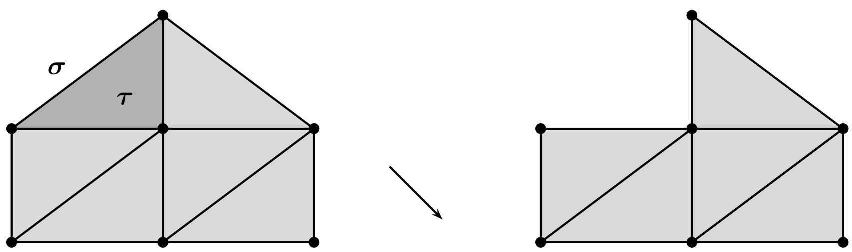

Figure 1. An elementary collapse.

Two simplicial complexes K, K' have the same strong homotopy type, denoted by $K \sim K'$, if they are related by a sequence of strong collapses and expansions. Surprisingly, this turns out to be intimately related to the classical notion of contiguity. More precisely, having the same strong homotopy type is equivalent to the existence of a strong equivalence.

A simplicial map $\varphi \colon K \to K'$ is called a strong equivalence if there exists $\psi \colon K' \to K$ such that $\varphi \circ \psi \sim id_{K'}$ and $\psi \circ \varphi \sim id_K$. The theory of strong homotopy types of simplicial complexes was introduced in [2]. Strong homotopy types can be described by elementary moves called strong collapses. From this theory, Barmak and Minian obtained new results for studying simplicial collapsibility.

A natural definition of Lusternik-Schnirelman (LS) category for simplicial complexes, that is invariant under strong equivalences, is given in [9], and a notion of discrete topological complexity in the setting of simplicial complexes by means of contiguous simplicial maps is given in [7].

Let K be a simplicial complex and $L \subseteq K$ a subcomplex. We say that L is *categorical*, provided there exists a vertex $v_0 \in K$ such that the inclusion map $i \colon L \hookrightarrow K$ and the constant map $c_{v_0} \colon L \to K$ are in the same contiguity class. The simplicial LS category, denoted by $scat(K)$, is defined as the least integer $n \geq 0$ such that K is covered by $(n+1)$ categorical subcomplexes [9]. Immediately from this definition, we can conclude that a simplicial complex K is strongly collapsible, i.e. has the strong homotopy type of a point if and only if $scat(K) = 0$.

The Cartesian product of two simplicial complexes may not satisfy the universal property of a product, so that it is not necessarily a simplicial complex. As in [11], we can define a product of two simplicial complexes, called categorical product, in such a way that their product is a simplicial complex and satisfy the universal property of a product. Let K and K' be two simplicial complexes. Then the categorical product of K and K', denoted by $K \prod K'$, is a simplicial complex such that

1. its vertices are pairs (v, ω) where v is a vertex of K and ω is a vertex of K', and
2. the projections $\mathrm{pr}_1 : K \prod K' \to K$ and $\mathrm{pr}_2 : K \prod K' \to K'$ are simplicial maps and are universal with the property.

Let K be a simplicial complex and $K^2 = K \prod K$ a categorical product. Then a simplicial subcomplex $\Omega \subset K^2$ is a Farber subcomplex, provided there exists a simplicial map $\sigma\colon \Omega \to K$ such that $\Delta \circ \sigma \sim \iota_\Omega$ where $\iota_\Omega\colon \Omega \hookrightarrow K^2$ is the inclusion map and $\Delta\colon K \to K^2$ is the diagonal map $\Delta(v) = (v, v)$.

Definition 1.6 *[7] The discrete topological complexity $TC(K)$ of the simplicial complex K is the least integer $n \geq 0$ such that K^2 can be covered by $(n+1)$ Farber subcomplexes.*

In other words, $TC(K) \leq n$ if and only if $K^2 = \Omega_0 \cup \ldots \cup \Omega_n$, and there exist simplicial maps $\sigma_j\colon \Omega_j \to K$ such that $\Delta \circ \sigma_j \sim \iota_j$ where $\iota_j\colon \Omega_j \hookrightarrow K^2$ are inclusions for $j = 0, \ldots, n$.

Before the end of this section, we remark that for a given simplicial complex K, $cat(|K|)$ and $TC(|K|)$ are lower bounds for $scat(K)$ and $TC(K)$, respectively.

2. Contiguity distance

Throughout the paper, a simplicial complex is meant to be an abstract simplicial complex, all simplicial complexes are assumed to be (edge-) path connected, and all maps between simplicial complexes are assumed to be simplicial maps.

Definition 2.1 *[13, Definition 8.1] For simplicial maps $\varphi, \psi\colon K \to K'$, the contiguity distance between φ and ψ, denoted by $SD(\varphi, \psi)$, is the least integer $n \geq 0$ such that there exists a covering of K by subcomplexes $K_0, K_1, \ldots, K_n$ with the property that $\varphi\big|_{K_j}, \psi\big|_{K_j}\colon K_j \to K'$ are in the same contiguity class for all $j = 0, 1, \ldots, n$.*

Remark 2.2 *There is another simplicial version of homotopic distance, also called contiguity distance which is introduced in [14] and given in the sense of Gonzalez [10]. According to [14], the contiguity distance in this paper is called "strict contiguity distance".*

It is easy to see that the contiguity distance defines a symmetric relation on the set of simplicial maps and the contiguity distance between two maps is zero if and only if they are in the same contiguity class. The next proposition tells us that this notion is well-defined on the set of equivalence classes of simplicial maps.

Proposition 2.3 *If $\varphi \sim \bar{\varphi}, \psi \sim \bar{\psi}\colon K \to K'$, then $SD(\varphi, \psi) = SD(\bar{\varphi}, \bar{\psi})$.*

Proof Suppose first that $SD(\varphi, \psi) = n$. By definition, this means that there exists a covering of K by subcomplexes $K_0, K_1, \ldots, K_n$ with the property that $\varphi\big|_{K_j}, \psi\big|_{K_j}\colon K \to K'$ are in the same contiguity class for all j. Since $\varphi \sim \bar{\varphi}$ and $\psi \sim \bar{\psi}$, their restrictions to K_j are also in the same contiguity classes for all j. Also recall that contiguity classes are equivalence classes, so we have $\bar{\varphi}\big|_{K_j} \sim \varphi\big|_{K_j} \sim \psi\big|_{K_j} \sim \bar{\psi}\big|_{K_j}$ for all j.

Therefore, $SD(\bar{\varphi}, \bar{\psi}) \leq n$. Starting with $SD(\bar{\varphi}, \bar{\psi})$ gives us $SD(\varphi, \psi) \leq SD(\bar{\varphi}, \bar{\psi})$, which completes the proof. □

We can use a finite covering of a complex K to produce an upper bound for the simplicial distance between maps.

Proposition 2.4 *Given two simplicial maps $\varphi, \psi \colon K \to K'$ and a finite covering of K by subcomplexes $K_0, K_1, \ldots, K_n$, we have*

$$SD(\varphi, \psi) \leq \sum_{i=0}^{n} SD(\varphi\big|_{K_j}, \psi\big|_{K_j}) + n.$$

Proof Suppose $SD(\varphi|_{K_j}, \psi|_{K_j}) = m_j$ for each $j = 0, 1, \ldots, n$. Thus, there exists a covering of K_j by subcomplexes $K_j^0, K_j^1, \ldots, K_j^{m_j}$ such that $\varphi|_{K_j^i} \sim \psi|_{K_j^i}$.

The collection $\mathcal{K} = \left\{K_0^0, \ldots, K_0^{m_0}, K_1^0, \ldots, K_1^{m_1}, \ldots, K_n^0, \ldots, K_n^{m_n}\right\}$ is a covering for K satisfying $\varphi|_L \sim \psi|_L$ for all $L \in \mathcal{K}$. Thus, since the cardinality of $\mathcal{K}$ is $(m_0 + m_1 + \ldots + m_n) + n + 1$, the required inequality holds. □

Next, we mention the relation between the simplicial LS-category and the contiguity distance between simplicial maps. First, note that for a subcomplex L of a simplicial complex K, if $id_K\big|_L$ and $c_{v_0}\big|_L$ are in the same contiguity class, then L is categorical in K. From this observation, it is easy to see that for a simplicial complex K and any vertex v_0 of K, we have

$$scat(K) = SD(id_K, c_{v_0}).$$

Let K be a simplicial complex and $|K|$ denote its geometric realization. We know that both $scat(K)$ and $TC(K)$ might differ from $cat(|K|)$ and $TC(|K|)$ (see, Theorem 2.15 and [7, Theorem 5.2]). Although the simplicial category and discrete topological complexity depend on both the simplicial structure and the geometric realization of the complex [7, 9], the particular considered triangulations play an important role. More precisely, for simplicial complexes K, K' and simplicial maps $\varphi, \psi \colon K \to K'$, we expect $SD(\varphi, \psi)$ is not necessarily the same as $\mathrm{D}(|\varphi|, |\psi|)$, where $|\varphi|, |\psi| \colon |K| \to |K'|$ are continuous maps between their corresponding geometric realizations [13].

Proposition 2.5 *For simplicial maps $\varphi, \psi : K \to L$, we have $\mathrm{D}(|\varphi|, |\psi|) \leq SD(\varphi, \psi)$.*

Proof Let $SD(\varphi, \psi) = n$ so that there exist subcomplexes $K_0, K_1, \ldots, K_n$ in such a way that the inclusion map $\iota_i \colon K_i \to K$ and the constant map $c_v \colon K_i \to L$ are in the same contiguity class, $\iota_i \sim c_v$. Note that the union of the closed subsets $|K_0|, |K_1|, \ldots, |K_n|$ of $|K|$ covers $|K|$ and the geometric realizations of ι_i and c_v,

$$|\iota_i|, |c_v| \colon |K_i| \to |K|$$

are homotopic continuous maps. □

The following is an example for the strict form of the inequality given in Proposition 2.5.

Example 2.6 *Consider the simplicial complex K given in Figure 2 [2]. Let id_K and c be the identity simplicial map and a constant simplicial map on K, respectively. We know that $scat(K) = 1$ ([9, Example 3.2]) so that $SD(id_K, c) = 1$. Notice that the homotopic distance $\mathrm{D}(|id_K|, |c|)$ is zero which follows from the fact that the geometric realization $|K|$ of K is contractible. Therefore, $\mathrm{D}(|id_K|, |c|) < SD(id_K, c)$.*

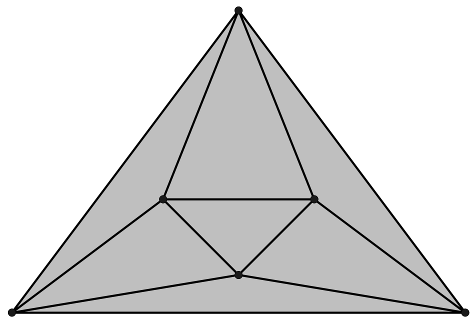

Figure 2. $|K|$ is contractible whereas K is not strongly collapsible.

Before we study the behaviour of contiguity distance under barycentric subdivision, we recall some basic notions and talk about how *scat* behaves under barycentric subdivision.

Definition 2.7 *The barycentric subdivison of a given simplicial complex* K *is the simplicial complex* $sd(K)$ *whose set of vertices is* K *and each* n*-simplex in* $sd(K)$ *is of the form* $\{\sigma_0, \sigma_1, \ldots, \sigma_n\}$ *where* $\sigma_0 \subsetneq \sigma_1 \subsetneq \ldots \subsetneq \sigma_n$.

Definition 2.8 *For a simplicial map* $\varphi\colon K \to L$, *the induced map* $sd(\varphi)\colon sd(K) \to sd(L)$ *is given by*

$$sd(\varphi)(\{\sigma_0, \ldots, \sigma_q\}) = \{\varphi(\sigma_0), \ldots, \varphi(\sigma_q)\}.$$

Notice that $sd(\varphi)$ is a simplicial map and $sd(id) = id$.

Proposition 2.9 *[8] If the simplicial maps* $\varphi, \psi\colon K \to L$ *are in the same contiguity class, so are* $sd(\varphi)$ *and* $sd(\psi)$.

The relation between the contiguity distance of two maps and the contiguity distance of their induced maps on barycentric subdivisions can be given as follows.

Theorem 2.10 *For simplicial maps* $\varphi, \psi\colon K \to K'$, $SD(sd(\varphi), sd(\psi)) \leq SD(\varphi, \psi)$.

Proof Let $SD(\varphi, \psi) = n$. Then there are subcomplexes $K_0, K_1, \ldots, K_n$ covering K such that $\varphi|_{K_i} \sim \psi|_{K_i}$ for all $i = 0, 1, \ldots, n$.

Take the cover $\{sd(K_0), sd(K_1), \ldots, sd(K_n)\}$ of $sd(K)$. By Proposition 2.9, if $\varphi|_{K_i} \sim \psi|_{K_i}$, then $sd(\varphi|_{K_i}) \sim sd(\psi|_{K_i})$.

On the other hand, $sd(\varphi|_{K_i}) = sd(\varphi|_{sd(K_i)})$. More precisely, if $\{\sigma_1, \ldots, \sigma_q\} \in sd(K_i)$,

$$sd(\varphi|_{K_i})(\{\sigma_1, \ldots, \sigma_q\}) = \big\{sd(\varphi|_{K_i})(\sigma_1), \ldots, sd(\varphi|_{K_i})(\sigma_q)\big\} = \big\{sd(\varphi)(\sigma_1), \ldots, sd(\varphi)(\sigma_q)\big\} = sd(\varphi\big|_{sd(K_i)}).$$

Hence, since we have $sd(\varphi|_{K_i}) \sim sd(\psi|_{K_i})$, it follows that $sd(\varphi\big|_{sd(K_i)}) \sim sd(\psi\big|_{sd(K_i)})$ for all i. □

Although the below corollary is given as a consequence of some theorems related to finite spaces in [9, Corollary 6.7] and a direct proof is given in [8, Theorem 3.1.1], we give the following alternative proof using the contiguity distance for the consistency of the paper.

Corollary 2.11 *For a simplicial complex K, $scat(sd(K)) \leq scat(K)$.*

Proof In Theorem 2.10, take $\varphi = id$ and $\psi = c$ as the identity map and a constant map, respectively. Thus, the induced maps $sd(id)$ and $sd(c)$ are also the identity and a constant map on $sd(K)$. Thus, the corollary follows. □

Observe that for a simplicial complex K being strongly collapsible is equivalent to saying that $scat(K) = SD(id_K, c_{v_0}) = 0$. Hence, for a strongly collapsible complex K, we have $id_K \sim c_{v_0}$. The following theorem tells us that the same is true for arbitrary maps.

Theorem 2.12 *For any maps $\varphi, \psi \colon K \to K'$, $SD(\varphi, \psi) = 0$, provided K or K' is strongly collapsible.*

Proof Suppose K is strongly collapsible, then we have $id_K \sim c_{v_0}$ where v_0 is a vertex in K. We have the following diagram

$$\begin{tikzcd} K \arrow[r, bend left, "id_K"] \arrow[r, bend right, "c_{v_0}"'] & K \arrow[r, "\varphi"] & K' \end{tikzcd}$$

which implies that

$$\varphi \circ id_K \sim \varphi \circ c_{v_0} \quad \text{(constant)}.$$

Similarly, we have

$$\begin{tikzcd} K \arrow[r, bend left, "id_K"] \arrow[r, bend right, "c_{v_0}"'] & K \arrow[r, "\psi"] & K' \end{tikzcd}$$

so that

$$\psi \circ id_K \sim \psi \circ c_{v_0} \quad \text{(constant)}.$$

Since K' is edge-path connected, all the constant maps are in the same contiguity class. Hence, we have $\varphi = \varphi \circ id_K \sim \psi \circ id_K = \psi$.

On the other hand, if K' is strongly collapsible

$$\text{scat}(K') = 0 = SD(id_{K'}, c_{\omega_0})$$

where ω_0 is a vertex in K'. That is,

$$id_{K'} \sim c_{\omega_0}.$$

This time, we have the following diagram:

$$\begin{tikzcd} K \arrow[r, "\varphi"] & K' \arrow[r, bend left, "id_{K'}"] \arrow[r, bend right, "c_{\omega_0}"'] & K' \end{tikzcd}$$

so that

$$id_{K'} \circ \varphi \sim c_{\omega_0} \circ \varphi \quad \text{(constant)}.$$

Similarly, we have

$$K \xrightarrow{\psi} K' \underset{c_{\omega_0}}{\overset{id_{K'}}{\rightrightarrows}} K'$$

so that

$$id_{K'} \circ \psi \sim c_{\omega_0} \circ \psi \quad \text{(constant)}.$$

Note that K' is edge-path connected since it is strongly collapsible. Hence, we have $\varphi = id_{K'} \circ \varphi \sim id_{K'} \circ \psi = \psi$. □

For the converse, we have the following result.

Corollary 2.13 *Let K be a simplicial complex. If $SD(\varphi, \psi) = 0$ for any pair of simplicial maps $\varphi, \psi \colon K \to K$, then K is strongly collapsible.*

Proof If we take $\varphi = id_K$ and $\psi = c_{v_0}$ on a fixed vertex $v_0 \in K$, our assumption $SD(id_K, c_{v_0}) = 0$ implies that $\mathrm{scat}(K) = 0$, which is equivalent to saying that K is strongly collapsible. □

Theorem 2.14 *Let v_0 be a vertex of the simplicial complex K. For the simplicial maps*

$$i_1, i_2 \colon K \to K^2$$

defined as $i_1(\sigma) = (\sigma, v_0)$ and $i_2(\sigma) = (v_0, \sigma)$, we have $scat(K) = SD(i_1, i_2)$.

Proof First, we prove that $SD(i_1, i_2) \leq scat(K)$. Let $L \subseteq K$ be categorical. That is, there exists a vertex v_0 of K such that the inclusion map $\iota \colon L \hookrightarrow K$ and the constant map $c_{v_0} \colon L \to K$ are in the same contiguity class. We want to show that $i_1\big|_L$ and $i_2\big|_L$ are also in the same contiguity class. Consider the following composition of simplicial maps

$$L \xrightarrow{\Delta_L} L^2 \underset{c_{v_0} \prod \iota}{\overset{\iota \prod c_{v_0}}{\rightrightarrows}} K^2 ,$$

where Δ_L is the diagonal map of L, defined on the set of vertices by $v \to (v, v)$, and $\iota \prod c_{v_0}$ and $c_{v_0} \prod \iota$ is the categorical product of ι and c_{v_0}. Then

$$i_1\big|_L = (\iota \prod c_{v_0}) \circ \Delta_L,$$

and

$$i_2\big|_L = (c_{v_0} \prod \iota) \circ \Delta_L.$$

Since L is categorical, then $\iota \sim c_{v_0}$. We have

$$\iota \prod c_{v_0} \sim c_{v_0} \prod c_{v_0},$$

$$c_{v_0} \prod \iota \sim c_{v_0} \prod c_{v_0}.$$

This implies

$$\iota \prod c_{v_0} \sim c_{v_0} \prod \iota$$

so that $(\iota \prod c_{v_0}) \circ \Delta_L \sim (c_{v_0} \prod \iota) \circ \Delta_L$, which proves our claim.

Next, we show that $scat(K) \leq SD(i_1, i_2)$. Assume that L is a subcomplex of K with $i_1\big|_L \sim i_2\big|_L$. Let $p_i \colon K^2 \to K$ be the projection maps for $i = 1, 2$. Then $p_1 \circ i_1\big|_L \sim p_1 \circ i_2\big|_L$ so that $\iota \sim c_{v_0}$. □

The Proposition 2.5 leads to the following theorem.

Theorem 2.15 *Let K be a simplicial complex and $|K|$ its geometric realization. $cat(|K|) \leq scat(K)$.*

Proof Consider the simplicial maps $i_1 : K \to K^2$ and $i_2 : K \to K^2$ defined in Theorem 2.14 so that $scat(K) = SD(i_1, i_2)$. In that case, their geometric realizations

$$|i_1|, |i_2| \colon |K| \to |K^2|$$

are continuous maps. By Lemma 5.1 in [7], we know that $|K^2|$ and $|K| \times |K|$ are homotopy equivalent spaces. Let $u : |K^2| \to |K| \times |K|$ be the homotopy equivalence. Therefore, the inclusion maps $\iota_1 : |K| \hookrightarrow |K| \times |K|$ and $\iota_2 : |K| \hookrightarrow |K| \times |K|$ are homotopic to $u \circ |i_1|$ and $u \circ |i_2|$, respectively. By Proposition 2.5 and [13, proposition 3.1], we have

$$cat(|K|) = \mathrm{D}(\iota_1, \iota_2) = \mathrm{D}(u \circ |i_1|, u \circ |i_2|) \leq \mathrm{D}(|i_1|, |i_2|) \leq SD(i_1, i_2) = scat(K).$$

□

Our next aim is to prove Theorem 2.20. Thus, we need Corollary 2.17 and Corollary 2.19, which follow from Proposition 2.16 and Proposition 2.18, respectively.

Proposition 2.16 *Let $\varphi, \psi \colon K \to K'$ and $\mu \colon M \to K$ be simplicial maps. Then we have*

$$SD(\varphi \circ \mu, \psi \circ \mu) \leq SD(\varphi, \psi).$$

Proof Let $SD(\varphi, \psi) = n$. Then there exist subcomplexes $K_0, \ldots, K_n$ of K such that $\varphi\big|_{K_j} \sim \psi\big|_{K_j}$ for all j. Define $M \supset M_j \colon = \mu^{-1}(K_j)$ and the restriction map $\mu_j \colon M_j \to K$. Then

$$(\varphi \circ \mu)_j = \varphi \circ \mu_j = \varphi \circ \iota_j \circ \bar{\mu}_j = \varphi_j \circ \bar{\mu}_j \sim \psi_j \circ \bar{\mu}_j = \psi \circ \iota_j \circ \bar{\mu}_j = \psi \circ \mu_j = (\psi \circ \mu)_j,$$

where $\iota_j \colon K_j \hookrightarrow K$ is the inclusion and $\bar{\mu}_j : M_j \to K_j$, $\bar{\mu}_j(x) = \mu_j(x)$ is a map satisfying $\mu_j = \iota_j \circ \bar{\mu}_j$. Therefore, $SD(\varphi \circ \mu, \psi \circ \mu) \leq n$. □

Corollary 2.17 *Let $\varphi, \psi \colon K \to K'$ be simplicial maps and $\beta \colon M \to K$ be a simplicial map which has a right strong equivalence (that is, β satisfies $\beta \circ \alpha \sim id_K$ where $\alpha \colon K \to M$). Then $SD(\varphi \circ \beta, \psi \circ \beta) = SD(\varphi, \psi)$.*

Proof Since $\beta \circ \alpha \sim id_K$, it follows that $\varphi \circ \beta \circ \alpha \sim \varphi$ and $\psi \circ \beta \circ \alpha \sim \psi$. Thus,

$$SD(\varphi, \psi) = SD(\varphi \circ \beta \circ \alpha, \psi \circ \beta \circ \alpha) \leq SD(\varphi \circ \beta, \psi \circ \beta) \leq SD(\varphi, \psi),$$

where the equality follows from Proposition 2.3 and the inequalities follow from Proposition 2.16. Hence, we have $SD(\varphi \circ \beta, \psi \circ \beta) = SD(\varphi, \psi)$. □

Proposition 2.18 *Let $\varphi, \psi \colon K \to K'$ and $\varphi, \varphi' \colon K' \to M$ be simplicial maps. If $\varphi \sim \varphi'$, then $SD(\varphi \circ \varphi, \varphi' \circ \psi) \leq SD(\varphi, \psi)$.*

Proof Suppose $SD(\varphi, \psi) = n$. Then there exist subcomplexes $K'_0, K'_1, \ldots, K'_n$ of K' such that $\varphi\big|_{K'_i}$ and $\psi\big|_{K'_j}$ are in the same contiguity class for all i, j. So

$$(\varphi \circ \varphi)\big|_{K'_i} = \varphi \circ \varphi\big|_{K'_i} \sim \varphi' \circ \varphi\big|_{K'_i} \sim \varphi' \circ \varphi\big|_{K'_j} = (\varphi' \circ \varphi)\big|_{K'_j}.$$

Hence, $SD(\varphi \circ \varphi, \varphi' \circ \psi) \leq n$. □

Corollary 2.19 *Let $\varphi, \psi \colon K \to K'$ be simplicial maps and $\alpha \colon K' \to M$ be a simplicial map which has a left strong equivalence (that is, α satisfies $\beta \circ \alpha \sim id_{K'}$, where $\beta \colon M \to K'$). Then $SD(\alpha \circ \varphi, \alpha \circ \psi) = SD(\varphi, \psi)$.*

Proof Since $\beta \circ \alpha \sim id_{K'}$, it follows that $\beta \circ \alpha \circ \varphi \sim \varphi$ and $\beta \circ \alpha \circ \psi \sim \psi$. Thus,

$$SD(\varphi, \psi) = SD(\beta \circ \alpha \circ \varphi, \beta \circ \alpha \circ \psi) \leq SD(\alpha \circ \varphi, \alpha \circ \psi) \leq SD(\varphi, \psi),$$

where the equality follows from Proposition 2.3 and the inequalities follow from Proposition 2.18. Hence, we have $SD(\alpha \circ \varphi, \alpha \circ \psi) = SD(\varphi, \psi)$. □

Theorem 2.20 *If $\beta : K' \sim K$ and $\alpha : L \sim L'$ have the same strong homotopy type and if simplicial maps $\varphi, \psi : K \to L$ and $\varphi', \psi' : K' \to L'$ make the following diagrams commutative with respect to f and g, respectively, in the sense of contiguity (that is, $\alpha \circ \varphi \circ \beta \sim \varphi'$ and $\alpha \circ \psi \circ \beta \sim \psi'$), then we have $SD(\varphi, \psi) = SD(\varphi', \psi')$.*

$$\begin{tikzcd} K \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & L \arrow[d, "\alpha"] \\ K' \arrow[u, "\beta"] \arrow[r, shift left, "\varphi'"] \arrow[r, shift right, "\psi'"'] & L' \end{tikzcd}$$

Proof $SD(\varphi', \psi') = SD(\alpha \circ \varphi \circ \beta, \alpha \circ \psi \circ \beta) = SD(\varphi \circ \beta, \psi \circ \beta) = SD(\varphi, \psi)$,

where the second equality follows from Corollary 2.19 and the last equality follows from Corollary 2.17. □

Remark 2.21 *Notice that the result of Theorem 2.20 is still valid even if we consider β and α as right and left strong equivalences, respectively.*

The simplicial LS category of a simplicial map is defined as in the following definition.

Definition 2.22 *[15] Let $\varphi \colon K \to K'$ be a simplicial map and ω_0 be a vertex of K'. Simplicial LS category $scat(\varphi)$ of φ is defined to be the least integer n such that there exists a covering of K by subcomplexes $K_0, K_1, \ldots, K_n$ such that $\varphi\big|_{K_j} \colon K_j \to K'$ and the constant map $c_{\omega_0} \colon K_j \to K$ are in the same contiguity class for all j.*

Corollary 2.23 *Let* $\varphi\colon K \to K'$ *be a simplicial map. Then* $scat(\varphi) \leq \min\{scat(K), scat(K')\}$.

Proof Let $id_K\colon K \to K$ be the identity map and $c_{v_0}\colon K \to K$ be the constant map at the vertex v_0 in K.

$$scat(\varphi) = SD(\varphi, \varphi \circ c_{v_0}) = SD(\varphi \circ id_K, \varphi \circ c_{v_0}) \leq SD(id_K, c_{\varphi(v_0)}) = scat(K),$$

where the inequality follows from Proposition 2.18. Hence, $scat(\varphi) \leq scat(K)$.

On the other hand, we have

$$scat(K') = SD(id_{K'}, c_{\omega_0}) \geq SD(id_{K'} \circ \varphi, c_{\omega_0} \circ \varphi) = scat(\varphi),$$

where $id_{K'}\colon K' \to K'$ is the identity map and $c_{\omega_0}\colon K' \to K'$ is the constant map at the vertex ω_0 in K'. Thus, $scat(\varphi) \leq scat(K')$. $\square$

Let K be a simplicial complex and $p_1, p_2\colon K^2 \to K$ projection maps onto the first and second factors, respectively. The following theorem is first proved in [7, theorem 3.4] (see also [13, example 8.2]). Here, we provide an alternative proof using contiguity distance.

Theorem 2.24 *For a simplicial complex* K*, we have* $SD(p_1, p_2) = TC(K)$.

Proof We first show that $TC(K) \leq SD(p_1, p_2)$. Suppose $TC(K) = n$. Then there is a covering for K^2 which consists of Farber subcomplexes $L_0, L_1, \ldots, L_n$. Since each L_i is a Farber subcomplex, there exists a simplicial map $\sigma_i\colon L_i \to K$ such that $\Delta \circ \sigma_i \sim \iota_{L_i}$.

$$\Delta \circ \sigma_i \sim \iota_{L_i}$$
$$p_1 \circ (\Delta \circ \sigma_i) \sim p_1 \circ \iota_{L_i} = p_1\big|_{L_i}$$
$$p_2 \circ (\Delta \circ \sigma_i) \sim p_2 \circ \iota_{L_i} = p_2\big|_{L_i}$$

Since $p_1 \circ (\Delta \circ \sigma_i) = p_2 \circ (\Delta \circ \sigma_i)$, we have $p_1\big|_{L_i} \sim p_2\big|_{L_i}$.

Next, we will show that $TC(K) \leq SD(p_1, p_2)$. Suppose $SD(p_1, p_2) = n$. Then there exist subcomplexes $L_0, L_1, \ldots, L_n$ which cover K^2 and $p_1\big|_{L_i} \sim p_2\big|_{L_i}$ for $i = 1, 2, \ldots, n$. By Definition 1.5, there exists a finite sequence of simplicial maps $\varphi^i_j : L^2_i \to L_i$ such that $p_1\big|_{L_i} = \varphi^i_1 \sim_c \varphi^i_2 \sim_c \ldots \sim_c \varphi^i_m = p_2\big|_{L_i}$. This means that for an element $([x], [y])$ in L_i where $[x] = \{x_1, x_2, \ldots, x_k\}$ and $[y] = \{y_1, y_2, \ldots, y_m\}$,

$$\varphi^i_1\big(([x], [y])\big) \cup \varphi^i_m\big(([x], [y])\big) = \{x_1, \ldots, x_k, y_1, \ldots, y_m\}$$

is a simplex in K.

We define a simplicial map $\sigma_i\colon L_i \to K$ so that

$$L_i \xrightarrow{\sigma} K \xrightarrow{\Delta} K^2$$

$\Delta \circ \sigma_i \sim_c \iota_{L_i}$.

Define

$$\sigma_i\big(([x],[y])\big) = \varphi_1^i\big(([x],[y])\big) \cup \varphi_1^m\big(([x],[y])\big) = \{x_1,\ldots,x_k,y_1,\ldots,y_m\} = \{x_1,\ldots,x_k,y_1,\ldots,y_m\}.$$

$$\Delta \circ \sigma_i\big(([x],[y])\big) = \big(\{x_1,\ldots,x_k,y_1,\ldots,y_m\},\{x_1,\ldots,x_k,y_1,\ldots,y_m\}\big).$$

$\iota_{L_i}\big(([x],[y])\big) = \big([x],[y]\big) = \big(\{x_1,\ldots,x_k\},\{y_1,\ldots,y_m\}\big)$. Thus, L_i is also a Farber subcomplex. □

There is a well-known inequality between topological complexity and LS-category of a topological space X. The same inequality holds for simplicial complexes (see [7, theorem 4.3]). In the following, we provide a proof in terms of contiguity distance.

Theorem 2.25 *For a simplicial complex K, we have* $\mathrm{scat}(K) \leq TC(K)$.

Proof Consider the following composition of maps

$$K \xrightarrow{i_1} K^2 \xrightarrow{p_1} K,$$

$$v \xrightarrow{i_1} (v,v_0) \xrightarrow{p_1} v,$$

and note that $p_1 \circ i_1 = id_K$. Similarly, consider the composition of maps

$$K \xrightarrow{i_1} K^2 \xrightarrow{p_2} K,$$

$$v \xrightarrow{i_1} (v,v_0) \xrightarrow{p_2} v_0,$$

and we have $p_2 \circ i_1 = c_{v_0}$. By Proposition 2.18,

$$\begin{aligned} & SD(p_1 \circ i_1, p_2 \circ i_2) \leq SD(p_1,p_2) = TC(K) \\ \Rightarrow\quad & SD(id_K, c_{v_0}) \leq TC(K) \\ \Rightarrow\quad & \mathrm{scat}(K) \leq TC(K). \end{aligned}$$

□

Corollary 2.26 *Let $\varphi,\psi\colon K \to K'$ be two simplicial maps (and K' be edge-path connected). Then $SD(\varphi,\psi) \leq scat(K)$.*

Proof If we take $K'' = K$, $\eta = id_K$ and $\eta' = c_{v_0}$ a constant map in Proposition 2.30, then the constant maps $\varphi \circ c_{v_0}$ and $\psi \circ c_{v_0}\colon K \to K'$ are in the same contiguity class since K' is edge-path connected. By Proposition 5 and Theorem 1, we have

$$SD(\varphi,\psi) = SD(\varphi \circ id_K, \psi \circ id_K) \leq SD(id_K, c_{v_0}) = scat(K).$$

□

Corollary 2.27 $TC(K) \leq scat(K^2)$.

Proof If we consider the projection maps $p_1, p_2 \colon K^2 \to K$, respectively, in Corollary 2.26, we have

$$SD(p_1, p_2) = TC(K) \leq scat(K^2).$$

□

Corollary 2.28 *Let $\varphi, \psi \colon K \to K'$ be two simplicial maps. Then $SD(\varphi, \psi) \leq TC(K')$.*

Proof Consider

$$K \prod K \xrightarrow{\varphi \prod \psi} K' \prod K' \overset{p_1}{\underset{p_2}{\rightrightarrows}} K'$$

where each p_i is a projection map for $i = 1, 2$. Then, using Proposition 2.5, we have

$$SD(\varphi, \psi) = SD\big(p_1 \circ (\varphi \prod \psi), p_2 \circ (\varphi \prod \psi)\big) \leq SD(p_1, p_2) = TC(K').$$

□

Remark 2.29 *Observe that Theorem 2.12 also follows from Corollaries 2.26 and 2.28.*

Proposition 2.30 *Let K, K', and K'' be simplicial complexes, $\eta, \eta' \colon K'' \to K$ and $\varphi, \psi \colon K \to K'$ be simplicial maps. If $\varphi \circ \eta' \sim \psi \circ \eta'$, then $SD(\varphi \circ \eta, \psi \circ \eta) \leq SD(\eta, \eta')$.*

Proof Let $SD(\eta, \eta') = n$. Then there exists a covering $\{L_0, L_1, \ldots, L_n\}$ for K'' such that $\eta\big|_{L_i} \sim \eta'\big|_{L_i}$ for $i = 1, 2, \ldots, n$. We have

$$\begin{gathered} \eta\big|_{L_i} \sim \eta'\big|_{L_i} \\ \varphi \circ \eta\big|_{L_i} \sim \varphi \circ \eta'\big|_{L_i}, \end{gathered}$$

$$\begin{gathered} \eta\big|_{L_i} \sim \eta'\big|_{L_i} \\ \psi \circ \eta\big|_{L_i} \sim \psi \circ \eta'\big|_{L_i}. \end{gathered}$$

Since $\varphi \circ \eta' \sim \psi \circ \eta'$, by the transitivity of $\sim$, we have $\varphi \circ \eta\big|_{L_i} \sim \psi \circ \eta\big|_{L_i}$, and this completes our proof. □

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